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# On a reverse Hardy–Hilbert-type integral inequality involving derivative functions of higher order

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**Abstract**

By means of the weight functions, the idea of introducing parameters and the technique of real analysis related to the beta and gamma functions, a new reverse Hardy–Hilbert-type integral inequality with the homogeneous kernel as $\frac{1}{(x+y)^{\lambda+m+n}}$ ($\lambda > 0$) involving two derivative functions of higher order is given. As applications, the equivalent statements of the best possible constant factor related to several parameters are considered, and some particular inequalities are obtained.

**MSC:** 26D15

**Keywords:** Weight function; Hardy–Hilbert-type integral inequality; Derivative function of higher order; Parameter; Beta function; Gamma function

## 1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the following Hardy–Hilbert inequality with the best possible constant factor $\pi / \sin(\frac{\pi}{p})$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

Suppose that $f(x), g(y) \geq 0$, $0 < \int_0^{\infty} f^p(x)\,dx < \infty$ and $0 < \int_0^{\infty} g^q(y)\,dy < \infty$. We have the integral analog of (1) named in the Hardy–Hilbert's integral inequality with the same best possible constant factor as follows (cf. [1], Theorem 316):

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}g^q(y)\,dy\right)^{\frac{1}{q}}. \tag{2}$$

Inequalities (1) and (2) play an important role in analysis and its applications (cf. [2–13]).

In 2006, by applying the Euler–Maclaurin summation formula, Krnic et al. [14] gave an extension of (1) with the kernel as $\frac{1}{(m+n)^{\lambda}}$ ($0 < \lambda \leq 4$). In 2019, by means of the result of

[14], Adiyasuren et al. [15] deduced an inequality involving the same kernel and two partial sums. In 2020, Mo et al. [16] gave an extension of (2) involving two upper limit functions. In 2016, Hong et al. [17] provided some equivalent statements of the extension of (1) with the best possible constant factor related to several parameters. Some other works may be consulted [18–23].

In this paper, following the way of [16] and [17], by means of the weight functions, the idea of introducing parameters and the technique of real analysis related to the beta and gamma functions, a new reverse Hardy–Hilbert-type integral inequality with the homogeneous kernel as $\frac{1}{(x+y)^{\lambda+m+n}}$ ($\lambda > 0$) involving two derivative functions of higher order is given. As applications, the equivalent statements of the best possible constant factor related to several parameters are considered, and some particular inequalities are obtained.

## 2 Some lemmas

In what follows, we suppose that $0 < p < 1$ ($q < 0$), $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda_i < \lambda$ ($i = 1,2$), $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$. $m,n \in \mathrm{N}_0 := \{0,1,\dots\}$, $f^{(i)}(t)$, $g^{(j)}(t)$ ($t > 0$) ($i = 0,1,\dots,m-1$; $j = 0,1,\dots,n-1$) are piecewise-smooth functions, and $f^{(i)}(0+) = g^{(j)}(0+) = 0$ ($i = 0,\dots,m-1$; $j = 0,\dots,n-1$),

$$f^{(m)}(u) = g^{(n)}(u) = o\left(e^{tu}\right) \quad (t > 0; u \to \infty),$$

$f^{(m)}(y), g^{(n)}(y) \geq 0$, such that

$$0 < \int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\left(f^{(m)}(x)\right)^q dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{q(1-\hat{\lambda}_2)-1}\left(g^{(n)}(y)\right)^q dy < \infty.$$

**Lemma 1** *For $t > 0$, $f(x) = f^{(0)}(x)$, $g(y) = g^{(0)}(y)$, we have the following expressions*:

$$\int_0^\infty e^{-tx} f(x)\,dx = \frac{1}{t^m}\int_0^\infty e^{-tx} f^{(m)}(x)\,dx, \tag{3}$$

$$\int_0^\infty e^{-ty} g(y)\,dy = \frac{1}{t^n}\int_0^\infty e^{-ty} g^{(n)}(y)\,dy. \tag{4}$$

*Proof* Since $f^{(i-1)}(0+) = 0$ ($i = 1,\dots,m$), on integration by parts, we have

$$\begin{aligned}\int_0^\infty e^{-tx} f^{(i)}(x)\,dx &= \int_0^\infty e^{-tx}\,df^{(i-1)}(x)\\ &= e^{-tx} f^{(i-1)}(x)\big|_0^\infty - \int_0^\infty f^{(i-1)}(x)\,de^{-tx}\\ &= \lim_{x\to\infty}\frac{f^{(i-1)}(x)}{e^{tx}} + t\int_0^\infty e^{-tx} f^{(i-1)}(x)\,dx.\end{aligned}$$

If $f^{(i-1)}(\infty) = \text{constant}$, then $\lim_{x\to\infty}\frac{f^{(i-1)}(x)}{e^{tx}} = 0$; if $f^{(i-1)}(\infty) = \infty$, then $\lim_{x\to\infty}\frac{f^{(i-1)}(x)}{e^{tx}} = \frac{1}{t}\lim_{x\to\infty}\frac{f^{(i)}(x)}{e^{tx}}$. Inductively, if there exist a $k_0 = \min_{k\in\{i-1,\dots,m-1\}}\{k; f^{(k)}(\infty) = \text{constant}\}$, then

$$\lim_{x\to\infty}\frac{f^{(i-1)}(x)}{e^{tx}}=\cdots=\frac{1}{t^{k_0-i+1}}\lim_{x\to\infty}\frac{f^{(k_0)}(x)}{e^{tx}}=0;$$

otherwise, for $f^{(m)}(x)=o(e^{tx})$ $(t>0; x\to\infty)$, we have

$$\lim_{x\to\infty}\frac{f^{(i-1)}(x)}{e^{tx}}=\cdots=\frac{1}{t^{m-i+1}}\lim_{x\to\infty}\frac{f^{(m)}(x)}{e^{tx}}=0.$$

It follows that

$$\int_0^\infty e^{-tx}f^{(i-1)}(x)\,dx=\frac{1}{t}\int_0^\infty e^{-tx}f^{(i)}(x)\,dx \quad (i=1,\ldots,m).$$

Hence, substitution of $i=1,\ldots,m$, we have (3). In the same way, we have (4).

The lemma is proved. □

**Lemma 2** *Define the following weight functions*:

$$\varpi(\lambda_2,x):=x^{\lambda-\lambda_2}\int_0^\infty\frac{t^{\lambda_2-1}}{(x+t)^\lambda}\,dt \quad (x\in\mathrm{R}_+), \tag{5}$$

$$\omega(\lambda_1,y):=y^{\lambda-\lambda_1}\int_0^\infty\frac{t^{\lambda_1-1}}{(t+y)^\lambda}\,dt \quad (y\in\mathrm{R}_+). \tag{6}$$

*We have the following expressions*:

$$\varpi(\lambda_2,x)=B(\lambda_2,\lambda-\lambda_2) \quad (x\in\mathrm{R}_+), \tag{7}$$

$$\omega(\lambda_1,y)=B(\lambda_1,\lambda-\lambda_1) \quad (y\in\mathrm{R}_+), \tag{8}$$

*where*, $B(u,v):=\int_0^\infty\frac{t^{u-1}}{(1+t)^{u+v}}\,dt$ $(u,v>0)$ *is the beta function* (*cf.* [24]).

*Proof* Setting $u=\frac{t}{x}$, we have

$$\varpi(\lambda_2,x)=x^{\lambda-\lambda_2}\int_0^\infty\frac{(ux)^{\lambda_2-1}}{(x+ux)^\lambda}x\,du=\int_0^\infty\frac{u^{\lambda_2-1}}{(1+u)^\lambda}\,du=B(\lambda_2,\lambda-\lambda_2),$$

namely, (7) follows. In the same way, we have (8).

The lemma is proved. □

Define the gamma function as follows (cf. [24]):

$$\Gamma(\alpha):=\int_0^\infty e^{-t}t^{\alpha-1}\,dt \quad (\alpha>0). \tag{9}$$

We have the following expression $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$ $(\alpha>0)$ and the formula related to the beta and gamma functions:

$$B(u,v)=\frac{1}{\Gamma(u+v)}\Gamma(u)\Gamma(v) \quad (u,v>0). \tag{10}$$

For $\lambda, x, y > 0$, by (9) we can obtain

$$\frac{1}{(x+y)^{\lambda+m+n}} = \frac{1}{\Gamma(\lambda+m+n)} \int_0^\infty t^{(\lambda+m+n)-1} e^{-(x+y)t}\, dt. \tag{11}$$

**Lemma 3** *We have the following reverse Hardy–Hilbert's integral inequality*:

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f^{(m)}(x) g^{(n)}(y)}{(x+y)^\lambda}\, dx\, dy &> B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1) \\ &\quad \times \left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1} \left(f^{(m)}(x)\right)^p dx\right]^{\frac{1}{p}} \\ &\quad \times \left[\int_0^\infty y^{q(1-\hat{\lambda}_2)-1} \left(g^{(n)}(y)\right)^q dy\right]^{\frac{1}{q}}. \end{aligned} \tag{12}$$

*Proof* By the reverse Hölder inequality (cf. [25]), we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f^{(m)}(x) g^{(n)}(y)}{(x+y)^\lambda}\, dx\, dy \\ &= \int_0^\infty \int_0^\infty \frac{1}{(x+y)^\lambda} \left[\frac{y^{(\lambda_2-1)/p}}{x^{(\lambda_1-1)/q}} f^{(m)}(x)\right] \left[\frac{x^{(\lambda_1-1)/q}}{y^{(\lambda_2-1)/p}} g^{(n)}(y)\right] dx\, dy \\ &\geq \left\{\int_0^\infty \left[\int_0^\infty \frac{1}{(x+y)^\lambda} \frac{y^{\lambda_2-1}\, dy}{x^{(\lambda_1-1)(p-1)}}\right] \left(f^{(m)}(x)\right)^p dx\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\int_0^\infty \left[\int_0^\infty \frac{1}{(x+y)^\lambda} \frac{x^{\lambda_1-1}\, dx}{y^{(\lambda_2-1)(q-1)}}\right] \left(g^{(n)}(y)\right)^q dy\right\}^{\frac{1}{q}} \\ &= \left[\int_0^\infty \varpi(\lambda_2, x) x^{p(1-\hat{\lambda}_1)-1} \left(f^{(m)}(x)\right)^p dx\right]^{\frac{1}{p}} \\ &\quad \times \left[\int_0^\infty \omega(\lambda_1, y) y^{q(1-\hat{\lambda}_2)-1} \left(g^{(n)}(y)\right)^q dy\right]^{\frac{1}{q}}. \end{aligned} \tag{13}$$

If (13) keeps the form of equality, then, there exist constants $A$ and $B$ such that they are not both zero and (cf. [25])

$$A \frac{y^{\lambda_2-1}}{x^{(\lambda_1-1)(p-1)}} \left(f^{(m)}(x)\right)^p = B \frac{x^{\lambda_1-1}}{y^{(\lambda_2-1)(q-1)}} \left(g^{(n)}(y)\right)^q \quad \text{a.e. in } (0,\infty) \times (0,\infty).$$

Assuming that $A \neq 0$, there exists a $y \in (0,\infty)$, such that

$$x^{p(1-\hat{\lambda}_1)-1} \left(f^{(m)}(x)\right)^p = \left[\frac{B}{A} y^{q(1-\lambda_2)} \left(g^{(n)}(y)\right)^q\right] x^{-1-(\lambda-\lambda_1-\lambda_2)} \quad \text{a.e. in } (0,\infty),$$

which contradicts the fact that $0 < \int_0^\infty x^{p(1-\hat{\lambda}_1)-1} (f^{(m)}(x))^p\, dx < \infty$. In fact, for $a = \lambda - \lambda_1 - \lambda_2 \in \mathbf{R}$, we have $\int_0^\infty x^{-1-a}\, dx = \infty$.

Then by (7), (8), and (13), we have (12).

The lemma is proved. □

## 3 Main results

**Theorem 1** *We have the following reverse Hardy–Hilbert-type integral inequality involving two derivative functions of higher order*:

$$\begin{aligned} I &:= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+m+n}}\,dx\,dy \\ &> \frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)} B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1) \\ &\quad \times \left[\int_0^\infty x^{p(1-\hat{\lambda}_1)-1}\left(f^{(m)}(x)\right)^p dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\hat{\lambda}_2)-1}\left(g^{(n)}(y)\right)^q dy\right]^{\frac{1}{q}}. \end{aligned} \tag{14}$$

*In particular, for* $\lambda_1 + \lambda_2 = \lambda$, (14) *reduces to*:

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+m+n}}\,dx\,dy \\ &> \frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)} B(\lambda_1, \lambda_2) \\ &\quad \times \left[\int_0^\infty x^{p(1-\lambda_1)-1}\left(f^{(m)}(x)\right)^p dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1}\left(g^{(n)}(y)\right)^q dy\right]^{\frac{1}{q}}, \end{aligned} \tag{15}$$

*where, the constant factor* $\frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)} B(\lambda_1, \lambda_2)$ *is the best possible. For* $m = n = 1$, *we have*:

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda+2}}\,dx\,dy > \frac{1}{\lambda(\lambda+1)} B(\lambda_1, \lambda_2) & \\ \times \left[\int_0^\infty x^{p(1-\lambda_1)-1} f'^{p}(x)\,dx\right]^{\frac{1}{p}} & \\ \times \left[\int_0^\infty y^{q(1-\lambda_2)-1} g'^{q}(y)\,dy\right]^{\frac{1}{q}}. & \end{aligned} \tag{16}$$

*Proof* By (11) and the Fubini theorem (cf. [26]), in view of (3) and (4), we have

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda+m+n)} \int_0^\infty \int_0^\infty f(x)g(y)\left[\int_0^\infty t^{(\lambda+m+n)-1} e^{-(x+y)t}\,dt\right] dx\,dy \\ &= \frac{1}{\Gamma(\lambda+m+n)} \int_0^\infty t^{(\lambda+m+n)-1} \left(\int_0^\infty e^{-xt} f(x)\,dx\right)\left(\int_0^\infty e^{-yt} g(y)\,dy\right) dt \\ &= \frac{1}{\Gamma(\lambda+m+n)} \int_0^\infty t^{(\lambda+m+n)-1} \left(\int_0^\infty t^{-m} e^{-xt} f^{(m)}(x)\,dx\right)\left(\int_0^\infty t^{-n} e^{-yt} g^{(n)}(y)\,dy\right) dt \\ &= \frac{1}{\Gamma(\lambda+m+n)} \int_0^\infty \int_0^\infty f^{(m)}(x) g^{(n)}(y)\left[\int_0^\infty t^{\lambda-1} e^{-(x+y)t}\,dt\right] dx\,dy \\ &= \frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)} \int_0^\infty \int_0^\infty \frac{f^{(m)}(x) g^{(n)}(y)}{(x+y)^\lambda}\,dx\,dy. \end{aligned} \tag{17}$$

Then by (12), we have (14).

When $\lambda_1 + \lambda_2 = \lambda$, $\hat{\lambda}_1 = \frac{\lambda_1}{p} + \frac{\lambda_1}{q} = \lambda_1$, $\hat{\lambda}_2 = \frac{\lambda_2}{q} + \frac{\lambda_2}{p} = \lambda_2$, (14) reduces to (15).

For any $0 < \varepsilon < \lambda_1 \min\{p, |q|\}$, we set the following functions:

$$\tilde{f}^{(m)}(x) := \begin{cases} 0, & 0 < x < 1, \\ \prod_{i=0}^{m-1}(\lambda_1 + i - \frac{\varepsilon}{p})x^{\lambda_1 - \frac{\varepsilon}{p} - 1}, & x \geq 1, \end{cases}$$

$$\tilde{g}^{(n)}(y) := \begin{cases} 0, & 0 < y < 1, \\ \prod_{j=0}^{n-1}(\lambda_2 + j - \frac{\varepsilon}{q})y^{\lambda_2 - \frac{\varepsilon}{q} - 1}, & y \geq 1, \end{cases}$$

$$\tilde{f}^{(k)}(x) := \int_0^x \left( \int_0^{t_{m-k}} \cdots \int_0^{t_2} \tilde{f}^{(m)}(t_1)\, dt_1 \cdots dt_{m-k-1} \right) dt_{m-k},$$

$$\tilde{g}^{(j)}(y) := \int_0^y \left( \int_0^{t_{n-j}} \cdots \int_0^{t_2} \tilde{g}^{(n)}(t_1)\, dt_1 \cdots dt_{n-j-1} \right) dt_{n-j},$$

where $\tilde{f}^{(m)}(u) = \tilde{g}^{(n)}(u) = o(e^{tu})$ $(t > 0;\ u \to \infty)$, $\tilde{f}^{(k)}(0^+) = \tilde{g}^{(j)}(0^+) = 0$ $(k = 0, \ldots, m-1;\ j = 0, \ldots, n-1)$. For $k = j = 0$, we have $\tilde{f}(x) = \tilde{g}(y) = 0$, $0 < x, y < 1$,

$$\begin{aligned}
\tilde{f}(x) &= \prod_{i=0}^{m-1}\left(\lambda_1 + i - \frac{\varepsilon}{p}\right) \int_1^x \left( \int_1^{t_m} \cdots \int_1^{t_2} t_1^{\lambda_1 - \frac{\varepsilon}{p} - 1}\, dt_1 \cdots dt_{m-1} \right) dt_m \\
&= x^{\lambda_1 - \frac{\varepsilon}{p} + m - 1} - O_1\left(x^{m-1}\right) \leq x^{\lambda_1 - \frac{\varepsilon}{p} + m - 1}, \quad x \geq 1, \\
\tilde{g}(y) &= \prod_{j=0}^{n-1}\left(\lambda_2 + j - \frac{\varepsilon}{q}\right) \int_1^y \left( \int_1^{t_n} \cdots \int_1^{t_2} t_1^{\lambda_2 - \frac{\varepsilon}{q} - 1}\, dt_1 \cdots dt_{n-1} \right) dt_n \\
&= y^{\lambda_2 - \frac{\varepsilon}{q} + n - 1} - O_2\left(y^{n-1}\right) \leq y^{\lambda_2 - \frac{\varepsilon}{q} + n - 1}, \quad y \geq 1,
\end{aligned}$$

where, for $m = n = 0$, $O_1(x^{m-1}) = O_2(y^{n-1}) = 0$; for $m, n \geq 1$, $O_1(x^{m-1})$ (resp. $O_2(y^{n-1})$) is a nonnegative polynomial of $m-1$ (resp. $n-1$)-order.

If there exists a constant $M(\geq \frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)} B(\lambda_1, \lambda_2))$, such that (15) is valid, when we replace $\frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)} B(\lambda_1, \lambda_2)$ by $M$, then in particular, we have

$$\begin{aligned}
\tilde{I} &:= \int_0^\infty \int_0^\infty \frac{\tilde{f}(x)\tilde{g}(y)}{(x+y)^{\lambda+m+n}}\, dx\, dy \\
&> M\left[\int_0^\infty x^{p(1-\lambda_1)-1}\left(\tilde{f}^{(m)}(x)\right)^p dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1}\left(\tilde{g}^{(n)}(y)\right)^q dy\right]^{\frac{1}{q}}.
\end{aligned} \tag{18}$$

We find that

$$\begin{aligned}
\tilde{J} &:= \left[\int_0^\infty x^{p(1-\lambda_1)-1}\left(\tilde{f}^{(m)}(x)\right)^p dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1}\left(\tilde{g}^{(n)}(y)\right)^q dy\right]^{\frac{1}{q}} \\
&= \prod_{i=0}^{m-1}\left(\lambda_1 - \frac{\varepsilon}{p} + i\right) \prod_{j=0}^{n-1}\left(\lambda_2 - \frac{\varepsilon}{q} + j\right) \left(\int_1^\infty x^{-\varepsilon-1}\, dx\right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\varepsilon-1}\, dy\right)^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon} \prod_{i=0}^{m-1}\left(\lambda_1 - \frac{\varepsilon}{p} + i\right) \prod_{j=0}^{n-1}\left(\lambda_2 - \frac{\varepsilon}{q} + j\right).
\end{aligned}$$

In view of the Fubini theorem (cf. [26]), we have

$$\begin{aligned}\tilde{I} &\le \int_1^\infty \left[\int_1^\infty \frac{y^{\lambda_2-\frac{\varepsilon}{q}+n-1}}{(x+y)^{\lambda+m+n}}\,dy\right] x^{\lambda_1-\frac{\varepsilon}{p}+m-1}\,dx = \int_1^\infty x^{-\varepsilon-1}\left[\int_{\frac{1}{x}}^\infty \frac{u^{\lambda_2-\frac{\varepsilon}{q}+n-1}}{(1+u)^{\lambda+m+n}}\,du\right]dx \\ &= \int_1^\infty x^{-\varepsilon-1}\left[\int_{\frac{1}{x}}^1 \frac{u^{\lambda_2-\frac{\varepsilon}{q}+n-1}}{(1+u)^{\lambda+m+n}}\,du\right]dx + \int_1^\infty x^{-\varepsilon-1}\left[\int_1^\infty \frac{u^{\lambda_2-\frac{\varepsilon}{q}+n-1}}{(1+u)^{\lambda+m+n}}\,du\right]dx \\ &= \int_0^1 \left(\int_{\frac{1}{u}}^\infty x^{-\varepsilon-1}\,dx\right)\frac{u^{\lambda_2-\frac{\varepsilon}{q}+n-1}}{(1+u)^{\lambda+m+n}}\,du + \frac{1}{\varepsilon}\int_1^\infty \frac{u^{\lambda_2-\frac{\varepsilon}{q}+n-1}}{(1+u)^{\lambda+m+n}}\,du \\ &= \frac{1}{\varepsilon}\left[\int_0^1 \frac{u^{\lambda_2+\frac{\varepsilon}{p}+n-1}}{(1+u)^{\lambda+m+n}}\,du + \int_1^\infty \frac{u^{\lambda_2-\frac{\varepsilon}{q}+n-1}}{(1+u)^{\lambda+m+n}}\,du\right].\end{aligned}$$

Then by (18), it follows that

$$\begin{aligned}&\int_0^1 \frac{u^{\lambda_2+\frac{\varepsilon}{p}+n-1}}{(1+u)^{\lambda+m+n}}\,du + \int_1^\infty \frac{u^{\lambda_2-\frac{\varepsilon}{q}+n-1}}{(1+u)^{\lambda+m+n}}\,du \ge \varepsilon\tilde{I} > \varepsilon M\tilde{J} \\ &\qquad = M\prod_{i=0}^{m-1}\left(\lambda_1-\frac{\varepsilon}{p}+i\right)\prod_{j=0}^{n-1}\left(\lambda_2-\frac{\varepsilon}{q}+j\right).\end{aligned}$$

Putting $\varepsilon \to 0^+$, in view of the continuity of the beta function, we have:

$$\begin{aligned}&\frac{\prod_{i=0}^{m-1}(\lambda_1+i)\prod_{j=0}^{n-1}(\lambda_2+j)}{\Gamma(\lambda+m+n)}\Gamma(\lambda)B(\lambda_1,\lambda_2) \\ &\quad = B(\lambda_1+m,\lambda_2+n) \ge M\prod_{i=0}^{m-1}(\lambda_1+i)\prod_{j=0}^{n-1}(\lambda_2+j).\end{aligned}$$

Namely, $\frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)}B(\lambda_1,\lambda_2) \ge M$. Hence, $M = \frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)}B(\lambda_1,\lambda_2)$ is the best possible constant of (15).

The theorem is proved. □

**Theorem 2** *If* $\lambda-\lambda_1-\lambda_2 \in (-p\lambda_1, p(\lambda-\lambda_1))$, *and the constant factor*

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)}B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$$

*in* (14) *is the best possible, then we have* $\lambda_1+\lambda_2=\lambda$.

*Proof* We have

$$\hat{\lambda}_1+\hat{\lambda}_2 = \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p} = \frac{\lambda}{p}+\frac{\lambda}{q} = \lambda.$$

For $\lambda-\lambda_1-\lambda_2 \in (-p\lambda_1, p(\lambda-\lambda_1))$, we find that

$$-p\lambda_1+\lambda_1 < \lambda-\lambda_2 < p(\lambda-\lambda_1)+\lambda_1$$

and then $0 < \hat{\lambda}_1 = \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q} < \lambda$, from which it follows that $0 < \hat{\lambda}_2 = \lambda-\hat{\lambda}_2 < \lambda$. Hence, we have $B(\hat{\lambda}_1,\hat{\lambda}_2) \in \mathrm{R}_+$.

By the reverse Hölder inequality (cf. [25]), we still have

$$\begin{aligned}
B(\hat{\lambda}_1,\hat{\lambda}_2) &= \int_0^{\infty} \frac{u^{\hat{\lambda}_1-1}}{(1+u)^{\lambda}}\,du = \int_0^{\infty} \frac{1}{(1+u)^{\lambda}} u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}\,du \\
&= \int_0^{\infty} \frac{1}{(1+u)^{\lambda}} \bigl(u^{\frac{\lambda-\lambda_2-1}{p}}\bigr)\bigl(u^{\frac{\lambda_1-1}{q}}\bigr)\,du \\
&\geq \left[\int_0^{\infty} \frac{u^{\lambda-\lambda_2-1}}{(1+u)^{\lambda}}\,du\right]^{\frac{1}{p}} \left[\int_0^{\infty} \frac{u^{\lambda_1-1}}{(1+u)^{\lambda}}\,du\right]^{\frac{1}{q}} \\
&= B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1). \qquad (19)
\end{aligned}$$

On substitution of $\lambda_i = \hat{\lambda}_i$ $(i = 1,2)$ in (15), we have

$$\begin{aligned}
\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{(x+y)^{\lambda+m+n}}\,dx\,dy &> \frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)} B(\hat{\lambda}_1,\hat{\lambda}_2) \\
&\quad \times \left[\int_0^{\infty} x^{p(1-\hat{\lambda}_1)-1}\bigl(f^{(m)}(x)\bigr)^p\,dx\right]^{\frac{1}{p}} \\
&\quad \times \left[\int_0^{\infty} y^{q(1-\hat{\lambda}_2)-1}\bigl(g^{(n)}(y)\bigr)^q\,dy\right]^{\frac{1}{q}}. \qquad (20)
\end{aligned}$$

Since $\frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)}B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$ in (14) is the best possible, we have the following inequality:

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)}B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \geq \frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)}B(\hat{\lambda}_1,\hat{\lambda}_2) \in \mathrm{R}_+,$$

namely, $B(\hat{\lambda}_1,\hat{\lambda}_2) \leq B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$.

Hence, (19) keeps the form of equality. Then (cf. [25]), there exist constants $A$ and $B$ such that they are not both zero, and $Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1}$ a.e. in $R_+$. Assuming that $A \neq 0$, we have $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ a.e. in $R_+$. It follows that $\lambda - \lambda_1 - \lambda_2 = 0$, and then $\lambda_1 + \lambda_2 = \lambda$.

The theorem is proved. □

**Theorem 3** *The following statements* (i), (ii), (iii), *and* (iv) *are equivalent*:

(i) *Both* $B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$ *and* $B(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})$ *are independent of* $p$, $q$;

(ii)

$$B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \geq B\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}\right); \qquad (21)$$

(iii) *if* $\lambda-\lambda_1-\lambda_2 \in (-p\lambda_1, p(\lambda-\lambda_1))$, *then* $\lambda_1+\lambda_2 = \lambda$;

(iv) *the constant factor* $\frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)}B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$ *in* (14) *is the best possible.*

*Proof* (i) ⇒ (ii). In view of (i) and the continuity of the beta function, we have

$$\begin{aligned}
&B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \\
&\quad = \lim_{q\to-\infty}\lim_{p\to1^-} B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) = B(\lambda_2,\lambda-\lambda_2),
\end{aligned}$$

$$B\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}\right)=\lim_{q\to-\infty}\lim_{p\to1^-}B\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}\right)$$
$$=B(\lambda_2,\lambda-\lambda_2).$$

Hence, we have (21).

(ii) ⇒ (iii). By (21), (19) keeps the form of equality. In view of the proof of Theorem 2, we have $\lambda_1+\lambda_2=\lambda$.

(iii) ⇒ (i). If $\lambda_1+\lambda_2=\lambda$, then

$$B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)=B(\lambda_1,\lambda_2),$$
$$B\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}\right)=B(\lambda_1,\lambda_2).$$

Hence, both $B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$ and $B(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})$ are independent of $p$, $q$.

(iii) ⇒ (iv). If $\lambda_1+\lambda_2=\lambda$, then by Theorem 1, the constant factor

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)}B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)\left(=\frac{\Gamma(\lambda)}{\Gamma(\lambda+m+n)}B(\lambda_1,\lambda_2)\right)$$

is the best possible in (14).

(iv) ⇒ (iii). By Theorem 2, if $\lambda-\lambda_1-\lambda_2\in(-p\lambda_1,p(\lambda-\lambda_1))$, then we have $\lambda_1+\lambda_2=\lambda$.

Therefore, statements (i), (ii), (iii), and (iv) are equivalent.

The theorem is proved. □

*Remark* 1 For $\lambda=1$, $\lambda_1=\frac{1}{r}$, $\lambda_2=\frac{1}{s}$ ($r>1$, $\frac{1}{r}+\frac{1}{s}=1$) in (15), we have

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{(x+y)^{1+m+n}}\,dx\,dy>\frac{\pi}{(m+n)!\sin(\pi/r)}\times\left[\int_0^\infty x^{\frac{p}{s}-1}\left(f^{(m)}(x)\right)^p\,dx\right]^{\frac{1}{p}}\times\left[\int_0^\infty y^{\frac{q}{r}-1}\left(g^{(n)}(y)\right)^q\,dy\right]^{\frac{1}{q}}.\tag{22}$$

In particular, for $r=s=2$, $m=n$, (22) reduces to

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{(x+y)^{1+2n}}\,dx\,dy>\frac{\pi}{(2n)!}\left[\int_0^\infty x^{\frac{p}{2}-1}\left(f^{(n)}(x)\right)^p\,dx\right]^{\frac{1}{p}}\times\left[\int_0^\infty y^{\frac{q}{2}-1}\left(g^{(n)}(y)\right)^q\,dy\right]^{\frac{1}{q}}.\tag{23}$$

The constant factors in the above inequalities are the best possible.

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**Availability of data and materials**
The data used to support the findings of this study are included within the article.

## Declarations

**Competing interests**
The authors declare no competing interests.

**Author contributions**
B.Y. carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. X.H. and C.H. participated in the design of the study and performed the numerical analysis. All authors reviewed the manuscript.

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