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Study on Hermite-Hadamard-type inequalities using a new generalized fractional integral operator

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Abstract

In this study, a new definition of the fractional integral operator is first proposed, which generalizes some well-known fractional integral operators. Then, by using this newly generalized fractional integral operator, we proved several new Hermite-Hadamard-type inequalities for convex functions. Finally, we provided some corollaries to show that the current results extend and enrich the previous results in the literature.

MSC: 26A09; 26A33; 26D10

Keywords: Fractional integral operator; Hermite-Hadamard-type inequality; Convex function

1 Introduction

Convexity plays a very important role in physics, statistics and engineering. In terms of inequality, convexity theory is also a powerful tool to prove many classical inequalities, which is closely related to the generation of new and useful inequalities.

Definition 1.1 ([1]) The function $f : [u, v] \to \mathbb{R}$ is said to be convex, if we have

$$f\bigl(\lambda t_1 + (1-\lambda)t_2\bigr) \le \lambda f(t_1) + (1-\lambda)f(t_2),$$

for all $t_1, t_2 \in [u, v]$ and $\lambda \in [0, 1]$.

The study of one of the celebrated inequalities for convex functions can be traced back to 1893. Hermite and Hadamard [2] proved the following result, which states that if $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function in I and $u, v \in I$, where $u < v$, then

$$f\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v f(\delta)\,d\delta \le \frac{f(u)+f(v)}{2}. \tag{1.1}$$

The inequality is known as Hermite-Hadamard inequality. Since then, the inequality (1.1) has attracted many mathematicians' attention. In particular, in recent decades, numerous generalizations, variants, and extensions of this inequality have been published based on the definition of convexity ([3–9]).

In the past decades, fractional calculus has gained remarkable popularity and importance due mainly to its application in diverse and widespread fields. As of late, it can be seen that popularize well-known integral inequalities using fractional integral operators has become an interesting topic.

First, let us recall the Riemann-Liouville fractional integral defined by [10], which will be further used in this paper.

Definition 1.2 Let $f \in L_1[u,v]$, the Riemann-Liouville integrals $I^{\alpha}_{u^+}f$ and $I^{\alpha}_{v^-}f$ of order α with $u \geq 0$ are defined by

$$I^{\alpha}_{u^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_u^x (x-\delta)^{\alpha-1}f(\delta)\,d\delta, \quad x > u, \tag{1.2}$$

$$I^{\alpha}_{v^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^v (\delta-x)^{\alpha-1}f(\delta)\,d\delta, \quad x < v. \tag{1.3}$$

Here, $\Gamma(\alpha)$ is the familiar Gamma function and $I^0_{u^+}f(x) = I^0_{v^-}f(x) = f(x)$.

In [11], Sarikaya first used the definitions of Riemann-Liouville integrals, and developed a new generalization of the inequality (1.1) as follows.

Theorem 1.1 *Let $f : [u,v] \to \mathbb{R}$ be a positive function with $u < v$ and $f \in L_1[u,v]$. If f is a convex function on $[u,v]$, then the following inequalities for fractional integrals with $\alpha > 0$ hold*:

$$f\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}\left[I^{\alpha}_{u^+}f(v) + I^{\alpha}_{v^-}f(u)\right] \leq \frac{f(u)+f(v)}{2}. \tag{1.4}$$

One year later, in [12], the same author obtained the analogous fractional Hermite-Hadamard-type inequality involving Riemann-Liouville integral.

Theorem 1.2 *Let $f : [u,v] \to \mathbb{R}$ be a positive function with $u < v$ and $f \in L_1[u,v]$. If f is a convex function on $[u,v]$, then the following inequalities for fractional integrals hold*:

$$f\left(\frac{u+v}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\left[I^{\alpha}_{(\frac{u+v}{2})^+}f(v) + I^{\alpha}_{(\frac{u+v}{2})^-}f(u)\right] \leq \frac{f(u)+f(v)}{2}, \tag{1.5}$$

where $\alpha > 0$.

Based on the above-mentioned two studies, the subject of fractional Hermite-Hadamard-type inequalities has received considerable attention in recent years. In fact, there are multiple definitions of fractional operators; Caputo, Caputo-Fabrizio, k-Riemann-Liouville, Katugampola, Conformable, Hadamard are just a few examples of fractional integral operators. The concept of each integral operator has its own conditions and rules. Therefore, it is of great significance to develop and explore Hermite-Hadamard inequality for several different types of convex functions via various fractional operators.

In 2017, Agarwal [13] obtained some Hermite-Hadamard type inequalities via generalized k-fractional integrals. Awan et al. [14] in (2018) by using harmonic convex functions derived some conformable fractional estimates for Hermite-Hadamard-type inequalities. Zhao [15] in (2020) employed s-convex functions to present Hermite-Hadamard-type inequalities using ψ-Riemann-Liouville fractional integrals. Later, Set [16] proved a new Hermite-Hadamard type inequalities for Atangana-Baleanu fractional integral operators based on a non-singular and non-local derivative operator. For more details, we refer the readers to [17–21].

Due to the diversity of fractional integral operators, it is unknown which integral operator is most suitable for studying Hermite-Hadamard type inequalities. The various classical fractional integral operators are also difficult to generalize and extend Hermite-Hadamard type inequalities. Therefore, the concept of using general fractional integral operators has been proposed by some researchers to meet the needs of modern mathematics. In [22], the author obtained the Hermite-Hadamard type inequalities for functions with certain conditions associated with the generalized fractional integral operators. Also, Ahmad et al. [23] in (2019), employing the fractional integral operators with exponential kernel, obtained some Hermite–Hadamard-type inequalities for convex functions. Recently, based on the generalized proportional fractional integral operators, Aljaaidi [24] in (2021) studied some new fractional integral Hermite-Hadamard inequalities. For more details, we refer the readers to [25–29].

We need to recall definitions of two generalized Riemann-Liouville fractional integrals and known results, which states that:

Definition 1.3 ([10]) Let $h:[u,v]\to\mathbb{R}$ be a positive monotone and increasing function on $[u,v]$, and $h'(\delta)$ is continuously differentiable on $[u,v]$. Then $I^{\alpha}_{u^+;h}w(x)$ and $I^{\alpha}_{v^-;h}w(x)$ fractional integrals of w with respect to the function h on $[u,v]$ of the order $\alpha>0$ are defined by

$$I^{\alpha}_{u^+;h}w(x)=\frac{1}{\Gamma(\alpha)}\int_u^x\frac{h'(\delta)w(\delta)}{[h(x)-h(\delta)]^{1-\alpha}}\,d\delta,\quad x>u, \tag{1.6}$$

$$I^{\alpha}_{v^-;h}w(x)=\frac{1}{\Gamma(\alpha)}\int_x^v\frac{h'(\delta)w(\delta)}{[h(\delta)-h(x)]^{1-\alpha}}\,d\delta,\quad x<v. \tag{1.7}$$

Observe that for $h(x)=x$, the fractional integrals (1.6) and (1.7) are reduced to Riemann-Liouville fractional integrals (1.2) and (1.3). On the other hand, for $h(x)=\ln x$, the fractional integrals (1.6) and (1.7) are reduced to Hadamard fractional integral.

In 2016, Jleli and Samet [25] proved that Hermite-Hamard type inequalities under the framework of integral operators (1.6) and (1.7). Besides, Budak [26] obtained the following integral inequalities of the Hermite-Hadamard type involving fractional integral operators (1.6) and (1.7).

Theorem 1.3 *Let $h:[u,v]\to\mathbb{R}$ be an increasing and positive monotone function on $[u,v]$ with $u<v$, $h'(\delta)$ being continuously differentiable on $[u,v]$ and $w:[u,v]\to\mathbb{R}$ is a positive and integrable function. If f is a convex function on $[u,v]$, then the following inequalities*

hold:

$$\left[I_{u^+h}^{\alpha}w(v)+I_{v^-h}^{\alpha}w(u)\right]f\left(\frac{u+v}{2}\right)\leq\frac{1}{2}\left[I_{u^+h}^{\alpha}wF(v)+I_{v^-h}^{\alpha}wF(u)\right]$$
$$\leq\frac{f(u)+f(v)}{2}\left[I_{u^+h}^{\alpha}w(v)+I_{v^-h}^{\alpha}w(u)\right],\quad(1.8)$$

where $F(\delta)=f(\delta)+\tilde{f}(\delta),\tilde{f}(\delta)=f(u+v-\delta)$, *for all* $\delta\in[u,v]$ *with* $\alpha>0$.

Later, Sarikaya et al. [27] defined a new fractional integral operator that can be degenerated to the fractional integral of Riemann, Riemann-Liuville, k-Riemann-Liouville, Katugampola and Hadamard, etc.

Definition 1.4 Let $f\in L[u,v]$, the generalized fractional integral operators $I_{u^+\phi}f$ and $I_{v^-\phi}f$ are defined by

$$I_{u^+\phi}f(x)=\int_u^x\frac{\phi(x-\delta)}{x-\delta}f(\delta)\,d\delta,\quad x>u,\quad(1.9)$$
$$I_{v^-\phi}f(x)=\int_x^v\frac{\phi(\delta-x)}{\delta-x}f(\delta)\,d\delta,\quad x<v,\quad(1.10)$$

where a function $\phi:[0,\infty)\to[0,\infty)$ satisfies the following conditions:

$$\int_0^1\frac{\phi(\delta)}{\delta}\,d\delta<\infty,\quad(1.11)$$
$$\frac{1}{T_1}\leq\frac{\phi(\theta)}{\phi(\xi)}\leq T_1,\quad\text{for }\frac{1}{2}\leq\frac{\theta}{\xi}\leq2,\quad(1.12)$$
$$\frac{\phi(\xi)}{\xi^2}\leq T_2\frac{\phi(\theta)}{\theta^2},\quad\text{for }\theta\leq\xi,\quad(1.13)$$
$$\left|\frac{\phi(\xi)}{\xi^2}-\frac{\phi(\theta)}{\theta^2}\right|\leq T_3|\xi-\theta|\frac{\phi(\xi)}{\xi^2},\quad\text{for }\frac{1}{2}\leq\frac{\theta}{\xi}\leq2,\quad(1.14)$$

where $T_1,T_2,T_3>0$ are independent of $\theta,\xi>0$. If $\phi(\xi)\xi^{\alpha}$ $(\alpha>0)$ is increasing and $\frac{\phi(\xi)}{\xi^{\beta}}$ $(\beta>0)$ is decreasing, then ϕ satisfies (1.11) to (1.14).

Later, some Hermite-Hadamard type inequalities were established under the framework of this fractional integral operator, as follows:

Theorem 1.4 ([27]) *Let* $f:[u,v]\to\mathbb{R}$ *be a convex function on* $[u,v]$ *with* $u<v$, *then the following inequalities for fractional integrals hold*:

$$f\left(\frac{u+v}{2}\right)\leq\frac{1}{2\Delta(1)}\left[I_{u^+\phi}f(v)+I_{v^-\phi}f(u)\right]\leq\frac{f(u)+f(v)}{2},\quad(1.15)$$

where $\Delta(1)=\int_0^1\frac{\varphi((v-u)\theta)}{\theta}\,d\theta$.

Theorem 1.5 ([28]) *Let* $g:[u,v]\to\mathbb{R}$ *be a positive and integrable function. If* $f:[u,v]\to\mathbb{R}$ *is a convex function on* $[u,v]$ *with* $u<v$. *Then the following inequalities for fractional*

integrals hold:

$$f\left(\frac{u+v}{2}\right) \le \frac{1}{2}\frac{[I_{u^+}{}_{\phi}gF(v) + I_{v^-}{}_{\phi}gF(u)]}{[I_{u^+}{}_{\phi}g(v) + I_{v^-}{}_{\phi}g(u)]} \le \frac{f(u)+f(v)}{2}, \tag{1.16}$$

where F is defined as in Theorem 1.3, for all $\delta \in [u,v]$.

It is known that the extension of the fractional Hermite-Hamard type inequalities can come from the new integral operator, and the obtained extension results can establish a new boundary for the fractional Hermite-Hamard type inequalities. Therefore, in this paper, the idea is to provide a more general integral operator and obtain some new fractional Hermite-Hadamard type inequalities.

This paper is motivated by the above argument and inspired by the combination of fractional integral operator in Sarikaya [28] and the fractional integral operator with respect to another function in Definition 1.3. In this paper, a more general definition of new generalized fractional integrals is first proposed (see Sect. 2) that can degenerate into the above two generalized fractional integrals. In addition, this new generalized fractional integral can be reduced to some well-known fractional integral operators. In terms of inequality, the new integral operator can be used to establish Hermite-Hamard type inequalities for various functions. The main aim is to establish some Hermite-Hadamard type inequalities for convex functions in the setting of new generalized integral operators. During the research, it was found that the obtained results can be regarded as an important extension and generalization of the previous works, which can be observed in the literature [11, 23, 25–29].

The remainder of the paper is organized as follows. In Sect. 2, a new definition of integral operator is introduced, and some classical special cases of this operator are obtained. The main results and some corollaries are given in Sect. 3. The two illustrative examples are discussed in Sect. 4. Finally, the results are summarized and new directions for future works are specified.

2 Generalized fractional integral operators

In this part, a new definition of the fractional integral is stated, which is extremely helpful for the following work. Let $h:[u,v]\to\mathbb{R}$ be an increasing and non-negative monotone function on $[u,v]$, where $h'(\delta)$ is continuously differentiable on $[u,v]$. The function g satisfies (1.11) to (1.14) and let $w:[u,v]\to\mathbb{R}$ be a positive and integrable function. Then ${}^{k}_{u^+}I^{\alpha}_{h\,g}w(x)$ and ${}^{k}_{v^-}I^{\alpha}_{h\,g}w(x)$ fractional integrals of w with respect to the function h on $[u,v]$ of order $\alpha>0$ are given by

$${}^{k}_{u^+}I^{\alpha}_{h\,g}w(x) = \frac{1}{k\Gamma_k(\alpha)}\int_u^x \frac{g(h(x)-h(\delta))}{h(x)-h(\delta)}h'(\delta)\big(h(x)-h(\delta)\big)^{\frac{\alpha}{k}-1}w(\delta)\,d\delta, \quad x>u, \tag{2.1}$$

and

$${}^{k}_{v^-}I^{\alpha}_{h\,g}w(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^v \frac{g(h(\delta)-h(x))}{h(\delta)-h(x)}h'(\delta)\big(h(\delta)-h(x)\big)^{\frac{\alpha}{k}-1}w(\delta)\,d\delta, \quad x<v. \tag{2.2}$$

The most outstanding feature of the generalized fractional integral is that it can be reduced to some types of fractional integrals, such as the Riemann-Liouville fractional integral, Hadamard fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integral, conformable fractional integral, and fractional integral with the exponential kernel, fractional integral with respect to another function, etc. It can be clearly seen that some special cases of integral operators (2.1) and (2.2) are as follows:

(1) If $h(\delta) = g(\delta) = \delta$ and $\alpha = k = 1$, then (2.1) and (2.2) are reduced to the usual Riemann integral, as shown below:

$$I_{u^+}f(x) = \int_u^x f(\delta)\,d\delta, \quad x > u,$$
$$I_{v^-}f(x) = \int_x^v f(\delta)\,d\delta, \quad x < v.$$

(2) If $g(\delta) = h(\delta) = \delta$ and $k = 1$, then (2.1) and (2.2) are reduced to the Definition 1.2.

(3) If $g(\delta) = h(\delta) = \delta$, then (2.1) and (2.2) are reduced to the k-Riemann-Liouville fractional integral, as shown below:

$$I^{\alpha}_{u^+k}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_u^x (x-\delta)^{\frac{\alpha}{k}-1} f(\delta)\,d\delta, \quad x > u,$$
$$I^{\alpha}_{v^-k}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^v (\delta-x)^{\frac{\alpha}{k}-1} f(\delta)\,d\delta, \quad x < v,$$

which is demonstrated by Mubeen in [30].

(4) If $h(\delta) = \delta$ and $\alpha = k = 1$, then (2.1) and (2.2) are reduced to the Definition 1.4.

(5) If $h(\delta) = \delta$, $\alpha = k = 1$, and $g(\delta) = \frac{\delta}{\alpha} e^{(-\frac{1-\alpha}{\alpha}\delta)}$, then (2.1) and (2.2) are reduced to the fractional integral with the exponential kernel, as shown below:

$$I^{\alpha}_{u}f(x) = \frac{1}{\alpha} \int_u^x e^{(-\frac{1-\alpha}{\alpha}(x-\delta))} f(\delta)\,d\delta, \quad x > u,$$
$$I^{\alpha}_{v}f(x) = \frac{1}{\alpha} \int_x^v e^{(-\frac{1-\alpha}{\alpha}(\delta-x))} f(\delta)\,d\delta, \quad x < v,$$

here, $\alpha \in (0,1)$.

(6) If take $k = 1$, and $g(\delta) = \delta$, then (2.1) and (2.2) are reduced to the Definition 1.3.

(7) If take $k = 1$, $g(\delta) = \delta$, and $h(\delta) = \ln\delta$, then (2.1) and (2.2) are reduced to the Hadamard fractional integral, as shown below:

$${}_HI^{\alpha}_{u^+}f(x) = \frac{1}{\Gamma(\alpha)} \int_u^x (\ln x - \ln\delta)^{\alpha-1} \frac{f(\delta)}{\delta}\,d\delta, \quad x > u,$$
$${}_HI^{\alpha}_{u^+}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^v (\ln\delta - \ln x)^{\alpha-1} \frac{f(\delta)}{\delta}\,d\delta, \quad x < v.$$

3 The new Hermite-Hadamard-type inequalities

In this section we provide some new Hermite-Hadamard-type inequalities by using generalized fractional integral operators. For simplicity, the following operator is denoted:

$$\begin{aligned} F^{\alpha}_{g,w,h}(m,n) = {} & \int_u^{\frac{u+v}{2}} \left(\int_{\delta}^{u+v-\delta} \frac{h'(\tau)w(\tau)g[h(n)-h(\tau)]}{|h(n)-h(\tau)|^{2-\frac{\alpha}{k}}}\,d\tau \right) |m-\delta|\,d\delta \\ & + \int_{\frac{u+v}{2}}^{v} \left(\int_{u+v-\delta}^{\delta} \frac{h'(\tau)w(\tau)g[h(n)-h(\tau)]}{|h(n)-h(\tau)|^{2-\frac{\alpha}{k}}}\,d\tau \right) |m-\delta|\,d\delta, \end{aligned} \tag{3.1}$$

for all $m, n \in [u, v]$ and

$$F^{\alpha}_{g,w,h}(v,u) = F^{\alpha}_{g,w,h}(u,u), \qquad F^{\alpha}_{g,w,h}(v,v) = F^{\alpha}_{g,w,h}(u,v). \tag{3.2}$$

Moreover, let $h(\delta) = \ln\delta$, $g(\delta) = \delta$ and $k = 1$, then

$$\begin{aligned} F^{\alpha}_{g,w,\ln}(m,n) = {} & \int_u^{\frac{u+v}{2}} \left(\int_\delta^{u+v-\delta} \left|\ln\frac{n}{\tau}\right|^{\alpha-1} w(\tau)\frac{d\tau}{\tau} \right) |m-\delta|\, d\delta \\ & + \int_{\frac{u+v}{2}}^{v} \left(\int_{u+v-\delta}^{\delta} \left|\ln\frac{n}{\tau}\right|^{\alpha-1} w(\tau)\frac{d\tau}{\tau} \right) |m-\delta|\, d\delta. \end{aligned} \tag{3.3}$$

Theorem 3.1 *Let $\alpha > 0$. Let $h : [u, v] \to \mathbb{R}$ be an increasing and non-negative monotone function on $[u, v]$, where $h'(\delta)$ is continuously differentiable on $[u, v]$. The function g satisfies (1.11) to (1.14) and let $w : [u, v] \to \mathbb{R}$ be positive and integrable function. If f is a convex function on $[u, v]$, then the following inequalities hold*:

$$\begin{aligned} f\left(\frac{u+v}{2}\right)\left[{}^{k}_{u^+}I^{\alpha}_{h\,g}w(v) + {}^{k}_{v^-}I^{\alpha}_{h\,g}w(u)\right] & \le \frac{1}{2}\left[{}^{k}_{u^+}I^{\alpha}_{h\,g}w(v)F(v) + {}^{k}_{v^-}I^{\alpha}_{h\,g}w(u)F(u)\right] \\ & \le \frac{f(u)+f(v)}{2}\left[{}^{k}_{u^+}I^{\alpha}_{h\,g}w(v) + {}^{k}_{v^-}I^{\alpha}_{h\,g}w(u)\right], \end{aligned} \tag{3.4}$$

where the mapping F is given as in (1.8).

Proof According to Definition 1.1,

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2},$$

for $a, b \in [u, v]$. Now, let $a = su + (1-s)v$ and $b = (1-s)u + sv$, for $s \in [0, 1]$. Then

$$f\left(\frac{u+v}{2}\right) \le \frac{1}{2} f\big(su + (1-s)v\big) + \frac{1}{2} f\big((1-s)u + sv\big). \tag{3.5}$$

Multiplying both sides of the (3.5) by

$$\frac{h'((1-s)u+sv)w((1-s)u+sv)g[h(v)-h((1-s)u+sv)]}{[h(v)-h((1-s)u+sv)]^{2-\frac{\alpha}{k}}},$$

and then integrating the resulting inequality with respect to s over [0,1] yields

$$\begin{aligned} & f\left(\frac{u+v}{2}\right) \int_0^1 \frac{h'((1-s)u+sv)w((1-s)u+sv)g[h(v)-h((1-s)u+sv)]}{[h(v)-h((1-s)u+sv)]^{2-\frac{\alpha}{k}}}\, ds \\ & \quad \le \frac{1}{2}\int_0^1 \frac{h'((1-s)u+sv)g[h(v)-h((1-s)u+sv)]}{[h(v)-h((1-s)u+sv)]^{2-\frac{\alpha}{k}}} w\big((1-s)u+sv\big) f\big((1-s)u+sv\big)\, ds \\ & \qquad + \frac{1}{2}\int_0^1 \frac{h'((1-s)u+sv)g[h(v)-h((1-s)u+sv)]}{[h(v)-h((1-s)u+sv)]^{2-\frac{\alpha}{k}}} \\ & \qquad \times w\big((1-s)u+sv\big) f\big(su+(1-s)v\big)\, ds, \end{aligned} \tag{3.6}$$

if a variable $r = (1-s)u + sv$ in (3.6) is changed, the following inequality holds

$$ {}_{u^+}^{k}I_{h\,g}^{\alpha} w(v) f\left(\frac{u+v}{2}\right) \le \frac{1}{2}\, {}_{u^+}^{k}I_{h\,g}^{\alpha} w(v)F(v). \tag{3.7}$$

Multiplying both sides of the (3.5) by

$$\frac{h'((1-s)u+sv)w((1-s)u+sv)g[h((1-s)u+sv))-h(u)]}{[h((1-s)u+sv)-h(u)]^{2-\frac{\alpha}{k}}},$$

and then integrating the resulting inequality with respect to s over [0,1] yields

$$ {}_{v^-}^{k}I_{h\,g}^{\alpha} w(u) f\left(\frac{u+v}{2}\right) \le \frac{1}{2}\, {}_{v^-}^{k}I_{h\,g}^{\alpha} w(u)F(u). \tag{3.8}$$

Adding the inequalities (3.7) and (3.8) results in the following

$$f\left(\frac{u+v}{2}\right)\left[{}_{u^+}^{k}I_{h\,g}^{\alpha} w(v) + {}_{v^-}^{k}I_{h\,g}^{\alpha} w(u)\right] \le \frac{1}{2}\left[{}_{u^+}^{k}I_{h\,g}^{\alpha} w(v)F(v) + {}_{v^-}^{k}I_{h\,g}^{\alpha} w(u)F(u)\right],$$

proving the first inequality asserted by Theorem 3.1.

To prove the second inequality of (3.4), the following is true according to the convexity of f.

$$f\big(su+(1-s)v\big) + f\big((1-s)u+sv\big) \le f(u) + f(v), \tag{3.9}$$

multiplying both sides of (3.9) by

$$\frac{h'((1-s)u+sv)w((1-s)u+sv)g[h(v)-h((1-s)u+sv)]}{[h(v)-h((1-s)u+sv)]^{2-\frac{\alpha}{k}}},$$

and then by using integration with respect to s over $[0,1]$ result in

$$\begin{aligned}
&\int_0^1 \frac{h'((1-s)u+sv)g[h(v)-h((1-s)u+sv)]}{[h(v)-h((1-s)u+sv)]^{2-\frac{\alpha}{k}}} w\big((1-s)u+sv\big) f\big(su+(1-s)v\big)\,ds \\
&\quad + \int_0^1 \frac{h'((1-s)u+sv))g[h(v)-h((1-s)u+sv)]}{[h(v)-h((1-s)u+sv)]^{2-\frac{\alpha}{k}}} w\big((1-s)u+sv\big) f\big((1-s)u+sv\big)\,ds \\
&\le \big[f(u)+f(v)\big]\int_0^1 \frac{h'((1-s)u+sv)g(h(v)-h((1-s)u+sv))}{[h(v)-h((1-s)u+sv)]^{2-\frac{\alpha}{k}}} w\big((1-s)u+sv\big)\,ds,
\end{aligned}$$

the above expression is simplified as follows,

$$ {}_{u^+}^{k}I_{h\,g}^{\alpha} w(v)F(v) \le \big[f(u)+f(v)\big]\, {}_{u^+}^{k}I_{h\,g}^{\alpha} w(v). \tag{3.10}$$

Multiplying both sides of (3.9) by

$$\frac{h'((1-s)u+sv)w((1-s)u+sv)g[h((1-s)u+sv))-h(u)]}{[h((1-s)u+sv)-h(u)]^{2-\frac{\alpha}{k}}},$$

and then by utilizing integration with respect to s over [0,1], we have

$$ {}_{\nu^-}^{k}I_{h\,g}^{\alpha}w(u)F(u) \le \left[f(u)+f(\nu)\right]{}_{\nu^-}^{k}I_{h\,g}^{\alpha}w(u). \tag{3.11}$$

According to (3.10) and (3.11),

$$\frac{1}{2}\left[{}_{u^+}^{k}I_{h\,g}^{\alpha}w(\nu)F(\nu)+{}_{\nu^-}^{k}I_{h\,g}^{\alpha}w(u)F(u)\right]\le\frac{f(u)+f(\nu)}{2}\left[{}_{u^+}^{k}I_{h\,g}^{\alpha}w(\nu)+{}_{\nu^-}^{k}I_{h\,g}^{\alpha}w(u)\right],$$

proving the second inequality is proved. □

Corollary 3.1 *Under the assumption of Theorem 3.1, if $k = 1$ and $h(\delta) = g(\delta) = \delta$, $w : [u,\nu] \to \mathbb{R}$ is positive, integrable, and symmetric to $\frac{u+\nu}{2}$. Then the inequality (3.4) is reduced to*:

$$f\left(\frac{u+\nu}{2}\right)\left[I_{u^+}^{\alpha}w(\nu)+I_{\nu^-}^{\alpha}w(u)\right]\le\left[I_{u^+}^{\alpha}wf(\nu)+L_{\nu^-}^{\alpha}wf(u)\right]$$
$$\le\frac{f(u)+f(\nu)}{2}\left[I_{u^+}^{\alpha}w(\nu)+I_{\nu^-}^{\alpha}w(u)\right],$$

where $I_{u^+}^{\alpha}w(\nu) = I_{\nu^-}^{\alpha}w(u) = \frac{1}{2}[I_{u^+}^{\alpha}w(\nu)+I_{\nu^-}^{\alpha}w(u)]$, Iscan proved the inequality in [29].

Corollary 3.2 *Under the assumption of Theorem 3.1, if $k = \alpha = 1$ and $h(\delta) = \delta$, then the inequality (3.4) becomes the inequality(1.15) of Theorem 1.4, according to Sarikaya in [27].*

Corollary 3.3 *Under the assumption of Theorem 3.1, if $k = 1$ and $g(\delta) = \delta$, then the inequality (3.4) is reduced to*:

$$f\left(\frac{u+\nu}{2}\right)\le\frac{\Gamma(\alpha+1)}{4[h(\nu)-h(u)]^{\alpha}}\left[I_{u^+\,h}^{\alpha}F(\nu)+I_{\nu^-\,h}^{\alpha}F(u)\right]\le\frac{f(u)+f(\nu)}{2},$$

where, the mapping F is given as in (1.8) and Jleli proved the inequalities in [25].

Corollary 3.4 *Under the assumption of Theorem 3.1, if $k = 1$ and $g(\delta) = \delta$, then the inequality (3.4) becomes inequality (1.8) of Theorem 1.3, according to Budak in [26].*

Corollary 3.5 *Under the assumption of Theorem 3.1, if $k = 1$, $g(\delta) = \delta$, and $h(\delta) = \ln\delta$, then the inequality (3.4) is reduced to*:

$$f\left(\frac{u+\nu}{2}\right)\left[{}_{H}I_{u^+}^{\alpha}w(\nu)+{}_{H}I_{\nu^-}^{\alpha}w(u)\right]\le\frac{1}{2}\left[{}_{H}I_{u^+}^{\alpha}wF(\nu)+{}_{H}I_{\nu^-}^{\alpha}wF(u)\right]$$
$$\le\frac{f(u)+f(\nu)}{2}\left[{}_{H}I_{u^+}^{\alpha}w(\nu)+{}_{H}I_{\nu^-}^{\alpha}w(u)\right],$$

where, ${}_{H}I_{u^+}^{\alpha}$ and ${}_{H}I_{\nu^-}^{\alpha}$ are given as in Sect. 2.

Theorem 3.2 *Let all conditions in the hypothesis of Theorem 3.1 hold true, then*

$$\frac{f(u)+f(\nu)}{2}\left[{}_{u^+}^{k}I_{h\,g}^{\alpha}w(\nu)+{}_{\nu^-}^{k}I_{h\,g}^{\alpha}w(u)\right]-\frac{1}{2}\left[{}_{u^+}^{k}I_{h\,g}^{\alpha}w(\nu)F(\nu)+{}_{\nu^-}^{k}I_{h\,g}^{\alpha}w(u)F(u)\right]$$
$$=\frac{1}{2k\Gamma_k(\alpha)}\int_u^{\nu}\Omega_{g,w,h}(\delta)f'(\delta)\,d\delta, \tag{3.12}$$

where,

$$\Omega_{g,w,h}(\delta) = \int_{u+v-\delta}^{\delta} \frac{h'(\tau)w(\tau)g(h(\tau)-h(u))}{h(\tau)-h(u)}\big[h(\tau)-h(u)\big]^{\frac{\alpha}{k}-1}\,d\tau + \int_{u+v-\delta}^{\delta} \frac{h'(\tau)w(\tau)g(h(v)-h(\tau))}{h(v)-h(\tau)}\big[h(v)-h(\tau)\big]^{\frac{\alpha}{k}-1}\,d\tau.$$

Proof With the analysis of integration by part, the following expression is obtained:

$$I = \int_u^v \Omega_{g,w,h}(\delta)f'(\delta)\,d\delta = \Omega_{g,w,h}(\delta)f(\delta)|_u^v - \int_u^v \Omega'_{g,w,h}(\delta)f(\delta)\,d\delta = I_1 - I_2. \tag{3.13}$$

Moreover, it can be concluded that:

$$\Omega_{g,w,h}(v) = -\Omega_{g,w,h}(u) = k\Gamma_k(\alpha)\big[{}^{k}_{u^+}I^{\alpha}_{h\,g}w(v) + {}^{k}_{v^-}I^{\alpha}_{h\,g}w(u)\big],$$

and

$$\begin{aligned}\Omega'_{g,w,h}(\delta) = {} & \frac{h'(\delta)w(\delta)g[h(\delta)-h(u)]}{h(\delta)-h(u)}\big[h(\delta)-h(u)\big]^{\frac{\alpha}{k}-1} \\ & + \frac{h'(u+v-\delta)f(u+v-\delta)g[h(u+v-\delta)-h(u)]}{h(u+v-\delta)-h(u)}\big[h(u+v-\delta)-h(u)\big]^{\frac{\alpha}{k}-1} \\ & + \frac{h'(\delta)w(\delta)g[h(v)-h(\delta)]}{h(v)-h(\delta)}\big[h(v)-h(\delta)\big]^{\frac{\alpha}{k}-1} \\ & + \frac{h'(u+v-\delta)w(u+v-\delta)g[h(v)-h(u+v-\delta)]}{h(v)-h(u+v-\delta)} \\ & \times \big[h(v)-h(u+v-\delta)\big]^{\frac{\alpha}{k}-1}.\end{aligned} \tag{3.14}$$

On the other hand, we have

$$I_1 = k\Gamma_k(\alpha)\big(f(u)+f(v)\big)\big[{}^{k}_{u^+}I^{\alpha}_{h\,g}w(v) + {}^{k}_{v^-}I^{\alpha}_{h\,g}w(u)\big], \tag{3.15}$$

and

$$\begin{aligned}I_2 = {} & \int_u^u \frac{h'(\delta)w(\delta)g[h(\delta)-h(u)]}{h(\delta)-h(u)}\big[h(\delta)-h(u)\big]^{\frac{\alpha}{k}-1}f(\delta)\,d\delta \\ & + \int_u^v \frac{h'(u+v-\delta)f(u+v-\delta)g[h(u+v-\delta)-h(h)]}{h(u+v-\delta)-h(u)}\big[h(u+v-\delta)-h(u)\big]^{\frac{\alpha}{k}-1}f(\delta)\,d\delta \\ & + \int_u^v \frac{h'(\delta)w(\delta)g[h(v)-h(\delta)]}{h(v)-h(\delta)}\big[h(v)-h(\delta)\big]^{\frac{\alpha}{k}-1}f(\delta)\,d\delta \\ & + \int_u^v \frac{h'(u+v-\delta)w(u+v-\delta)g[h(v)-h(u+v-\delta)]}{h(v)-h(u+v-\delta)}\big[h(v)-h(u+v-\delta)\big]^{\frac{\alpha}{k}-1}f(\delta)\,d\delta \\ = {} & k\Gamma_k(\alpha)\big[{}^{k}_{v^-}I^{\alpha}_{h\,g}w(u)f(u) + {}^{k}_{v^-}I^{\alpha}_{h\,g}w(u)\tilde{f}(u) + {}^{k}_{u^+}I^{\alpha}_{h}w(v)f(v) + {}^{k}_{u^+}I^{\alpha}_{h}w(v)\tilde{f}(v)\big] \\ = {} & k\Gamma_k(\alpha)\big[{}^{k}_{v^-}I^{\alpha}_{h\,g}w(u)F(u) + {}^{k}_{u^+}I^{\alpha}_{h}w(v)F(v)\big].\end{aligned} \tag{3.16}$$

If we plug (3.15) and (3.16) back into (3.13), the expected result is immediately obtained. □

Corollary 3.6 *Under the assumption of Theorem 3.2 with $\Omega_{g,w,h}(\delta)$, if $k = 1$ and $g(\delta) = \delta$, then*

$$\frac{f(u)+f(v)}{2}\left[I^{\alpha}_{u^+ h}w(v) + I^{\alpha}_{v^- h}w(u)\right] - \frac{1}{2}\left[I^{\alpha}_{u^+ h}w(v)F(v) + I^{\alpha}_{v^- h}w(u)F(u)\right]$$
$$= \frac{1}{2\Gamma(\alpha)}\int_u^v \Omega_{g,w,h}(\delta)f'(\delta)\,d\delta, \tag{3.17}$$

where,

$$\Omega_{g,w,h}(\delta) = \int_{u+v-\delta}^{\delta}\frac{h'(\tau)w(\tau)}{[h(\tau)-h(u)]^{1-\alpha}}\,d\tau + \int_{u+v-\delta}^{\delta}\frac{h'(\tau)w(\tau)}{[h(v)-h(\tau)]^{1-\alpha}}\,d\tau,$$

which is provided by Budak in [26].

Corollary 3.7 *Under the assumption of Theorem 3.2 with $\Omega_{g,k,h}(\delta)$, if $k = 1$ and $h(\delta) = g(\delta) = \delta$, then*

$$\frac{f(u)+f(v)}{2}\left[I^{\alpha}_{u^+}w(v) + I^{\alpha}_{v^-}w(u)\right] - \frac{1}{2}\left[I^{\alpha}_{u^+}w(v)F(v) + I^{\alpha}_{v^-}w(u)F(v)\right]$$
$$= \frac{1}{2\Gamma(\alpha)}\int_u^v \Omega_{g,w,h}(\delta)f'(\delta)\,d\delta,$$

where,

$$\Omega_{g,w,h}(\delta) = \int_{u+v-\delta}^{\delta}(\tau-u)^{\alpha-1}w(\tau)\,d\tau + \int_{u+v-\delta}^{\delta}(v-\tau)^{\alpha-1}w(\tau)\,d\tau.$$

Proof With the analysis of integration by part, we have

$$I = \int_u^v \Omega_{g,w,h}(\delta)f'(\delta)\,d\delta = \Omega_{g,w,h}(\delta)f(\delta)|_u^v - \int_u^v \Omega'_{g,w,h}(\delta)f(\delta)\,d\delta$$
$$= I_1 - I_2. \tag{3.18}$$

Hence,

$$\Omega_{g,w,h}(v) = -\Omega_{g,w,h}(u) = \Gamma(\alpha)\left[I^{\alpha}_{u^+}w(v) + I^{\alpha}_{v-}w(v)\right],$$

and

$$\Omega'_{g,w,h}(\delta) = w(\delta)(\delta-u)^{\alpha-1} + w(u+v-\delta)(v-\delta)^{\alpha-1}$$
$$+ w(\delta)(v-\delta)^{\alpha-1} + w(u+v-\delta)(\delta-u)^{\alpha-1}.$$

It can be concluded that,

$$I_1 = \Gamma(\alpha)\big(f(u)+f(v)\big)\left[I^{\alpha}_{u^+}w(v) + I^{\alpha}_{v-}w(u)\right], \tag{3.19}$$

and

$$\begin{aligned} I_2 &= \int_u^v (\delta-u)^{\alpha-1} w(\delta) f(\delta)\,d\delta + \int_u^v (v-\delta)^{\alpha-1} w(u+v-\delta) f(\delta)\,d\delta \\ &\quad + \int_u^v (v-\delta)^{\alpha-1} w(\delta) f(\delta)\,d\delta + \int_u^v (\delta-u)^{\alpha-1} w(u+v-\delta) f(\delta)\,d\delta \\ &= \Gamma(\alpha)\bigl[I_{v^-}^{\alpha} w(u)F(u) + I_{u^+}^{\alpha} w(v)F(v)\bigr]. \end{aligned} \tag{3.20}$$

The proof is completed by substituting (3.19) and (3.20) into (3.18). □

Corollary 3.8 *Under the assumption of Theorem 3.2 with $\Omega_{g,w,h}(\delta)$, if $k=1$, $g(\delta)=\delta$, and $h(\delta)=\ln\delta$, then*

$$\begin{aligned} &\frac{f(u)+f(v)}{2}\bigl[{}_HI_{u^+}^{\alpha} w(v) + {}_HI_{v^-}^{\alpha} w(u)\bigr] - \frac{1}{2}\bigl[{}_HI_{u^+}^{\alpha} w(v)F(v) + {}_HI_{v^-}^{\alpha} w(u)F(u)\bigr] \\ &\quad = \frac{1}{2\Gamma(\alpha)} \int_u^v \Omega_{g,w,h}(\delta) f'(\delta)\,d\delta, \end{aligned} \tag{3.21}$$

where,

$$\Omega_{g,w,h}(\delta) = \int_{u+v-\delta}^{\delta} (\ln\tau - \ln u)^{\alpha-1} w(\tau) \frac{d\tau}{\tau} + \int_{u+v-\delta}^{\delta} (\ln v - \ln\tau)^{\alpha-1} w(\tau) \frac{d\tau}{\tau}.$$

Proof With the analysis of integration by part, we have

$$\begin{aligned} I &= \int_u^v \Omega_{g,w,h}(\delta) f'(\delta)\,d\delta = \Omega_{g,w,h}(\delta) f(\delta)\big|_u^v - \int_u^v {\Omega'}_{g,w,h}(\delta) f(\delta)\,d\delta \\ &= I_1 - I_2. \end{aligned} \tag{3.22}$$

Thus,

$$\Omega_{g,w,h}(v) = -\Omega_{g,w,h}(u) = \Gamma(\alpha)\bigl[{}_HI_{u^+}^{\alpha} w(v) + {}_HI_{v^-}^{\alpha} w(v)\bigr],$$

and

$$\begin{aligned} {\Omega'}_{g,w,h}(\delta) &= (\ln\delta - \ln u)^{\alpha-1} \frac{w(\delta)}{\delta} + \bigl(\ln(u+v-\delta) - \ln u\bigr)^{\alpha-1} \frac{w(u+v-\delta)}{u+v-\delta} \\ &\quad + (\ln v - \ln\delta)^{\alpha-1} \frac{w(\delta)}{\delta} + \bigl(\ln(u+v-\delta) - \ln\delta\bigr)^{\alpha-1} \frac{w(u+v-\delta)}{u+v-\delta}. \end{aligned}$$

Owing to

$$I_1 = \Gamma(\alpha)\bigl(f(u)+f(v)\bigr)\bigl[{}_HI_{u^+}^{\alpha} w(v) + {}_HI_{v^-}^{\alpha} w(v)\bigr], \tag{3.23}$$

and

$$\begin{aligned} I_2 &= \int_u^v (\ln\delta - \ln u)^{\alpha-1} \frac{w(\delta)}{\delta} f(\delta)\,d\delta \\ &\quad + \int_u^v \bigl(\ln(u+v-\delta) - \ln u\bigr)^{\alpha-1} \frac{w(u+v-\delta)}{u+v-\delta} f(\delta)\,d\delta \end{aligned}$$

$$
\begin{aligned}
&+\int_u^\nu (\ln\nu-\ln\delta)^{\alpha-1}\frac{w(\delta)}{\delta}f(\delta)\,d\delta \\
&+\int_u^\nu \bigl(\ln(u+\nu-\delta)-\ln\delta\bigr)^{\alpha-1}\frac{w(u+\nu-\delta)}{u+\nu-\delta}f(\delta)\,d\delta \\
&=\Gamma(\alpha)\bigl[{}_HI^\alpha_{\nu^-}w(u)F(u)+{}_HI^\alpha_{u^+}w(\nu)F(\nu)\bigr].
\end{aligned}
\tag{3.24}
$$

By adding (3.23) and (3.24) into (3.22), which is completed the proof. □

Theorem 3.3 *Let all conditions in the hypothesis of Theorem 3.1 hold true. If $|f'|$ is a convex mapping on $[u,\nu]$ with $\alpha>0$, then the following inequality hold:*

$$
\begin{aligned}
&\left|\frac{f(u)+f(\nu)}{2}\bigl[{}^k_{u^+}I^\alpha_{h\,g}w(\nu)+{}^k_{\nu^-}I^\alpha_{h\,g}w(u)\bigr]-\frac{1}{2}\bigl[{}^k_{u^+}I^\alpha_{h\,g}w(\nu)F(\nu)+{}^k_{\nu^-}I^\alpha_{h\,g}w(u)F(u)\bigr]\right| \\
&\quad\le\frac{F^\alpha_{g,w,h}(u,u)+F^\alpha_{g,w,h}(\nu,\nu)}{2k\Gamma_k(\alpha)(\nu-u)}\bigl[|f'(u)|+|f'(\nu)|\bigr],
\end{aligned}
\tag{3.25}
$$

where the mapping $F^\alpha_{g,w,h}$ is defined as in (3.1).

Proof Using the use of Theorem 3.2 and the convexity of $|f'|$, we acquire

$$
\begin{aligned}
&\left|\frac{f(u)+f(\nu)}{2}\bigl[{}^k_{u^+}I^\alpha_{h g}w(\nu)+{}^k_{\nu^-}I^\alpha_{h g}w(u)\bigr]-\frac{1}{2}\bigl[{}^k_{u^+}I^\alpha_{h g}w(\nu)F(\nu)+{}^k_{\nu^-}I^\alpha_{h g}w(u)F(u)\bigr]\right| \\
&\quad\le\frac{1}{2k\Gamma_k(\alpha)}\int_u^\nu\bigl|\Omega_{g,w,h}(\delta)\bigr|\bigl|f'(\delta)\bigr|\,d\delta,
\end{aligned}
$$

and

$$
\bigl|f'(\delta)\bigr|=\left|f'\left(\frac{\nu-\delta}{\nu-u}u+\frac{\delta-u}{\nu-u}\nu\right)\right|\le\frac{\nu-\delta}{\nu-u}\bigl|f'(u)\bigr|+\frac{\delta-u}{\nu-u}\bigl|f'(\nu)\bigr|.
$$

Thus,

$$
\begin{aligned}
&\left|\frac{f(u)+f(\nu)}{2}\bigl[{}^k_{u^+}I^\alpha_{h g}w(\nu)+{}^k_{\nu^-}I^\alpha_{h g}w(u)\bigr]-\frac{1}{2}\bigl[{}^k_{u^+}I^\alpha_{h g}w(\nu)F(\nu)+{}^k_{\nu^-}I^\alpha_{h g}w(u)F(u)\bigr]\right| \\
&\quad\le\frac{|f'(u)|}{2k\Gamma_k(\alpha)(\nu-u)}\int_u^\nu\bigl|\Omega_{g,w,h}(\delta)\bigr|(\nu-\delta)\,d\delta \\
&\qquad+\frac{|f'(\nu)|}{2k\Gamma_k(\alpha)(\nu-u)}\int_u^\nu\bigl|\Omega_{g,w,h}(\delta)\bigr|(\delta-u)\,d\delta.
\end{aligned}
\tag{3.26}
$$

Since h is an monotonically increasing function and w is non-negative, the definition of g satisfies (1.11) to (1.14). Hence, $\Omega_{g,w,h}$ is a increasing function on $[u,\nu]$, we get

$$
\begin{aligned}
\Omega_{g,w,h}(u)={}&-\int_u^\nu\frac{h'(\tau)w(\tau)g[h(\tau)-h(u)]}{h(\tau)-h(u)}\bigl[h(\tau)-h(u)\bigr]^{\frac{\alpha}{k}-1}\,d\tau \\
&-\int_u^\nu\frac{h'(\tau)w(\tau)g[h(\nu)-h(\tau)]}{h(\nu)-h(\tau)}\bigl[h(\nu)-h(\tau)\bigr]^{\frac{\alpha}{k}-1}\,d\tau<0,
\end{aligned}
$$

and

$$
\Omega_{g,w,h}\left(\frac{u+\nu}{2}\right)=0.
$$

As a result, we get

$$\begin{cases} \Omega_{g,w,h}(\delta) \leq 0, & u \leq \delta \leq \frac{u+v}{2}, \\ \Omega_{g,w,h}(\delta) > 0, & \frac{u+v}{2} \leq \delta \leq v. \end{cases}$$

Therefore, it follows that

$$\begin{aligned} &\int_u^v \left|\Omega_{g,w,h}(\delta)\right|(v-u)\,d\delta \\ &= \int_u^{\frac{u+v}{2}} \left(\int_\delta^{u+v-\delta} \frac{h'(\tau)w(\tau)g[h(\tau)-h(u)]}{[h(\tau)-h(u)]}\big[h(\tau)-h(u)\big]^{\frac{\alpha}{k}-1}\,d\tau\right)(v-\delta)\,d\delta \\ &\quad + \int_{\frac{u+v}{2}}^v \left(\int_{u+v-\delta}^{\delta} \frac{h'(\tau)w(\tau)g[h(\tau)-h(u)]}{[h(\tau)-h(u)]}\big[h(\tau)-h(u)\big]^{\frac{\alpha}{k}-1}\,d\tau\right)(v-\delta)\,d\delta \\ &\quad + \int_u^{\frac{u+v}{2}} \left(\int_\delta^{u+v-\delta} \frac{h'(\tau)w(\tau)g[h(v)-h(\tau)]}{[h(v)-h(\tau)]}\big[h(v)-h(\tau)\big]^{\frac{\alpha}{k}-1}\,d\tau\right)(v-\delta)\,d\delta \\ &\quad + \int_{\frac{u+v}{2}}^v \left(\int_{u+v-\delta}^{\delta} \frac{h'(\tau)w(\tau)g[h(v)-h(\tau)]}{[h(v)-h(\tau)]}\big[h(v)-h(\tau)\big]^{\frac{\alpha}{k}-1}\,d\tau\right)(v-\delta)\,d\delta \\ &= F^{\alpha}_{g,w,h}(v,u) + F^{\alpha}_{g,w,h}(v,v). \end{aligned}$$

From (3.26), we acquire

$$\int_u^v \left|\Omega_{g,w,h}(\delta)\right|(v-\delta)\,d\delta = F^{\alpha}_{g,w,h}(u,u) + F^{\alpha}_{g,w,h}(v,v). \tag{3.27}$$

And similarly, it is clear that

$$\int_u^v \left|\Omega_{g,w,h}(\delta)\right|(\delta-u)\,d\delta = F^{\alpha}_{g,w,h}(u,u) + F^{\alpha}_{g,w,h}(v,v). \tag{3.28}$$

If we plug (3.27) and (3.28) back into (3.26), the proof is complete. □

Corollary 3.9 *Under the assumption of Theorem 3.3, if $k = 1$ and $h(\delta) = g(\delta) = \delta$, w is a symmetric to $\frac{(u+v)}{2}$, then*

$$\begin{aligned} &\left|\frac{f(u)+f(v)}{2}\big[I^{\alpha}_{u^+}w(v) + I^{\alpha}_{v^-}w(u)\big] - \big[I^{\alpha}_{u^+}w(v)F(v) + I^{\alpha}_{v^-}w(u)F(u)\big]\right| \\ &\quad \leq \frac{(v-u)^{\alpha+1}\|w\|_\infty}{\Gamma(\alpha+2)}\left(1-\frac{1}{2^\alpha}\right)\big[\left|f'(u)\right| + \left|f'(v)\right|\big], \end{aligned}$$

which is proved by Iscan in [29].

Corollary 3.10 *Under the assumption of Theorem 3.3, if $k = 1$, and $g(\delta) = \delta$, then*

$$\begin{aligned} &\left|\frac{f(u)+f(v)}{2}\big[I^{\alpha}_{u^+h}w(v) + I^{\alpha}_{v^-h}w(u)\big] - \frac{1}{2}\big[I^{\alpha}_{u^+h}w(v)F(v) + I^{\alpha}_{v^-h}w(u)F(u)\big]\right| \\ &\quad \leq \frac{F^{\alpha}_{w,h}(u,u) + F^{\alpha}_{w,h}(v,v)}{2\Gamma(\alpha)(v-u)}\big[\left|f'(u)\right| + \left|f'(v)\right|\big], \end{aligned}$$

Budak proved this inequality in [26].

Corollary 3.11 *Under the assumption of Theorem 3.3, if $k = 1$, $g(\delta) = \delta$, and $h(\delta) = \ln\delta$, then the following fractional Hadamard inequality holds*:

$$\begin{aligned}&\frac{f(u)+f(v)}{2}\left[{}_HI^{\alpha}_{u^+}w(v) + {}_HI^{\alpha}_{v^-}w(u)\right] - \frac{1}{2}\left[{}_HI^{\alpha}_{u^+}w(v)F(v) + {}_HI^{\alpha}_{v^-}w(u)F(u)\right]\\&\quad\leq \frac{F^{\alpha}_{w,\ln}(u,u) + F^{\alpha}_{w,\ln}(v,v)}{2\Gamma(\alpha)(v-u)}\left[\left|f'(u)\right| + \left|f'(v)\right|\right].\end{aligned}$$

Proof The proof of this result is quiet similar to that given earlier for the Theorem 3.3 and so is omitted. □

Corollary 3.12 *Under the assumption of Theorem 3.3, if $k = 1$ and $g(\delta) = \delta$, then the following inequality holds*:

$$\begin{aligned}&\left|\frac{f(u)+f(v)}{2} - \frac{\Gamma(\alpha+1)}{4[h(b)-h(a)]^{\alpha}}\left(I^{\alpha}_{u^+;h}F(v) + I^{\alpha}_{v^-;h}F(u)\right)\right|\\&\quad\leq \frac{L^{\alpha}_h(b,b) + L^{\alpha}_h(b,b) - L^{\alpha}_h(b,a) - L^{\alpha}_h(a,a)}{4[h(b)-h(a)]^{\alpha}(v-u)}\left(\left|f'(u)\right| + \left|f'(v)\right|\right),\end{aligned}$$

where the mapping L^{α}_h is defined by

$$L^{\alpha}_h(m,n) = \int_u^{\frac{u+v}{2}} \left|h(n)-h(\delta)\right|^{\alpha}|x-m|\,d\delta + \int_{\frac{u+v}{2}}^{v} \left|h(n)-h(\delta)\right|^{\alpha}|x-m|\,d\delta.$$

Jleli and Samet proved this inequality in [25].

Corollary 3.13 *Under the assumption of Theorem 3.3, if $k = 1$, $w(\delta) = 1$, and $g(\delta) = h(\delta) = \delta$, then the following inequality holds*:

$$\left|\frac{f(u)+f(v)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(I^{\alpha}_{u^+}f(v) + I^{\alpha}_{v^-}f(u)\right)\right| \leq \frac{(v-u)}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(f'(u)+f'(v)\right)$$

which is proved by Sarikaya in [11].

4 Application examples

In this section, we provide two examples to illustrate the applications of results obtained in the previous section.

Example 1 Let $0 < u < v$, we can get the following special inequalities

$$\frac{(u+v)^2}{4} \leq \frac{(u^2+uv+v^2)}{3} \leq \frac{u^2+v^2}{2}. \tag{4.1}$$

Proof Taking $\phi(x) = x^2$, $x \in (0,\infty)$ and $f = x^2$. So, according to the result in Corollary 3.2, we have

$$\frac{(u+v)^2}{4} \leq \frac{1}{(v-u)^2}\left(\int_u^v (v-\delta)\delta^2\,d\delta + \int_u^v (\delta-u)\delta^2\,d\delta\right) \leq \frac{u^2+v^2}{2}.$$

Simplifying the above inequalities, we can obtain (4.1). □

Example 2 Let $0 < u < v$, we can get the following special inequalities

$$\left|\frac{u^2+v^2}{2} - \frac{3\pi}{16}\left(\frac{2(u^2+v^2)}{3} - \frac{24(v-u)^2}{105}\right)\right| \le \frac{(4-\sqrt{2})(v^2-u^2)}{10}. \tag{4.2}$$

Proof Taking $\alpha = \frac{3}{2}$ and $f = x^2$. So, according to the result in Corollary 3.13, we have

$$\left|\frac{u^2+v^2}{2} - \frac{\Gamma(5/2)\Gamma(3/2)}{2(v-u)^{3/2}}\left(\int_u^v \left[(v-\delta)^{1/2} + (\delta-u)^{1/2}\right]\delta^2\, d\delta\right)\right| \le \frac{(4-\sqrt{2})(v^2-u^2)}{10}.$$

By computation, we can obtain (4.2). □

5 Conclusions

Convexity and fractional integral operators are the most important tools to deal with Hermite-Hadamard inequality problems. In the present paper, a more general fractional operator was proposed (see Sect. 2), which generalizes some well-known fractional integral operators, such as Riemann-Liouville fractional integral, Katugampola fractional integral, k-Riemann-Liouville fractional integral, Hadamard fractional integral, conformable fractional integral, fractional integral with the exponential kernel, and fractional integral with respect to another function. New inequalities of the Hermite-Hadamard type for convex function were established under the framework of this integral operator. Noteworthy, the obtained results can be reduced to some known results in the literature, see for instance [11, 23, 25–29].

In recent years, the fractional Hermite-Hadamard-type inequalities have been one of the hot research topics in fractional calculus theory. Therefore, this research is valuable and meaningful. Finally, it was pointed out that there is still some work to be done in the future, such as: discussing the Hermite-Hadamard-type inequalities for s-convex functions involving fractional integral (2.1)–(2.2), studying Hermite-Jensen-Mercer type inequalities via integral (2.1)–(2.2), and considering Hermite-Hadamard-type inequalities for MT-convex functions using integral (2.1)–(2.2).

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Author contributions
JN was a major contributor to writing the manuscript and funding acquisition. GC and HD made the formal analysis, writing–review, and editing. All authors read and approved the final manuscript.

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References

1. Peajcariaac, J.E., Tong, Y.L.: Convex Functions, Partial Orderings, and Statistical Applications. Academic Press, San Diego (1992)
2. Hadamard, J.: Etude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. J. Math. Pures Appl. **58**, 171–216 (1893)

3. Dragomir, S.S.: Hermite-Hadamard's type inequalities for operator convex functions. Appl. Math. Comput. **218**(3), 766–772 (2011)
4. Wang, J.R., Zhu, C., Zhou, Y.: New generalized Hermite-Hadamard type inequalities and applications to special means. J. Inequal. Appl. **2013**(1), 325 (2013)
5. Awan, M.U., Noor, M.A., Safdar, F., et al.: Hermite-Hadamard type inequalities with applications. Miskolc Math. Notes **21**(2), 593–614 (2020)
6. Budak, H., Khan, S., Ali, M.A., et al.: Refinements of quantum Hermite-Hadamard-type inequalities. Open Math. **19**(1), 724–734 (2021)
7. Sarikaya, M.Z.: On new Hermite Hadamard Fejér type integral inequalities. Stud. Univ. Babeş–Bolyai, Math. **57**(3), 377–386 (2012)
8. Awan, M.U., Talib, S., Kashuri, A., et al.: Some new (p,q)-Dragomir-Agarwal and Iyengar type integral inequalities and their applications. AIMS Math. **7**(4), 5728–5751 (2022)
9. Du, L., Xu, R.: Some new Pachpatte type inequalities on time scales and their applications. J. Math. Inequal. **6**(2), 229–240 (2012)
10. Kilbas, A.A., Marichev, O.I., Samko, S.G.: Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach, Switzerland (1993)
11. Sarikaya, M.Z., Set, E., Yaldiz, H., et al.: Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**(10), 2403–2407 (2013)
12. Sarikaya, M.Z., Yaldiz, H.: On generalized Hermite-Hadamard type integral inequalities involving Riemann-Liouville fractional integrals. Nihonkai Math. J. **25**(2), 93–104 (2014)
13. Agarwal, P., Jleli, M., Tomar, M.: Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals. J. Inequal. Appl. **2017**(1), 55 (2017)
14. Awan, M.U., Noor, M.A., Mihai, M.V., et al.: Inequalities via harmonic convex functions: conformable fractional calculus approach. J. Math. Inequal. **12**(1), 143–153 (2018)
15. Zhao, Y., Sang, H., Xiong, W., et al.: Hermite-Hadamard-type inequalities involving ψ-Riemann-Liouville fractional integrals via s-convex functions. J. Inequal. Appl. **2020**(1), 128 (2020)
16. Set, E., Akdemir, A.O., Karaoğlan, A., et al.: On new generalizations of Hermite-Hadamard type inequalities via Atangana-Baleanu fractional integral operators. Axioms **10**(3), 1–13 (2021)
17. Sahoo, S.K., Jarad, F., Kodamasingh, B., et al.: Hermite-Hadamard type inclusions via generalized Atangana-Baleanu fractional operator with application. AIMS Math. **7**(7), 12303–12321 (2022)
18. Mehreen, N., Anwar, M.: Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities for p-convex functions via conformable fractional integrals. J. Inequal. Appl. **2020**(1), 107 (2020)
19. Noor, M.A., Noor, K.I., Awan, M.U.: Generalized fractional Hermite-Hadamard inequalities. Miskolc Math. Notes **21**(2), 1001–1011 (2020)
20. Ali, M.A., Sarikaya, M.Z., Budak, H.: Fractional Hermite-Hadamard type inequalities for subadditive functions. Filomat **36**(11), 3715–3729 (2022)
21. Aljaaidi, T.A., Pachpatte, D.B.: Reverse Hermite-Hadamard's inequalities using ψ-fractional integral. Eng. App. Sci. Lett. **3**(4), 75–84 (2020)
22. Set, E., Choi, J., Celik, B.: Certain Hermite-Hadamard type inequalities involving generalized fractional integral operators. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **112**(4), 1539–1547 (2018)
23. Ahmad, B., Alsaedi, A., Kirane, M., et al.: Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals. J. Comput. Appl. Math. **353**(2019), 120–129 (2019)
24. Aljaaidi, T.A., Pachpatte, D.B., Abdeljawad, T., et al.: Generalized proportional fractional integral Hermite-Hadamard's inequalities. Adv. Differ. Equ. **2021**(1), 493 (2021)
25. Jleli, M., Samet, B.: On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function. J. Nonlinear Sci. Appl. **9**(3), 1252–1260 (2016)
26. Budak, H.: On Fejér type inequalities for convex mappings utilizing fractional integrals of a function with respect to another function. Results Math. **74**(1), 1–15 (2019)
27. Sarikaya, M.Z., Ertugral, F.: On the generalized Hermite-Hadamard inequalities. An. Univ. Craiova, Ser. Mat. Inform. **47**(1), 193–213 (2020)
28. Kashuri, A., Liko, R.: On Fejér type inequalities for convex mappings utilizing generalized fractional integrals. Appl. Appl. Math. **15**(1), 240–255 (2020)
29. Iscan, I.: Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals. Stud. Univ. Babeş–Bolyai, Math. **60**(3), 355–366 (2015)
30. Mubeen, S., Habibullah, G.M.: k-Fractional integrals and application. Int. J. Contemp. Math. Sci. **7**(2), 89–94 (2012)

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