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A reverse extended Hardy–Hilbert's inequality with parameters

Ricai Luo[1*], Bicheng Yang[2] and Xingshou Huang[1]

*Correspondence:
hcxylor@126.com
[1]School of Mathematics and Statistics, Hechi University, Yizhou, Guangxi 456300, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, by virtue of the symmetry principle, applying the techniques of real analysis and Euler–Maclaurin summation formula, we construct proper weight coefficients and use them to establish a reverse extended Hardy–Hilbert's inequality with multi-parameters. Then, we obtain the equivalent forms and some equivalent statements of the best possible constant factor related to several parameters. Finally, we illustrate how the obtained results can generate some new reverse Hardy–Hilbert-type inequalities.

MSC: 26D15; 26D10; 47A05

Keywords: Weight coefficient; Hardy–Hilbert's inequality; Equivalent statement; Parameter; Reverse

1 Introduction

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. We have the following well-known Hardy–Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

In 2006, by introducing parameters $\lambda_i \in (0,2]$ $(i = 1,2)$, $\lambda_1 + \lambda_2 = \lambda \in (0,4]$, using Euler–Maclaurin summation formula, an extension of (1) was provided by Krnić et al. [2] as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}, \tag{2}$$

where the constant factor $B(\lambda_1,\lambda_2)$ is the best possible.

$$B(u,v) = \int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}\,dt \quad (u,v > 0)$$

is the beta function. For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, inequality (2) reduces to (1); for $p = q = 2, \lambda_1 = \lambda_2 = \frac{\lambda}{2}$, (2) reduces to Yang's inequality in [3]. Recently, applying inequality (2), Adiyasuren et al. [4] gave a new Hardy–Hilbert's inequality with the kernel $\frac{1}{(m+n)^{\lambda}}$ involving two partial sums.

If $f(x), g(y) \geq 0$, $0 < \int_0^\infty f^p(x)\,dx < \infty$, and $0 < \int_0^\infty g^q(y)\,dy < \infty$, then we still have the following Hardy–Hilbert's integral inequality (cf. [1], Theorem 316):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)\,dy\right)^{\frac{1}{q}}, \tag{3}$$

where the constant factor $\pi/\sin(\frac{\pi}{p})$ is still the best possible. Inequalities (1), (2), and (3) with their extensions and reverses play an important role in the analysis and its applications (cf. [5–15]).

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [1], Theorem 351): If $K(t)$ $(t > 0)$ is a decreasing function, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi(s) = \int_0^\infty K(t)t^{s-1}\,dt < \infty$, $a_n \geq 0$, $0 < \sum_{n=1}^\infty a_n^p < \infty$, then we have

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^\infty K(nx)a_n\right)^p dx < \phi^p\left(\frac{1}{q}\right)\sum_{n=1}^\infty a_n^p. \tag{4}$$

In recent years, some new extensions of (4) with the reverses were provided by [16–23].

In 2016, by means of the technique of real analysis and the weight coefficients, Hong et al. [24] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to a few parameters. Other similar works about the extensions of (1), (2), (3), and (4) with the reverses were given by [25–32].

In this paper, following the way of [2, 24], by means of the weight coefficients, the idea of introduced parameters, the techniques of real analysis, and the Euler–Maclaurin summation formula, a new reverse of the extension of (1) with parameters as well as the equivalent forms are given. The equivalent statements of the best possible constant factor related to several parameters are obtained, and some particular inequalities are provided.

2 Some lemmas

In what follows, we suppose that $0 < p < 1$ $(q < 0)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in (0, \frac{5}{2}]$, $\lambda_i \in (0, \frac{5}{4}] \cap (0, \lambda)$ $(i = 1, 2)$, $a_m, b_n \geq 0$, $m, n \in \mathrm{N} = \{1, 2, \ldots\}$, such that

$$0 < \sum_{m=1}^\infty m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p < \infty, \quad \text{and} \quad 0 < \sum_{n=1}^\infty n^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}b_n^q < \infty.$$

Lemma 1 (cf. [5], (2.2.13)) *If $g(t)$ is a positive strictly decreasing function in $[m, \infty)$ $(m \in \mathrm{N})$ with $g(\infty) = 0$, $P_i(t)$ and B_i $(i \in \mathrm{N})$ are the Bernoulli functions and the Bernoulli numbers of i-order, then we have*

$$\int_m^\infty P_{2q-1}(t)g(t)\,dt = \varepsilon\frac{B_{2q}}{q}\left(\frac{1}{2^{2q}} - 1\right)g(m) \quad (0 < \varepsilon < 1; q = 1, 2, \ldots). \tag{5}$$

In particular, for $q = 1$, in view of $B_2 = \frac{1}{6}$, we have

$$-\frac{1}{8}g(m) < \int_m^\infty P_1(t)g(t)\,dt < 0. \tag{6}$$

Lemma 2 *Define the following weight coefficient*:

$$\varpi(\lambda_2, m) := m^{\lambda-\lambda_2} \sum_{n=1}^{\infty} \frac{n^{\lambda_2-1}}{m^\lambda + n^\lambda} \quad (m \in \mathrm{N}). \tag{7}$$

We have the following inequalities:

$$\frac{\pi}{\lambda \sin(\pi\lambda_2/\lambda)}\bigl(1-\theta_m(\lambda_2)\bigr) < \varpi(\lambda_2, m) < k_\lambda(\lambda_2) := \frac{\pi}{\lambda\sin(\pi\lambda_2/\lambda)} \quad (m \in \mathrm{N}), \tag{8}$$

where $\theta_m(\lambda_2)$ *is indicated by*

$$\theta_m(\lambda_2) := \frac{\sin(\pi\lambda_2/\lambda)}{\pi} \int_0^{\frac{1}{m^\lambda}} \frac{u^{(\lambda_2/\lambda)-1}}{1+u}\,du = O\left(\frac{1}{m^{\lambda_2}}\right) \in (0,1) \quad (m \in \mathrm{N}). \tag{9}$$

Proof For fixed $m \in \mathrm{N}$, we set the following function:

$$g(m,t) := \frac{t^{\lambda_2-1}}{m^\lambda + t^\lambda} \quad (t > 0).$$

By the use of Euler–Maclaurin summation formula (cf. [2, 3]), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} g(m,n) &= \int_1^\infty g(m,t)\,dt + \frac{1}{2}g(m,1) + \int_1^\infty P_1(t)g'(m,t)\,dt \\
&= \int_0^\infty g(m,t)\,dt - h(m), \\
h(m) &:= \int_0^1 g(m,t)\,dt - \frac{1}{2}g(m,1) - \int_1^\infty P_1(t)g'(m,t)\,dt.
\end{aligned}$$

We find $\frac{1}{2}g(m,1) = \frac{1}{2(m^\lambda+1)}$,

$$\begin{aligned}
\int_0^1 g(m,t)\,dt &= \int_0^1 \frac{t^{\lambda_2-1}}{m^\lambda + t^\lambda}\,dt = \frac{1}{\lambda_2}\int_0^1 \frac{dt^{\lambda_2}}{m^\lambda + t^\lambda} \\
&= \frac{1}{\lambda_2}\frac{t^{\lambda_2}}{m^\lambda + t^\lambda}\Bigg|_0^1 + \frac{\lambda}{\lambda_2}\int_0^1 \frac{t^{\lambda+\lambda_2-1}}{(m^\lambda + t^\lambda)^2}\,dt > \frac{1}{\lambda_2}\frac{1}{m^\lambda+1}.
\end{aligned}$$

We also obtain

$$\begin{aligned}
g'(m,t) &= \frac{(\lambda_2-1)t^{\lambda_2-2}}{m^\lambda + t^\lambda} - \frac{\lambda t^{\lambda+\lambda_2-2}}{(m^\lambda+t^\lambda)^2} = -\frac{(1-\lambda_2)t^{\lambda_2-2}}{m^\lambda+t^\lambda} - \frac{\lambda(m^\lambda+t^\lambda-m^\lambda)t^{\lambda_2-2}}{(m^\lambda+t^\lambda)^2} \\
&= -\frac{(\lambda+1-\lambda_2)t^{\lambda_2-2}}{m^\lambda+t^\lambda} + \frac{\lambda m^\lambda t^{\lambda_2-2}}{(m^\lambda+t^\lambda)^2}.
\end{aligned}$$

For $0 < \lambda_2 \le \frac{5}{4}$, $\lambda_2 < \lambda \le \frac{5}{2}$, it follows that

$$\frac{d}{dt}\left[\frac{t^{\lambda_2-2}}{(m^\lambda+t^\lambda)^i}\right] < 0 \quad (i = 1,2).$$

By (6), we obtain

$$(\lambda+1-\lambda_2)\int_1^\infty P_1(t)\frac{t^{\lambda_2-2}}{m^\lambda+t^\lambda}\,dt > -\frac{\lambda+1-\lambda_2}{8(m^\lambda+1)},$$
$$-m^\lambda\lambda\int_1^\infty P_1(t)\frac{t^{\lambda_2-2}}{(m^\lambda+t^\lambda)^2}\,dt > 0,$$

and then we find

$$-\int_1^\infty P_1(t)g'(m,t)\,dt > -\frac{\lambda+1-\lambda_2}{8(m^\lambda+1)}+0 = -\frac{\lambda+1-\lambda_2}{8(m^\lambda+1)}.$$

Hence, it follows that

$$\begin{aligned} h(m) &> \frac{1}{\lambda_2}\frac{1}{m^\lambda+1}-\frac{1}{2(m^\lambda+1)}-\frac{\lambda+1-\lambda_2}{8(m^\lambda+1)} = \frac{8-(5+\lambda)\lambda_2+\lambda_2^2}{8\lambda_2(m^\lambda+1)} \\ &\geq \frac{8-(5+\frac{5}{2})\lambda_2+\lambda_2^2}{8\lambda_2(m^\lambda+1)} = \frac{16-15\lambda_2+2\lambda_2^2}{16\lambda_2(m^\lambda+1)}. \end{aligned}$$

Since $(6-15\lambda_2+2\lambda_2^2)' = -15+4\lambda_2 < 0$ $(\lambda_2 \in (0,\frac{5}{4}])$, we have

$$h(m) > \frac{16-15(\frac{5}{4})+2(\frac{5}{4})^2}{16\lambda_2(m^\lambda+1)} = \frac{3}{128\lambda_2(m^\lambda+1)} > 0.$$

Setting $t = mu^{1/\lambda}$, we find

$$\begin{aligned} \varpi(\lambda_2,m) &= m^{\lambda-\lambda_2}\sum_{n=1}^\infty g(m,n) < m^{\lambda-\lambda_2}\int_0^\infty g(m,t)\,dt \\ &= m^{\lambda-\lambda_2}\int_0^\infty \frac{t^{\lambda_2-1}}{m^\lambda+t^\lambda}\,dt = \frac{1}{\lambda}\int_0^\infty \frac{u^{(\lambda_2/\lambda)-1}}{1+u}\,du = \frac{\pi}{\lambda\sin(\pi\lambda_2/\lambda)}. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} \sum_{n=1}^\infty g(m,n) &= \int_1^\infty g(m,t)\,dt + \frac{1}{2}g(m,1) + \int_1^\infty P_1(t)g'(m,t)\,dt \\ &= \int_1^\infty g(m,t)\,dt + H(m), \end{aligned}$$
$$H(m) := \frac{1}{2}g(m,1) + \int_1^\infty P_1(t)g'(m,t)\,dt.$$

Since we find $\frac{1}{2}g(m,1) = \frac{1}{2(m^\lambda+1)}$ and

$$g'(m,t) = -\frac{(\lambda+1-\lambda_2)t^{\lambda_2-2}}{m^\lambda+t^\lambda} + \frac{\lambda m^\lambda t^{\lambda_2-2}}{(m^\lambda+t^\lambda)^2},$$

in view of (6), we obtain

$$-(\lambda+1-\lambda_2)\int_1^\infty P_1(t)\frac{t^{\lambda_2-2}}{m^\lambda+t^\lambda}\,dt > 0, \quad \text{and}$$

$$\lambda m^{\lambda}\int_{1}^{\infty}P_{1}(t)\frac{t^{\lambda_{2}-2}}{(m^{\lambda}+t^{\lambda})^{2}}\,dt>-\frac{\lambda m^{\lambda}}{8(m^{\lambda}+1)^{2}}.$$

Hence, we have

$$H(m)>\frac{1}{2(m^{\lambda}+1)}-\frac{\lambda m^{\lambda}}{8(m^{\lambda}+1)^{2}}>\frac{4}{8(m^{\lambda}+1)}-\frac{5/2}{8(m^{\lambda}+1)}>0.$$

Setting $t=mu^{1/\lambda}$, we obtain

$$\begin{aligned}\varpi(\lambda_{2},m)&=m^{\lambda-\lambda_{2}}\sum_{n=1}^{\infty}g(m,n)>m^{\lambda-\lambda_{2}}\int_{1}^{\infty}g(m,t)\,dt\\&=m^{\lambda-\lambda_{2}}\int_{0}^{\infty}g(m,t)\,dt-m^{\lambda-\lambda_{2}}\int_{0}^{1}g(m,t)\,dt\\&=\frac{\pi}{\lambda\sin(\pi\lambda_{2}/\lambda)}\left[1-\frac{\lambda\sin(\pi\lambda_{2}/\lambda)}{\pi}m^{\lambda-\lambda_{2}}\int_{0}^{1}\frac{t^{\lambda_{2}-1}}{m^{\lambda}+t^{\lambda}}\,dt\right]\\&=\frac{\pi}{\lambda\sin(\pi\lambda_{2}/\lambda)}\bigl(1-\theta_{m}(\lambda_{2})\bigr)>0,\end{aligned}$$

where $\theta_{m}(\lambda_{2})=\frac{\sin(\pi\lambda_{2}/\lambda)}{\pi}\int_{0}^{\frac{1}{m^{\lambda}}}\frac{u^{(\lambda_{2}/\lambda)-1}}{1+u}\,du$. Since we find

$$0<\int_{0}^{\frac{1}{m^{\lambda}}}\frac{u^{(\lambda_{2}/\lambda)-1}}{1+u}\,du<\int_{0}^{\frac{1}{m^{\lambda}}}u^{(\lambda_{2}/\lambda)-1}\,du=\frac{\lambda}{\lambda_{2}m^{\lambda_{2}}},$$

namely, $\theta_{m}(\lambda_{2})=O(\frac{1}{m^{\lambda_{2}}})\in(0,1)$ $(m\in\mathrm{N})$. Therefore, inequalities (8) with (9) follow.

The lemma is proved. □

Lemma 3 *We have the following reverse extended Hardy–Hilbert's inequality with parameters*:

$$\begin{aligned}I={}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_{m}b_{n}}{m^{\lambda}+n^{\lambda}}>k_{\lambda}^{\frac{1}{p}}(\lambda_{2})k_{\lambda}^{\frac{1}{q}}(\lambda_{1})\\&\times\left\{\sum_{m=1}^{\infty}\bigl(1-\theta_{m}(\lambda_{2})\bigr)m^{p[1-(\frac{\lambda-\lambda_{2}}{p}+\frac{\lambda_{1}}{q})]-1}a_{m}^{p}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\lambda_{2}}{p}+\frac{\lambda-\lambda_{1}}{q})]-1}b_{n}^{q}\right\}^{\frac{1}{q}}.\end{aligned}\tag{10}$$

Proof In the same way, for $n\in\mathbf{N}$, we have the following inequalities for another weight coefficient:

$$\begin{aligned}&\omega(\lambda_{1},n):=n^{\lambda-\lambda_{1}}\sum_{m=1}^{\infty}\frac{m^{\lambda_{1}-1}}{m^{\lambda}+n^{\lambda}}\quad(n\in\mathrm{N}),\\&\frac{\pi}{\lambda\sin(\pi\lambda_{1}/\lambda)}\bigl(1-\theta_{n}(\lambda_{1})\bigr)<\omega(\lambda_{1},n)<k_{\lambda}(\lambda_{1})=\frac{\pi}{\lambda\sin(\pi\lambda_{1}/\lambda)},\end{aligned}\tag{11}$$

$$\theta_{n}(\lambda_{1})=\frac{\sin(\pi\lambda_{1}/\lambda)}{\pi}\int_{0}^{\frac{1}{n^{\lambda}}}\frac{u^{(\lambda_{1}/\lambda)-1}}{1+u}\,du=O\biggl(\frac{1}{n^{\lambda_{1}}}\biggr)\in(0,1)\quad(n\in\mathrm{N}).\tag{12}$$

By the reverse Hölder inequality (cf. [33]), we obtain

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^{\lambda}+n^{\lambda}}\left[\frac{n^{(\lambda_2-1)/p}}{m^{(\lambda_1-1)/q}}a_m\right]\left[\frac{m^{(\lambda_1-1)/q}}{n^{(\lambda_2-1)/p}}b_n\right]$$
$$\geq \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{\lambda}+n^{\lambda}}\frac{n^{\lambda_2-1}}{m^{(\lambda_1-1)(p-1)}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^{\lambda}+n^{\lambda}}\frac{m^{\lambda_1-1}}{n^{(\lambda_2-1)(q-1)}}b_n^q\right]^{\frac{1}{q}}$$
$$= \left\{\sum_{m=1}^{\infty}\varpi(\lambda_2,m)m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega(\lambda_1,n)n^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}b_n^q\right\}^{\frac{1}{q}}.$$

Then, by (8) and (11), in view of $0 < p < 1$ ($q < 0$), we have (10).

The lemma is proved. □

Remark 1 By (10), for $\lambda_1 + \lambda_2 = \lambda \in (0, \frac{5}{2}]$, $0 < \lambda_i \leq \frac{5}{4}$ ($i = 1, 2$), we find

$$0 < \sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p < \infty, \qquad 0 < \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q < \infty$$

and the following reverse inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^{\lambda}+n^{\lambda}} > k_{\lambda}(\lambda_1)\left[\sum_{m=1}^{\infty}\bigl(1-\theta_m(\lambda_2)\bigr)m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{13}$$

Lemma 4 *The constant factor* $k_{\lambda}(\lambda_1) = \frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}$ *in* (13) *is the best possible.*

Proof For any $0 < \varepsilon < p\lambda_1$, we set

$$\tilde{a}_m := m^{\lambda_1-\frac{\varepsilon}{p}-1}, \qquad \tilde{b}_n := n^{\lambda_2-\frac{\varepsilon}{q}-1} \quad (m, n \in \mathrm{N}).$$

If there exists a constant $M \geq k_{\lambda}(\lambda_1)$ such that (13) is valid when we replace $k_{\lambda}(\lambda_1)$ by M, then in particular, by substitution of $a_m = \tilde{a}_m$ and $b_n = \tilde{b}_n$ in (13), we have

$$\tilde{I} := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{m^{\lambda}+n^{\lambda}}$$
$$> M\left[\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{\lambda_2}}\right)\right)m^{p(1-\lambda_1)-1}\tilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}\tilde{b}_n^q\right]^{\frac{1}{q}}. \tag{14}$$

By (14) and the decreasingness property of series, for $0 < p < 1$, $q < 0$, we obtain

$$\begin{aligned}
\tilde{I} &> M\left[\sum_{m=1}^{\infty}\left(1 - O\left(\frac{1}{m^{\lambda_2}}\right)\right)m^{p(1-\lambda_1)-1}m^{p\lambda_1-\varepsilon-p}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}n^{q\lambda_2-\varepsilon-q}\right]^{\frac{1}{q}} \\
&= M\left[\sum_{m=1}^{\infty}\left(1 - O\left(\frac{1}{m^{\lambda_2}}\right)\right)m^{-\varepsilon-1}\right]^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{-\varepsilon-1}\right)^{\frac{1}{q}} \\
&= M\left(\sum_{m=1}^{\infty}m^{-\varepsilon-1} - \sum_{m=1}^{\infty}O\left(\frac{1}{m^{\lambda_2+\varepsilon+1}}\right)\right)^{\frac{1}{p}}\left(1 + \sum_{n=2}^{\infty}n^{-\varepsilon-1}\right)^{\frac{1}{q}} \\
&> M\left(\int_1^{\infty}x^{-\varepsilon-1}\,dx - O(1)\right)^{\frac{1}{p}}\left(1 + \int_1^{\infty}y^{-\varepsilon-1}\,dy\right)^{\frac{1}{q}} \\
&= \frac{M}{\varepsilon}\big(1 - \varepsilon O(1)\big)^{\frac{1}{p}}(\varepsilon + 1)^{\frac{1}{q}}.
\end{aligned}$$

By (11) and (12), setting $\hat{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p} \in (0, \frac{5}{4}) \cap (0, \lambda)$ $(0 < \hat{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p} = \lambda - \hat{\lambda}_1 < \lambda)$, we find

$$\begin{aligned}
\tilde{I} &= \sum_{n=1}^{\infty}\left[n^{(\lambda_2+\frac{\varepsilon}{p})}\sum_{m=1}^{\infty}\frac{1}{m^{\lambda} + n^{\lambda}}m^{(\lambda_1-\frac{\varepsilon}{p})-1}\right]n^{-\varepsilon-1} \\
&= \sum_{n=1}^{\infty}\omega(\hat{\lambda}_1, n)n^{-\varepsilon-1} < k_{\lambda}(\hat{\lambda}_1)\left(1 + \sum_{n=2}^{\infty}n^{-\varepsilon-1}\right) \\
&< k_{\lambda}(\hat{\lambda}_1)\left(1 + \int_1^{\infty}x^{-\varepsilon-1}\,dx\right) = \frac{1}{\varepsilon}k_{\lambda}(\hat{\lambda}_1)(\varepsilon + 1).
\end{aligned}$$

Then we have

$$k_{\lambda}\left(\lambda_1 - \frac{\varepsilon}{p}\right)(\varepsilon + 1) > \varepsilon\tilde{I} > M\big(1 - \varepsilon O(1)\big)^{\frac{1}{p}}(\varepsilon + 1)^{\frac{1}{q}}.$$

For $\varepsilon \to 0^{+}$, we find $k_{\lambda}(\lambda_1) \geq M$. Hence, $M = k_{\lambda}(\lambda_1)$ is the best possible constant factor of (13). The lemma is proved. □

Remark 2 Setting $\tilde{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\tilde{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we find

$$\tilde{\lambda}_1 + \tilde{\lambda}_2 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p} = \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda.$$

If we add the condition that $\lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1))$,then we can find

$$0 < \tilde{\lambda}_1 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} < \lambda, \qquad 0 < \tilde{\lambda}_2 = \lambda - \tilde{\lambda}_1 < \lambda;$$

if we add the condition that $\lambda - \lambda_1 - \lambda_2 \in [p(\lambda - \lambda_1 - \frac{5}{4}, p(\frac{5}{4} - \lambda_1)]$,then we have $\tilde{\lambda}_1, \tilde{\lambda}_2 \leq \frac{5}{4}$.

Then, with regard to the above assumptions, we can rewrite (13) as follows:

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^{\lambda}+n^{\lambda}}$$
$$> k_{\lambda}(\tilde{\lambda}_1)\left[\sum_{m=1}^{\infty}\left(1-\theta_m(\tilde{\lambda}_2)\right)m^{p(1-\tilde{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\tilde{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{15}$$

Lemma 5 *If the constant factor* $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ *in* (10) *is the best possible, then for*

$$\lambda-\lambda_1-\lambda_2 \in \left(-p\lambda_1, p(\lambda-\lambda_1)\right)\cap\left[p(\lambda-\lambda_1-\frac{5}{4}, p\left(\frac{5}{4}-\lambda_1\right)\right](\supset\{0\}),$$

we have $\lambda_1+\lambda_2=\lambda$.

Proof For $0<\tilde{\lambda}_1<\lambda$,we have $k_{\lambda}(\tilde{\lambda}_1)=\frac{\pi}{\lambda\sin(\pi\tilde{\lambda}_1/\lambda)}\in \mathrm{R}_+=(0,\infty)$.

If the constant factor $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ in (10) is the best possible, then for $\tilde{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\tilde{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, in view of the assumption and (15), we have the following inequality:

$$k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\ge k_{\lambda}(\tilde{\lambda}_1).$$

By the reverse Hölder inequality with weight (cf. [33]), we find

$$\begin{aligned}
k_{\lambda}(\tilde{\lambda}_1) &= k_{\lambda}\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)\\
&= \int_0^{\infty}\frac{1}{1+u^{\lambda}}u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}\,du = \int_0^{\infty}\frac{1}{1+u^{\lambda}}\left(u^{\frac{\lambda-\lambda_2-1}{p}}\right)\left(u^{\frac{\lambda_1-1}{q}}\right)du\\
&\ge \left(\int_0^{\infty}\frac{1}{1+u^{\lambda}}u^{\lambda-\lambda_2-1}\,du\right)^{\frac{1}{p}}\left(\int_0^{\infty}\frac{1}{1+u^{\lambda}}u^{\lambda_1-1}\,du\right)^{\frac{1}{q}}\\
&= \left(\int_0^{\infty}\frac{1}{1+v^{\lambda}}v^{\lambda_2-1}\,dv\right)^{\frac{1}{p}}\left(\int_0^{\infty}\frac{1}{1+u^{\lambda}}u^{\lambda_1-1}\,du\right)^{\frac{1}{q}}\\
&= k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1).
\end{aligned} \tag{16}$$

Hence, we find $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)=k_{\lambda}(\tilde{\lambda}_1)$, namely, (16) keeps the form of equality.

We observe that (16) keeps the form of equality if and only if there exist constants A and B such that they are not all zero and (cf. [33])

$$Au^{\lambda-\lambda_2-1}=Bu^{\lambda_1-1}\quad a.e. \text{ in } \mathrm{R}_+.$$

Assuming that $A\neq 0$, we have $u^{\lambda-\lambda_2-\lambda_1}=\frac{B}{A}$ *a.e.* in R_+, and then $\lambda-\lambda_2-\lambda_1=0$, namely, $\lambda_1+\lambda_2=\lambda$.

The lemma is proved. □

3 Main results

Theorem 1 *Inequality* (10) *is equivalent to the following inequalities*:

$$J := \left[\sum_{n=1}^{\infty} n^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}\left(\sum_{m=1}^{\infty}\frac{1}{m^{\lambda}+n^{\lambda}}a_m\right)^p\right]^{\frac{1}{p}}$$

$$> k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\left\{\sum_{m=1}^{\infty}\bigl(1-\theta_m(\lambda_2)\bigr)m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}}, \tag{17}$$

$$J_1 := \left[\sum_{m=1}^{\infty}\frac{m^{q(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})-1}}{(1-\theta_m(\lambda_2))^{q-1}}\left(\sum_{n=1}^{\infty}\frac{1}{m^{\lambda}+n^{\lambda}}b_n\right)^q\right]^{\frac{1}{q}}$$

$$> k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\left\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\right\}^{\frac{1}{q}}. \tag{18}$$

If the constant factor in (10) *is the best possible, then so is the constant factor in* (17) *and* (18).

Proof Suppose that (17) is valid. By the reverse Hölder inequality (cf. [33]), we have

$$I = \sum_{n=1}^{\infty}\left[n^{\frac{-1}{p}+(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}\sum_{m=1}^{\infty}\frac{1}{m^{\lambda}+n^{\lambda}}a_m\right]\left[n^{\frac{1}{p}-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}b_n\right]$$

$$\geq J\left\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\right\}^{\frac{1}{q}}. \tag{19}$$

Then by (17) we obtain (10).

On the other hand, assuming that (10) is valid, we set

$$b_n := n^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}\left(\sum_{m=1}^{\infty}\frac{1}{m^{\lambda}+n^{\lambda}}a_m\right)^{p-1}, \quad n \in \mathbf{N}.$$

If $J = \infty$, then (17) is naturally valid; if $J = 0$, then it is impossible to make (17) valid, namely, $J > 0$. Suppose that $0 < J < \infty$. By (10), we have

$$\sum_{n=1}^{\infty}n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q$$

$$= J^p = I > k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$$

$$\times\left\{\sum_{m=1}^{\infty}\bigl(1-\theta_m(\lambda_2)\bigr)m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\right\}^{\frac{1}{q}},$$

$$J = \left\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\right\}^{\frac{1}{p}}$$

$$> k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\left\{\sum_{m=1}^{\infty}\bigl(1-\theta_m(\lambda_2)\bigr)m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}},$$

namely, (17) follows, which is equivalent to (10).

Suppose that (18) is valid. By the reverse Hölder inequality, we have

$$\begin{aligned} I &= \sum_{m=1}^{\infty}\Big[\big(1-\theta_m(\lambda_2)\big)^{\frac{1}{p}} m^{\frac{1}{q}-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})} a_m\Big]\left[\frac{m^{\frac{-1}{q}+(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})}}{(1-\theta_m(\lambda_2))^{1/p}}\sum_{n=1}^{\infty}\frac{1}{m^\lambda+n^\lambda}b_n\right] \\ &\geq \left\{\sum_{m=1}^{\infty}\big(1-\theta_m(\lambda_2)\big) m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}} J_1. \end{aligned} \tag{20}$$

Then by (18) we obtain (10). On the other hand, assuming that (10) is valid, we set

$$a_m := \frac{m^{q(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})-1}}{(1-\theta_m(\lambda_2))^{q-1}}\left(\sum_{n=1}^{\infty}\frac{1}{m^\lambda+n^\lambda}b_n\right)^{q-1}, \quad m \in \mathrm{N}.$$

If $J_1 = \infty$, then (18) is naturally valid; if $J_1 = 0$, then it is impossible to make (18) valid, namely, $J_1 > 0$. Suppose that $0 < J_1 < \infty$. By (10), we have

$$\begin{aligned} &\sum_{m=1}^{\infty}\big(1-\theta_m(\lambda_2)\big) m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p \\ &\quad = J_1^q = I > k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \\ &\qquad \times \left\{\sum_{m=1}^{\infty}\big(1-\theta_m(\lambda_2)\big) m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{q}}, \\ &J_1 = \left\{\sum_{m=1}^{\infty}\big(1-\theta_m(\lambda_2)\big) m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{q}} \\ &\quad > k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{q}}, \end{aligned}$$

namely, (18) follows, which is equivalent to (10).

Hence, inequalities (10), (17), and (18) are equivalent.

If the constant factor in (10) is the best possible, then so is the constant factor in (17) and (18). Otherwise, by (19) (or (20)), we would reach a contradiction that the constant factor in (10) is not the best possible.

The theorem is proved. □

Theorem 2 *The following statements* (*i*), (*ii*), (*iii*), *and* (*iv*) *are equivalent*:

(i) *Both* $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ *and* $k_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ *are independent of* p, q;

(ii) $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$;

(iii) $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ *in* (10) *is the best possible constant factor*;

(iv) *If* $\lambda-\lambda_1-\lambda_2 \in (-p\lambda_1, p(\lambda-\lambda_1)) \cap [p(\lambda-\lambda_1-\frac{5}{4}, p(\frac{5}{4}-\lambda_1)]$, *then* $\lambda_1+\lambda_2=\lambda$.

If the statement (iv) follows, namely, $\lambda_1 + \lambda_2 = \lambda$, then we have (13) and the following equivalent inequalities with the best possible constant factor $k_\lambda(\lambda_1)$:

$$\left[\sum_{n=1}^{\infty} n^{p\lambda_2-1}\left(\sum_{m=1}^{\infty}\frac{1}{m^\lambda+n^\lambda}a_m\right)^p\right]^{\frac{1}{p}} > k_\lambda(\lambda_1)\left[\sum_{m=1}^{\infty}\bigl(1-\theta_m(\lambda_2)\bigr)m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}, \qquad (21)$$

$$\left[\sum_{m=1}^{\infty}\frac{m^{q\lambda_1-1}}{(1-\theta_m(\lambda_2))^{q-1}}\left(\sum_{n=1}^{\infty}\frac{1}{m^\lambda+n^\lambda}b_n\right)^q\right]^{\frac{1}{q}} > k_\lambda(\lambda_1)\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \qquad (22)$$

Proof (i) $\Rightarrow$ (ii). By (i), since $\tilde{\lambda}_1 + \tilde{\lambda}_2 = \lambda$, we have

$$k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = \lim_{p\to 1^-}\lim_{q\to-\infty} k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\lambda_2),$$

$$k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right) = k_\lambda\left(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}\right) = \lim_{p\to 1^-}\lim_{q\to-\infty} k_\lambda\left(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}\right) = k_\lambda(\lambda_2),$$

namely, $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$.

(ii) $\Rightarrow$ (iv). If $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$, then (16) keeps the form of equality. In view of the proof of Lemma 5, it follows that $\lambda_1 + \lambda_2 = \lambda$.

(iv) $\Rightarrow$ (i). If $\lambda_1 + \lambda_2 = \lambda$, then

$$k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right) = k_\lambda(\lambda_1),$$

which is independent of p, q. Hence, it follows that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv).

(iii) $\Rightarrow$ (iv). By the assumption and Lemma 5, we have $\lambda_1 + \lambda_2 = \lambda$.

(iv) $\Rightarrow$ (iii). By Lemma 4, for $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)(= k_\lambda(\lambda_1))$ is the best possible constant factor of (10). Therefore, we have (iii) $\Leftrightarrow$ (iv).

Hence, the statements (i), (ii), (iii), and (iv) are equivalent.

The theorem is proved. □

Remark 3 (i) For $\lambda_1 = \lambda_2 = \frac{\lambda}{2} \in (0, \frac{5}{4}]$ $(0 < \lambda \le \frac{5}{2})$ in (13), (21), and (22), we have the following equivalent inequalities with the best possible constant factor $\frac{\pi}{\lambda}$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda+n^\lambda} > \frac{\pi}{\lambda}\left[\sum_{m=1}^{\infty}\left(1-\theta_m\left(\frac{\lambda}{2}\right)\right)m^{p(1-\frac{\lambda}{2})-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{2})-1}b_n^q\right]^{\frac{1}{q}}, \qquad (23)$$

$$\left[\sum_{n=1}^{\infty}n^{\frac{p\lambda}{2}-1}\left(\sum_{m=1}^{\infty}\frac{1}{m^\lambda+n^\lambda}a_m\right)^p\right]^{\frac{1}{p}} > \frac{\pi}{\lambda}\left\{\sum_{m=1}^{\infty}\left(1-\theta_m\left(\frac{\lambda}{2}\right)\right)m^{p(1-\frac{\lambda}{2})-1}a_m^p\right]^{\frac{1}{p}}, \qquad (24)$$

$$\left[\sum_{m=1}^{\infty}\frac{m^{\frac{q\lambda}{2}-1}}{(1-\theta_m(\frac{\lambda}{2}))^{q-1}}\left(\sum_{n=1}^{\infty}\frac{1}{m^\lambda+n^\lambda}b_n\right)^q\right]^{\frac{1}{q}} > \frac{\pi}{\lambda}\left[\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{2})-1}b_n^q\right]^{\frac{1}{q}}. \qquad (25)$$

In particular, for $\lambda = \frac{5}{2}$,we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^{5/2}+n^{5/2}} > \frac{2\pi}{5}\left[\sum_{m=1}^{\infty}\left(1-\theta_m\left(\frac{5}{4}\right)\right)\frac{a_m^p}{m^{1+p/4}}\right]^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{n^{1+q/4}}\right)^{\frac{1}{q}}, \qquad (26)$$

$$\left[\sum_{n=1}^{\infty} n^{\frac{5p}{4}-1}\left(\sum_{m=1}^{\infty} \frac{1}{m^{5/2}+n^{5/2}} a_m\right)^p\right]^{\frac{1}{p}} > \frac{2\pi}{5}\left[\sum_{m=1}^{\infty}\left(1-\theta\left(\frac{5}{4}\right)\right)\frac{a_m^p}{m^{1+p/4}}\right]^{\frac{1}{p}}, \tag{27}$$

$$\left[\sum_{m=1}^{\infty} \frac{m^{\frac{5q}{4}-1}}{(1-\theta_m(\frac{5}{4}))^{q-1}}\left(\sum_{n=1}^{\infty} \frac{1}{m^{5/2}+n^{5/2}} b_n\right)^q\right]^{\frac{1}{q}} > \frac{2\pi}{5}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{n^{1+q/4}}\right)^{\frac{1}{q}}. \tag{28}$$

(ii) For $\lambda = 1$, $\lambda_1 = \frac{1}{r}$, $\lambda_2 = \frac{1}{s}$ ($r > 1$, $\frac{1}{r}+\frac{1}{s} = 1$) in (13), (21), and (22), we have the following equivalent inequalities with the best possible constant factor $\frac{\pi}{\sin(\pi/r)}$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} > \frac{\pi}{\sin(\pi/r)}\left[\sum_{m=1}^{\infty}\left(1-\hat{\theta}_m\left(\frac{1}{r}\right)\right)m^{\frac{p}{s}-1}a_m^p\right]^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{\frac{q}{r}-1}b_n^q\right)^{\frac{1}{q}}, \tag{29}$$

$$\left[\sum_{n=1}^{\infty} n^{\frac{p}{s}-1}\left(\sum_{m=1}^{\infty} \frac{1}{m+n} a_m\right)^p\right]^{\frac{1}{p}} > \frac{\pi}{\sin(\pi/r)}\left[\sum_{m=1}^{\infty}\left(1-\hat{\theta}_m\left(\frac{1}{s}\right)\right)m^{\frac{p}{s}-1}a_m^p\right]^{\frac{1}{p}}, \tag{30}$$

$$\left[\sum_{m=1}^{\infty} \frac{m^{\frac{q}{r}-1}}{(1-\hat{\theta}_m(\frac{1}{s}))^{q-1}}\left(\sum_{n=1}^{\infty} \frac{1}{m+n} b_n\right)^q\right]^{\frac{1}{q}} > \frac{\pi}{\sin(\pi/r)}\left(\sum_{n=1}^{\infty} n^{\frac{q}{r}-1}b_n^q\right)^{\frac{1}{q}}, \tag{31}$$

where $\hat{\theta}_m(\frac{1}{s}) := \frac{\sin(\pi/s)}{\pi}\int_0^{\frac{1}{m}} \frac{u^{-1/r}}{1+u}\,du = O(\frac{1}{m^{1/s}}) \in (0,1)$ ($m \in \mathrm{N}$).

Inequality (29) is a reverse of (1).

4 Conclusions

In this paper, by virtue of the symmetry principle, applying the techniques of real analysis and Euler–Maclaurin summation formula, we construct proper weight coefficients and use them to establish a reverse extended Hardy–Hilbert's inequality with multi-parameters in Lemma 3. Then, we obtain the equivalent forms and some equivalent statements of the best possible constant factor related to several parameters in Theorem 1 and Theorem 2. Finally, we illustrate how the obtained results can generate some new reverse Hardy–Hilbert-type inequalities in Remark 3. The lemmas and theorems provide an extensive account of this type of reverse inequalities.

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Availability of data and materials
The data used to support the findings of this study are included within the article.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author contributions
BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. RL and XH participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Hechi University, Yizhou, Guangxi 456300, P.R. China. [2]Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 51003, China.

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