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Some inequalities related to Csiszár divergence via diamond integral on time scales

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Abstract

In this paper, Csiszár f-divergence via diamond integral is introduced and some inequalities related to Csiszár f-divergence involving diamond integrals are presented. Some examples are presented for different divergence measures by fixing time scales. Some divergence measures are estimated in terms of logarithmic, identric, geometric, and arithmetic means. The obtained results generalize some known results in the literature and provide new bounds for some divergence measures in q-calculus.

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1 Introduction

Well over a century ago, measures were developed to estimate the distance between two probability distributions. Divergence measures are important in many statistical inference and data processing problems, such as estimation, compression and classification. For example, a group of biologists have visited the immense outer-space and observed that some space worms have variable number of teeth. Now they want to post this information back to Earth. But posting information from space to Earth is high in price. So they need to send their observations with a minimum amount of information. An efficient way is to convert their observations to a probability distribution.

The f-divergence is the distance in between two probability distribution by making an average value, which is weighted by a specified function. Some special cases of which are K–L divergence, Hellinger distance, Bhattacharyya discrimination, χ^2-divergence, and triangular distance. Anwar et al. [7] estimated the difference between the two sides of the relevant f-divergence and Shannon's inequality. Khan et al. [15] proposed new bounds for Csiszár and relevant divergences with the help of Jensen–Mercer's inequality. They also obtained several results for Zipf–Mandelbrot entropy. In [14], the authors have presented new findings for the Shannon and Zipf–Mandelbrot entropies. They have also estimated different bounds for these entropies.

The aim behind the mathematical theory of time scales is to merge continuous and discrete analysis presented by S. Hilger in 1988 (see [8, 9]). This theory has developed very rapidly in last three decades. Many authors have established time scale versions of integral inequalities. Ansari et al. [1] presented some inequalities for Csiszár divergence between two probability measures for delta integrals on time scales. Ansari et al. [1–3, 6] provided estimation of divergence measures for delta integrals via weighted Jensen inequality, Taylor's polynomial, Green's function, and Fink's identity. In [4], the authors have obtained new entropic bounds via delta integrals using Hermite interpolating polynomial.

There are many significant inequalities which have been proved with the help of convex functions. In [12], authors have provided the f-divergence functional given as follows:

Let $f:\mathbb{R}^+ \to (0,\infty)$ be a convex function. If $\tilde{\mathbf{x}} = (x_1, x_2, \ldots, x_n)$ and $\tilde{\mathbf{y}} = (y_1, y_2, \ldots, y_n)$ are such that $\sum_{j=1}^n x_j = 1$ and $\sum_{j=1}^n y_j = 1$, then

$$I_f(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) := \sum_{j=1}^n y_j f\left(\frac{x_j}{y_j}\right)$$

with $f(0) := \lim_{\delta\to 0^+} f(\delta)$, $0f(\frac{0}{0}) := 0$, and $0f(\frac{c}{0}) := \lim_{\delta\to 0^+} \delta f(\frac{c}{\delta})$, $c > 0$ is f-divergence functional.

The Csiszár's f-divergence can be used to find the difference between two probability densities.

In this study the flow of work is given as follows: In Sect. 2, the mathematical theory of time scales is presented. Next, in Sect. 3, bounds for Csiszár divergence via diamond integrals are presented. In order to illustrate the theoretical results, some examples are given for some fixed time scales. Lastly, in Sect. 4, bounds of some divergence measures are estimated in terms of special means.

2 Preliminaries

Now we introduce some basic definitions and results related to the mathematical theory of time scales. A nonempty closed subset of real numbers is called a time scale, denoted by $\mathbb{T}$. For example, Cantor set, $\mathbb{N}$, and $\mathbb{R}$. Furthermore, readers are referred to [8] for some essentials on time scales, including continuity and differentiability.

Definition 1 (Delta integral [8, Definition 1.71]) A mapping $H:\mathbb{T} \to (-\infty,\infty)$ is called the delta antiderivative of $h:[b_1,b_2]_{\mathbb{T}} = [b_1,b_2]\cap\mathbb{T} \to (-\infty,\infty)$ if $H^{\triangle}(\zeta) = h(\zeta)$ holds true $\forall\zeta \in \mathbb{T}^{\kappa}$. The delta integral of h is

$$\int_{b_1}^{b_2} h(\zeta)\triangle\zeta = H(b_2) - H(b_1). \tag{1}$$

Definition 2 (Nabla integral [8, Definition 8.42]) A mapping $G:\mathbb{T} \to (-\infty,\infty)$ is called the nabla antiderivative of $g:[b_1,b_2]_{\mathbb{T}} \to (-\infty,\infty)$ if $G^{\nabla}(\zeta) = g(\zeta)$ $\forall\zeta \in \mathbb{T}^{\kappa}$. The nabla integral of g is

$$\int_{b_1}^{b_2} g(\zeta)\nabla\zeta = G(b_2) - G(b_1). \tag{2}$$

In [16], the authors have defined the diamond-alpha integral as follows:

Let $l : [c_1, c_2]_{\mathbb{T}} \to \mathbb{R}$ be a continuous mapping and $c_1, c_2 \in \mathbb{T}$ ($c_1 < c_2$). The diamond alpha integral of l is given as

$$\int_{c_1}^{c_2} l(\zeta)\diamondsuit_\alpha \zeta := \int_{c_1}^{c_2} \alpha l(\zeta)\triangle\zeta + \int_{c_1}^{c_2} (1-\alpha)l(\zeta)\nabla\zeta, \quad 0 \le \alpha \le 1, \tag{3}$$

if γl is $\triangle$- and $(1-\gamma)l$ is ∇-integrable on $[c_1, c_2]_{\mathbb{T}}$.

In case $\alpha = 0$, we have the nabla-integral and, for $\alpha = 1$, we have the delta-integral.

In [10], a real-valued function γ is given as follows:

$$\gamma(x) = \lim_{y\to x} \frac{\sigma(x) - y}{\sigma(x) + 2x - 2y - \rho(x)}. \tag{4}$$

Clearly,

$$\gamma(x) = \begin{cases} \frac{1}{2}, & \text{if } x \text{ is dense;} \\ \frac{\sigma(x)-x}{\sigma(x)-\rho(x)}, & \text{if } x \text{ is not dense.} \end{cases}$$

In general, $0 \le \gamma(x) \le 1$.

In [11], diamond integral is defined as follows.

Definition 3 (Diamond integral [11]) Assume that $g : [b_1, b_2]_{\mathbb{T}} \to \mathbb{R}$ is a continuous function and $b_1, b_2 \in \mathbb{T}$ ($b_1 < b_2$). The $\diamondsuit$-integral of g is given as

$$\int_{b_1}^{b_2} g(\zeta)\diamondsuit\zeta = \int_{b_1}^{b_2} \gamma(\zeta)g(\zeta)\triangle\zeta + \int_{b_1}^{b_2} \big(1-\gamma(\zeta)\big)g(\zeta)\nabla\zeta, \quad 0 \le \gamma(\zeta) \le 1, \tag{5}$$

where γg is $\triangle$- and $(1-\gamma)g$ is ∇-integrable on $[b_1, b_2]_{\mathbb{T}}$.

For monotonicity, additivity, reflexivity, and multiplicativity properties of $\diamondsuit$-integrals, see [11].

Throughout the paper, we assume that:

(A1) $\Theta := [a_1, a_2]_{\mathbb{T}}$, with $a_1, a_2 \in \mathbb{T}$ and $a_1 < a_2$;

(A2) $\Lambda := \{l | l : \Theta \to \mathbb{R}^+, \int_\Theta l(\xi)\diamondsuit\xi = 1\}$.

3 Csiszár divergence via diamond integral

Csiszár divergence via diamond integral is defined as follows.

Definition 4 Assume that $l_2, l_1 \in \Lambda$ and ϕ is a convex function on $(0, \infty)$. If

$$D_\phi(l_1, l_2) := \int_\Theta l_2(\zeta)\phi\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\diamondsuit\zeta, \tag{6}$$

then $D_\phi(l_1, l_2)$ is called Csiszár divergence.

If we use $\phi(\zeta) = \zeta^2 - 1$ in (6), then Karl Pearson χ^2-divergence via diamond integral can be given as follows:

$$D_{\chi^2}(l_1, l_2) := \int_\Theta l_2(\zeta)\left[\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)^2 - 1\right]\diamondsuit\zeta. \tag{7}$$

A new bound for Csiszár divergence is obtained in the following result.

Theorem 1 *Assume that the mapping* $\phi : [0,\infty) \to (-\infty,\infty)$ *is convex on* $[\mu_1,\mu_2] \subset [0,\infty)$ *and* $\mu_1 \le 1 \le \mu_2$. *If*

$$\mu_1 \le \frac{l_2(\zeta)}{l_1(\zeta)} \le \mu_2, \quad \forall \zeta \in \mathbb{T}, \tag{8}$$

then

$$I_\phi(l_1,l_2) = \int_\Theta l_2(\zeta)\phi\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\diamond\zeta \le \frac{\mu_2 - 1}{\mu_2-\mu_1}\phi(\mu_1) + \frac{1-\mu_1}{\mu_2-\mu_1}\phi(\mu_2). \tag{9}$$

Proof Since ϕ is convex on $[\mu_1,\mu_2]$,

$$\phi\big(u\mu_1 + (1-u)\mu_2\big) \le u\phi(\mu_1) + (1-u)\phi(\mu_2), \tag{10}$$

for every $u \in [0,1]$. Put $u = \frac{\mu_2-\nu}{\mu_2-\mu_1}$, $1-u = 1 - \frac{\mu_2-\nu}{\mu_2-\mu_1} = \frac{\nu-\mu_1}{\mu_2-\mu_1}$ in (10) to obtain

$$\phi(\nu) \le \frac{\mu_2-\nu}{\mu_2-\mu_1}\phi(\mu_1) + \frac{\nu-\mu_1}{\mu_2-\mu_1}\phi(\mu_2). \tag{11}$$

Use $\nu = \frac{l_2(\zeta)}{l_1(\zeta)}$, in (11) to obtain

$$\phi\left(\frac{l_2(\zeta)}{l_1(\zeta)}\right) \le \frac{\mu_2 - \frac{l_2(\zeta)}{l_1(\zeta)}}{\mu_2-\mu_1}\phi(\mu_1) + \frac{\frac{l_2(\zeta)}{l_1(\zeta)} - \mu_1}{\mu_2-\mu_1}\phi(\mu_2). \tag{12}$$

Multiply (12) by $l_1(\zeta)$ to obtain

$$l_1(\zeta)\phi\left(\frac{l_2(\zeta)}{l_1(\zeta)}\right) \le \frac{\mu_2 l_1(\zeta) - l_2(\zeta)}{\mu_2-\mu_1}\phi(\mu_1) + \frac{l_2(\zeta) - \mu_1 l_1(\zeta)}{\mu_2-\mu_1}\phi(\mu_2). \tag{13}$$

Integrating (13) over Θ and since $l_2, l_1 \in \Lambda$, we get

$$I_\phi(l_1,l_2) = \int_\Theta l_2(\zeta)\phi\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\diamond\zeta \le \frac{\mu_2 - 1}{\mu_2-\mu_1}\phi(\mu_1) + \frac{1-\mu_1}{\mu_2-\mu_1}\phi(\mu_2),$$

which is the desired result. □

Example 1 Choose the set of real numbers as time scale in Theorem 1 to obtain [13, Theorem 1, p. 2].

Example 2 If we take $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then for $\zeta = hy \in h\mathbb{Z}$,

$$\gamma(\zeta) = \frac{\sigma(\zeta)-\zeta}{\sigma(\zeta)-\rho(\zeta)} = \frac{h(y+1)-hy}{h(y+1)-h(y-1)} = \frac{h}{2h} = \frac{1}{2}, \qquad 1-\gamma(\zeta) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Then from Theorem 1 we get

$$\begin{aligned} I_\phi(l_1,l_2) &= \int_\Theta l_2(\zeta)\phi\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\diamond\zeta \\ &= \int_\Theta \gamma(\zeta)l_2(\zeta)\phi\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\Delta\zeta + \int_\Theta (1-\gamma(\zeta))l_2(\zeta)\phi\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\nabla\zeta \\ &= \frac{1}{2}\left[\sum_{j=\frac{a_1}{h}}^{\frac{a_2}{h}-1} hl_2(jh)\phi\left(\frac{l_1(jh)}{l_2(jh)}\right) + \sum_{j=\frac{a_1}{h}+1}^{\frac{a_2}{h}} hl_2(jh)\phi\left(\frac{l_1(jh)}{l_2(jh)}\right)\right] \\ &\leq \frac{\mu_2-1}{\mu_2-\mu_1}\phi(\mu_1) + \frac{1-\mu_1}{\mu_2-\mu_1}\phi(\mu_2). \end{aligned} \tag{14}$$

Remark 1 Inequality (14) is generalization of the specific bound for Csiszár divergence obtained by Ansari *et al.* [5].

Example 3 Choose the set of integers as time scale. Then (9) takes the form

$$\frac{1}{2}\left[\sum_{j=a_1}^{a_2-1} l_2(j)\phi\left(\frac{l_1(j)}{l_2(j)}\right) + \sum_{j=a_1+1}^{a_2} l_2(j)\phi\left(\frac{l_1(j)}{l_2(j)}\right)\right] \leq \frac{\mu_2-1}{\mu_2-\mu_1}\phi(\mu_1) + \frac{1-\mu_1}{\mu_2-\mu_1}\phi(\mu_2).$$

Example 4 Choose $\mathbb{T} = q^{\mathbb{N}_0}$ $(q > 1)$. Then for $\zeta = q^n \in q^{\mathbb{N}_0}$ we have

$$\gamma(\zeta) = \frac{\sigma(\zeta)-\zeta}{\sigma(\zeta)-\rho(\zeta)} = \frac{q^{n+1}-q^n}{q^{n+1}-q^{n-1}} = \frac{q^2-q}{q^2-1} = \frac{q}{q+1}$$

and

$$1-\gamma(\zeta) = 1-\frac{q}{q+1} = \frac{1}{q+1}.$$

In Theorem 1, use $a_1 = q^r$ and $a_2 = q^s$ $(r < s)$ to obtain

$$\begin{aligned} &\frac{q-1}{q+1}\left[\sum_{j=r}^{s-1} q^{j+1}l_2(q^j)\phi\left(\frac{l_1(q^j)}{l_2(q^j)}\right) + \sum_{j=r+1}^{s} q^{j-1}l_2(q^j)\phi\left(\frac{l_1(q^j)}{l_2(q^j)}\right)\right] \\ &\quad\leq \frac{\mu_2-1}{\mu_2-\mu_1}\phi(\mu_1) + \frac{1-\mu_1}{\mu_2-\mu_1}\phi(\mu_2). \end{aligned} \tag{15}$$

Remark 2 Inequality (15) provides a new bound for Csiszár divergence in q-calculus.

Theorem 2 *If the assumptions of Theorem 1 are true and ϕ is differentiable on $[\mu_1,\mu_2]$, then*

$$0 \leq \frac{\mu_2-1}{\mu_2-\mu_1}\phi(\mu_1) + \frac{1-\mu_1}{\mu_2-\mu_1}\phi(\mu_2) - I_\phi(l_1,l_2) \tag{16}$$

$$\leq \frac{\phi'(\mu_2)-\phi'(\mu_1)}{\mu_2-\mu_1}\big((1-\mu_1)(\mu_2-1) - D_{\chi^2}(l_1,l_2)\big) \tag{17}$$

$$\leq \frac{(\phi'(\mu_2)-\phi'(\mu_1))(\mu_2-\mu_1)}{4}. \tag{18}$$

Proof Given that ϕ is a differentiable convex function, we obtain

$$\phi(\nu_1) - \phi(\nu_2) \geq \phi'(\nu_2)(\nu_1 - \nu_2), \quad \forall \nu_1, \nu_2 \in (\mu_1, \mu_2). \tag{19}$$

Let $b_1, b_2 \in [\mu_1, \mu_2]$, $\beta_1, \beta_2 \geq 0$, and $\beta_1 + \beta_2 > 0$. Put $\nu_1 = \frac{b_1\beta_1 + b_2\beta_2}{\beta_1+\beta_2}$ and $\nu_2 = b_1$ in (19) to obtain

$$\begin{aligned}\phi\left(\frac{b_1\beta_1 + b_2\beta_2}{\beta_1 + \beta_2}\right) - \phi(b_1) &\geq \phi'(b_1)\left(\frac{b_1\beta_1 + b_2\beta_2}{\beta_1 + \beta_2} - b_1\right) \\ &= \frac{\beta_2\phi'(b_1)(b_2 - b_1)}{\beta_1 + \beta_2}.\end{aligned} \tag{20}$$

Use $\nu_1 = \frac{b_1\beta_1 + b_2\beta_2}{\beta_1+\beta_2}$ and $\nu_2 = b_2$ in (19) to get

$$\begin{aligned}\phi\left(\frac{b_1\beta_1 + b_2\beta_2}{\beta_1 + \beta_2}\right) - \phi(b_2) &\geq \phi'(b_2)\left(\frac{b_1\beta_1 + b_2\beta_2}{\beta_1 + \beta_2} - b_2\right) \\ &= \frac{-\beta_1\phi'(b_2)(b_2 - b_1)}{\beta_1 + \beta_2}.\end{aligned} \tag{21}$$

Multiply (20) with β_1 and (21) with β_2 and add the results to obtain

$$\begin{aligned}&(\beta_1 + \beta_2)\phi\left(\frac{b_1\beta_1 + b_2\beta_2}{\beta_1 + \beta_2}\right) - \beta_2\phi(b_2) - \beta_1\phi(b_1) \\ &\quad\geq \frac{\beta_1\beta_2(\phi'(b_1) - \phi'(b_2))(b_2 - b_1)}{\beta_1 + \beta_2}.\end{aligned} \tag{22}$$

Divide (22) by $-(\beta_1 + \beta_2)$ to get

$$\begin{aligned}0 &\leq \frac{\beta_2\phi(b_2) + \beta_1\phi(b_1)}{(\beta_1 + \beta_2)} - \phi\left(\frac{b_1\beta_1 + b_2\beta_2}{\beta_1 + \beta_2}\right) \\ &\leq \frac{\beta_1\beta_2(\phi'(b_2) - \phi'(b_1))(b_2 - b_1)}{(\beta_1 + \beta_2)^2}.\end{aligned} \tag{23}$$

Use $\beta_1 = \mu_2 - y$, $\beta_2 = y - \mu_1$, $b_1 = \mu_1$, and $b_2 = \mu_2$ in (23) to obtain

$$\begin{aligned}0 &\leq \frac{(y - \mu_1)\phi(\mu_2) + (\mu_2 - y)\phi(\mu_1)}{(\mu_2 - \mu_1)} - \phi(y) \\ &\leq \frac{(\mu_2 - y)(y - \mu_1)(\phi'(\mu_2) - \phi'(\mu_1))}{(\mu_2 - \mu_1)}.\end{aligned} \tag{24}$$

Use $y = \frac{l_2(\zeta)}{l_1(\zeta)}$ in (24) and multiply by $l_1(\zeta)$ to get

$$\begin{aligned}0 &\leq \frac{(l_2(\zeta) - \mu_1 l_1(\zeta))\phi(\mu_2) + (\mu_2 l_1(\zeta) - l_2(\zeta))\phi(\mu_1)}{(\mu_2 - \mu_1)} - l_1(\zeta)\phi\left(\frac{l_2(\zeta)}{l_1(\zeta)}\right) \\ &\leq \frac{(\mu_2 l_1(\zeta) - l_2(\zeta))(l_2(\zeta) - \mu_1 l_1(\zeta))(\phi'(\mu_2) - \phi'(\mu_1))}{(\mu_2 - \mu_1)}.\end{aligned} \tag{25}$$

Take diamond integral on both sides of (25) with $l_2, l_1 \in \Lambda$ to obtain

$$\begin{aligned}
&\frac{(1-\mu_1)\phi(\mu_2)+(\mu_2-1)\phi(\mu_1)}{(\mu_2-\mu_1)} - I_\phi(l_1,l_2) \\
&\le \frac{(\phi'(\mu_2)-\phi'(\mu_1))}{(\mu_2-\mu_1)} \int_\Theta \frac{(\mu_2 l_1(\zeta)-l_2(\zeta))(l_2(\zeta)-\mu_1 l_1(\zeta))}{l_1(\zeta)} \diamond\zeta \\
&= \frac{(\phi'(\mu_2)-\phi'(\mu_1))}{(\mu_2-\mu_1)} \left(\mu_2 - \int_\Theta \frac{l_2^2(\zeta)}{l_1(\zeta)}\diamond\zeta - \mu_2\mu_1 + \mu_1\right) \\
&= \frac{(\phi'(\mu_2)-\phi'(\mu_1))}{(\mu_2-\mu_1)} \\
&\quad \times \left(\mu_2 - \int_\Theta l_1(\zeta)\left(\frac{l_2(\zeta)}{l_1(\zeta)}\right)^2 \diamond\zeta + \int_\Theta l_1(\zeta)\diamond\zeta - \int_\Theta l_1(\zeta)\diamond\zeta - \mu_2\mu_1 + \mu_1\right) \\
&= \frac{(\phi'(\mu_2)-\phi'(\mu_1))}{(\mu_2-\mu_1)} \left(\mu_2 - D_{\chi^2}(l_1,l_2) - 1 - \mu_2\mu_1 + \mu_1\right) \\
&= \frac{(\phi'(\mu_2)-\phi'(\mu_1))}{(\mu_2-\mu_1)} \left((\mu_2-1)(1-\mu_1) - D_{\chi^2}(l_1,l_2)\right),
\end{aligned}$$

therefore (17) is proved. Since $(\mu_2-1)(1-\mu_1) \le \frac{1}{4}(\mu_2-\mu_1)^2$ and $D_{\chi^2}(l_1,l_2) \ge 0$, (18) is obvious. □

Remark 3 Choose the set of real numbers as a time scale in Theorem 2 to obtain [13, Theorem 2].

Example 5 If we take $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then from Theorem 2 we get

$$\begin{aligned}
0 &\le \frac{\mu_2-1}{\mu_2-\mu_1}\phi(\mu_1) + \frac{1-\mu_1}{\mu_2-\mu_1}\phi(\mu_2) \\
&\quad - \sum_{j=\frac{a_1}{h}}^{\frac{a_2}{h}-1} h l_1(jh)\phi\left(\frac{l_2(jh)}{l_1(jh)}\right) - \sum_{j=\frac{a_1}{h}+1}^{\frac{a_2}{h}} h l_1(jh)\phi\left(\frac{l_2(jh)}{l_1(jh)}\right) \\
&\le \frac{\phi'(\mu_2)-\phi'(\mu_1)}{\mu_2-\mu_1}\left\{(1-\mu_1)(\mu_2-1) - \sum_{j=\frac{a_1}{h}}^{\frac{a_2}{h}-1} h l_1(jh)\left[\left(\frac{l_2(jh)}{l_1(jh)}\right)^2 - 1\right]\right. \\
&\quad \left. - \sum_{j=\frac{a_1}{h}+1}^{\frac{a_2}{h}} h l_1(jh)\left[\left(\frac{l_2(jh)}{l_1(jh)}\right)^2 - 1\right]\right\} \\
&\le \frac{(\phi'(\mu_2)-\phi'(\mu_1))(\mu_2-\mu_1)}{4}.
\end{aligned}$$

Example 6 Select $\mathbb{T} = q^{\mathbb{N}_0}$ where $q > 1$. Then $\zeta = q^m$ for some $m \in \mathbb{N}_0$. Additionally, use $a_1 = q^r$ and $a_2 = q^s$ ($r < s$) in Theorem 2 to obtain

$$\begin{aligned}
0 &\le \frac{\mu_2-1}{\mu_2-\mu_1}\phi(\mu_1) + \frac{1-\mu_1}{\mu_2-\mu_1}\phi(\mu_2) \\
&\quad - \frac{q-1}{q+1}\sum_{j=r}^{s-1} q^{j+1} l_1(q^j)\phi\left(\frac{l_2(q^j)}{l_1(q^j)}\right) - \frac{q-1}{q+1}\sum_{j=r+1}^{s} q^{j-1} l_1(q^j)\phi\left(\frac{l_2(q^j)}{l_1(q^j)}\right)
\end{aligned}$$

$$\leq \frac{\phi'(\mu_2)-\phi'(\mu_1)}{\mu_2-\mu_1}\left\{(1-\mu_1)(\mu_2-1)-\frac{q-1}{q+1}\sum_{j=r}^{s-1}q^{j+1}l_1(q^j)\left[\left(\frac{l_2(q^j)}{l_1(q^j)}\right)^2-1\right]\right.$$
$$\left.-\frac{q-1}{q+1}\sum_{j=r+1}^{s}q^{j-1}l_1(q^j)\left[\left(\frac{l_2(q^j)}{l_1(q^j)}\right)^2-1\right]\right\}$$
$$\leq \frac{(\phi'(\mu_2)-\phi'(\mu_1))(\mu_2-\mu_1)}{4}.$$

Theorem 3 *Let the assumptions of Theorem 1 be true and suppose ϕ is twice differentiable on $[\mu_1,\mu_2]$. If $n\leq\phi''(t)\leq N$ for each $t\in[\mu_1,\mu_2]$, then*

$$\frac{n}{2}\left[(1-\mu_1)(\mu_2-1)-D_{\chi^2}(l_1,l_2)\right]$$
$$\leq \frac{\mu_2-1}{\mu_2-\mu_1}\phi(\mu_1)+\frac{1-\mu_1}{\mu_2-\mu_1}\phi(\mu_2)-I_\phi(l_1,l_2) \tag{26}$$
$$\leq \frac{N}{2}\big((1-\mu_1)(\mu_2-1)-D_{\chi^2}(l_1,l_2)\big). \tag{27}$$

Proof Consider the mapping $\zeta_n:[0,\infty)\to(-\infty,\infty)$ as $\zeta_n(\xi)=\phi(\xi)-\frac{n\xi^2}{2}$. Since $\zeta_n''(\xi)=\phi''(\xi)-n\geq 0$ for each $\xi\in[\mu_1,\mu_2]$, ζ_n is convex on $[\mu_1,\mu_2]$. Use ζ_n in (9) to obtain

$$I_{\phi-\frac{n}{2}(\cdot)^2}(l_1,l_2)\leq\frac{\mu_2-1}{\mu_2-\mu_1}\left[\phi(\mu_1)-\frac{n}{2}(\mu_1)^2\right]+\frac{1-\mu_1}{\mu_2-\mu_1}\left[\phi(\mu_2)-\frac{n}{2}(\mu_2)^2\right]. \tag{28}$$

Also,

$$I_{\phi-\frac{n}{2}(\cdot)^2}(l_1,l_2)=I_\phi(l_1,l_2)-\frac{n}{2}\int_\Theta l_2(\zeta)\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)^2\diamond\zeta$$
$$=I_\phi(l_1,l_2)-\frac{n}{2}\int_\Theta\left[l_2(\zeta)\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)^2+1-1\right]\diamond\zeta$$
$$=I_\phi(l_1,l_2)-\frac{n}{2}D_{\chi^2}(l_1,l_2)-\frac{n}{2}.$$

Therefore (28) gives

$$\frac{\mu_2-1}{\mu_2-\mu_1}\frac{n}{2}\mu_1^2+\frac{1-\mu_1}{\mu_2-\mu_1}\frac{n}{2}\mu_2^2-\frac{n}{2}D_{\chi^2}(l_1,l_2)-\frac{n}{2}$$
$$\leq\frac{\mu_2-1}{\mu_2-\mu_1}\phi(\mu_1)+\frac{1-\mu_1}{\mu_2-\mu_1}\phi(\mu_2)-I_\phi(l_1,l_2). \tag{29}$$

Since

$$\frac{\mu_2-1}{\mu_2-\mu_1}\frac{n}{2}\mu_1^2+\frac{1-\mu_1}{\mu_2-\mu_1}\frac{n}{2}\mu_2^2-\frac{n}{2}D_{\chi^2}(l_1,l_2)-\frac{n}{2}$$
$$=\frac{n}{2}\left(\frac{\mu_2-1}{\mu_2-\mu_1}\mu_1^2+\frac{1-\mu_1}{\mu_2-\mu_1}\mu_2^2-D_{\chi^2}(l_1,l_2)-1\right)$$
$$=\frac{n}{2}\left(\frac{\mu_2^2-\mu_1^2-\mu_1\mu_2(\mu_2-\mu_1)}{\mu_2-\mu_1}-D_{\chi^2}(l_1,l_2)-1\right)$$
$$=\frac{n}{2}\big(\mu_2+\mu_1-\mu_1\mu_2-1-D_{\chi^2}(l_1,l_2)\big)$$

$$= \frac{n}{2}\big((\mu_2 - 1)(1 - \mu_1) - D_{\chi^2}(l_1, l_2)\big),$$

inequality (26) is proved. The proof of (27) is similar, here we take $\zeta_n(\xi) = \frac{N\xi^2}{2} - \phi(\xi)$. □

Remark 4 Select the set of real numbers as a time scale in Theorem 3 to obtain [13, Theorem 3].

Example 7 If we take $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then from Theorem 3 we get

$$\begin{aligned}
&\frac{n}{2}\Bigg\{(1-\mu_1)(\mu_2-1) - \sum_{j=\frac{a_1}{h}}^{\frac{a_2}{h}-1} hl_1(jh)\Bigg[\bigg(\frac{l_2(jh)}{l_1(jh)}\bigg)^2 - 1\Bigg] \\
&\quad - \sum_{j=\frac{a_1}{h}+1}^{\frac{a_2}{h}} hl_1(jh)\Bigg[\bigg(\frac{l_2(jh)}{l_1(jh)}\bigg)^2 - 1\Bigg]\Bigg\} \\
&\le \frac{\mu_2-1}{\mu_2-\mu_1}\phi(\mu_1) + \frac{1-\mu_1}{\mu_2-\mu_1}\phi(\mu_2) \\
&\quad - \sum_{j=\frac{a_1}{h}}^{\frac{a_2}{h}-1} hl_1(jh)\phi\bigg(\frac{l_2(jh)}{l_1(jh)}\bigg) - \sum_{j=\frac{a_1}{h}+1}^{\frac{a_2}{h}} hl_1(jh)\phi\bigg(\frac{l_2(jh)}{l_1(jh)}\bigg) \\
&\le \frac{N}{2}\Bigg((1-\mu_1)(\mu_2-1) - \sum_{j=\frac{a_1}{h}}^{\frac{a_2}{h}-1} hl_1(jh)\Bigg[\bigg(\frac{l_2(jh)}{l_1(jh)}\bigg)^2 - 1\Bigg] \\
&\quad - \sum_{j=\frac{a_1}{h}+1}^{\frac{a_2}{h}} hl_1(jh)\Bigg[\bigg(\frac{l_2(jh)}{l_1(jh)}\bigg)^2 - 1\Bigg]\Bigg).
\end{aligned}$$

Example 8 Select $\mathbb{T} = q^{\mathbb{N}_0}$ where $q > 1$, then $\zeta = q^m$ for some $m \in \mathbb{N}_0$. Additionally, use $a_1 = q^r$ and $a_2 = q^s$ $(r < s)$ in Theorem 3 to obtain

$$\begin{aligned}
&\frac{n}{2}\Bigg\{(1-\mu_1)(\mu_2-1) - \frac{q-1}{q+1}\sum_{j=r}^{s-1} q^{j+1}l_1(q^j)\Bigg[\bigg(\frac{l_2(q^j)}{l_1(q^j)}\bigg)^2 - 1\Bigg] \\
&\quad - \frac{q-1}{q+1}\sum_{j=r+1}^{s} q^{j-1}l_1(q^j)\Bigg[\bigg(\frac{l_2(q^j)}{l_1(q^j)}\bigg)^2 - 1\Bigg]\Bigg\} \\
&\le \frac{\mu_2-1}{\mu_2-\mu_1}\phi(\mu_1) + \frac{1-\mu_1}{\mu_2-\mu_1}\phi(\mu_2) \\
&\quad - \frac{q-1}{q+1}\sum_{j=r}^{s-1} q^{j+1}l_1(q^j)\phi\bigg(\frac{l_2(q^j)}{l_1(q^j)}\bigg) - \frac{q-1}{q+1}\sum_{j=r+1}^{s} q^{j-1}l_1(q^j)\phi\bigg(\frac{l_2(q^j)}{l_1(q^j)}\bigg) \\
&\le \frac{N}{2}\Bigg((1-\mu_1)(\mu_2-1) - \frac{q-1}{q+1}\sum_{j=r}^{s-1} q^{j+1}l_1(q^j)\Bigg[\bigg(\frac{l_2(q^j)}{l_1(q^j)}\bigg)^2 - 1\Bigg] \\
&\quad - \frac{q-1}{q+1}\sum_{j=r+1}^{s} q^{j-1}l_1(q^j)\Bigg[\bigg(\frac{l_2(q^j)}{l_1(q^j)}\bigg)^2 - 1\Bigg]\Bigg).
\end{aligned}$$

Corollary 1 *If the assumptions of Theorem 3 are true and $n \geq 0$, then*

$$0 \leq \frac{n}{2}\left[(1-\mu_1)(\mu_2-1) - D_{\chi^2}(l_1,l_2)\right] \leq \frac{\mu_2-1}{\mu_2-\mu_1}\phi(\mu_1) + \frac{1-\mu_1}{\mu_2-\mu_1}\phi(\mu_2) - I_\phi(l_1,l_2).$$

Proof The statement follows from Theorem 3. Indeed, since

$$\mu_1 \leq \frac{l_2(\zeta)}{l_1(\zeta)} \leq \mu_2,$$

one has

$$\begin{aligned} 0 &\leq \int_\Theta \frac{(\mu_2 l_1(\zeta) - l_2(\zeta))(l_2(\zeta) - \mu_1 l_1(\zeta))}{l_1(\zeta)} \diamond\zeta \\ &= (1-\mu_1)(\mu_2-1) - D_{\chi^2}(l_1,l_2). \end{aligned}$$

Hence, the proof is complete. □

Remark 5 Choose $\gamma = 1$ in Corollary 1 to obtain [5, Corollary 1].

Remark 6 Select the set of real numbers as a time scale in Corollary 1 to obtain [13, Corollary 1].

4 Some bounds in terms of special means

In this section, first of all we recall a few special means:

Geometric mean

$$G(\xi_1,\xi_2) = \pm\sqrt{\xi_1\xi_2}.$$

Arithmetic mean

$$A(\xi_1,\xi_2) = \frac{\xi_1+\xi_2}{2}.$$

Logarithmic mean

$$L(\xi_1,\xi_2) = \begin{cases} \xi_2, & \text{if } \xi_1 = \xi_2, \\ \frac{\xi_2-\xi_1}{\ln\xi_2 - \ln\xi_2}, & \text{if } \xi_1 \neq \xi_2 \text{ and } \xi_1,\xi_2 > 0. \end{cases}$$

Identric mean

$$L(\xi_1,\xi_2) = \begin{cases} \xi_2, & \text{if } \xi_1 = \xi_2, \\ \frac{1}{e}\left(\frac{\xi_2^{\xi_2}}{\xi_1^{\xi_1}}\right)^{\frac{1}{\ln\xi_2-\ln\xi_2}}, & \text{if } \xi_1 \neq \xi_2. \end{cases}$$

Now we discuss some special cases of f-divergence such as Bhattacharyya distance, K–L divergence, Hellinger distance, triangular discrimination, and Jeffreys distance.

4.1 Bhattacharyya distance via diamond integral

If we use $\phi(\zeta) = -\sqrt{\zeta}$ in (6), then we obtain Bhattacharyya distance.

Definition 5 Bhattacharyya distance via diamond integral can be defined as follows:

$$D_B(l_1, l_2) := -\int_\Theta \sqrt{l_1(\zeta)l_2(\zeta)}\diamond\zeta.$$

Proposition 1 *If the assumptions of Theorem 1 are true, then*

$$D_B(l_1, l_2) \leq \frac{-1 - G^2(\mu_1, \mu_2)}{2A(\sqrt{\mu_1}, \sqrt{\mu_2})}.$$

Proof Use $\phi(\zeta) = -\sqrt{\zeta}$ in Theorem 1 to get

$$\begin{aligned} D_B(l_1, l_2) &\leq \frac{(\mu_2 - 1)(-\sqrt{\mu_1}) + (1 - \mu_1)(-\sqrt{\mu_2})}{\mu_2 - \mu_1} \\ &= \frac{-\sqrt{\mu_2} + \sqrt{\mu_1} - \mu_2\sqrt{\mu_1} + \mu_1\sqrt{\mu_2}}{\mu_2 - \mu_1} \\ &= \frac{-1(\sqrt{\mu_2} - \sqrt{\mu_1}) - \sqrt{\mu_1}\sqrt{\mu_2}(\sqrt{\mu_2} - \sqrt{\mu_1})}{\mu_2 - \mu_1} \\ &= \frac{-1 - \sqrt{\mu_1}\sqrt{\mu_2}}{(\sqrt{\mu_2} + \sqrt{\mu_1})} = \frac{-1 - G^2(\mu_1, \mu_2)}{2A(\sqrt{\mu_1}, \sqrt{\mu_2})}. \end{aligned}$$

□

The next result provides a new bound for Bhattacharyya discrimination in q-calculus.

Example 9 Select $\mathbb{T} = q^{\mathbb{N}_0}$ where $q > 1$, then $\zeta = q^m$ for some $m \in \mathbb{N}_0$. In Proposition 1, use $a_1 = q^r$ and $a_2 = q^s$ $(r < s)$ to obtain

$$-\sum_{j=r}^{s-1} q^{j+1}\sqrt{l_1(q^j)l_2(q^j)} - \sum_{j=r+1}^{s} q^{j-1}\sqrt{l_1(q^j)l_2(q^j)} \leq \frac{-1 - G^2(\mu_1, \mu_2)}{2A(\sqrt{\mu_1}, \sqrt{\mu_2})}.$$

Proposition 2 *If the assumptions of Theorem 2 are true, then we have*

$$\begin{aligned} 0 &\leq \frac{-1 - G^2(\mu_1, \mu_2)}{2A(\sqrt{\mu_1}, \sqrt{\mu_2})} - D_B(l_1, l_2) \\ &\leq \frac{1}{2G^2(\mu_1, \mu_2)A(\sqrt{\mu_1}, \sqrt{\mu_2})}\big[(\mu_2 - 1)(1 - \mu_1) - D_{\chi^2}(l_1, l_2)\big]. \end{aligned}$$

Proof Use $\phi(\zeta) = -\sqrt{\zeta}$ in Theorem 2 to obtain the desired result, since in this case

$$\frac{\phi'(\mu_2) - \phi'(\mu_1)}{\mu_2 - \mu_1} = \frac{1}{2\sqrt{\mu_1\mu_2}(\sqrt{\mu_2} + \sqrt{\mu_1})}.$$

□

Proposition 3 *If the assumptions of Theorem 3 are true, then we have*

$$\begin{aligned} 0 &\le \frac{1}{8\sqrt{\mu_2^3}}\big[(\mu_2-1)(1-\mu_1)-D_{\chi^2}(l_1,l_2)\big] \\ &\le \frac{-1-G^2(\mu_1,\mu_2)}{2A(\sqrt{\mu_1},\sqrt{\mu_2})}-D_B(l_1,l_2) \\ &\le \frac{1}{8\sqrt{\mu_1^3}}\big[(\mu_2-1)(1-\mu_1)-D_{\chi^2}(l_1,l_2)\big]. \end{aligned}$$

Proof Use $\phi(\zeta)=-\sqrt{\zeta}$ in Theorem 3 to obtain the desired result, since in this case

$$\phi''(\zeta)=\frac{1}{4\sqrt{\zeta^3}} \quad \text{and} \quad \frac{1}{4\sqrt{\mu_2^3}} \le \phi''(\zeta) \le \frac{1}{4\sqrt{\mu_1^3}}$$

for each $\zeta \in [\mu_1,\mu_2]$. □

4.2 Kullback–Leibler divergence via diamond integral

If we use $\phi(\zeta)=\zeta\ln\zeta$ in (6), then we obtain Kullback–Leibler divergence.

Definition 6 Kullback–Leibler divergence via diamond integral can be given as follows:

$$D(l_1,l_2) := \int_\Theta l_1(\zeta)\ln\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\diamond\zeta.$$

Proposition 4 *If*

$$\mu_1 \le \frac{l_2(\zeta)}{l_1(\zeta)} \le \mu_2,$$

for each $\zeta \in \mathbb{T}$, then

$$D(l_2,l_1) := \int_\Theta l_1(\zeta)\ln\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\diamond\zeta \le \ln I(\mu_1,\mu_2)-\frac{G^2(\mu_1,\mu_2)}{L(\mu_1,\mu_2)}+1.$$

Proof Use $\phi(\zeta)=\zeta\ln\zeta$ in Theorem 1 to get

$$\begin{aligned} D(l_2,l_1) &= \int_\Theta l_2(\zeta)\phi\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\diamond\zeta \\ &\le \frac{\mu_2-1}{\mu_2-\mu_1}\mu_1\ln\mu_1+\frac{1-\mu_1}{\mu_2-\mu_1}\mu_2\ln\mu_2 \\ &= \frac{\mu_2\mu_1\ln\mu_1-\mu_1\ln\mu_1+\mu_2\ln\mu_2-\mu_2\mu_1\ln\mu_2}{\mu_2-\mu_1} \\ &= \frac{\mu_2\ln\mu_2-\mu_1\ln\mu_1}{\mu_2-\mu_1}+\mu_1\mu_2\frac{\ln\mu_2-\ln\mu_1}{\mu_2-\mu_1} \\ &= \frac{\mu_2\ln\mu_2-\mu_1\ln\mu_1}{\mu_2-\mu_1}-1+1+(\sqrt{\mu_1\mu_2})^2\frac{\ln\mu_2-\ln\mu_1}{\mu_2-\mu_1} \\ &= \ln I(\mu_1,\mu_2)+1-\frac{G^2(\mu_1,\mu_2)}{L(\mu_1,\mu_2)}. \end{aligned}$$

□

Remark 7 Choose $\gamma = 1$ in Proposition 4 to obtain [5, Proposition 1].

Remark 8 Select the set of real numbers as a time scale in Proposition 4 to obtain [13, Proposition 1].

Example 10 If we take $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then from Proposition 4 we get

$$\begin{aligned} &\sum_{j=\frac{a_1}{h}}^{\frac{a_2}{h}-1} hl_1(jh)\ln\left(\frac{l_1(jh)}{l_2(jh)}\right) + \sum_{j=\frac{a_1}{h}+1}^{\frac{a_2}{h}} hl_1(jh)\ln\left(\frac{l_1(jh)}{l_2(jh)}\right) \\ &\quad\le \ln I(\mu_1,\mu_2) - \frac{G^2(\mu_1,\mu_2)}{L(\mu_1,\mu_2)} + 1. \end{aligned} \tag{30}$$

Example 11 Select $\mathbb{T} = q^{\mathbb{N}_0}$ where $q > 1$, then $\zeta = q^m$ for some $m \in \mathbb{N}_0$. Additionally, use $a_1 = q^r$ and $a_2 = q^s$ $(r < s)$ in Proposition 4 to obtain

$$\begin{aligned} &\frac{q-1}{q+1}\left[\sum_{j=r}^{s-1} q^{j+1} l_1(q^j)\ln\left(\frac{l_1(q^j)}{l_2(q^j)}\right) + \sum_{j=r+1}^{s} q^{j-1} l_1(q^j)\ln\left(\frac{l_1(q^j)}{l_2(q^j)}\right)\right] \\ &\quad\le \ln I(\mu_1,\mu_2) - \frac{G^2(\mu_1,\mu_2)}{L(\mu_1,\mu_2)} + 1. \end{aligned} \tag{31}$$

Remark 9 Inequality (31) provides a new upper bound for K–L divergence in q-calculus.

Proposition 5 *If the assumptions of Proposition 4 are true, then we have*

$$\begin{aligned} 0 &\le \ln I(\mu_1,\mu_2) - \frac{G^2(\mu_1,\mu_2)}{L(\mu_1,\mu_2)} + 1 - \int_\Theta l_2(\zeta)\phi\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\diamond\zeta \\ &\le \frac{(\mu_2-1)(1-\mu_1) - D_{\chi^2}(l_1,l_2)}{L(\mu_1,\mu_2)}. \end{aligned} \tag{32}$$

Proof Use $\phi(\zeta) = \zeta\ln\zeta$ in Theorem 2 to get

$$\frac{\phi'(\mu_2)-\phi'(\mu_1)}{\mu_2-\mu_1} = \frac{\ln\mu_2 + 1 - 1 - \ln\mu_1}{\mu_2-\mu_1} = \frac{1}{L(\mu_1,\mu_2)}.$$ □

Remark 10 Select the set of real numbers as a time scale in Proposition 5 to obtain [13, Proposition 2].

Example 12 Select $\mathbb{T} = q^{\mathbb{N}_0}$ where $q > 1$, then $\zeta = q^m$ for some $m \in \mathbb{N}_0$. Further, use $a_1 = q^r$ and $a_2 = q^s$ $(r < s)$ in Proposition 5 to obtain

$$\begin{aligned} 0 \le{}& \ln I(\mu_1,\mu_2) - \frac{G^2(\mu_1,\mu_2)}{L(\mu_1,\mu_2)} + 1 \\ &- \frac{q-1}{q+1}\sum_{j=r}^{s-1} q^{j+1} l_2(q^j)\phi\left(\frac{l_1(q^j)}{l_2(q^j)}\right) - \frac{q-1}{q+1}\sum_{j=r+1}^{s} q^{j-1} l_2(q^j)\phi\left(\frac{l_1(q^j)}{l_2(q^j)}\right) \end{aligned}$$

$$\leq \frac{(\mu_2-1)(1-\mu_1)}{L(\mu_1,\mu_2)} - \frac{\frac{q-1}{q+1}\{\sum_{j=r}^{s-1} q^{j+1} l_1(q^j)[(\frac{l_2(q^j)}{l_1(q^j)})^2-1] + \sum_{j=r+1}^{s} q^{j-1} l_1(q^j)[(\frac{l_2(q^j)}{l_1(q^j)})^2-1]\}}{L(\mu_1,\mu_2)}.$$

In the following result we use Theorem 3 to improve (32).

Proposition 6 *If* l_1, l_2 *satisfy* (8), *then*

$$\begin{aligned}
&\frac{1}{2l_1}\big[(1-\mu_1)(\mu_2-1) - D_{\chi^2}(l_1,l_2)\big] \\
&\quad\leq \ln I(\mu_1,\mu_2) - \frac{G^2(\mu_1,\mu_2)}{L(\mu_1,\mu_2)} + 1 - \int_\Theta l_2(\zeta)\phi\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\diamond\zeta \\
&\quad\leq \frac{1}{2l_2}\big((1-\mu_1)(\mu_2-1) - D_{\chi^2}(l_1,l_2)\big).
\end{aligned}$$

Proof Use $\phi(\zeta) = \zeta \ln \zeta$ in Theorem 3, then $\phi''(\zeta) = \zeta^{-1}$. Since $\mu_1 \leq \zeta \leq \mu_2$, one gets

$$\frac{1}{\mu_2} \leq \frac{1}{\zeta} \leq \frac{1}{\mu_1},$$

which implies

$$\frac{1}{\mu_2} \leq \phi''(\zeta) \leq \frac{1}{\mu_1},$$

completing the proof. □

Example 13 Select the set of real numbers as a time scale in Proposition 6 to obtain [13, Proposition 3].

Example 14 Select $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, and $\zeta = q^m$ for some $m \in \mathbb{N}_0$. Also, use $a_1 = q^r$ and $a_2 = q^s$ $(r < s)$ in Proposition 6 to obtain

$$\begin{aligned}
&\frac{1}{2l_1}\left[(1-\mu_1)(\mu_2-1) - \frac{q-1}{q+1}\left\{\sum_{j=r}^{s-1} q^{j+1} l_1(q^j)\left[\left(\frac{l_2(q^j)}{l_1(q^j)}\right)^2 - 1\right]\right.\right. \\
&\qquad \left.\left. + \sum_{j=r+1}^{s} q^{j-1} l_1(q^j)\left[\left(\frac{l_2(q^j)}{l_1(q^j)}\right)^2 - 1\right]\right\}\right] \\
&\quad\leq \ln I(\mu_1,\mu_2) - \frac{G^2(\mu_1,\mu_2)}{L(\mu_1,\mu_2)} + 1 \\
&\qquad - \frac{q-1}{q+1}\sum_{j=r}^{s-1} q^{j+1} l_2(q^j)\phi\left(\frac{l_1(q^j)}{l_2(q^j)}\right) - \frac{q-1}{q+1}\sum_{j=r+1}^{s} q^{j-1} l_2(q^j)\phi\left(\frac{l_1(q^j)}{l_2(q^j)}\right) \\
&\quad\leq \frac{1}{2l_2}\left((1-\mu_1)(\mu_2-1) - \frac{q-1}{q+1}\left\{\sum_{j=r}^{s-1} q^{j+1} l_1(q^j)\left[\left(\frac{l_2(q^j)}{l_1(q^j)}\right)^2 - 1\right]\right.\right. \\
&\qquad \left.\left. + \sum_{j=r+1}^{s} q^{j-1} l_1(q^j)\left[\left(\frac{l_2(q^j)}{l_1(q^j)}\right)^2 - 1\right]\right\}\right).
\end{aligned}$$

Remark 11 Use $\phi(\zeta) = -\ln\zeta$ in (6) to obtain

$$\begin{aligned} D_\phi(l_1,l_2) &= -\int_\Theta l_2(\zeta)\ln\left(\frac{l_1(\zeta)}{l_2(\zeta)}\right)\diamond\zeta \\ &= \int_\Theta l_2(\zeta)\ln\left(\frac{l_2(\zeta)}{l_1(\zeta)}\right)\diamond\zeta = D(l_2,l_1). \end{aligned}$$

Proposition 7 *If* l_1, l_2 *satisfy* (8), *then*

$$D(l_2,l_1) = \int_\Theta l_2(\zeta)\ln\left(\frac{l_2(\zeta)}{l_1(\zeta)}\right)\diamond\zeta \le \ln I\left(\frac{1}{\mu_1},\frac{1}{\mu_2}\right) - \frac{1}{L(\mu_1,\mu_2)} + 1.$$

Proof Use $\phi(\zeta) = -\ln\zeta$ in (9) to obtain

$$\begin{aligned} D(l_2,l_1) &= \int_\Theta l_2(\zeta)\ln\left(\frac{l_2(\zeta)}{l_1(\zeta)}\right)\diamond\zeta \\ &\le \frac{(\mu_2-1)(-\ln\mu_1)}{\mu_2-\mu_1} + \frac{(1-\mu_1)(-\ln\mu_2)}{\mu_2-\mu_1} \\ &= \frac{(\mu_2-1)(-\ln\mu_1)+(1-\mu_1)(-\ln\mu_2)}{\mu_2-\mu_1} \\ &= \frac{(\mu_1\ln\mu_2-\mu_2\ln\mu_1)}{\mu_2-\mu_1} - \frac{\ln\mu_2-\ln\mu_1}{\mu_2-\mu_1} \\ &= \frac{\mu_1\mu_2(\frac{1}{\mu_2}\ln\mu_2-\frac{1}{\mu_1}\ln\mu_1)}{\mu_2-\mu_1} - \frac{\ln\mu_2-\ln\mu_1}{\mu_2-\mu_1} \\ &= \frac{\frac{1}{\mu_1}\ln\frac{1}{\mu_1}-\frac{1}{\mu_2}\ln\frac{1}{\mu_2}}{\frac{1}{\mu_1}-\frac{1}{\mu_2}} - \frac{1}{L(\mu_1,\mu_2)} \\ &= \ln I(\mu_1,\mu_2) - \frac{1}{L(\mu_1,\mu_2)} + 1, \end{aligned}$$

which completes the proof. □

Example 15 Select the set of real numbers as a time scale in Proposition 7 to obtain [13, Proposition 4].

Remark 12 Choose $\gamma = 1$ in Proposition 7 to obtain [5, Proposition 4].

Proposition 8 *If* l_1, l_2 *satisfy* (8), *then*

$$\begin{aligned} 0 &\le \ln I(\mu_1,\mu_2) - \frac{1}{L(\mu_1,\mu_2)} + 1 - D(l_2,l_1) \\ &= \frac{1}{G^2(\mu_1,\mu_2)}\big[(\mu_2-1)(1-\mu_1) - D_{\chi^2}(l_1,l_2)\big]. \end{aligned} \tag{33}$$

Proof Use $\phi(\zeta) = -\ln\zeta$ in Theorem 2 to obtain the desired result, since in this case

$$\frac{\phi'(\mu_2)-\phi'(\mu_1)}{\mu_2-\mu_1} = \frac{1}{\mu_1\mu_2} = \frac{1}{G^2(\mu_1,\mu_2)}.$$

□

Example 16 Select the set of real numbers as a time scale in Proposition 8 to obtain [13, Proposition 5].

Remark 13 Choose $\gamma = 1$ in Proposition 8 to obtain [5, Proposition 5].

In the following result, Theorem 3 is used to improve (33).

Proposition 9 *If the assumptions of Theorem 3 are true, then*

$$\frac{1}{2\mu_2^2}\big[(\mu_2-1)(1-\mu_1)-D_{\chi^2}(l_1,l_2)\big] \le \ln I(\mu_1,\mu_2) - \frac{1}{L(\mu_1,\mu_2)} + 1 - D(l_2,l_1) = \frac{1}{2\mu_1^2}\big[(\mu_2-1)(1-\mu_1)-D_{\chi^2}(l_1,l_2)\big].$$

Proof Use $\phi(\zeta) = -\ln\zeta$ in Theorem 3 to obtain the desired result, since in this case

$$\phi''(\zeta) = \frac{1}{\zeta^2} \quad \text{and} \quad \frac{1}{2\mu_2^2} \le \phi''(\zeta) \le \frac{1}{2\mu_1^2},$$

for each $\zeta \in [\mu_1,\mu_2]$. □

Example 17 Select the set of real numbers as a time scale in Proposition 9 to obtain [13, Proposition 6].

Remark 14 Choose $\gamma = 1$ in Proposition 9 to obtain [5, Proposition 6].

4.3 Triangular discrimination via diamond integral

If we use $\phi(\zeta) = \frac{(\zeta-1)^2}{\zeta+1}$ in (6), then we obtain triangular discrimination.

Definition 7 Triangular discrimination via diamond integral can be defined as follows:

$$D_{\triangle}(l_1,l_2) := \int_{\Theta} \frac{(l_2(\zeta)-l_1(\zeta))^2}{l_2(\zeta)+l_1(\zeta)} \diamond\zeta. \tag{34}$$

Proposition 10 *If the assumptions of Theorem 1 are true, then we have*

$$D_{\triangle}(l_1,l_2) \le \frac{4A(\mu_1,\mu_2) - 2G^2(\mu_1,\mu_2) - 2}{2A(\mu_1,\mu_2) + G^2(\mu_1,\mu_2) + 1}.$$

Proof Use $\phi(\zeta) = \frac{(\zeta-1)^2}{\zeta+1}$ in Theorem 1 to obtain

$$\begin{aligned} D_{\triangle}(l_1,l_2) &\le \frac{(\mu_2-1)(\mu_1-1)^2(\mu_2+1) + (1-\mu_1)(\mu_2-1)^2(\mu_1+1)}{(\mu_2-\mu_1)(\mu_2+1)(\mu_1+1)} \\ &\le \frac{(\mu_2^2-\mu_1^2) - (2\mu_1\mu_2^2 - 2\mu_1^2\mu_2) - (\mu_2-\mu_1)}{(\mu_2-\mu_1)(\mu_2+1)(\mu_1+1)} \\ &\le \frac{2(\mu_2-\mu_1)(\mu_2+\mu_1-\mu_2\mu_1-1)}{(\mu_2-\mu_1)(\mu_2+1)(\mu_1+1)} \\ &= \frac{2(\mu_2+\mu_1-\mu_2\mu_1-1)}{(\mu_2+1)(\mu_1+1)} \end{aligned}$$

$$= \frac{2(\mu_2 + \mu_1 - \mu_2\mu_1 - 1)}{\mu_2 + \mu_1 + \mu_2\mu_1 + 1}$$
$$= \frac{4A(\mu_1, \mu_2) - 2G^2(\mu_1, \mu_2) - 2}{2A(\mu_1, \mu_2) + G^2(\mu_1, \mu_2) + 1}. \qquad \square$$

Example 18 Select the set of real numbers as time scale in Proposition 10 to obtain

$$\int_\Theta \frac{(l_2(\zeta) - l_1(\zeta))^2}{l_2(\zeta) + l_1(\zeta)} d\zeta \leq \frac{4A(\mu_1, \mu_2) - 2G^2(\mu_1, \mu_2) - 2}{2A(\mu_1, \mu_2) + G^2(\mu_1, \mu_2) + 1}.$$

Remark 15 Choose $\gamma = 1$ in Proposition 10 to obtain [5, Proposition 7].

The next result provides a new bound for triangular discrimination in q-calculus.

Example 19 Select $\mathbb{T} = q^{\mathbb{N}_0}$ where $q > 1$, then $\zeta = q^m$ for some $m \in \mathbb{N}_0$. In Proposition 10, use $a_1 = q^r$ and $a_2 = q^s$ ($r < s$) to obtain

$$\sum_{j=r}^{s-1} q^{j+1} \frac{(l_2(q^j) - l_1(q^j))^2}{l_2(q^j) + l_1(q^j)} + \sum_{j=r+1}^{s} q^{j-1} \frac{(l_2(q^j) - l_1(q^j))^2}{l_2(q^j) + l_1(q^j)}$$
$$\leq \frac{4A(\mu_1, \mu_2) - 2G^2(\mu_1, \mu_2) - 2}{2A(\mu_1, \mu_2) + G^2(\mu_1, \mu_2) + 1}.$$

Proposition 11 *If the assumptions of Theorem 2 are true, then we have*

$$0 \leq \frac{4A(\mu_1, \mu_2) - 2G^2(\mu_1, \mu_2) - 2}{2A(\mu_1, \mu_2) + G^2(\mu_1, \mu_2) + 1} - D_\triangle(l_1, l_2)$$
$$\leq \frac{8A(\mu_1, \mu_2) + 8}{[2A(\mu_1, \mu_2) + G^2(\mu_1, \mu_2) + 1]^2} \big[(\mu_2 - 1)(1 - \mu_1) - D_{\chi^2}(l_1, l_2)\big].$$

Proof Use $\phi(\zeta) = \frac{(\zeta-1)^2}{\zeta+1}$ in Theorem 2 to obtain the desired result, since in this case

$$\frac{\phi'(\mu_2) - \phi'(\mu_1)}{\mu_2 - \mu_1} = \frac{4(\mu_2 + \mu_1 + 2)}{((\mu_2 + 1)(\mu_1 + 1))^2} = \frac{8A(\mu_1, \mu_2) + 8}{[2A(\mu_1, \mu_2) + G^2(\mu_1, \mu_2) + 1]^2}. \qquad \square$$

Proposition 12 *If the assumptions of Theorem 3 are true, then we have*

$$0 \leq \frac{8}{[1 + \mu_2]^3} \big[(\mu_2 - 1)(1 - \mu_1) - D_{\chi^2}(l_1, l_2)\big]$$
$$\leq \frac{4A(\mu_1, \mu_2) - 2G^2(\mu_1, \mu_2) - 2}{2A(\mu_1, \mu_2) + G^2(\mu_1, \mu_2) + 1} - D_\triangle(l_1, l_2)$$
$$\leq \frac{8}{[1 + \mu_1]^3} \big[(\mu_2 - 1)(1 - \mu_1) - D_{\chi^2}(l_1, l_2)\big].$$

Proof Use $\phi(\zeta) = \frac{(\zeta-1)^2}{\zeta+1}$ in Theorem 3 to obtain the desired result, since in this case

$$\phi''(\zeta) = \frac{8}{[1 + \zeta]^3} \text{and} \frac{8}{[1 + \mu_2]^3} \leq \phi''(\zeta) \leq \frac{8}{[1 + \mu_1]^3}$$

for each $\zeta \in [\mu_1, \mu_2]$. $\square$

4.4 Hellinger distance via diamond integral

If we use $\phi(\zeta) = \frac{(\sqrt{\zeta}-1)^2}{2}$ in (6), then we obtain Hellinger distance.

Definition 8 Hellinger distance via diamond integral can be defined as follows:

$$h^2(u,v) := \frac{1}{2}\int_{\Theta}\bigl(\sqrt{v(\zeta)}-\sqrt{u(\zeta)}\bigr)^2\diamond\zeta.$$

Proposition 13 *If the assumptions of Theorem 1 are true, then we have*

$$h^2(l_1,l_2) \le \frac{2A(\sqrt{\mu_1},\sqrt{\mu_2})-G^2(\mu_1,\mu_2)-1}{2A(\sqrt{\mu_1},\sqrt{\mu_2})}.$$

Proof Use $\phi(\zeta) = \frac{(\sqrt{\zeta}-1)^2}{2}$ in Theorem 1 to obtain

$$\begin{aligned}
h^2(l_1,l_2) &\le \frac{\frac{1}{2}(\mu_2-1)(\sqrt{\mu_1}-1)^2+\frac{1}{2}(1-\mu_1)(\sqrt{\mu_2}-1)^2}{(\mu_2-\mu_1)} \\
&= \frac{\frac{1}{2}(\sqrt{\mu_2}-1)(1-\sqrt{\mu_1})}{(\mu_2-\mu_1)}\bigl[(\sqrt{\mu_2}+1)(1-\sqrt{\mu_1})+(\sqrt{\mu_2}-1)(1+\sqrt{\mu_1})\bigr] \\
&= \frac{(\sqrt{\mu_2}-1)(1-\sqrt{\mu_1})(\sqrt{\mu_2}-\sqrt{\mu_1})}{(\mu_2-\mu_1)} = \frac{(\sqrt{\mu_2}-1)(1-\sqrt{\mu_1})}{(\sqrt{\mu_2}+\sqrt{\mu_1})} \\
&= \frac{\sqrt{\mu_2}+\sqrt{\mu_1}-\sqrt{\mu_1\mu_2}-1}{\sqrt{\mu_1}+\sqrt{\mu_2}} = \frac{2A(\sqrt{\mu_1},\sqrt{\mu_2})-G^2(\mu_1,\mu_2)-1}{2A(\sqrt{\mu_1},\sqrt{\mu_2})}.
\end{aligned}$$

□

Remark 16 Choose $\gamma = 1$ in Proposition 13 to obtain [5, Proposition 10].

The next example provides a new bound for Hellinger discrimination in q-calculus.

Example 20 Select $\mathbb{T} = q^{\mathbb{N}_0}$ where $q > 1$, then $\zeta = q^m$ for some $m \in \mathbb{N}_0$. In Proposition 13, use $a_1 = q^r$ and $a_2 = q^s$ $(r < s)$ to obtain

$$\begin{aligned}
&\frac{1}{2}\left[\sum_{j=r}^{s-1} q^{j+1}\bigl(\sqrt{v(q^j)}-\sqrt{u(q^j)}\bigr)^2+\sum_{j=r+1}^{s} q^{j-1}\bigl(\sqrt{v(q^j)}-\sqrt{u(q^j)}\bigr)^2\right] \\
&\quad\le \frac{2A(\sqrt{\mu_1},\sqrt{\mu_2})-G^2(\mu_1,\mu_2)-1}{2A(\sqrt{\mu_1},\sqrt{\mu_2})}.
\end{aligned}$$

Proposition 14 *If the assumptions of Theorem 2 are true, then we have*

$$\begin{aligned}
0 &\le \frac{(\sqrt{\mu_2}-1)(1-\sqrt{\mu_1})}{(\sqrt{\mu_2}+\sqrt{\mu_1})}-h^2(l_1,l_2) \\
&\le \frac{1}{4\sqrt{\mu_1\mu_2}A(\sqrt{\mu_1},\sqrt{\mu_2})}\bigl[(\mu_2-1)(1-\mu_1)-D_{\chi^2}(l_1,l_2)\bigr].
\end{aligned}$$

Proof Use $\phi(\zeta) = \frac{(\sqrt{\zeta}-1)^2}{2}$ in Theorem 2 to obtain the desired result, since in this case

$$\frac{\phi'(\mu_2)-\phi'(\mu_1)}{\mu_2-\mu_1} = \frac{1}{2\sqrt{\mu_1\mu_2}(\sqrt{\mu_1}+\sqrt{\mu_2})}.$$

□

Example 21 Select the set of real numbers as a time scale in Proposition 14 to obtain [13, Proposition 8].

Remark 17 Choose $\gamma = 1$ in Proposition 14 to obtain [5, Proposition 11].

Proposition 15 *If the assumptions of Theorem 3 are true, then we have*

$$\begin{aligned}0 &\le \frac{1}{8\sqrt{\mu_2^3}}\big[(\mu_2-1)(1-\mu_1)-D_{\chi^2}(l_1,l_2)\big]\\ &\le \frac{(\sqrt{\mu_2}-1)(1-\sqrt{\mu_1})}{(\sqrt{\mu_2}+\sqrt{\mu_1})}-h^2(l_1,l_2)\\ &\le \frac{1}{8\sqrt{\mu_1^3}}\big[(\mu_2-1)(1-\mu_1)-D_{\chi^2}(l_1,l_2)\big].\end{aligned}$$

Proof Use $\phi(\zeta)=\frac{(\sqrt{\zeta}-1)^2}{2}$ in Theorem 3 to obtain the desired result, since in this case

$$\phi''(\zeta)=\frac{1}{4\sqrt{\zeta^3}} \quad \text{and} \quad \frac{1}{8\sqrt{\mu_2^3}} \le \phi''(\zeta) \le \frac{1}{8\sqrt{\mu_1^3}},$$

for each $\zeta \in [\mu_1,\mu_2]$. □

Remark 18 Choose $\gamma = 1$ in Proposition 15 to obtain [5, Proposition 12].

4.5 Jeffreys distance via diamond integral

If we use $\phi(\zeta)=(\zeta-1)\ln\zeta$ in (6), then we obtain Jeffreys distance.

Definition 9 Jeffreys distance via diamond integral can be defined as follows:

$$D_J(l_1,l_2) := \int_\Theta \big(l_1(\zeta)-l_2(\zeta)\big)\ln\left[\frac{l_1(\zeta)}{l_2(\zeta)}\right]\diamond\zeta.$$

Proposition 16 *If*

$$\mu_1 \le \frac{l_2(\zeta)}{l_1(\zeta)} \le \mu_2,$$

for each $\zeta \in \mathbb{T}$, then

$$\begin{aligned}D_J(l_1,l_2) &= \int_\Theta \big(l_1(\zeta)-l_2(\zeta)\big)\ln\left[\frac{l_1(\zeta)}{l_2(\zeta)}\right]\diamond\zeta\\ &\le \ln I(\mu_1,\mu_2)-\frac{G^2(\mu_1,\mu_2)+1}{L(\mu_1,\mu_2)}+\ln I\left(\frac{1}{\mu_1},\frac{1}{\mu_2}\right)+2.\end{aligned}$$

Proof Use $\phi(\zeta) = (\zeta - 1)\ln\zeta$ in Theorem 1 to get

$$\begin{aligned}
D_J(l_1,l_2) &= \int_{\Theta} \bigl(l_1(\zeta) - l_2(\zeta)\bigr) \ln\left[\frac{l_1(\zeta)}{l_2(\zeta)}\right] \diamond\zeta \\
&\le \frac{\mu_2 - 1}{\mu_2 - \mu_1}(\mu_1 - 1)\ln\mu_1 + \frac{1-\mu_1}{\mu_2 - \mu_1}(\mu_2 - 1)\ln\mu_2 \\
&= \frac{(\mu_2 - 1)(\mu_1 - 1)\ln\mu_1 + (1-\mu_1)(\mu_2 - 1)\ln\mu_2]}{\mu_2 - \mu_1} \\
&= \frac{\mu_2\ln\mu_2 - \mu_1\ln\mu_1}{\mu_2 - \mu_1} - \frac{\mu_1\mu_2(\ln\mu_2 - \ln\mu_1)}{\mu_2 - \mu_1} \\
&\quad + \frac{\mu_1\ln\mu_2 - \mu_2\ln\mu_1}{\mu_2 - \mu_1} - \frac{\ln\mu_2 - \ln\mu_1}{\mu_2 - \mu_1} \\
&= \ln\left(\frac{\mu_2^{\mu_2}}{\mu_1^{\mu_1}}\right)^{\frac{1}{\mu_2 - \mu_1}} - \ln e + \ln e \\
&\quad - \frac{(\mu_1\mu_2 + 1)(\ln\mu_2 - \ln\mu_1)}{\mu_2 - \mu_1} + \frac{(\mu_1\mu_2)(\frac{\ln\mu_2}{\mu_2} - \frac{\ln\mu_1}{\mu_1})}{\mu_2 - \mu_1} \\
&= \ln I(\mu_1,\mu_2) - \frac{G^2(\mu_1,\mu_2) + 1}{L(\mu_1,\mu_2)} + \ln I\left(\frac{1}{\mu_1}, \frac{1}{\mu_2}\right) + 2.
\end{aligned}$$

□

Example 22 Select $q^{\mathbb{N}_0}$ with $q > 1$ as a time scale, then $\zeta = q^m$ for some $m \in \mathbb{N}_0$. Further, use $a_1 = q^r$ and $a_2 = q^s$ $(r < s)$ in Proposition 16 to obtain

$$\begin{aligned}
&\sum_{j=r}^{s-1} q^{j+1}\bigl(l_1(q^j) - l_2(q^j)\bigr)\ln\left[\frac{l_1(q^j)}{l_2(q^j)}\right] + \sum_{j=r+1}^{s} q^{j-1}\bigl(l_1(q^j) - l_2(q^j)\bigr)\ln\left[\frac{l_1(q^j)}{l_2(q^j)}\right] \\
&\quad \le \ln I(\mu_1,\mu_2) - \frac{G^2(\mu_1,\mu_2) + 1}{L(\mu_1,\mu_2)} + \ln I\left(\frac{1}{\mu_1}, \frac{1}{\mu_2}\right) + 2.
\end{aligned}$$

Proposition 17 *If the assumptions of Theorem 2 are true, then we have*

$$\begin{aligned}
0 &\le \ln I(\mu_1,\mu_2) - \frac{G^2(\mu_1,\mu_2) + 1}{L(\mu_1,\mu_2)} + \ln I\left(\frac{1}{\mu_1}, \frac{1}{\mu_2}\right) + 2 - D_J(l_1,l_2) \\
&\le \left[\frac{1}{G^2(\mu_1,\mu_2)} + \frac{1}{L(\mu_1,\mu_2)}\right]\bigl[(\mu_2 - 1)(1 - \mu_1) - D_{\chi^2}(l_1,l_2)\bigr].
\end{aligned}$$

Proof Use $\phi(\zeta) = (\zeta - 1)\ln\zeta$ in Theorem 2 to obtain the desired result, since in this case

$$\frac{\phi'(\mu_2) - \phi'(\mu_1)}{\mu_2 - \mu_1} = \frac{1}{\mu_1\mu_2} + \frac{\ln\mu_2 - \ln\mu_1}{\mu_2 - \mu_1} = \frac{1}{G^2(\mu_1,\mu_2)} + \frac{1}{L(\mu_1,\mu_2)}.$$

□

Proposition 18 *If the assumptions of Theorem 3 are true, then we have*

$$\begin{aligned}
0 &\le \frac{\mu_2 + 1}{\mu_2^2}\bigl[(\mu_2 - 1)(1 - \mu_1) - D_{\chi^2}(l_1,l_2)\bigr] \\
&\le \ln I(\mu_1,\mu_2) - \frac{G^2(\mu_1,\mu_2) + 1}{L(\mu_1,\mu_2)} + \ln I\left(\frac{1}{\mu_1}, \frac{1}{\mu_2}\right) + 2 - D_J(l_1,l_2) \\
&\le \frac{\mu_1 + 1}{\mu_1^2}\bigl[(\mu_2 - 1)(1 - \mu_1) - D_{\chi^2}(l_1,l_2)\bigr].
\end{aligned}$$

Proof Use $\phi(\zeta) = (\zeta - 1)\ln\zeta$ in Theorem 3 to obtain the desired result, since in this case

$$\phi''(\zeta) = \frac{\zeta + 1}{\zeta^2} \quad \text{and} \quad \frac{\mu_2 + 1}{\mu_2^2} \leq \phi''(\zeta) \leq \frac{\mu_1 + 1}{\mu_1^2},$$

for each $\zeta \in [\mu_1, \nu_2]$. □

5 Conclusion

In this work, Csiszár f-divergence for diamond integral has been introduced. Some inequalities for Csiszár f-divergence have been proved. Bounds of different divergence measures have been obtained in terms of some special means by using particular convex functions. The proved results are generalizations of the results provided in [5, 13]. This idea can be used to study different divergence notions on time scales like Jensen–Shannon divergence and Rényi divergence, etc.

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Competing interests
The authors declare no competing interests.

Author contributions
MB initiated the work and made calculations. KAK supervised and validated the draft. AN deduced the existing results and finalized the draft. JP dealt with the formal analysis and investigation. All the authors read and approved the final manuscript.

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