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On the Durrmeyer variant of q-Bernstein operators based on the shape parameter λ

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Abstract

In this work, we consider several approximation properties of a Durrmeyer variant of q-Bernstein operators based on Bézier basis with the shape parameter $\lambda \in [-1, 1]$. First, we calculate some moment estimates and show the uniform convergence of the proposed operators. Next, we investigate the degree of approximation with regard to the usual modulus of continuity, for elements of Lipschitz-type class and Peetre's K-functional, respectively. Finally, to compare the convergence behavior and consistency of the related operators, we demonstrate some convergence and error graphs for certain $\lambda \in [-1, 1]$ and q-integers.

MSC: 41A25; 41A35; 41A36

Keywords: Durrmeyer operators; q-Bernstein operators; Shape parameter λ; Lipschitz-type function; Peetre's K-functional

1 Introduction

In [1], Bernstein presented one of the most elegant and simplest proofs of Weierstrass's approximation theorem. An integral modification of the linear positive operators Bernstein was proposed and studied by Durrmeyer [2] as follows:

$$D_m(\mu;z) = (m+1)\sum_{r=0}^{m} b_{m,r}(z)\int_0^1 \mu(u)b_{m,r}(u)\,dt, \tag{1.1}$$

where $b_{m,r}(z) := \binom{m}{r} z^r(1-z)^{m-r}$ are Bernstein basis functions.

In 1997, Phillips [3] introduced several approximation theorems and also proved a Voronovskaya-type asymptotic relation of Bernstein operators including q-integers. Afterwards, Gupta et el. [4] established q-Durrmeyer operators and Mursaleen et al. [5] investigated some valuable approximation theorems on generalized q-Bernstein–Schurer operators. In 2008, Gupta [6] discussed a Durrmeyer type of q-Bernstein operators as:

$$D_{m,q}(\mu;z) = [m+1]_q \sum_{r=0}^{m} b_{m,r}(z;q)q^{-r}\int_0^1 b_{m,r}(qu;q)\mu(u)\,d_q u, \quad \bigl(z \in [0,1]\bigr), \tag{1.2}$$

where $\mu \in C[0,1]$, $q \in (0,1)$ and $b_{m,r}(z;q) := \left[{m \atop r}\right]_q z^r \prod_{r=0}^{m-r-1}(1-q^r z)$.

He estimated several moment estimates, order of approximation, and established a direct result of operators (1.2).

Now, we recall the following definitions of q-calculus (for details, see [7]). Let the integer $\nu > 0$ and $q \in (0,1]$. The q-integer $[\nu]_q$ is given by

$$[\nu]_q := \begin{cases} \frac{1-q^\nu}{1-q}, & q \neq 1, \\ \nu, & q = 1. \end{cases}$$

The q-factorial and q-binomial are given as follows, respectively:

$$[\nu]_q! := \begin{cases} [\nu]_q[\nu-1]_q..[1]_q, & \nu = 1,2,.., \\ 1, & \nu = 0 \end{cases}$$

and

$$\begin{bmatrix} \nu \\ l \end{bmatrix}_q := \frac{[\nu]_q!}{[l]_q![\nu-l]_q!}, \quad (\nu \geq l \geq 0).$$

The q-analog of the integration on $[0,A]$ is given by

$$\int_0^A \mu(u)\,d_q u := A(1-q)\sum_{j=0}^{\infty} \mu\bigl(Aq^j\bigr)q^j, \quad 0 < q < 1.$$

One of the fundamental topics of computer graphics (CG) and computer-aided geometric design (CAGD) is constructing basis functions with certain properties such as nonnegativity, total positivity, and partition of unity. This is why basis functions with attractive properties have a major role in construction of surfaces and curves. Many spline curves having shape parameters have been introduced for the purpose of controlling the shape of curves flexibly. In the 1970s, Pierre Bézier, who was an expert on the computer representations geometric design, considered Bernstein–Bézier basis functions to improve the shape modeling of car bodies. Nowadays, this type of basis function is extensively discussed in many applications fields such as; shock capturing, numerical solutions of hyperbolic conservation laws, 3D modeling, animation, seismic imaging, industrial CAGD, and font design. One may see the recent applications on CAGD: in [8], Khan et al. studied Bézier curves based on the (p,q)-analog of Bernstein operators, Kanat et al. [9, 10] investigated some important approximation theorems for (p,q)-analog Kantorovich and Stancu variants of Lupaş–Schurer operators, respectively. Oruç et al. [11] discussed q-Bernstein polynomials and Bézier curves and also one can see some valuable results in the book of Sederberg [12].

Very recently, Ye et al. [13] proposed a new type of Bézier basis functions dependent on a shape parameter $\lambda \in [-1,1]$. Inspired by this work, many authors added new papers to the literature in a short time. For instance; Cai et al. [14] discussed λ-Bernstein operators and investigated several approximation theorems, namely, the Korovkin-type approximation, the local approximation results, and the Voronovskaya-type theorem of these newly defined operators. In 2020, Özger [15] considered some statistical convergence propetries of the univariate and bivariate of Bernstein operators for $\lambda \in [-1,1]$. Moreover, Acu et al.

[16] examined the Kantorovich type of λ-Bernstein operators and calculated the degree of convergence in respect of Ditzian–Totik moduli of smoothness. Mursaleen et al. [17] constructed and obtained some approximation results of the Stancu–Chlodowsky kind of (λ, q)-Bernstein operators. A Kantorovich and Durrmeyer type of λ-Szász–Mirakjan operators is constructed with [18, 19] and also new classes of λ-Bernstein and its Kantorovich-type operators were established by Aslan [20, 21]. He studied several approximation properties such as; some direct theorems and local approximation results of these operators. Aliaga et al. [22] investigated some valuable convergence properties of the bivariate generalizations of the (λ, q)-Bernstein operators and its GBS (Generalized Boolean Sum)-type operators. Also, Cai et al. [23, 24] proposed and introduced a new generalization of q-Bernstein operators and Kantorovich-type q-Bernstein operators including the shape parameter λ. Rahman et al. [25] proposed and studied λ-Bernstein–Kantorovich operators with shifted knots. Özger [26] obtained several statistical approximation theorems of one- and two-dimesional λ-Kantorovich operators. In [27, 28], Özger et al. introduced Kantorovich-type λ-Schurer operators and the rate of weighted statistical convergence results of generalized blending-type Bernstein–Kantorovich operators. In 2021, Cai et al. [29] investigated some approximation properties and the q-statistical convergence of Stancu-type generalized Baskakov–Szász operators. Recently in [30], the Szász-type operators involving Charlier polynomials were studied and in [31], the convergence and other related properties of q-Bernstein-Kantorovich operators including the shifted knots of real positive numbers are investigated.

In the recent past, Cai et al. [32] obtained some statistical convergence properties of the following (λ, q)-Bernstein operators as:

$$\widetilde{B}_{m,q,\lambda}(\mu;z)=\sum_{r=0}^{m}\widetilde{b}_{m,r}(z;q,\lambda)\mu\left(\frac{[r]_q}{[m]_q}\right), \tag{1.3}$$

where

$$\begin{aligned}\widetilde{b}_{m,0}(z;q,\lambda)&=b_{m,0}(z;q)-\frac{\lambda}{[m]_q+1}b_{m+1,1}(z;q),\\ \widetilde{b}_{m,r}(z;q,\lambda)&=b_{m,r}(z;q)+\lambda\left(\frac{[m]_q-2[r]_q+1}{[m]_q^2-1}b_{m+1,r}(z;q)\right.\\ &\quad\left.-\frac{[m]_q-2[r]_q-1}{[m]_q^2-1}b_{m+1,r+1}(z;q)\right)\quad(r=1,2,\dots,m-1),\\ \widetilde{b}_{m,m}(z;q,\lambda)&=b_{m,m}(z;q)-\frac{\lambda}{[m]_q+1}b_{m+1,m}(z;q),\end{aligned} \tag{1.4}$$

$z\in[0,1]$, $m\geq 2$, $\lambda\in[-1,1]$, $q\in(0,1)$, and $b_{m,r}(z;q)$ are given by (1.1).

Encouraged by the above studies, we aim to study the following Durrmeyer-type λ-Bernstein operators including q-integers:

$$D_{m,q}^{\lambda}(\mu;z)=[m+1]_q\sum_{r=0}^{m}q^{-r}\widetilde{b}_{m,r}(z;q,\lambda)\int_0^1 b_{m,r}(qt;q)\mu(t)\,d_qt. \tag{1.5}$$

Here, $b_{m,r}(z;q)$ are given by (1.1) and $\widetilde{b}_{m,r}(z;q,\lambda)$ are given by (1.4).

Remark 1.1 For the operators given by (1.5) the following relations hold true:

▷ For $\lambda = 0$, operators (1.5) reduce to the Durrmeyer type of q-Bernstein operators constructed by Gupta [6].

▷ For $\lambda = 0$ and $q = 1$, operators (1.5) reduce to the Durrmeyer operators [2].

▷ For $q = 1$, operators (1.5) reduce to the Durrmeyer variant of λ-Bernstein-type operators constructed by Radu et al. [33].

This paper is organized as follows: In Sect. 2, we investigate some moment estimates. Next, in Sect. 3 we study the uniform convergence of operators (1.5) and investigate the pointwise error estimates with the aid of the usual modulus of continuity, elements of Lipschitz-type class, and Peetre's K-functional, respectively. Finally, the illustrative graphics that demonstrate the convergence behavior and consistency of the proposed operators are provided by MATLAB program.

2 Preliminaries

Lemma 2.1 (See [32]) *Let* $e_s(t) = t^s$, $s \in \mathbb{N} \cup \{0\}$, $q \in (0,1)$, $m > 1$, $z \in [0,1]$, *and* $\lambda \in [-1,1]$. *Then operators* (1.3) *satisfy*:

$$\widetilde{B}_{m,q,\lambda}(e_0;z) = 1, \tag{2.1}$$

$$\begin{aligned}\widetilde{B}_{m,q,\lambda}(e_1;z) &= z + \frac{[m+1]_q\lambda z(1-z^m)}{[m]_q([m]_q-1)} \\ &\quad - \frac{2[m+1]_q\lambda z}{[m]_q^2-1}\left(\frac{1-z^m}{[m]_q} + qz\left(1-z^{m-1}\right)\right) \\ &\quad + \frac{\lambda}{q[m]_q([m]_q+1)}\left(1-\prod_{r=0}^{m}\left(1-q^rz\right) - z^{m+1} - [m+1]_qz\left(1-z^m\right)\right) \\ &\quad + \frac{\lambda}{[m]_q^2-1}\left\{2[m+1]_qz^2\left(1-z^{m-1}\right) - \frac{2[m+1]_qz(1-z^m)}{q[m]_q}\right. \\ &\quad \left. + \frac{2}{q[m]_q}\left(1-\prod_{r=0}^{m}\left(1-q^rz\right) - z^{m+1}\right)\right\},\end{aligned} \tag{2.2}$$

$$\begin{aligned}\widetilde{B}_{m,q,\lambda}(e_2;z) &= z^2 + \frac{z(1-z)}{[m]_q} + \frac{[m+1]_q\lambda z}{[m]_q([m]_q-1)}\left(qz\left(1-z^{m-1}\right) + \frac{1-z^m}{[m]_q}\right) \\ &\quad - \frac{2[m+1]_q\lambda}{[m]_q([m]_q^2-1)}\left\{\frac{z(1-z^m)}{[m]_q} + q(q+2)z^2\left(1-z^{m-1}\right)\right. \\ &\quad \left. + q^3[m-1]_qz^3\left(1-y^{m-2}\right)\right\} - \frac{\lambda}{q[m]_q([m]_q+1)}\left\{[m+1]_qz^2\left(1-z^{m-1}\right)\right. \\ &\quad \left. - \frac{[m+1]_qz(1-z^m)}{q[m]_q} + \frac{1-\prod_{r=0}^{m}(1-q^rz)-z^{m+1}}{q[m]_q}\right\} + \frac{2\lambda}{[m]_q([m]_q^2-1)} \\ &\quad \times\left\{q[m-1]_q[m+1]_qz^3\left(1-z^{m-2}\right) - \frac{(1-q)[m+1]_qz^2(1-z^{m-1})}{q}\right. \\ &\quad \left. + \frac{[m+1]_qz(1-z^m)}{q^2[m]_q} - \frac{1-\prod_{r=0}^{m}(1-q^rz)-z^{m+1}}{q^2[m]_q}\right\}.\end{aligned} \tag{2.3}$$

Lemma 2.2 *Let $q \in (0,1)$, $\lambda \in [-1,1]$, $z \in [0,1]$, and $m > 1$. Then, for the operators (1.5) we arrive at*

$$D^{\lambda}_{m,q}(1;z) = 1, \tag{2.4}$$

$$\begin{aligned}
D^{\lambda}_{m,q}(t;z) &= \frac{z}{q} + \frac{q-(1+q)z}{q[m+2]_q} + \frac{\lambda zq(1-z^m)[m+1]_q}{[m+2]_q([m]_q-1)} \\
&\quad - \frac{2[m+1]_q\lambda zq}{[m+2]_q([m]_q^2-1)}\bigl[1-z^m+qz\bigl(1-z^{m-1}\bigr)[m]_q\bigr] \\
&\quad + \frac{\lambda}{[m+2]_q([m]_q+1)}\left[1-\prod_{r=0}^{m}\bigl(1-q^r z\bigr)-z^{m+1}-[m+1]_q z\bigl(1-z^m\bigr)\right] \\
&\quad + \frac{2\lambda}{[m+2]_q([m]_q^2-1)}\left[q[m]_q[m+1]_q z^2\bigl(1-z^{m-1}\bigr)-[m+1]_q z\bigl(1-z^m\bigr)\right. \\
&\quad \left. + 1-\prod_{r=0}^{m}\bigl(1-q^r z\bigr)-z^{m+1}\right],
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
D^{\lambda}_{m,q}\bigl(t^2;z\bigr) &= \frac{q^3[m]_q([m]_q-1)}{[m+2]_q[m+3]_q}z^2 + \frac{q^3 z(1-z)([m]_q-1)}{[m+2]_q[m+3]_q} \\
&\quad + \frac{q^3\lambda z(1-z)[m+1]_q}{[m+2]_q[m+3]_q}\left[q[m]_q z\bigl(1-z^{m-1}\bigr)+\frac{(1-z^m)}{[m]_q}\right] \\
&\quad - \frac{2q^3\lambda[m+1]_q}{[m+2]_q[m+3]_q([m]_q+1)}\left[\frac{z(1-z^m)}{[m]_q}+q(q+2)z^2\bigl(1-z^{m-1}\bigr)\right. \\
&\quad \left. + q^3 z^3[m-1]_q\bigl(1-z^{m-2}\bigr)\right] \\
&\quad - \frac{q^2\lambda([m]_q-1)}{[m+2]_q[m+3]_q([m]_q+1)}\left[[m+1]_q z^2\bigl(1-z^{m-1}\bigr)\right. \\
&\quad \left. - \frac{[m+1]_q z(1-z^m)+1-\prod_{r=0}^{m}(1-q^r z)-z^{m+1}}{q[m]_q}\right] \\
&\quad + \frac{2q^3\lambda}{[m+2]_q[m+3]_q([m]_q+1)} \\
&\quad \times \left[q[m-1]_q[m+1]_q z^3\bigl(1-z^{m-2}\bigr)-\frac{(1-q)[m+1]_q z^2(1-z^{m-1})}{q}\right. \\
&\quad \left. + \frac{[m+1]_q z(1-z^m)}{q^2[m]_q}-\frac{1-\prod_{r=0}^{m}(1-q^r z)-z^{m+1}}{q^2[m]_q}\right] + \frac{q(1+q)^2 z[m]_q}{[m+2]_q[m+3]_q} \\
&\quad + \frac{q(1+q)^2\lambda z[m+1]_q(1-z^m)}{[m+2]_q[m+3]_q([m]_q-1)} \\
&\quad - \frac{2q(1+q)^2\lambda z[m+1]_q}{[m+2]_q[m+3]_q([m]_q^2-1)}\bigl[1-z^m+qz[m]_q\bigl(1-z^{m-1}\bigr)\bigr] \\
&\quad - \frac{(1+q)^2\lambda}{[m+2]_q[m+3]_q([m]_q+1)} \\
&\quad \times \left[1-\prod_{r=0}^{m}\bigl(1-q^r z\bigr)-z^{m+1}-[m+1]_q z\bigl(1-z^m\bigr)\right]
\end{aligned}$$

$$+\frac{2(1+q)^2\lambda}{[m+2]_q[m+3]_q([m]_q^2-1)}$$
$$\times\left[q[m]_q[m+1]_q z^2\left(1-z^{m-1}\right)-[m+1]_q z\left(1-z^m\right)\right.$$
$$\left.+1-\prod_{r=0}^{m}\left(1-q^r z\right)-z^{m+1}\right]+\frac{1+q}{[m+2]_q[m+3]_q}. \tag{2.6}$$

Proof Using the definition of q-Beta functions (see [7]), for $u = 0, 1, 2, 3,..$ it follows that

$$\int_0^1 b_{m,r}(qt;q)t^u\,d_qt=\frac{q^r[m]_q![r+u]_q!}{[m+u+1]_q![r]_q!}.$$

Further, we obtain

$$\begin{aligned}D^{\lambda}_{m,q}(1;z)&=[m+1]_q\sum_{r=0}^{m}q^{-r}\widetilde{b}_{m,r}(z;q,\lambda)\int_0^1 b_{m,r}(qt;q)\,d_qt\\&=[m+1]_q\sum_{r=0}^{m}q^{-r}\widetilde{b}_{m,r}(z;q,\lambda)\frac{q^r[m]_q![r]_q!}{[m+1]_q![r]_q!}\\&=\sum_{r=0}^{m}\widetilde{b}_{m,r}(z;q,\lambda)\\&=\widetilde{B}_{m,q,\lambda}(1;y)=1.\end{aligned}$$

Therefore, we find (2.4),

$$\begin{aligned}D^{\lambda}_{m,q}(t;z)&=[m+1]_q\sum_{r=0}^{m}q^{-r}\widetilde{b}_{m,r}(z;q,\lambda)\int_0^1 b_{m,r}(qt;q)t\,d_qt\\&=[m+1]_q\sum_{r=0}^{m}q^{-r}\widetilde{b}_{m,r}(z;q,\lambda)\frac{q^r[m]_q![r+1]_q!}{[m+2]_q![r]_q!}\\&=\sum_{r=0}^{m}\widetilde{b}_{m,r}(z;q,\lambda)\frac{[r+1]_q}{[m+2]_q}.\end{aligned}$$

Using $[r+1]_q = 1+q[r]_q$, we obtain

$$\begin{aligned}D^{\lambda}_{m,q}(t;z)&=\frac{1}{[m+2]_q}\sum_{r=0}^{m}\widetilde{b}_{m,r}(z;q,\lambda)\left(1+q[r]_q\right)\\&=\frac{q[m]_q}{[m+2]_q}\sum_{r=0}^{m}\widetilde{b}_{m,r}(z;q,\lambda)\frac{[r]_q}{[m]_q}+\frac{1}{[m+2]_q}\sum_{r=0}^{m}\widetilde{b}_{m,r}(z;q,\lambda)\\&=\frac{q[m]_q}{[m+2]_q}\widetilde{B}_{m,q,\lambda}(t;z)+\frac{1}{[m+2]_q}\widetilde{B}_{m,q,\lambda}(1;z).\end{aligned}$$

Taking into consideration (2.1) and (2.2), we obtain (2.5),

$$D^{\lambda}_{m,q}\left(t^2;z\right)=[m+1]_q\sum_{r=0}^{m}q^{-r}\widetilde{b}_{m,r}(z;q,\lambda)\int_0^1 b_{m,r}(qt;q)t^2\,d_qt$$

$$= [m+1]_q \sum_{r=0}^{m} q^{-r}\widetilde{b}_{m,r}(z;q,\lambda)\frac{q^r[m]_q![r+2]_q!}{[m+3]_q![r]_q!}$$
$$= \sum_{r=0}^{m} \widetilde{b}_{m,r}(z;q,\lambda)\frac{[r+2]_q[r+1]_q}{[m+2]_q[m+3]_q}.$$

By the fact that $[r+1]_q = 1 + q[r]_q$ and $[r]_q^2 = [r]_q(q[r-1]_q + 1)$, we find that

$$\begin{aligned}
D_{m,q}^{\lambda}(t^2;z) &= \frac{1}{[m+2]_q[m+3]_q}\left\{\sum_{r=0}^{m}\widetilde{b}_{m,r}(z;q,\lambda)\big(1+q[r+1]_q\big)\big(1+q[r]_q\big)\right\} \\
&= \frac{1}{[m+2]_q[m+3]_q}\left\{\sum_{r=0}^{m}\widetilde{b}_{m,r}(z;q,\lambda)\big(1+q+q^2[r]_q\big)\big(1+q[r]_q\big)\right\} \\
&= \frac{1}{[m+2]_q[m+3]_q}\left\{\sum_{r=0}^{m}\widetilde{b}_{m,r}(z;q,\lambda)\big(q^4[r]_q[r-1]_q + q(1+q)^2[r]_q + 1 + q\big)\right\} \\
&= \frac{q^3[m]_q([m]_q-1)}{[m+2]_q[m+3]_q}\widetilde{B}_{m,q,\lambda}(t^2;z) + \frac{q(1+q)^2[m]_q}{[m+2]_q[m+3]_q}\widetilde{B}_{m,q,\lambda}(t;z) \\
&\quad + \frac{1+q}{[m+2]_q[m+3]_q}\widetilde{B}_{m,q,\lambda}(1;z).
\end{aligned}$$

Consequently, by (2.1), (2.2), and (2.3), we arrive at (2.6). Thus, we arrive at the desired result. □

Lemma 2.3 *Let $q \in (0,1)$, $\lambda \in [-1,1]$, $z \in [0,1]$, and $m > 1$. From Lemma 2.2, we arrive at*

$$\begin{aligned}
D_{m,q}^{\lambda}(t-z;z) &= \frac{1-q}{q}z + \frac{(1+q)z}{[m+2]_q} + \frac{\lambda qz(1-z^m)}{[m+2]_q([m]_q-1)} \\
&\quad - \frac{2\lambda q[m+1]_q z}{[m+2]_q([m]_q^2-1)}\big[1 - z^m + q[m]_q z\big(1-z^{m-1}\big)\big] \\
&\quad + \frac{\lambda}{[m+2]_q([m]_q+1)}\left[1 - \prod_{r=0}^{m}\big(1-q^r z\big) - z^{m+1} - [m+1]_q z\big(1-z^m\big)\right] \\
&\quad + \frac{2\lambda}{[m+2]_q([m]_q^2-1)}\left[q[m]_q[m+1]_q z^2\big(1-z^{m-1}\big) - 2[m+1]_q z\big(1-z^m\big)\right. \\
&\quad \left. + 1 - \prod_{r=0}^{m}\big(1-q^r z\big) - z^{m+1}\right] \\
&=: \gamma_m(z,q) \qquad (2.7) \\
&\leq \frac{1-q}{q} + \frac{1}{[m+2]_q} + \frac{4[m]_q}{[m]_q^2-1} + \frac{1}{[m+2]_q([m]_q+1)} \\
&=: \alpha_m(q), \qquad (2.8)
\end{aligned}$$

$$\begin{aligned}
D_{m,q}^{\lambda}\big((t-z)^2;z\big) &\leq \frac{(1-q)^2}{q^2} + \frac{2[2]_q}{q[m+2]_q} + \frac{[2]_q(1+2q+q^3)}{q[m+2]_q[m+3]_q} \\
&\quad + \frac{3}{[m+3]_q} + \frac{1}{[m]_q[m+3]_q} + \frac{12}{[m+3]_q([m]_q+1)}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{3}{[m]_q[m+3]_q([m]_q+1)} + \frac{16}{[m+3]_q([m]_q^2-1)} \\
&+ \frac{8}{[m+2]_q[m+3]_q([m]_q+1)} + \frac{8}{[m]_q^2-1} + \frac{6}{[m]_q-1} + \frac{2}{[m]_q+1} \\
&=: \beta_m(q).
\end{aligned} \tag{2.9}
$$

Proof One can obtain (2.7) easily by simple computation from (2.4) and (2.5). When $\lambda \in [0,1]$, we arrive at

$$
\begin{aligned}
D_{m,q}^{\lambda}(t-z;z) \leq{}& \frac{1-q}{q} + \frac{2}{[m+2]_q} + \frac{q[m+1]_q}{[m+2]_q([m]_q-1)} \\
&+ \frac{1}{[m+2]_q([m]_q+1)} + \frac{2q[m]_q[m+1]_q}{[m+2]_q([m]_q^2-1)}.
\end{aligned} \tag{2.10}
$$

Also, for $\lambda \in [-1,0]$, we obtain

$$
\begin{aligned}
D_{m,q}^{\lambda}(t-z;z) \leq{}& \frac{1-q}{q} + \frac{2}{[m+2]_q} + \frac{2q^2[m]_q[m+1]_q}{[m+2]_q([m]_q^2-1)} \\
&+ \frac{1+[m+1]_q}{[m+2]_q([m]_q+1)} + 2\frac{1+[m+1]_q}{[m+2]_q([m]_q^2-1)}.
\end{aligned} \tag{2.11}
$$

If we combine (2.10) and (2.11), this becomes

$$
D_{m,q}^{\lambda}(t-z;z) \leq \frac{1-q}{q} + \frac{2}{[m+2]_q} + \frac{4[m]_q}{[m]_q^2-1} + \frac{1}{[m+2]_q([m]_q+1)}.
$$

On the other hand, applying the linearity of operator (1.5), we obtain

$$
\begin{aligned}
D_{m,q}^{\lambda}\big((t-z)^2;z\big) ={}& D_{m,q}^{\lambda}\big(t^2;z\big) - 2zD_{m,q}^{\lambda}(t;z) + z^2 D_{m,q}^{\lambda}(1;z) \\
\leq{}& \left(\frac{q^3[m]_q([m]_q-1)}{[m+2]_q[m+3]_q} - \frac{2}{q} + 1\right)z^2 + \frac{q^3 z(1-z)([m]_q-1)}{[m+2]_q[m+3]_q} \\
&+ \frac{q(1+q)^2 z[m]_q}{[m+2]_q[m+3]_q} + \frac{1+q}{[m+2]_q[m+3]_q} + \frac{2(1+q)z}{q[m+2]_q} \\
\leq{}& \frac{(1-q)^2}{q^2} + \frac{2[2]_q}{q[m+2]_q} + \frac{[2]_q(1+2q+q^3)}{q[m+2]_q[m+3]_q} + \phi_{m,q}^{\lambda}(z),
\end{aligned}
$$

where the $\phi_{m,q}^{\lambda}(z)$ function depends on λ, $[m]_q$ and z. Now, we will calculate $\phi_{m,q}^{\lambda}(z)$ for $\lambda \in [0,1]$ and $\lambda \in [-1,0]$. When $\lambda \in [0,1]$, we arrive at

$$
\begin{aligned}
\phi_{m,q}^{\lambda}(z) \leq{}& \frac{3}{[m+3]_q} + \frac{1}{[m]_q[m+3]_q} + \frac{1}{[m+3]_q([m]_q+1)} \\
&+ \frac{2}{[m]_q[m+3]_q([m]_q+1)} + \frac{8}{[m+3]_q([m]_q-1)} \\
&+ \frac{4}{[m+2]_q[m+3]_q([m]_q+1)} + \frac{8}{[m]_q^2-1} + \frac{1}{[m]_q-1} + \frac{2}{[m]_q+1}.
\end{aligned} \tag{2.12}
$$

When $\lambda \in [-1,0]$, we obtain

$$\begin{aligned}\phi_{m,q}^{\lambda}(z) \leq{}& \frac{3}{[m+3]_q} + \frac{12}{[m+3]_q([m]_q+1)} + \frac{3}{[m]_q[m+3]_q([m]_q+1)}\\&+ \frac{8}{[m+2]_q[m+3]_q([m]_q+1)} + \frac{2}{[m]_q[m+3]_q([m]_q+1)}\\&+ \frac{16}{[m+3]_q([m]_q^2-1)} + \frac{8}{[m]_q^2-1} + \frac{6}{[m]_q-1}\\&+ \frac{2}{[m+2]_q([m]_q+1)}.\end{aligned} \tag{2.13}$$

According to (2.12) and (2.13), therefore

$$\begin{aligned}\phi_{m,q}^{\lambda}(z) \leq{}& \frac{3}{[m+3]_q} + \frac{1}{[m]_q[m+3]_q}\frac{12}{[m+3]_q([m]_q+1)}\\&+ \frac{3}{[m]_q[m+3]_q([m]_q+1)} + \frac{16}{[m+3]_q([m]_q^2-1)}\\&+ \frac{8}{[m+2]_q[m+3]_q([m]_q+1)} + \frac{8}{[m]_q^2-1}\\&+ \frac{6}{[m]_q-1} + \frac{2}{[m]_q+1}.\end{aligned} \tag{2.14}$$

If we combine (2.12), (2.13), and (2.14), we have

$$\begin{aligned}D_{m,q}^{\lambda}\big((t-z)^2;z\big) \leq{}& \frac{(1-q)^2}{q^2} + \frac{2[2]_q}{q[m+2]_q} + \frac{[2]_q(1+2q+q^3)}{q[m+2]_q[m+3]_q}\\&+ \frac{3}{[m+3]_q} + \frac{1}{[m]_q[m+3]_q} + \frac{12}{[m+3]_q([m]_q+1)}\\&+ \frac{3}{[m]_q[m+3]_q([m]_q+1)} + \frac{16}{[m+3]_q([m]_q^2-1)}\\&+ \frac{8}{[m+2]_q[m+3]_q([m]_q+1)} + \frac{8}{[m]_q^2-1} + \frac{6}{[m]_q-1} + \frac{2}{[m]_q+1}.\end{aligned}$$

Then, the desired result of (2.9) is proved. □

Remark 2.4 It is known that for $0 < q < 1$, $\lim_{m\to\infty}[m]_q = \frac{1}{1-q}$. To supply the approximation outcomes, we use a sequence $q := \{q_m\}$ so that $0 < q_m < 1$, $q_m \to 1$, $\frac{1}{[m]_{q_m}} \to 0$ as $m \to \infty$.

3 Order of convergence of operators $D_{m,q}^{\lambda}$

In this section, we discuss the uniform approximation of operators $D_{m,q}^{\lambda}(\mu;z)$. Let $C[0,1]$ be the space of all real-valued continuous functions on $[0,1]$ endowed with the sup-norm $\|\mu\|_{C[0,1]} = \sup_{z\in[0,1]}|\mu(z)|$.

Theorem 3.1 *Suppose that the sequence $q := \{q_m\}$ satisfies the terms given by Remark 2.4. Then, for the operators $D_{m,q}^{\lambda}(\mu;z)$, we arrive at*

$$\lim_{m\to\infty}\big\|D_{m,q_m}^{\lambda}(\mu;.)-\mu\big\|_{C[0,1]} = 0. \tag{3.1}$$

Proof Consider the sequence of functions $e_r(z) = z^r$, where $r \in \{0,1,2\}$ and $z \in [0,1]$. According to the Bohman–Korovkin theorem [34], it is sufficient to verify that

$$\lim_{m\to\infty} \left\| D^{\lambda}_{m,q_m}(e_r;.) - e_r \right\|_{C[0,1]} = 0, \quad \text{for } r = 0,1,2. \tag{3.2}$$

The proof of (3.2) follows easily, using equations (2.4), (2.5), and (2.6). □

Moreover, we investigate the degree of approximation in respect of the usual modulus of continuity, for the class of the Lipschitz continuous functions and the K-functional of Peetre. We denote the usual modulus of continuity as:

$$\omega(\mu;\nu) := \sup_{0<s\leq\nu} \sup_{z\in[0,1]} \left|\mu(z+s) - \mu(z)\right|.$$

For $\omega(\mu;\nu)$ one has some detailed properties with [35]. In addition, we define an element of a Lipschitz function with $Lip_L(\zeta)$, where $L > 0$ and $0 < \zeta \leq 1$. A function μ is related to $Lip_L(\zeta)$, if

$$\left|\mu(t) - \mu(z)\right| \leq L|t-z|^{\zeta}, \quad (t,z \in \mathbb{R}).$$

The K-functional of Peetre is given by

$$K_2(\mu,\nu) = \inf_{\tau\in C^2[0,1]} \left\{ \|\mu - \tau\|_{C[0,1]} + \nu \left\|\tau''\right\|_{C[0,1]} \right\},$$

where $\nu > 0$ and $C^2[0,1] = \{\tau \in C[0,1] : \tau', \tau'' \in C[0,1]\}$.

According to [36], there exists a constant $C > 0$, then

$$K_2(\mu;\nu) \leq C\omega_2(\mu;\sqrt{\nu}), \quad \nu > 0, \tag{3.3}$$

where

$$\omega_2(\mu;\nu) = \sup_{0<l\leq\nu} \sup_{z\in[0,1]} \left|\mu(z+2l) - 2\mu(z+l) + \mu(z)\right|,$$

is the second degree moduli of smoothness.

Theorem 3.2 *Suppose that* $z \in [0,1]$, $\mu \in C[0,1]$, $\lambda \in [-1,1]$, *and* $m > 1$. *Then, one has*

$$\left|D^{\lambda}_{m,q_m}(\mu;z) - \mu(z)\right| \leq 2\omega\left(\mu;\sqrt{\beta_m(q_m)}\right),$$

where $\beta_m(q_m)$ *is defined in* (2.9).

Proof Using the property $|\mu(t) - \mu(z)| \leq (1 + \frac{|t-z|}{\delta})\omega(\mu;\delta)$ and applying $D^{\lambda}_{m,q_m}(.;z)$, yields

$$\left|D^{\lambda}_{m,q_m}(\mu;z) - \mu(z)\right| \leq \left(1 + \frac{1}{\delta} D^{\lambda}_{m,q_m}\left(|t-z|;z\right)\right)\omega(\mu;\delta).$$

By the Cauchy–Bunyakovsky–Schwarz inequality, we arrive at

$$\begin{aligned}\left|D^{\lambda}_{m,q_m}(\mu;z)-\mu(z)\right| &\leq \left(1+\frac{1}{\delta}\sqrt{D^{\lambda}_{m,q_m}\left((t-z)^2;z\right)}\right)\omega(\mu;\delta)\\ &\leq \left(1+\frac{1}{\delta}\sqrt{\beta_m(q_m)}\right)\omega(\mu;\delta).\end{aligned}$$

Taking $\delta=\sqrt{\beta_m(q_m)}$ gives the desired result. □

Theorem 3.3 *Suppose that $q:=\{q_m\}$ satisfies the terms given by Remark 2.4. For all $\mu\in Lip_L(\zeta)$, $\lambda\in[-1,1]$, $z\in[0,1]$, $0<\zeta\leq 1$, and $m>1$, we obtain*

$$\left|D^{\lambda}_{m,q_m}(\mu;z)-\mu(z)\right|\leq L\left(\beta_m(q_m)\right)^{\frac{\zeta}{2}}.$$

Proof Let $\mu\in Lip_L(\zeta)$. Taking into account operators (1.4), we arrive at

$$\begin{aligned}&\left|D^{\lambda}_{m,q_m}(\mu;z)-\mu(z)\right|\\ &\quad\leq D^{\lambda}_{m,q_m}\left(\left|\mu(t)-\mu(z)\right|;z\right)\\ &\quad=[m+1]_{q_m}\sum_{r=0}^{m}\widetilde{b}_{m,q_m,\lambda}(z)q_m^{-r}\int_0^1 b_{m,r}(q_mt;q_m)\left|\mu(t)-\mu(z)\right|d_{q_m}t\\ &\quad\leq L\left[[m+1]_{q_m}\sum_{r=0}^{m}\widetilde{b}_{m,q_m,\lambda}(z)q_m^{-r}\int_0^1 b_{m,r}(q_mt;q_m)|t-z|^{\zeta}\,d_{q_m}t\right].\end{aligned}$$

Utilizing Hölder's inequality and for $p_1=\frac{2}{\zeta}$ and $p_2=\frac{2}{2-\zeta}$, one has $\frac{1}{p_1}+\frac{1}{p_2}=1$. Therefore, we may write

$$\begin{aligned}&\left|D^{\lambda}_{m,q}(\mu;z)-\mu(z)\right|\\ &\quad\leq L\left[[m+1]_{q_m}\sum_{r=0}^{m}\widetilde{b}_{m,q_m,\lambda}(z)q_m^{-r}\int_0^1 b_{m,r}(qt;q)\,d_{q_m}t\right]^{\frac{2-\zeta}{2}}\\ &\qquad\cdot\left[[m+1]_{q_m}\sum_{r=0}^{m}\widetilde{b}_{m,q_m,\lambda}(z)q_m^{-r}\int_0^1 b_{m,r}(qt;q)(t-z)^2\,d_{q_m}t\right]^{\frac{\zeta}{2}}\\ &\quad=L\left(D^{\lambda}_{m,q}\left((t-z)^2;z\right)\right)^{\frac{\zeta}{2}}\leq L\left(\beta_m(q_m)\right)^{\frac{\zeta}{2}}.\end{aligned}$$

Then, we find the desired result. □

Theorem 3.4 *Let $z\in[0,1]$, $\mu\in C[0,1]$, and $\lambda\in[-1,1]$. Then, we obtain*

$$\left|D^{\lambda}_{m,q}(\mu;z)-\mu(z)\right|\leq C\omega_2\left(\mu;\frac{1}{2}\sqrt{\alpha_m(q)^2+\beta_m(q)}\right)+\omega\left(\mu;\alpha_m(q)\right),$$

where a positive constant $C>0$, $\alpha_m(q)$, and $\beta_m(q)$ are given by Lemma 2.3.

Proof First, we use the auxiliary operators below:

$$\widetilde{D}_{m,q}^{\lambda}(\mu;z) = D_{m,q}^{\lambda}(\mu;z) - \mu\big(D_{m,q}^{\lambda}(t;z)\big) + \mu(z). \tag{3.4}$$

According to (2.4) and (2.5), we find $\widetilde{D}_{m,q}^{\lambda}(t-z;z) = 0$.

From Taylor's formula, we have

$$\zeta(t) = \zeta(z) + (t-z)\zeta'(z) + \int_z^t (t-u)\zeta''(u)\,du, \quad \big(\zeta \in C^2[0,1]\big). \tag{3.5}$$

Operating $D_{m,q}^{\lambda}(.;z)$ on (3.5) gives

$$\begin{aligned}
\widetilde{D}_{m,q}^{\lambda}(\zeta;z) - \zeta(z) &= \widetilde{D}_{m,q}^{\lambda}\big((t-z)\zeta'(z);z\big) + \widetilde{D}_{m,q}^{\lambda}\left(\int_z^t (t-u)\zeta''(u)\,du;z\right) \\
&= \zeta'(z)\widetilde{D}_{m,q}^{\lambda}(t-z;z) + D_{m,q}^{\lambda}\left(\int_z^t (t-u)\zeta''(u)\,du;z\right) \\
&\quad - \int_z^{D_{m,q}^{\lambda}(t;z)} \big(D_{m,q}^{\lambda}(t;z) - u\big)\zeta''(u)\,du \\
&= D_{m,q}^{\lambda}\left(\int_z^t (t-u)\zeta''(u)\,du;z\right) - \int_z^{D_{m,q}^{\lambda}(t;z)} \big(D_{m,q}^{\lambda}(t;z) - u\big)\zeta''(u)\,du.
\end{aligned}$$

From (3.4), we obtain

$$\begin{aligned}
&\big|\widetilde{D}_{m,q}^{\lambda}(\zeta;z) - \zeta(z)\big| \\
&\quad\le \left|D_{m,q}^{\lambda}\left(\int_z^t (t-u)\zeta''(u)\,du;z\right)\right| + \left|\int_z^{D_{m,q}^{\lambda}(t;z)} \big(D_{m,q}^{\lambda}(t;z) - u\big)\zeta''(u)\,du\right| \\
&\quad\le D_{m,q}^{\lambda}\left(\left|\int_z^t (t-u)\right|\big|\zeta''(u)\big|\,du;z\right) + \int_z^{D_{m,q}^{\lambda}(t;z)} \big|D_{m,q}^{\lambda}(t;z) - u\big|\big|\zeta''(u)\big|\,du \\
&\quad\le \|\zeta''\|\big\{D_{m,q}^{\lambda}\big((t-z)^2;z\big) + \big(D_{m,q}^{\lambda}(t;z) - z\big)^2\big\} \\
&\quad\le \big\{\beta_m(q) + \big(\alpha_m(q)\big)^2\big\}\|\zeta''\|_{C[0,1]}.
\end{aligned}$$

Moreover by (2.4), (2.5), and (3.4), we yield

$$\begin{aligned}
\big|\widetilde{D}_{m,q}^{\lambda}(\mu;z)\big| &\le \big|D_{m,q}^{\lambda}(\mu;z)\big| + 2\|\mu\|_{C[0,1]} \le \|\mu\|_{C[0,1]} D_{m,q}^{\lambda}(1;z) + 2\|\mu\|_{C[0,1]} \\
&\le 3\|\mu\|_{C[0,1]}.
\end{aligned} \tag{3.6}$$

In view of (3.5) and (3.6), we obtain

$$\begin{aligned}
\big|D_{m,q}^{\lambda}(\mu;z) - \mu(z)\big| &\le \big|\widetilde{D}_{m,q}^{\lambda}(\mu-\zeta;z) - (\mu-\zeta)(z)\big| \\
&\quad + \big|\widetilde{D}_{m,q}^{\lambda}(\zeta;z) - \zeta(z)\big| + \big|\mu(z) - \mu\big(D_{m,q}^{\lambda}(t;z)\big)\big| \\
&\le 4\|\mu-\zeta\|_{C[0,1]} + \big\{\beta_m(q) + \big(D_{m,q}^{\lambda}(t;z)\big)^2\big\}\|\zeta''\|_{C[0,1]} + \omega\big(\mu;\alpha_m(q)\big).
\end{aligned}$$

Applying the infimum over $\zeta \in C^2[0,1]$ and by (3.3), we have

$$\begin{aligned} \left|D_{m,q}^{\lambda}(\mu;z) - \mu(z)\right| &\le 4K_2\left(\mu; \frac{\{\beta_m(q) + (\alpha_m(q))^2\}}{4}\right) + \omega\bigl(\mu;\alpha_m(q)\bigr) \\ &\le C\omega_2(\mu; \frac{1}{2}\sqrt{\alpha_m(q)^2 + \beta_m(q)} + \omega\bigl(\mu;\alpha_m(q)\bigr). \end{aligned}$$

Thus, the desired result is obtained. □

4 Graphical analysis

In this section, we demonstrate certain numerical experiments including some numerical comparative results to see the approximation accuracy of the operators $D_{m,q}^{\lambda}(\mu;y)$.

Example 4.1 Let $\mu(y) = (y - 1/4)\sin(2\pi y)$, $q := \{q_m\} = 1 - 1/m$, and $|D_{m,q}^{\lambda}(\mu;y) - \mu(y)|$ be the error function of operators (λ,q)-Durrmeyer.

In Fig. 1, we show the graphs of function $\mu(y)$ and operators $D_{m,q}^{\lambda}$ for $\lambda = 0.5$, $m = 50$, $m = 60$, and $m = 80$, respectively. Also, we illustrate its error of the approximation process in Fig. 2.

For $\lambda = 0$, the convergences of operators $D_{m,q}^{\lambda}$ to $\mu(y)$ for $m = 50, 60, 80$ are shown in Fig. 3 and its error of the approximation process is illustrated by Fig. 4.

For $\lambda = -0.5$, the convergences of operators $D_{m,q}^{\lambda}$ to $\mu(y)$ for $m = 50, 60, 80$ are shown in Fig. 5 and its error of the approximation process is illustrated by Fig. 6.

From the graphs below, we can conclude: As the value of m increases at certain points of the given interval, then the approximation of the operators $D_{m,q}^{\lambda}$ to the function μ becomes better. With the different values of shape parameter λ, one has more modeling flexibility.

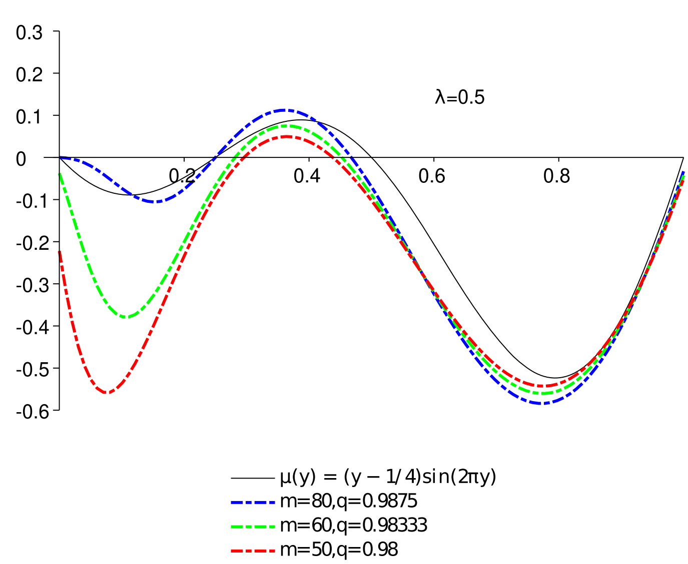

Figure 1 The approximation of $D_{m,q}^{0.5}(\mu;y)$ to $\mu(y)$ for fixed values of q and m

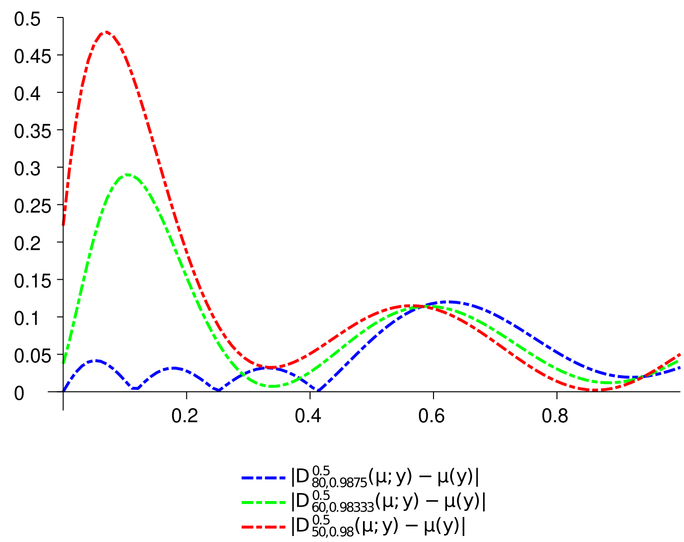

Figure 2 The error of approximation $D^{0.5}_{m,q}(\mu;y)$ to $\mu(y)$ for fixed values of q and m

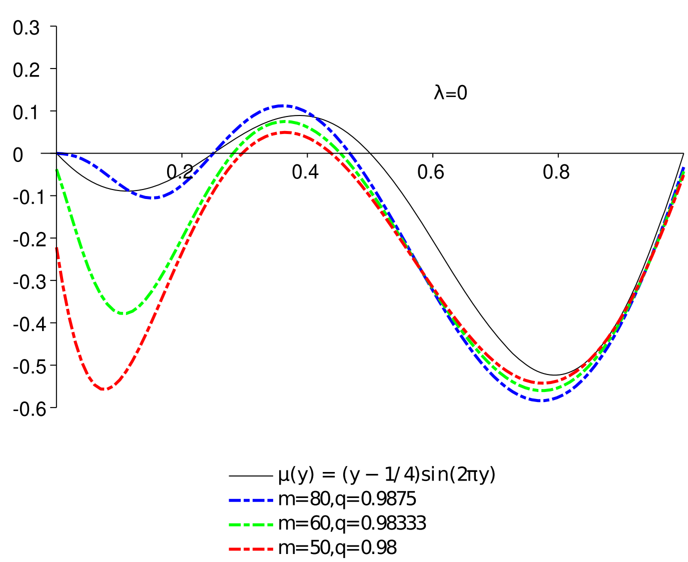

Figure 3 The approximation of $D^{0}_{m,q}(\mu;y)$ to $\mu(y)$ for fixed values of q and m

5 Conclusion

In the present work, we established certain convergence properties of the Durrmeyer-type q-Bernstein operators that includes the shape parameter λ on $[-1,1]$. We studied Korovkin type convergence theorem and estimated the order of convergence by means of the usual modulus of continuity, for the elements of the Lipschitz-type class and Peetre's K-functional. To make our research more intuitive, several graphical illustrations are presented. As a result, the $\lambda \in [-1,1]$ and $q \in (0,1]$ give us more modeling in flexibility.

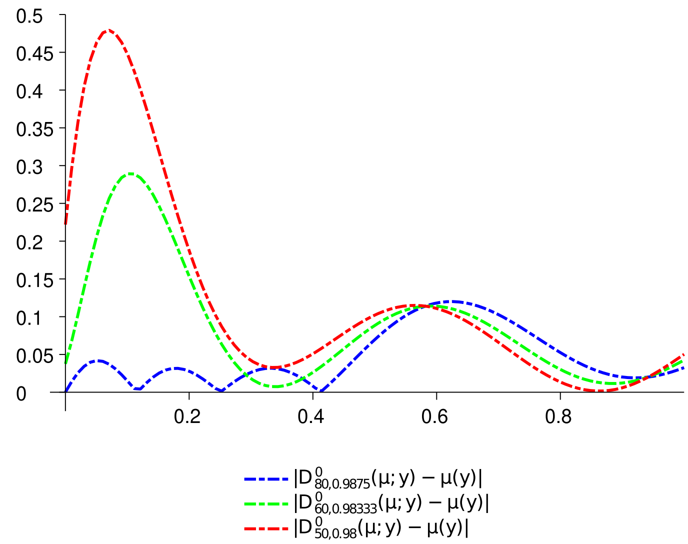

Figure 4 The error of approximation $D^{0}_{m,q}(\mu;y)$ to $\mu(y)$ for fixed values of q and m

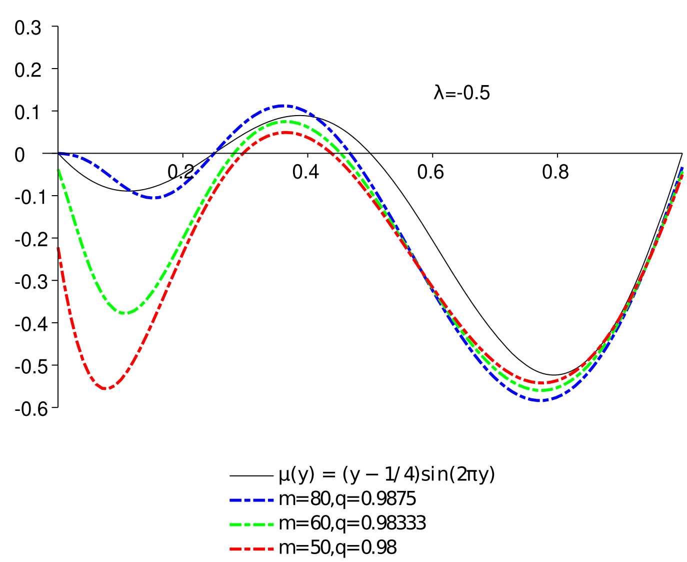

Figure 5 The approximation of $D^{-0.5}_{m,q}(\mu;y)$ to $\mu(y)$ for fixed values of q and m

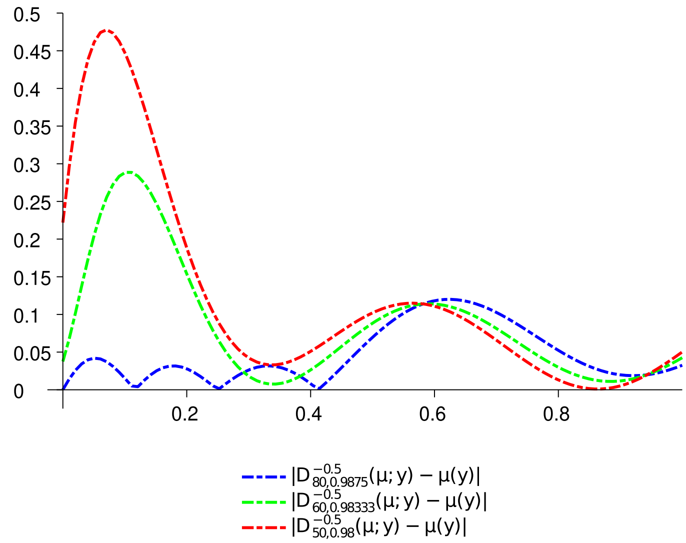

Figure 6 The error of approximation $D^{-0.5}_{m,q}(\mu;y)$ to $\mu(y)$ for fixed values of q and m

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
L.S. and F.Z. managed funding and participated in discussing the proposal. R.A. wrote the main manuscript text and M.M. edited and prepared the final draft of the manuscript. All authors read and approved the final manuscript.

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