Caraballo *Journal of Inequalities and Applications* (2023) 2023:72
https://doi.org/10.1186/s13660-023-02962-w

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Sharp bounds on moments of random variables

David Caraballo[1*]

*Correspondence:
dgc3@georgetown.edu
[1]Department of Mathematics and Statistics, Georgetown University, St. Mary's Hall, 3rd floor, 37th and O Streets, N.W., Washington D.C. 20057, USA

Abstract

We give new inequalities involving the expectation and variance of a random variable defined on a finite interval and having an essentially bounded probability density function. Our inequalities sharpen previous inequalities by N.S. Barnett and S.S. Dragomir. We prove our bounds are optimal, and we explicitly give the cases in which equality occurs.

MSC: 60E15; 49J40; 26D15

Keywords: Random variable; Expectation; Variance; Inequalities; Probability density function; Finite interval

1 Introduction

There has been considerable interest in the study of absolutely continuous random variables X that take values in a compact interval $[a,b]$ and whose probability density functions ρ are essentially bounded. See, for example, [1–6], and [7].

One result of particular importance, introduced by Barnett and Dragomir in [5], and discussed in [1–3, 5, 7], and elsewhere, gives good general bounds involving the mean and variance of X:

Theorem 1 ([5, Theorem 1(ii)]) *If X is an absolutely continuous random variable whose pdf ρ satisfies $m \le \rho(x) \le M$ for $\mathcal{L}^1$ a.e. $x \in [a,b]$ and $\int_a^b \rho(x)\,dx = 1$, then*

$$\frac{m(b-a)^3}{6} \le \big[b - E(X)\big]\big[E(X) - a\big] - \sigma^2(X) \le \frac{M(b-a)^3}{6} \tag{1}$$

and

$$\left|\big[b - E(X)\big]\big[E(X) - a\big] - \sigma^2(X) - \frac{(b-a)^3}{6}\right| \le \frac{\sqrt{5}(b-a)^3(M-m)}{60}. \tag{2}$$

In this paper we establish, for the first time, optimal lower and upper bounds for

$$\big[b - E(X)\big]\big[E(X) - a\big] - \sigma^2(X)$$

© The Author(s) 2023. **Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

for such random variables X, in particular sharpening the key results (1) and (2). Moreover, we show that the distributions yielding the optimal bounds are unique, and we give them explicitly.

Since pdfs that differ on sets having Lebesgue measure zero correspond to identically distributed random variables, without loss of generality we will identify such pdfs. In what follows, $\mathcal{L}^1(E) = |E|$ denotes the Lebesgue measure of the set E.

2 Results

Theorem 2 *Suppose X is an absolutely continuous random variable whose pdf ρ satisfies $0 \le m \le \rho(x) \le M < \infty$ for $\mathcal{L}^1$ a.e. $x \in [a,b]$ and $\int_a^b \rho(x)\,dx = 1$.*

(1) *If $m = M$, then*

$$\big[b - E(X)\big]\big[E(X) - a\big] - \sigma^2(X) = \frac{(b-a)^2}{6}.$$

(2) *If $m < M$, then*

$$\begin{aligned}&\big[b - E(X)\big]\big[E(X) - a\big] - \sigma^2(X)\\ &\quad\ge \frac{1}{6}m(b-a)^3 + \frac{1}{12}\big(2(b-a)(M-m) + M(b-a) - 1\big)\frac{(1-m(b-a))^2}{(M-m)^2}.\end{aligned}$$

This lower bound is sharp, and we have equality if and only if the function

$$\rho(x) = \begin{cases} M, & \text{if } x \in [a, a+\frac{G}{2}) \cup (b-\frac{G}{2}, b], \\ m, & \text{if } x \in [a+\frac{G}{2}, b-\frac{G}{2}], \end{cases}$$

with

$$G = \frac{1-m(b-a)}{M-m},$$

is a pdf of X.

(3) *If $m < M$, then*

$$\begin{aligned}&\big[b - E(X)\big]\big[E(X) - a\big] - \sigma^2(X)\\ &\quad\le \frac{1}{4}(b-a)^2 - \frac{1}{12}\left(m(b-a)^3 + \frac{(1-m(b-a))^3}{(M-m)^2}\right).\end{aligned}$$

This upper bound is sharp, and we have equality if and only if the function

$$\rho(x) = \begin{cases} M, & \text{if } x \in ((\frac{1}{2}a+\frac{1}{2}b)-\frac{G}{2}, (\frac{1}{2}a+\frac{1}{2}b)+\frac{G}{2}), \\ m, & \text{if } x \in [a, (\frac{1}{2}a+\frac{1}{2}b)-\frac{G}{2}] \cup [(\frac{1}{2}a+\frac{1}{2}b)+\frac{G}{2}, b], \end{cases}$$

with G defined as in part (2), is a pdf of X.

3 Proofs

We first record a useful lemma, proven using a clever algebraic manipulation by Barnett and Dragomir in [5].

Lemma 3 ([5, (2.5)]) *Suppose X is as stated in our theorem. Then,*

$$\big[b - E(X)\big]\big[E(X) - a\big] - \sigma^2(X) = \int_a^b (b-x)(x-a)\rho(x)\,dx.$$

Proof of (1) In this case, X is a continuous uniform random variable on $[a,b]$, and so its expected value and variance are $E(X) = (a+b)/2$ and $\sigma^2(X) = (b-a)^2/12$. Substituting these values yields the result immediately.

Proof of (2) Suppose $m < M$. Let $c = b - a$, and define

$$\begin{aligned} f(x) &= x(c-x) \quad \text{for all } x \in [0,c], \\ \tau_a(x) &= x + a \quad \text{for all real } x, \\ \phi(x) &= \frac{x-m}{M-m} \quad \text{for all } x \in [m,M]. \end{aligned}$$

The translation operator τ_a is invertible, with $\tau_a^{-1}(x) = x - a$ for all real x. Since $1/(M - m) > 0$, the affine transformation $\phi : [m,M] \to [0,1]$ is invertible with $\phi^{-1}(x) = (M-m)x + m$ for all $x \in [0,1]$. Define

$$\begin{aligned} \mathcal{C}_\rho &= \left\{\rho : m \le \rho(x) \le M \text{ for } \mathcal{L}^1 \text{ a.e. } x \in [a,b], \text{and } \int_a^b \rho(x)\,dx = 1\right\}, \\ \mathcal{C}_h &= \left\{h : m \le h(x) \le M \text{ for } \mathcal{L}^1 \text{ a.e. } x \in [0,c], \text{and } \int_0^c h(x)\,dx = 1\right\}, \\ \mathcal{C}_g &= \left\{g : 0 \le g(x) \le 1 \text{ for } \mathcal{L}^1 \text{ a.e. } x \in [0,c], \text{and } \int_0^c g(x)\,dx = G\right\}. \end{aligned}$$

We say that a function ρ, h, or g is *admissible* provided it is an element of $\mathcal{C}_\rho$, $\mathcal{C}_h$, or $\mathcal{C}_g$, respectively. Whenever ρ_*, h_*, and g_* are admissible, define

$$\begin{aligned} I_{\rho_*} &= \int_a^b (b-x)(x-a)\rho_*(x)\,dx, \\ I_{h_*} &= \int_0^c (c-x)x \cdot h_*(x)\,dx = \int_0^c f(x)\cdot h_*(x)\,dx, \\ I_{g_*} &= \int_0^c (c-x)x \cdot g_*(x)\,dx = \int_0^c f(x)\cdot g_*(x)\,dx. \end{aligned}$$

If $\rho \in \mathcal{C}_\rho$, $h = \rho \circ \tau_a$, and $g = \phi \circ h$, then $m \le h(x) \le M$ for $\mathcal{L}^1$ a.e. $x \in [0,c]$, $0 \le g(x) \le 1$ for $\mathcal{L}^1$ a.e. $x \in [0,c]$, $\rho = h \circ \tau_a^{-1}$, $h = \phi^{-1} \circ g$, and

$$\begin{aligned} 1 &= \int_a^b \rho(x)\,dx = \int_0^c h(x)\,dx = \int_0^c \big[(M-m)g(x) + m\big]\,dx \\ &= (M-m)\int_0^c g(x)\,dx + mc, \end{aligned}$$

from which it follows that $h \in \mathcal{C}_h$ and $g \in \mathcal{C}_g$.

If $g \in \mathcal{C}_g$, $h = \phi^{-1} \circ g$, and $\rho = h \circ \tau_a^{-1}$, then $m \leq h(x) \leq M$ for $\mathcal{L}^1$ a.e. $x \in [0, c]$, $m \leq \rho(x) \leq M$ for $\mathcal{L}^1$ a.e. $x \in [a, b]$, $g = \phi \circ h$, $h = \rho \circ \tau_a$, and

$$\frac{1 - mc}{M - m} = G = \int_0^c g(x)\,dx = \int_0^c \frac{h(x) - m}{M - m}\,dx = \frac{1}{M - m}\left(\int_0^c h(x)\,dx - mc\right),$$

from which it follows that

$$1 = \int_0^c h(x)\,dx = \int_a^b \rho(x)\,dx,$$

and hence, $h \in \mathcal{C}_h$ and $\rho \in \mathcal{C}_\rho$.

We say that a function ρ, h, or g is a *minimizer* provided it is an element of $\mathcal{C}_\rho$, $\mathcal{C}_h$, or $\mathcal{C}_g$, respectively, and provided it results in the lowest possible value for I_{ρ_*}, I_{h_*}, or I_{g_*}, respectively, among all admissible functions ρ_*, h_*, or g_*.

If ρ is a minimizer and $h = \rho \circ \tau_a$, then h is admissible, as shown above, and

$$I_h = \int_0^c (c - x)x \cdot h(x)\,dx = \int_0^c (c - x)x \cdot \rho(x + a)\,dx = \int_a^b (b - u)(u - a) \cdot \rho(u)\,du = I_\rho,$$

and so h is a minimizer. If, additionally, $g = \phi \circ h$, then g is admissible, as shown above, and we have

$$\begin{aligned} I_g &= \int_0^c (c - x)x \cdot g(x)\,dx \\ &= \int_0^c (c - x)x \cdot \frac{h(x) - m}{M - m}\,dx \\ &= \frac{1}{M - m}\left(I_h - \frac{1}{6}mc^3\right). \end{aligned}$$

Thus, g is a minimizer as well, as I_g is a strictly increasing affine function of I_h.

Similarly, if g is a minimizer and $h = \phi^{-1} \circ g$, then h is admissible, as shown above, and we have

$$\begin{aligned} I_h &= \int_0^c (c - x)x \cdot h(x)\,dx \\ &= \int_0^c (c - x)x \cdot \big((M - m)g(x) + m\big)\,dx \\ &= (M - m)I_g + \frac{1}{6}mc^3. \end{aligned}$$

Then, h is a minimizer, as I_h is a strictly increasing affine function of I_g. If, additionally, $\rho = h \circ \tau_a^{-1}$, then ρ is admissible, as shown above, and we have

$$I_\rho = \int_a^b (b - x)(x - a)\rho(x)\,dx = \int_0^c (c - u)u \cdot h(u)\,du = I_h,$$

and so ρ is a minimizer.

By Lemma 3, X minimizes

$$\big[b - E(X)\big]\big[E(X) - a\big] - \sigma^2(X)$$

if and only if ρ is admissible and minimizes I_ρ. By the argument above, that occurs if and only if the function g defined by $g = \phi \circ h$, where $h = \rho \circ \tau_a$, minimizes I_g.

We will now use the Bathtub Principle [8, Theorem 1.14] with Ω, μ, f, G, $\mathcal{C}$, and I there replaced by $[0,c]$, the Lebesgue measure, f, G, $\mathcal{C}_g$, and I_g, respectively, to find a function g that minimizes I_g and to prove that it is unique among minimizers of I_g. It will then follow from our reasoning above that the function ρ defined by $\rho = h \circ \tau_a^{-1}$, where $h = \phi^{-1} \circ g$, uniquely minimizes I_ρ, and hence solves our problem. Let

$$\begin{aligned}
s &= \sup\bigl\{t : \bigl|\bigl\{x \in [0,c] : f(x) < t\bigr\}\bigr| \le G\bigr\} \\
&= \sup\left\{t : \left|\left\{x \in [0,c] : \frac{1}{4}c^2 - \left(x - \frac{c}{2}\right)^2 < t\right\}\right| \le G\right\} \\
&= \sup\left\{t : \left|\left\{x \in [0,c] : x > \frac{c}{2} + \sqrt{\frac{1}{4}c^2 - t} \text{ or } x < \frac{c}{2} - \sqrt{\frac{1}{4}c^2 - t}\right\}\right| \le G\right\} \\
&= \sup\bigl\{t : c - \sqrt{c^2 - 4t} \le G\bigr\} \\
&= \sup\left\{t : t \le \frac{1}{4}\bigl(c^2 - (c-G)^2\bigr)\right\} \\
&= \frac{1}{4}\bigl(c^2 - (c-G)^2\bigr).
\end{aligned}$$

For this s, we have $|\{x \in [0,c] : f(x) = s\}| = 0$ and

$$\begin{aligned}
&\bigl|\bigl\{x \in [0,c] : f(x) < s\bigr\}\bigr| \\
&\quad = \left|\left\{x \in [0,c] : \frac{1}{4}c^2 - \left(x - \frac{c}{2}\right)^2 < \frac{1}{4}\bigl(c^2 - (c-G)^2\bigr)\right\}\right| \\
&\quad = \left|\left\{x \in [0,c] : \left(x - \frac{c}{2}\right)^2 > \frac{1}{4}(c-G)^2\right\}\right| \\
&\quad = \left|\left[0, \frac{G}{2}\right) \cup \left(c - \frac{G}{2}, c\right]\right| \\
&\quad = G.
\end{aligned}$$

Since $|\{x \in [0,c] : f(x) < s\}| = G$, by the Bathtub Principle [8, Theorem 1.14] there is a unique $g \in \mathcal{C}_g$ that minimizes the integral I_g, and it is given by $g(x) = \chi_{\{f<s\}}(x)$.

Thus,

$$g(x) = \begin{cases} 1, & \text{if } f(x) < s, \\ 0, & \text{if } f(x) \ge s \end{cases} = \begin{cases} 1, & \text{if } x \in [0, \frac{G}{2}) \cup (c - \frac{G}{2}, c], \\ 0, & \text{if } x \in [\frac{G}{2}, c - \frac{G}{2}]. \end{cases}$$

We then calculate

$$h(x) = (M - m)g(x) + m = \begin{cases} M, & \text{if } x \in [0, \frac{G}{2}) \cup (c - \frac{G}{2}, c], \\ m, & \text{if } x \in [\frac{G}{2}, c - \frac{G}{2}] \end{cases}$$

and

$$\rho(x) = h(x-a) = \begin{cases} M, & \text{if } x \in [a, a+\frac{G}{2}) \cup (a+c-\frac{G}{2}, a+c], \\ m, & \text{if } x \in [a+\frac{G}{2}, a+c-\frac{G}{2}] \end{cases}$$
$$= \begin{cases} M, & \text{if } x \in [a, a+\frac{G}{2}) \cup (b-\frac{G}{2}, b], \\ m, & \text{if } x \in [a+\frac{G}{2}, b-\frac{G}{2}]. \end{cases}$$

This ρ is the desired minimizer. Finally, we will calculate the minimum value of I_ρ,

$$\begin{aligned} I_\rho &= I_h \\ &= \int_0^c f(x)h(x)\,dx = \int_0^{G/2} Mf(x)\,dx + \int_{G/2}^{c-G/2} mf(x)\,dx + \int_{c-G/2}^{c} Mf(x)\,dx \\ &= \frac{1}{24}MG^2(3c-G) + \frac{1}{12}m\bigl(2c^3 - G^2(3c-G)\bigr) + \frac{1}{24}MG^2(3c-G) \\ &= \frac{1}{12}G^2(3c-G)(M-m) + \frac{1}{6}mc^3. \end{aligned}$$

We have

$$G(M-m) = 1 - mc$$

and also

$$\begin{aligned} G(3c-G) &= \frac{1-mc}{M-m}\left(3c - \frac{1-mc}{M-m}\right) \\ &= \frac{1}{(M-m)^2}(1-mc)(3Mc-2mc-1). \end{aligned}$$

Substituting these expressions, we obtain

$$\begin{aligned} &\int_0^c f(x)h(x)\,dx \\ &\quad = \frac{1}{6}mc^3 + \frac{1}{12}(1-mc)\frac{1}{(M-m)^2}(1-mc)(3Mc-2mc-1) \\ &\quad = \frac{1}{6}mc^3 + \frac{1}{12}\bigl(2c(M-m) + Mc - 1\bigr)\frac{(1-mc)^2}{(M-m)^2} \\ &\quad = \frac{1}{6}m(b-a)^3 + \frac{1}{12}\bigl(2(b-a)(M-m) + M(b-a) - 1\bigr)\frac{(1-m(b-a))^2}{(M-m)^2}, \end{aligned}$$

as asserted in (2).

Proof of (3) Suppose $m < M$. Let c, G, f, τ_a, ϕ, $\mathcal{C}_p$, $\mathcal{C}_h$, $\mathcal{C}_g$, I_{ρ_*}, I_{h_*}, and I_{g_*} be as in the proof of (2). We say that a function ρ, h, or g is a *maximizer* provided it is an element of $\mathcal{C}_\rho$, $\mathcal{C}_h$, or $\mathcal{C}_g$, respectively, and provided it results in the highest possible value for I_{ρ_*}, I_{h_*}, or I_{g_*}, respectively, among all admissible functions ρ_*, h_*, or g_*.

If ρ is a maximizer and $h = \rho \circ \tau_a$, then h is admissible and $I_h = I_\rho$, as shown above, and so h is a maximizer. If, additionally, $g = \phi \circ h$, then g is admissible and

$$I_g = \frac{1}{M-m}\left(I_h - \frac{1}{6}mc^3\right),$$

as shown above, and so g is a maximizer, as I_g is a strictly increasing affine function of I_h.

Similarly, if g is a maximizer and $h = \phi^{-1} \circ g$, then h is admissible and

$$I_h = (M-m)I_g + \frac{1}{6}mc^3,$$

as shown above, and so h is a maximizer, as I_h is a strictly increasing affine function of I_g. If, additionally, $\rho = h \circ \tau_a^{-1}$, then ρ is admissible and $I_\rho = I_h$, as shown above, and so ρ is a maximizer.

By Lemma 3, X maximizes

$$\bigl[b - E(X)\bigr]\bigl[E(X) - a\bigr] - \sigma^2(X)$$

if and only if ρ is admissible and maximizes I_ρ. By the argument above, that occurs if and only if the function g defined by $g = \phi \circ h$, where $h = \rho \circ \tau_a$, maximizes I_g.

Define

$$F(x) = \frac{c^2}{4} - f(x) = \left(x - \frac{c}{2}\right)^2 \quad \text{for all } x \in [0,c].$$

Whenever g_* is admissible, define

$$I'_{g_*} = \int_0^c F(x) \cdot g_*(x)\,dx.$$

For each admissible g_*, we have

$$I_{g_*} + I'_{g_*} = \int_0^c f(x) \cdot g_*(x)\,dx + \int_0^c F(x) \cdot g_*(x)\,dx = \int_0^c \frac{c^2}{4} \cdot g_*(x)\,dx = \frac{c^2}{4}G.$$

Since this sum is constant, g maximizes I_g, as needed, if and only if g minimizes I'_g.

We will now use the Bathtub Principle [8, Theorem 1.14] with Ω, μ, f, G, $\mathcal{C}$, and I there replaced by $[0,c]$, the Lebesgue measure, F, G, $\mathcal{C}_g$, and I'_g, respectively, to find a function g that minimizes I'_g and to prove that it is unique among minimizers of I_g. It will then follow from our reasoning above that the function ρ defined by $\rho = h \circ \tau_a^{-1}$, where $h = \phi^{-1} \circ g$, uniquely maximizes I_ρ, and hence solves our problem. Let

$$\begin{aligned} s &= \sup\bigl\{t : \bigl|\{x \in [0,c] : F(x) < t\}\bigr| \le G\bigr\} \\ &= \sup\left\{t : \left|\left\{x \in [0,c] : \left(x - \frac{c}{2}\right)^2 < t\right\}\right| \le G\right\} \\ &= \sup\left\{t : \left|\left(\frac{c}{2} - \sqrt{t}, \frac{c}{2} + \sqrt{t}\right)\right| \le G\right\} \end{aligned}$$

$$= \sup\{t : 2\sqrt{t} \leq G\}$$
$$= \frac{G^2}{4}.$$

For this s, we have $|\{x \in [0,c] : F(x) = s\}| = 0$ and

$$\begin{aligned}
&\left|\{x \in [0,c] : F(x) < s\}\right| \\
&\quad = \left|\left\{x \in [0,c] : \left(x - \frac{c}{2}\right)^2 < \frac{G^2}{4}\right\}\right| \\
&\quad = \left|\left(\frac{c}{2} - \frac{G}{2}, \frac{c}{2} + \frac{G}{2}\right)\right| \\
&\quad = G.
\end{aligned}$$

Since $|\{x \in [0,c] : F(x) < s\}| = G$, by the Bathtub Principle [8, Theorem 1.14] there is a unique $g \in \mathcal{C}_g$ that minimizes the integral I'_g, and it is given by $g(x) = \chi_{\{F<s\}}(x)$.

Thus,

$$g(x) = \begin{cases} 1, & \text{if } F(x) < s, \\ 0, & \text{if } F(x) \geq s \end{cases} = \begin{cases} 1, & \text{if } x \in (\frac{c}{2} - \frac{G}{2}, \frac{c}{2} + \frac{G}{2}), \\ 0, & \text{if } x \in [0, \frac{c}{2} - \frac{G}{2}] \cup [\frac{c}{2} + \frac{G}{2}, c]. \end{cases}$$

We then calculate

$$h(x) = (M - m)g(x) + m = \begin{cases} M, & \text{if } x \in (\frac{c}{2} - \frac{G}{2}, \frac{c}{2} + \frac{G}{2}), \\ m, & \text{if } x \in [0, \frac{c}{2} - \frac{G}{2}] \cup [\frac{c}{2} + \frac{G}{2}, c] \end{cases}$$

and

$$\rho(x) = h(x - a) = \begin{cases} M, & \text{if } x \in ((\frac{1}{2}a + \frac{1}{2}b) - \frac{G}{2}, (\frac{1}{2}a + \frac{1}{2}b) + \frac{G}{2}), \\ m, & \text{if } x \in [a, (\frac{1}{2}a + \frac{1}{2}b) - \frac{G}{2}] \cup [(\frac{1}{2}a + \frac{1}{2}b) + \frac{G}{2}, b]. \end{cases}$$

This ρ is the desired maximizer. Finally, we will calculate the maximum value of I_ρ,

$$\begin{aligned}
I_\rho &= I_h \\
&= \int_0^{\frac{c}{2}-\frac{G}{2}} mf(x)\,dx + \int_{\frac{c}{2}-\frac{G}{2}}^{\frac{c}{2}+\frac{G}{2}} Mf(x)\,dx + \int_{\frac{c}{2}+\frac{G}{2}}^{c} mf(x)\,dx \\
&= \frac{1}{24}m(c-G)^2(2c+G) + \frac{1}{12}GM\left(3c^2 - G^2\right) + \frac{1}{24}m(c-G)^2(2c+G) \\
&= \frac{1}{12}m(c-G)^2(2c+G) + \frac{1}{12}GM\left(3c^2 - G^2\right).
\end{aligned}$$

We now substitute $GM = Gm + 1 - mc$ into the previous expression.

$$\begin{aligned}
I_\rho &= \frac{1}{12}m(c-G)^2(2c+G) + \frac{1}{12}(Gm + 1 - mc)\left(3c^2 - G^2\right) \\
&= \frac{1}{4}c^2 - \frac{1}{12}mc^3 - \frac{1}{12}G^2 + \frac{1}{12}G^2mc
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}c^2 - \frac{1}{12}mc^3 - \frac{1}{12}G^2(1-mc) \\
&= \frac{1}{4}c^2 - \frac{1}{12}\left(mc^3 + \frac{(1-mc)^3}{(M-m)^2}\right) \\
&= \frac{1}{4}(b-a)^2 - \frac{1}{12}\left(m(b-a)^3 + \frac{(1-m(b-a))^3}{(M-m)^2}\right),
\end{aligned}
$$

as asserted in (3). □

Finally, we will comment on the choice of f and F in the proof. The proof makes essential use of the relationship between I_ρ and I_g. I_h is an intermediate quantity, related to I_ρ by the translation τ_a and related to I_g by the affine transformation ϕ. f is simply what results after starting with $(b-x)(x-a)$ in I_ρ and translating by a to obtain I_h:

$$I_\rho = \int_a^b (b-x)(x-a)\rho(x)\,dx = \int_0^c (c-x)x \cdot h(x)\,dx = \int_0^c f(x)\cdot h(x)\,dx = I_h.$$

In the proof of part (3) of the theorem, we need to maximize an integral, but the version of the Bathtub Principle that we use is stated for minimizations. It is for this reason that we apply the Bathtub Principle with the complementary function $F(x) = -f(x) + c^2/4$. Here, the $c^2/4$ constant term ensures that $F(x)$ is nonnegative on $[0,c]$. The specific choices of f and F do not affect the bounds in the theorem, which are absolute.

Acknowledgements
Not applicable.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
D.C. wrote the manuscript. This is a single-author work.

Authors' information
This is given on the title page.

Received: 6 April 2022 Accepted: 30 March 2023 Published online: 16 May 2023

References

1. Agarwal, R.P., Barnett, N.S., Cerone, P., Dragomir, S.S.: A survey on some inequalities for expectation and variance. Comput. Math. Appl. **49**, 429–480 (2005)
2. Barnett, N.S., Cerone, P., Dragomir, S.S.: Further inequalities for the expectation and variance of a random variable defined on a finite interval. Math. Inequal. Appl. **6**(1) (2000)
3. Barnett, N.S., Cerone, P., Dragomir, S.S. (eds.): Inequalities for Random Variables over a Finite Interval RGMIA Monographs (2004)
4. Barnett, N.S., Cerone, P., Dragomir, S.S., Roumeliotis, J.: Some inequalities for the dispersion of a random variable whose pdf is defined on a finite interval. J. Inequal. Pure Appl. Math. **2**(1), Art. 1 (2001)
5. Barnett, N.S., Dragomir, S.S.: Some elementary inequalities for the expectation and variance of a random variable whose pdf is defined on a finite interval. RGMIA Res. Rep. Collect. **2**(7), Art. 12 (1999)

6. Cerone, P., Dragomir, S.S.: On some inequalities for the expectation and variance. Korean J. Comput. Appl. Math. **8**(2), 357–380 (2001)
7. Kumar, P.: Moments inequalities of a random variable defined over a finite interval. J. Inequal. Pure Appl. Math. **3**(3), Art. 41 (2002)
8. Lieb, E.H., Loss, M.: Analysis. Am. Math. Soc., Providence (1997)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Submit your manuscript to a SpringerOpen journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com