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Hermite–Hadamard and Fejér-type inequalities for strongly reciprocally (p, h)-convex functions of higher order

Han Li[1*], Muhammad Shoaib Saleem[2*], Imran Ahmed[3] and Kiran Naseem Aslam[2]

*Correspondence:
woshilihanlaoshi@sina.com;
shaby455@yahoo.com
[1]School of Mathematics, ChengDu Normal University, ChengDu, 611130, China
[2]Department of Mathematics, University of Okara, Okara, Pakistan
Full list of author information is available at the end of the article

Abstract

In this paper, we investigate the properties of a newly introduced class of functions, strongly reciprocally (p, h)-convex functions of higher order. We establish Hermite–Hadamard-type and Fejér-type inequalities for this class of functions. Additionally, we present fractional integral inequalities applicable to strongly reciprocally (p, h)-convex functions of higher order.

Keywords: Strongly reciprocally convex function; Strongly convex functions of higher order; (p, h)-convex function; Hermite–Hadamard-type inequality; Fejér-type inequality

1 Introduction

Convexity is a fundamental and widely applicable notion. Its enormous practical impact on industry and business makes it an important aspect of our daily lives. In fact, convexity plays a crucial role in solving many real-world problems, particularly those related to constrained control and estimation. From a geometric perspective, a real-valued function is considered convex if the line segment connecting any two of its points lies entirely on or above the function graph in the Euclidean space.

The convexity of a function in the classical sense is defined as follows: a function $f_1 : M \to \mathbb{R}$ is convex if

$$f_1\big(jx + (1-j)y\big) \le jf_1(x) + (1-j)f_1(y) \quad \forall j \in [0,1]. \tag{1.1}$$

If the above inequality is reversed, then the function is said to be concave.

Convexity is a constantly evolving notion, and its extensions and generalizations are growing rapidly with the advancement of modern engineering, optimization, economics, and nonlinear programming [1–3]. Recent generalizations of convexity include those found in [4–11], among others.

The notion of convex functions has been used to prove various inequalities, including the Schur, Hermite–Hadamard, and Fejér-type inequalities. Among these, the Hermite–

Hadamard inequality is arguably the most important and most studied. It is stated as follows.

Let $f_1 : M \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, and let $a_1, b_1 \in M$ with $a_1 < b_1$. Then the following Hermite–Hadamard double inequality holds:

$$f_1\left(\frac{a_1+b_1}{2}\right) \leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1} f_1(x)\,dx \leq \frac{f_1(a_1)+f_1(b_1)}{2}. \tag{1.2}$$

This inequality has numerous applications in various fields of mathematics, including analysis, geometry, and optimization. Other applications of convexity include optimization problems, game theory, and statistics, among others.

Fejér [11] presented a generalization of the Hermite–Hadamard inequality using a weight function $w(x)$. Let $f_1 : [a_1, b_1] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, and let $w : [a_1, b_1] \to \mathbb{R}$ be a nonnegative integrable function that is symmetric about $\frac{a_1+b_1}{2}$. Then we have

$$f_1\left(\frac{a_1+b_1}{2}\right)\int_{a_1}^{b_1} w(x)\,dx \leq \int_{a_1}^{b_1} f_1(x)w(x)\,dx \leq \frac{f_1(a_1)+f_1(b_1)}{2}\int_{a_1}^{b_1} w(x)\,dx \tag{1.3}$$

This inequality can be seen as a weighted generalization of the Hermite–Hadamard inequality and has numerous applications in various areas of mathematics, including analysis, approximation theory, and numerical integration. The weight function $w(x)$ can be chosen to reflect various properties of the function being integrated, leading to more accurate approximations and more informative results.

There are several results present in the literature about advancement of inequality theory. For example, in 2021, Kizil and Ardic [12] presented new inequalities for strongly convex functions using the Atangana–Baleanu integral operators. In 2020, Akdemir and Özata [13] established Grüss-type inequalities for fractional integral operators involving the extended generalized Mittag-Leffler function. In 2019, Özdemir [14] refined the Hadamard integral inequality via k-fractional integrals for p-convex functions. In 2022, Avci Ardic, Akdemir, and Set [15] introduced new integral inequalities via geometric-arithmetic convex functions and provided various applications. Moreover, Ekinci, Özdemir, and Set [16] developed in 2020 new integral inequalities of Ostrowski type for quasi-convex functions and provided applications in numerical analysis, and Wang and Shi [17] in 2006 established Hermite–Hadamard-type inequalities for n times differentiable and GA-convex functions with applications to means. These studies provided useful results for researchers in the field of inequalities and their applications in various mathematical problems.

Set [18] (2018), studies the integral operators of strongly (p,h)-convex functions. The author presents some new properties of these operators and proves some important results concerning the (p,h)-convex functions. Sahiner and Sarikaya (2020), in [19] investigated some Fejér-type inequalities for $s^{(k)}$-convex functions. The author established some new inequalities, including some refinements of known results, and presented some interesting applications of their findings. Malik et al. [20] in 2019 was concerned with inequalities for generalized (p,q)-Euler numbers and polynomials. The authors established some new inequalities involving these numbers and polynomials, and discuss some interesting consequences of their results. Orhan and Sahiner [20] in 2020 presented Hermite–Hadamard-type inequalities for s-convex functions via fractional integrals. The authors established some new inequalities and showed that their results generalize and refine some known results. Zhang and Zhu [21] in 2021 investigated Hermite–Hadamard-type inequalities for strongly s-convex functions. The authors proved some new inequalities, including some refinements of known results, and discussed some interesting applications of their findings. Overall, [18–22] contributed to the theory of convex functions and their applications and provided new and interesting results in this area.

In this paper, we delve into the study of strongly reciprocally (p, h)-convex functions of higher order. The concept of (p, h)-convexity is a generalization of ordinary convexity, which has important applications in optimization, numerical analysis, and other fields of mathematics. Strong reciprocal (p, h)-convexity is a further refinement of (p, h)-convexity, which has been shown to be a useful tool in the study of inequalities involving means and related functions. We begin by introducing the notion of strongly reciprocally (p, h)-convex functions of higher order and exploring some of their basic properties. We then establish a number of new inequalities for this class of functions, including Hermite–Hadamard-type and Fejér-type inequalities. These inequalities are important in their own right and have a variety of applications in different areas of mathematics. Finally, we present fractional integral inequalities that are applicable to strongly reciprocally (p, h)-convex functions of higher order. These inequalities extend the classical Hermite–Hadamard- and Fejér-type inequalities and have potential applications in the study of fractional calculus and related areas of mathematics. Overall, the results presented in this paper contribute to the ongoing study of (p, h)-convexity and its generalizations and have potential applications in a wide range of mathematical fields.

The paper is structured as follows. First, we provide an introduction to strongly reciprocally (p, h)-convex functions of higher order, including their basic properties and preliminary material. Next, we present our main results: Hermite–Hadamard- and Fejér-type inequalities for this generalization. Finally, we demonstrate a practical application of this new class of convex functions by deriving some fractional integral inequalities.

2 Preliminaries

In this section, we give an overview of known definitions and results.

Definition 2.1 ([23]) A set $M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\}$ is a p-convex set if

$$\big(jx^p + (1-j)y^p\big)^{\frac{1}{p}} \in M \tag{2.1}$$

for all $x, y \in M$ and $j \in [0, 1]$, where $p = 2u + 1$ or $p = \frac{d}{c}$, $d = 2v + 1$, $c = 2w + 1$, and $u, v, w \in N$.

Definition 2.2 ([23]) Let $M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\}$ be a p-convex set. A function $f_1 : M = [a_1, b_1] \to \mathbb{R}$ is p-convex if

$$f_1\big(\big(jx^p + (1-j)y^p\big)^{\frac{1}{p}}\big) \leq jf_1(x) + (1-j)f_1(y) \tag{2.2}$$

for all $x, y \in M$ and $j \in [0, 1]$.

Definition 2.3 ([3]) A function $f_1 : M = [a_1, b_1] \to \mathbb{R}$ is a strongly convex function with modulus $\mu \geq 0$ on M if

$$f_1\big(jx + (1-j)y\big) \leq jf_1(x) + (1-j)f_1(y) - \mu j(1-j)(y-x)^2 \tag{2.3}$$

for all $x, y \in M$ and $j \in [0,1]$.

Definition 2.4 ([24]) A function $f_1 : M = [a_1, b_1] \to \mathbb{R}$ is strongly p-convex if

$$f_1\big(\big(jx^p + (1-j)y^p\big)^{\frac{1}{p}}\big) \leq jf_1(x) + (1-j)f_1(y) - \mu j(1-j)\big(y^p - x^p\big)^2 \tag{2.4}$$

for all $x, y \in M$ and $j \in [0,1]$.

Definition 2.5 ([25]) A set $M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\}$ is harmonic convex if

$$\frac{xy}{jx + (1-j)y} \in M \tag{2.5}$$

for all $x, y \in M$ and $j \in [0,1]$.

Definition 2.6 Let $M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\}$ be a harmonic convex set. A function $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is harmonic convex if

$$f_1\left(\frac{xy}{jx + (1-j)y}\right) \leq (1-j)f_1(x) + jf_1(y) \tag{2.6}$$

for all $x, y \in M$ and $j \in [0,1]$.

Definition 2.7 ([26]) A set $M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\}$ is a p-harmonic convex set if

$$\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{\frac{1}{p}} \in M \tag{2.7}$$

for all $x, y \in M$ and $j \in [0,1]$.

Definition 2.8 ([26]) Let $M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\}$ be a p-harmonic convex set. A function $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is p-harmonic convex if

$$f_1\left(\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{\frac{1}{p}}\right) \leq (1-j)f_1(x) + jf_1(y) \tag{2.8}$$

for all $x, y \in M$ and $j \in [0,1]$.

Definition 2.9 ([27]) Let M be an interval, and let $\mu \in (0, \infty)$. A function $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is strongly reciprocally convex with modulus μ on M if

$$f_1\left(\frac{xy}{jx + (1-j)y}\right) \leq (1-j)f_1(x) + jf_1(y) - \mu j(1-j)\left(\frac{1}{x} - \frac{1}{y}\right)^2 \tag{2.9}$$

for all $x, y \in M$ and $j \in [0,1]$.

Definition 2.10 ([28]) Let M be a p-convex set, and let $\mu \in (0,\infty)$. A function $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a strongly reciprocally p-convex function with modulus μ on M if

$$f_1\left(\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{\frac{1}{p}}\right) \leq (1-j)f_1(x) + jf_1(y) - \mu j(1-j)\left(\frac{1}{x^p} - \frac{1}{y^p}\right)^2, \tag{2.10}$$

for all $x, y \in M$ and $j \in [0,1]$.

Definition 2.11 ([29]) Let $f_1, h : I = [a_1, b_1] \to \mathbb{R}$ be nonnegative functions. Then f_1 is h-convex if

$$f_1\bigl(jx + (1-j)y\bigr) \leq h(j)f_1(x) + h(1-j)f_1(y) \tag{2.11}$$

for all $x, y \in M$ and $j \in [0,1]$.

Definition 2.12 ([30]) Let $M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\}$ be a p-convex set. A function $f_1 : M = [a_1, b_1] \to \mathbb{R}$ is (p, h)-convex if f_1 is nonnegative and

$$f_1\bigl(\bigl(jx^p + (1-j)y^p\bigr)^{\frac{1}{p}}\bigr) \leq h(j)f_1(x) + h(1-j)f_1(y) \tag{2.12}$$

for all $x, y \in M$ and $j \in [0,1]$.

Definition 2.13 ([31]) Let M be an interval, and let $\mu \in (0,\infty)$. A function $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is higher-order strongly convex with modulus μ on M if

$$f_1\bigl(jx + (1-j)v\bigr) \leq jf_1(x) + (1-j)f_1(y) - \mu\phi(j)\|y - x\|^l \tag{2.13}$$

for all $x, y \in M$, $j \in [0,1]$ and $l \geq 1$, where

$$\phi(j) = j(1-j).$$

Definition 2.14 Let M be an interval, and let $\mu \in (0,\infty)$. A function $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is strongly reciprocally (p, h)-convex with modulus μ on M if

$$f_1\left(\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{\frac{1}{p}}\right) \leq h(1-j)f_1(x) + h(j)f_1(y) - \mu j(1-j)\left(\frac{1}{x^p} - \frac{1}{y^p}\right)^2 \tag{2.14}$$

for all $x, y \in M$ and $j \in [0,1]$.

Now we are ready to introduce a new class of convex functions, strongly reciprocally (p, h)-convex functions of higher order.

Definition 2.15 Let M be an interval, and let $\mu \in (0,\infty)$. A function $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is strongly reciprocally (p, h)-convex of higher order with modulus μ on M if

$$f_1\left(\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{\frac{1}{p}}\right) \leq h(1-j)f_1(x) + h(j)f_1(y) - \mu\phi(j)\left\|\frac{1}{x^p} - \frac{1}{y^p}\right\|^l \tag{2.15}$$

for all $x, y \in M$, $j \in [0,1]$ and $l \geq 1$, where

$$\phi(j) = j(1-j).$$

Remark 2.16 Inserting $l = 2$ into Definition 2.15 with $\phi(j)$ as above, we obtain Definition 2.14. Similarly, inserting $l = 2$ and $h(j) = j$ into Definition 2.15, we obtain Definition 2.10, and for $l = 2$, $h(j) = j$, and $p = 1$, Definition 2.15 reduces to Definition 2.9.

As we know, $\mathbb{R}$ is a normed space under the usual modulus norm. Thus for any $x \in \mathbb{R}$,

$$\|x\| = |x|. \tag{2.16}$$

Using (2.16), inequality 2.15 can be written as

$$f_1\left(\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{\frac{1}{p}}\right) \leq h(1-j)f_1(x) + h(j)f_1(y) - \mu\phi(j)\left|\frac{1}{x^p} - \frac{1}{y^p}\right|^l \tag{2.17}$$

for all $x, y \in M$, $j \in [0,1]$, and $l \geq 1$, where

$$\phi(j) = j(1-j).$$

3 Basic results

In the following sections, we delve into the fascinating world of strongly reciprocally (p, h)-convex functions of higher order, which we will refer to as $SR(ph)$ throughout the rest of this paper. To begin with, we explore some intriguing insights, which can be obtained by performing simple algebraic operations on $SR(ph)$.

The following proposition is concerned about the addition of two functions from $SR(ph)$.

Proposition 3.1 *Let $f_1, g_1 : M \to \mathbb{R}$ be two two functions from $SR(ph)$ with modulus μ on M. Then $f_1 + g_1 : M \to \mathbb{R}$ is also in $SR(ph)$ with modulus μ^* on M, where $\mu^* = 2\mu$.*

Proof We start by definition:

$$\begin{aligned} & f_1 + g_1\left(\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{\frac{1}{p}}\right) \\ &\quad = f_1\left(\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{\frac{1}{p}}\right) + g_1\left(\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{\frac{1}{p}}\right) \\ &\quad \leq h(j)f_1(x) + h(1-j)f_1(y) - \mu\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l \\ &\qquad + h(j)g_1(x) + h(1-j)g_1(y) - \mu\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l, \end{aligned} \tag{3.1}$$

which in turns implies that

$$= h(j)(f_1 + g_1)(x) + h(1-j)(f_1 + g_1)(y) - 2\mu j(1-j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l$$

$$= h(j)(f_1 + g_1)(x) + h(1-j)(f_1 + g_1)(y) - \mu^* \phi(j) \left\| \frac{1}{y^p} - \frac{1}{x^p} \right\|^l,$$

where $\mu^* = 2\mu, \mu \geq 0$, and $\phi(j) = j(1-j)$. This completes the proof. □

Our next result is concerned with the scalar multiplication in $SR(ph)$.

Proposition 3.2 *Let $f_1 : M \to \mathbb{R}$ be a function in $SR(ph)$ with modulus $\mu \geq 0$. Then for any $\lambda \geq 0$, $\lambda f_1 : M \to \mathbb{R}$ is also in $SR(ph)$ with modulus ν^* on M, where $\nu^* = \lambda\mu$.*

Proof Let $\lambda \geq 0$. Then by the definition of f_1 we obtain

$$\begin{aligned}
\lambda f_1 \left(\left(\frac{x^p y^p}{j x^p + (1-j) y^p} \right)^{\frac{1}{p}} \right) &= \lambda \left(f_1 \left[\left(\frac{x^p y^p}{j x^p + (1-j) y^p} \right)^{\frac{1}{p}} \right] \right) \\
&\leq \lambda \left[h(j) f_1(x) + h(1-j) f_1(y) - \mu \phi(j) \left\| \frac{1}{y^p} - \frac{1}{x^p} \right\|^l \right] \\
&= h(j) \lambda f_1(x) + h(1-j) \lambda f_1(y) - \lambda \mu \phi(j) \left\| \frac{1}{y^p} - \frac{1}{x^p} \right\|^l \\
&= h(j) \lambda f_1(x) + h(1-j) \lambda f_1(y) - \nu^* \phi(j) \left\| \frac{1}{y^p} - \frac{1}{x^p} \right\|^l,
\end{aligned}$$

where $\nu^* = \lambda\mu, \mu \geq 0$, and $\phi(j) = j(1-j)$. This completes the proof. □

Proposition 3.3 *Let $f_{1i} : M \to \mathbb{R}$, $1 \leq i \leq n$, be in $SR(ph)$ with modulus μ. Then for $\lambda_i \geq 0$, $1 \leq i \leq n$, the function $f_1 : M \to \mathbb{R}$, where $f_1 = \sum_{i=1}^n \lambda_i f_{1i}$, is also in $SR(ph)$ with modulus $\gamma \geq 0$, where $\gamma = \sum_{i=1}^n \lambda_i \mu$.*

Proof Let M be a p-harmonic convex set. Then for all $x, y \in M$ and $j \in [0,1]$, we have

$$\begin{aligned}
f_1 \left(\left(\frac{x^p y^p}{j x^p + (1-j) y^p} \right)^{\frac{1}{p}} \right) &= \sum_{i=1}^n \lambda_i f_{1i} \left(\left(\frac{x^p y^p}{j x^p + (1-j) y^p} \right)^{\frac{1}{p}} \right) \\
&\leq \sum_{i=1}^n \lambda_i \left[h(j) f_{1i}(x) + h(1-j) f_{1i}(y) - \mu \phi(j) \left\| \frac{1}{y^p} - \frac{1}{x^p} \right\|^l \right] \\
&= h(j) \sum_{i=1}^n \lambda_i f_{1i}(x) + h(1-j) \sum_{i=1}^n \lambda_i f_{1i}(y) \\
&\quad - \sum_{i=1}^n \lambda_i \left[\mu \phi(j) \left\| \frac{1}{y^p} - \frac{1}{x^p} \right\|^l \right] \\
&= h(j) f_1(x) + h(1-j) f_1(y) - \gamma \phi(j) \left\| \frac{1}{y^p} - \frac{1}{x^p} \right\|^l,
\end{aligned}$$

where $\gamma = \sum_{i=1}^n \lambda_i \mu$. This completes the proof. □

Proposition 3.4 *Let $f_{1i} : M \to \mathbb{R}$, $1 \leq i \leq n$, be in $SR(ph)$ with modulus μ. Then $f_1 = \max\{f_{1i}, i = 1, 2, \ldots, n\}$ is also in $SR(ph)$ with modulus μ.*

Proof Let M be a p-harmonic convex set. Then for all $x, y \in M$ and $j \in [0,1]$, we have

$$\begin{aligned}
&f_1\left(\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{\frac{1}{p}}\right) \\
&= \max\left\{f_{1i}\left(\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{\frac{1}{p}}\right), i = 1,2,3,\dots,n\right\} \\
&= f_c\left(\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)\right) \\
&\leq h(j)f_c(x) + h(1-j)f_c(y) - \mu\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l \\
&= h(j)\max\{f_{1i}(x)\} + h(1-j)\max\{f_{1i}(y)\} - \mu\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l \\
&= h(j)f_1(x) + h(1-j)f_1(y) - \mu\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l. \qquad (3.2)
\end{aligned}$$

This completes the proof. □

4 Hermite–Hadamard-type inequality

In this section, we establish a Hermite–Hadamard-type inequality for the function belonging to the space $SR(ph)$.

Theorem 4.1 *Let $M \subset \mathbb{R} \setminus \{0\}$ be an interval. If $f_1 : M \to \mathbb{R}$ is in $SR(ph)$ with modulus $\mu \geq 0$ and $f_1 \in L[a_1, b_1]$, then for $h(\frac{1}{2}) \neq 0$, we have*

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})}\left[f_1\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{\frac{1}{p}} + \mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p - a_1^p}{a_1^p b_1^p}\right|^l\left[\frac{1-(-1)^{2l+1}}{2(l+1)}\right]\right] \\
&\leq \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}\,dx \\
&\leq \int_0^1\left[h(1-j)f_1(a_1) + h(j)f_1(b_1)\right]dj - \mu\left|\frac{b_1^p - a_1^p}{a_1^p b_1^p}\right|^l\int_0^1\phi(j)\,dj. \qquad (4.1)
\end{aligned}$$

Proof Substituting $j = \frac{1}{2}$ into Definition 2.15 gives

$$f_1\left(\left(\frac{2x^p y^p}{x^p + y^p}\right)^{\frac{1}{p}}\right) \leq h\left(\frac{1}{2}\right)f_1(x) + h\left(\frac{1}{2}\right)f_1(y) - \mu\phi\left(\frac{1}{2}\right)\left\|\frac{1}{x^p} - \frac{1}{y^p}\right\|^l. \qquad (4.2)$$

Let $x = [(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p})^{\frac{1}{p}}]$ and $y = [(\frac{a_1^p b_1^p}{jb_1^p + (1-j)a_1^p})^{\frac{1}{p}}]$. Integrating (4.2) with respect to j over $[0,1]$, we have

$$\begin{aligned}
f_1\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{\frac{1}{p}} \leq{}& h\left(\frac{1}{2}\right)f_1\left(\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{\frac{1}{p}}\right) + h\left(\frac{1}{2}\right)f_1\left(\left(\frac{a_1^p b_1^p}{jb_1^p + (1-j)a_1^p}\right)^{\frac{1}{p}}\right) \\
&- \mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p - a_1^p}{a_1^p b_1^p}\right|^l|1-2j|^l,
\end{aligned}$$

$$\int_0^1 f_1\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{\frac{1}{p}} dj \le \int_0^1 h\left(\frac{1}{2}\right) f_1\left(\left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p}\right)^{\frac{1}{p}}\right) dj + \int_0^1 h\left(\frac{1}{2}\right) f_1\left(\left(\frac{a_1^p b_1^p}{j b_1^p + (1-j) a_1^p}\right)^{\frac{1}{p}}\right) dj - \mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p - a_1^p}{a_1^p b_1^p}\right|^l \int_0^1 |1-2j|^l \, dj,$$

$$f_1\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{\frac{1}{p}} \le 2h\left(\frac{1}{2}\right)\frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}\, dx - \mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p - a_1^p}{a_1^p b_1^p}\right|^l \left[\frac{1-(-1)^{2l+1}}{2(l+1)}\right],$$

and

$$f_1\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{\frac{1}{p}} + \mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p - a_1^p}{a_1^p b_1^p}\right|^l \left[\frac{1-(-1)^{2l+1}}{2(l+1)}\right] \le 2h\left(\frac{1}{2}\right)\frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}\, dx,$$

$$\frac{1}{2h(\frac{1}{2})}\left[f_1\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{\frac{1}{p}} + \mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p - a_1^p}{a_1^p b_1^p}\right|^l \left[\frac{1-(-1)^{2l+1}}{2(l+1)}\right]\right] \le \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}\, dx,$$

which is the left side of inequality (4.1).

Finally, for the right side of inequality (4.1), setting $x = a_1$ and $y = b_1$ in Definition 2.15 gives

$$f_1\left(\left(\frac{a_1^p b_1^p}{t a_1^p + (1-t) b_1^p}\right)^{\frac{1}{p}}\right) \le h(1-j) f_1(a_1) + h(j) f_1(b_1) - \mu\phi(j)\left\|\frac{1}{a_1^p} - \frac{1}{b_1^p}\right\|^l. \tag{4.3}$$

Integrating (4.3) with respect to j over $[0,1]$, we have

$$\int_0^1 f_1\left[\left(\frac{a_1^p b_1^p}{t a_1^p + (1-t) b_1^p}\right)^{\frac{1}{p}}\right] dj \le \int_0^1 h(1-j) f_1(a_1)\, dj + \int_0^1 h(j) f_1(b_1)\, dj - \mu\left|\frac{b_1^p - a_1^p}{a_1^p b_1^p}\right|^l \int_0^1 \phi(j)\, dj$$

and

$$\frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}\, dx \le \int_0^1 \left[h(1-j) f_1(a_1) + h(j) f_1(b_1)\right] dj - \mu\left|\frac{b_1^p - a_1^p}{a_1^p b_1^p}\right|^l \int_0^1 \phi(j)\, dj,$$

which is the right side of inequality (4.1). This completes the proof. □

Remark 4.2

1. Inserting $l = 2$, $h(j) = j$, and $p = 1$ with $\phi(j) = j(1-j)$ into Theorem 4.1, we obtain the Hermite–Hadamard inequality for strongly reciprocally convex function; see [27, Theorem 3.1].

2. Insertion of $l=2$, $h(j)=j$, $p=1$, and $\mu=0$ with $\phi(j)=j(1-j)$ into Theorem 4.1 yields the Hermite–Hadamard inequality for harmonic convex functions; see [27, Theorem 2.4].

5 Fejér-type inequality

Now we are going to develop a Fejér-type inequality for the functions belonging to $SR(ph)$.

Theorem 5.1 *Let $M\subset\mathbb{R}\setminus\{0\}$ be an interval. If $f_1:M\to\mathbb{R}$ is in $SR(ph)$ with modulus $\mu\geq 0$, then for $h(\frac{1}{2})\neq 0$, we have*

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})}\left[f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{\frac{1}{p}}\int_{a_1}^{b_1}\frac{w(x)}{x^{1+p}}\,dx\right.\\
&\qquad\left.+\frac{\mu}{|a_1^pb_1^p|^l}\phi\left(\frac{1}{2}\right)\int_{a_1}^{b_1}\frac{|2a_1^pb_1^p-(a_1^p+b_1^p)x^p|^lw(x)}{|x^p|^lx^{1+p}}\,dx\right]\\
&\quad\leq\int_{a_1}^{b_1}\frac{f_1(x)w(x)}{x^{1+p}}\,dx\\
&\quad\leq\left[f_1(a_1)+f_1(b_1)\right]\int_{a_1}^{b_1}h\left(\frac{a_1^p(b_1^p-x^p)}{x^p(b_1^p-a_1^p)}\right)\frac{w(x)}{x^{1+p}}\,dx\\
&\qquad-\mu\left\|\frac{b_1^p-a_1^p}{a_1^pb_1^p}\right\|^l\int_{a_1}^{b_1}\phi\left(\frac{a_1^p(b_1^p-x^p)}{x^p(b_1^p-a_1^p)}\right)\frac{w(x)}{x^{1+p}}\,dx
\end{aligned}\tag{5.1}$$

for $a_1,b_1\in M$ with $a_1\leq b_1$ and $f_1\in L[a_1,b_1]$, where $w:M\to\mathbb{R}$ is a nonnegative integrable function satisfying

$$w\left(\frac{a_1^pb_1^p}{x^p}\right)^{\frac{1}{p}}=w\left[\left(\frac{a_1^pb_1^p}{a_1^p+b_1^p-x^p}\right)^{\frac{1}{p}}\right].$$

Proof Substituting $j=\frac{1}{2}$ into Definition 2.15 yields

$$f_1\left(\left(\frac{2x^py^p}{x^p+y^p}\right)^{\frac{1}{p}}\right)\leq h\left(\frac{1}{2}\right)f_1(x)+h\left(\frac{1}{2}\right)f_1(y)-\mu\phi\left(\frac{1}{2}\right)\left\|\frac{1}{x^p}-\frac{1}{y^p}\right\|^l.\tag{5.2}$$

Let $x=[(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p})^{\frac{1}{p}}]$ and $y=[(\frac{a_1^pb_1^p}{jb_1^p+(1-j)a_1^p})^{\frac{1}{p}}]$. Integrating (5.2) with respect to j over $[0,1]$, we have

$$\begin{aligned}
f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{\frac{1}{p}}&\leq h\left(\frac{1}{2}\right)f_1\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)+h\left(\frac{1}{2}\right)f_1\left(\left(\frac{a_1^pb_1^p}{jb_1^p+(1-j)a_1^p}\right)^{\frac{1}{p}}\right)\\
&\qquad-\mu\phi\left(\frac{1}{2}\right)\left|\frac{ja_1^p+(1-j)b_1^p}{a_1^pb_1^p}-\frac{jb_1^p+(1-j)a_1^p}{a_1^pb_1^p}\right|^l.
\end{aligned}$$

Since w is a nonnegative symmetric and integrable function, we have

$$f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{\frac{1}{p}}w\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)$$

$$\begin{aligned}&\le h\left(\frac{1}{2}\right)f_1\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)w\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)\\&\quad+h\left(\frac{1}{2}\right)f_1\left(\left(\frac{a_1^pb_1^p}{jb_1^p+(1-j)a_1^p}\right)^{\frac{1}{p}}\right)w\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)\\&\quad-\mu\phi\left(\frac{1}{2}\right)\left|\frac{ja_1^p+(1-j)b_1^p}{a_1^pb_1^p}-\frac{jb_1^p+(1-j)a_1^p}{a_1^pb_1^p}\right|^lw\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right).\end{aligned}\tag{5.3}$$

Integrating inequality (5.3) with respect to j over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{\frac{1}{p}}w\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)dj\\&\le\int_0^1 h\left(\frac{1}{2}\right)f_1\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)w\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)dj\\&\quad+\int_0^1 h\left(\frac{1}{2}\right)f_1\left(\left(\frac{a_1^pb_1^p}{jb_1^p+(1-j)a_1^p}\right)^{\frac{1}{p}}\right)w\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)dj\\&\quad-\mu\phi\left(\frac{1}{2}\right)\left|\frac{ja_1^p+(1-j)b_1^p}{a_1^pb_1^p}-\frac{jb_1^p+(1-j)a_1^p}{a_1^pb_1^p}\right|^lw\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)dj\end{aligned}$$

and

$$\begin{aligned}&f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{\frac{1}{p}}\int_{a_1}^{b_1}\frac{w(x)}{x^{1+p}}\,dx+\frac{\mu}{|a_1^pb_1^p|^l}\phi\left(\frac{1}{2}\right)\int_{a_1}^{b_1}\frac{|2a_1^pb_1^p-(a_1^p+b_1^p)x^p|^lw(x)}{|x^p|^lx^{1+p}}\,dx\\&\le 2h\left(\frac{1}{2}\right)\int_{a_1}^{b_1}\frac{f_1(x)w(x)}{x^{1+p}}\,dx,\\&\frac{1}{2h(\frac{1}{2})}\left(f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{\frac{1}{p}}\int_{a_1}^{b_1}\frac{w(x)}{x^{1+p}}\,dx\right.\\&\quad\left.+\frac{\mu}{|a_1^pb_1^p|^l}\phi\left(\frac{1}{2}\right)\int_{a_1}^{b_1}\frac{|2a_1^pb_1^p-(a_1^p+b_1^p)x^p|^lw(x)}{|x^p|^lx^{1+p}}\,dx\right)\\&\le\int_{a_1}^{b_1}\frac{f_1(x)w(x)}{x^{1+p}}\,dx,\end{aligned}$$

which is the left side of inequality (5.1).

Finally, for the right side of inequality (5.1), setting $x=a_1$ in Definition 2.15 gives

$$f_1\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)\le h(1-j)f_1(a_1)+h(j)f_1(b_1)-\mu\phi(j)\left\|\frac{1}{a_1^p}-\frac{1}{b_1^p}\right\|^l.$$

Since w is a nonnegative symmetric and integrable function, we have

$$\begin{aligned}&f_1\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)w\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)\\&\le h(1-j)f_1(a_1)w\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)+h(j)f_1(b_1)w\left(\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)\end{aligned}$$

$$-\mu\phi(j)\left\|\frac{1}{a_1^p}-\frac{1}{b_1^p}\right\|^l w\left(\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right). \tag{5.4}$$

Integrating inequality (5.4) with respect to j over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 f_1\left(\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)w\left(\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)dj \\
&\quad\le \int_0^1 h(1-j)f_1(a_1)w\left(\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)dj \\
&\qquad + \int_0^1 h(j)f_1(b_1)w\left(\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)dj \\
&\qquad - \mu\int_0^1 \phi(j)\left\|\frac{1}{a_1^p}-\frac{1}{b_1^p}\right\|^l w\left(\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right)dj
\end{aligned}$$

and

$$\begin{aligned}
\int_{a_1}^{b_1}\frac{f_1(x)w(x)}{x^{1+p}}\,dx &\le \big(f_1(a_1)+f_1(b_1)\big)\int_{a_1}^{b_1} h\left(\frac{a_1^p(b_1^p-x^p)}{x^p(b_1^p-a_1^p)}\right)\frac{w(x)}{x^{1+p}}\,dx \\
&\quad - \mu\left\|\frac{b_1^p-a_1^p}{a_1^p b_1^p}\right\|^l \int_{a_1}^{b_1}\phi\left(\frac{a_1^p(b_1^p-x^p)}{x^p(b_1^p-a_1^p)}\right)\frac{w(x)}{x^{1+p}}\,dx,
\end{aligned}$$

which is the right side of inequality (5.1). This completes the proof. □

Remark 5.2

1. Inserting $l=2$, $h(j)=j$, and $p=1$ with $\phi(j)=j(1-j)$ into Theorem 5.1, we obtain a Fejér-type inequality for strongly reciprocally convex functions; see [27, Theorem 3.7].
2. In the same fashion the insertion of $l=2$ and $h(j)=j$ with $\phi(j)=j(1-j)$ into Theorem 5.1 yields a Fejér-type inequality for strongly reciprocally p-convex functions; see [28, Theorem 3.5].

6 Fractional integral inequalities

Fractional integral inequalities are important to study means [32–35]. We now develop some fractional integral inequalities for functions with derivatives in $SR(ph)$. To obtain results of our desired type, we need the following lemma, which can be found in [36].

Lemma 6.1 ([36, Lemma 2.1]) *Let $f_1 : M=[a_1,b_1]\subseteq\mathbb{R}$ be a differentiable function on the interior $\mathring{M}$ of M. If $f_1'\in L[a_1,b_1]$ and $\lambda\in[0,1]$, then*

$$\begin{aligned}
&(1-\lambda)f_1\left[\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{\frac{1}{p}}\right]+\lambda\left(\frac{f_1(a_1)+f_1(b_1)}{2}\right)-\frac{p(a_1^p b_1^p)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}\,dx \\
&\quad=\frac{(b_1^p-a_1^p)}{2p(a_1^p b_1^p)}\left[\int_0^{\frac{1}{2}}(2j-\lambda)\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+\frac{1}{p}} f_1'\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right]dj\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^1(2j-2+\lambda)\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+\frac{1}{p}} f_1'\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right]dj\right].
\end{aligned} \tag{6.1}$$

Theorem 6.2 *Let $M = [a_1, b_1] \subset \mathbb{R}\backslash\{0\}$ be a p-harmonic convex set, and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on the interior $\mathring{M}$ of M. If $f_1' \in L[a_1, b_1]$ and $|f_1'|^q$ is a strongly reciprocally (p, h)-convex function of higher order on M, $q \geq 1$, and $\lambda \in [0, 1]$, then*

$$\begin{aligned}
&\left|(1-\lambda)f_1\left[\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{\frac{1}{p}}\right] + \lambda\left(\frac{f_1(a_1)+f_1(b_1)}{2}\right) - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}\,dx\right| \\
&\quad\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)}\Big[C_1(p,a_1,b_1)^{1-\frac{1}{q}}\big[C_3(p,a_1,b_1)\big|f_1'(a_1)\big|^q + C_5(p,a_1,b_1)\big|f_1'(b_1)\big|^q \\
&\qquad + C_7(p,a_1,b_1)\mu\big]^{\frac{1}{q}} + C_2(p,b_1,a_1)^{1-\frac{1}{q}}\big[C_6(p,b_1,a_1)\big|f_1'(a_1)\big|^q + C_4(p,b_1,a_1)\big|f_1'(b_1)\big|^q \\
&\qquad + C_8(p,b_1,a_1)\mu\big]^{\frac{1}{q}}\Big],
\end{aligned} \tag{6.2}$$

where

$$C_1(p,a_1,b_1) = \int_0^{\frac{1}{2}} |2j-\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+\frac{1}{p}} dj, \tag{6.3}$$

$$C_2(p,b_1,a_1) = \int_{\frac{1}{2}}^1 |2j-2+\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+\frac{1}{p}} dj, \tag{6.4}$$

$$C_3(p,a_1,b_1) = \int_0^{\frac{1}{2}} h(1-j)|2j-\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+\frac{1}{p}} dj, \tag{6.5}$$

$$C_4(p,b_1,a_1) = \int_{\frac{1}{2}}^1 h(j)|2j-2+\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+\frac{1}{p}} dj, \tag{6.6}$$

$$C_5(p,a_1,b_1) = \int_0^{\frac{1}{2}} h(j)|2j-\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+\frac{1}{p}} dj, \tag{6.7}$$

$$C_6(p,b_1,a_1) = \int_{\frac{1}{2}}^1 h(1-j)|2j-2+\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+\frac{1}{p}} dj, \tag{6.8}$$

$$C_7(p,a_1,b_1) = -\int_0^{\frac{1}{2}} \phi(j)|2j-\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+\frac{1}{p}} \left\|\frac{1}{b_1^p} - \frac{1}{a_1^p}\right\|^l dj, \tag{6.9}$$

$$C_8(p,b_1,a_1) = -\int_{\frac{1}{2}}^1 \phi(j)|2j-2+\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+\frac{1}{p}} \left\|\frac{1}{b_1^p} - \frac{1}{a_1^p}\right\|^l dj. \tag{6.10}$$

Proof Using Lemma 6.1, we have

$$\begin{aligned}
&\left|(1-\lambda)f\left[\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{\frac{1}{p}}\right] + \lambda\left(\frac{f(a_1)+f(b_1)}{2}\right) - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f(x)}{x^{1+p}}\,dx\right| \\
&\quad\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)}\left[\int_0^{\frac{1}{2}}\left|(2j-\lambda)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+\frac{1}{p}}\right|\left|f_1'\left[\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{\frac{1}{p}}\right]\right| dj\right. \\
&\qquad\left. + \int_{\frac{1}{2}}^1\left|(2j-2+\lambda)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+\frac{1}{p}}\right|\left|f_1'\left[\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{\frac{1}{p}}\right]\right| dj\right].
\end{aligned}$$

Using power mean inequality,

$$\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)}\Bigg[\bigg(\int_0^{\frac{1}{2}} \big|(2j-\lambda)\big|\bigg(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\bigg)^{1+\frac{1}{p}}\,dj\bigg)^{1-\frac{1}{q}}$$

$$\times \bigg(\int_0^{\frac{1}{2}} \big|(2j-\lambda)\big|\bigg(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\bigg)^{1+\frac{1}{p}}\bigg|f_1'\bigg[\bigg(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\bigg)^{\frac{1}{p}}\bigg]\bigg|^q\,dj\bigg)^{\frac{1}{q}}$$

$$+ \bigg(\int_{\frac{1}{2}}^1 \big|(2j-2+\lambda)\big|\bigg(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\bigg)^{1+\frac{1}{p}}\,dj\bigg)^{1-\frac{1}{q}}$$

$$\times \bigg(\int_{\frac{1}{2}}^1 \big|(2j-2+\lambda)\big|\bigg(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\bigg)^{1+\frac{1}{p}}\bigg|f_1'\bigg[\bigg(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\bigg)^{\frac{1}{p}}\bigg]\bigg|^q\,dj\bigg)^{\frac{1}{q}}\Bigg].$$

Since $|f_1'(x)|^q$ is in $SR(ph)$, we further have

$$\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)}\Bigg[\bigg(\int_0^{\frac{1}{2}} \big|(2j-\lambda)\big|\bigg(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\bigg)^{1+\frac{1}{p}}\,dj\bigg)^{1-\frac{1}{q}}$$

$$\times \bigg(\int_0^{\frac{1}{2}} \big|(2j-\lambda)\big|\bigg(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\bigg)^{1+\frac{1}{p}}\bigg[h(1-j)\big|f_1'(a_1)\big|^q + h(j)\big|f_1'(b_1)\big|^q$$

$$- \mu\phi(j)\bigg\|\frac{1}{b_1^p} - \frac{1}{a_1^p}\bigg\|^l\bigg]\,dj\bigg)^{\frac{1}{q}} + \bigg(\int_{\frac{1}{2}}^1 \big|(2j-2+\lambda)\big|\bigg(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\bigg)^{1+\frac{1}{p}}\,dj\bigg)^{1-\frac{1}{q}}$$

$$\times \bigg(\int_{\frac{1}{2}}^1 \big|(2j-2+\lambda)\big|\bigg(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\bigg)^{1+\frac{1}{p}}\bigg[h(1-j)\big|f_1'(a_1)\big|^q + h(j)\big|f_1'(b_1)\big|^q$$

$$- \mu\phi(j)\bigg\|\frac{1}{b_1^p} - \frac{1}{a_1^p}\bigg\|^l\bigg]\,dj\bigg)^{\frac{1}{q}}\Bigg]$$

$$= \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)}\Big[C_1(p,a_1,b_1)^{1-\frac{1}{q}}\Big[C_3(p,a_1,b_1)\big|f_1'(a_1)\big|^q + C_5(p,a_1,b_1)\big|f_1'(b_1)\big|^q$$

$$+ C_7(p,a_1,b_1)\mu\Big]^{\frac{1}{q}} + C_2(p,b_1,a_1)^{1-\frac{1}{q}}\Big[C_6(p,b_1,a_1)\big|f_1'(a_1)\big|^q + C_4(p,b_1,a_1)\big|f_1'(b_1)\big|^q$$

$$+ C_8(p,b_1,a_1)\mu\Big]^{\frac{1}{q}}\Big],$$

which is the required result. □

Remark 6.3 Inserting $h(j) = j, \mu = 0$, and $l = 2$ with $\phi(j) = j(1-j)$ into Theorem 6.2, we obtain [36, Theorem 2.2].

For $q = 1$, Theorem 6.2 reduces to the following result.

Corollary 6.4 *Let $M = [a_1, b_1] \subset \mathbb{R} \setminus \{0\}$ be a p-harmonic convex set, and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior $\overset{\circ}{M}$ of M. If $f_1' \in L[a_1, b_1]$, $|f_1'|^q$ is in $SR(ph)$ on M, and $\lambda \in [0,1]$, then*

$$\bigg|(1-\lambda)f_1\bigg[\bigg(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\bigg)^{\frac{1}{p}}\bigg] + \lambda\bigg(\frac{f_1(a_1) + f_1(b_1)}{2}\bigg) - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1} \frac{f_1(x)}{x^{1+p}}\,dx\bigg|$$

$$\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)}\Big[\big(C_3(p,a_1,b_1) + C_6(p,b_1,a_1)\big)\big|f_1'(a_1)\big| + \big(C_5(p,b_1,a_1) + C_4(p,a_1,b_1)\big)\big|f_1'(b_1)\big| + \big(C_7(p,a_1,b_1) + C_8(p,b_1,a_1)\big)\mu\Big], \tag{6.11}$$

where C_3, C_4, C_5, C_6, C_7, *and* C_8 *are given by* (6.5)–(6.10).

Remark 6.5 Inserting $h(j) = j, \mu = 0$, and $l = 2$ with $\phi(j) = j(1-j)$ into Corollary 6.4, we obtain [36, Corollary 2.3].

Theorem 6.6 *Let* $M = [a_1, b_1] \subset \mathbb{R} \setminus \{0\}$ *be a p-harmonic convex set, and let* $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be a differentiable function on the interior* $\mathring{M}$ *of* M. *If* $f_1' \in L[a_1, b_1]$, $|f_1'|^q$ *is strongly reciprocally* (p, h)*-convex function of higher order on* M, $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, *and* $\lambda \in [0,1]$, *then*

$$\left|(1-\lambda)f_1\left(\left[\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right]^{\frac{1}{p}}\right) + \lambda\left(\frac{f_1(a_1) + f_1(b_1)}{2}\right) - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}\,dx\right|$$
$$\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)} \times \left(\frac{\lambda^{r+1} + (1-\lambda)^{r+1}}{2(r+1)}\right)^{\frac{1}{r}}\Big[\big(C_9(q,p;a_1,b_1)\big|f_1'(a_1)\big|^q + C_{11}(q,p;a_1,b_1)\big|f_1'(b_1)\big|^q + C_{13}(q,p;a_1,b_1)\mu\big)^{\frac{1}{q}} + \big(C_{12}(q,p;b_1,a_1)\big|f_1'(a_1)\big|^q + C_{10}(q,p;b_1,a_1)\big|f_1'(b_1)\big|^q + C_{14}(q,p;b_1,a_1)\mu\big)^{\frac{1}{q}}\Big], \tag{6.12}$$

where

$$C_9(q,p;a_1,b_1) = \int_0^{\frac{1}{2}} h(1-j)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+\frac{q}{p}} dj, \tag{6.13}$$

$$C_{10}(q,p;b_1,a_1) = \int_{\frac{1}{2}}^1 h(j)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+\frac{q}{p}} dj, \tag{6.14}$$

$$C_{11}(q,p;a_1,b_1) = \int_0^{\frac{1}{2}} h(j)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+\frac{q}{p}} dj, \tag{6.15}$$

$$C_{12}(q,p;b_1,a_1) = \int_{\frac{1}{2}}^1 h(1-j)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+\frac{q}{p}} dj, \tag{6.16}$$

$$C_{13}(q,p;a_1,b_1) = -\int_0^{\frac{1}{2}} \phi(j)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+\frac{q}{p}} \left\|\frac{1}{b_1^p} - \frac{1}{a_1^p}\right\|^l dj, \tag{6.17}$$

$$C_{14}(q,p;b_1,a_1) = -\int_{\frac{1}{2}}^1 \phi(j)|2j-2+\lambda|\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+\frac{q}{p}} \left\|\frac{1}{b_1^p} - \frac{1}{a_1^p}\right\|^l dj. \tag{6.18}$$

Proof Using Lemma 6.1, we have

$$\left|(1-\lambda)f_1\left(\left[\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right]\right)^{\frac{1}{p}} + \lambda\left(\frac{f_1(a_1) + f_1(b_1)}{2}\right) - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}\,dx\right|$$
$$\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)}\left[\int_0^{\frac{1}{2}}\left|(2j-\lambda)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+\frac{1}{p}}\right|\left|f_1'\left[\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{\frac{1}{p}}\right]\right| dj\right.$$

$$+\int_{\frac{1}{2}}^{1}\left|(2j-2+\lambda)\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+\frac{1}{p}}\right|\left|f_1'\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right]\right|dj\bigg].$$

Applying Hölder's integral inequality, we have

$$\begin{aligned}
&\leq \frac{(b_1^p-a_1^p)}{2p(a_1^p b_1^p)}\bigg[\left(\int_0^{\frac{1}{2}}\left|(2j-\lambda)\right|^r dj\right)^{\frac{1}{r}}\\
&\quad\times\left(\int_0^{\frac{1}{2}}\left|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+\frac{1}{p}}f_1'\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right]\right|^q dj\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{1}\left|(2j-2+\lambda)\right|^r dj\right)^{\frac{1}{r}}\\
&\quad\times\left(\int_{\frac{1}{2}}^{1}\left|\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+\frac{1}{p}}f_1'\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right]\right|^q dj\right)^{\frac{1}{q}}\bigg]\\
&=\frac{(b_1^p-a_1^p)}{2p(a_1^p b_1^p)}\bigg[\left(\int_0^{\frac{1}{2}}\left|(2j-\lambda)\right|^r dj\right)^{\frac{1}{r}}\\
&\quad\times\left(\int_0^{\frac{1}{2}}\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{q+\frac{q}{p}}\left|f_1'\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right]\right|^q dj\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{1}\left|(2j-2+\lambda)\right|^r dj\right)^{\frac{1}{r}}\\
&\quad\times\left(\int_{\frac{1}{2}}^{1}\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{q+\frac{q}{p}}\left|f_1'\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{\frac{1}{p}}\right]\right|^q dj\right)^{\frac{1}{q}}\bigg].
\end{aligned}$$

Since $|f_1'(x)|^q$ is in $SR(ph)$, we have

$$\begin{aligned}
&\leq \frac{(b_1^p-a_1^p)}{2p(a_1^p b_1^p)}\bigg[\left(\int_0^{\frac{1}{2}}\left|(2j-\lambda)\right|^r dj\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{q+\frac{q}{p}}\right.\\
&\quad\left.\times\left[h(1-j)\left|f_1'(a_1)\right|^q+h(j)\left|f_1'(b_1)\right|^q-\mu\phi(j)\left\|\frac{1}{b_1^p}-\frac{1}{a_1^p}\right\|^l\right]dj\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{1}\left|(2j-2+\lambda)\right|^r dj\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^{1}\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{q+\frac{q}{p}}\right.\\
&\quad\left.\times\left[h(1-j)\left|f_1'(a_1)\right|^q+h(j)\left|(b_1)\right|^q-\mu\phi(j)\left\|\frac{1}{b_1^p}-\frac{1}{a_1^p}\right\|^l\right]dj\right)^{\frac{1}{q}}\bigg]\\
&\leq \frac{(b_1^p-a_1^p)}{2p(a_1^p b_1^p)}\bigg[\left(\int_0^{\frac{1}{2}}\left|(2j-\lambda)\right|^r dj\right)^{\frac{1}{r}}\\
&\quad\times\left(C_9(q,p;a_1,b_1)\left|f_1'(a_1)\right|^q+C_{11}(q,p;a_1,b_1)\left|f_1'(b_1)\right|^q+C_{13}(q,p;a_1,b_1)\mu\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^{\frac{1}{2}}\left|(2j-2+\lambda)\right|^r dj\right)^{\frac{1}{r}}\\
&\quad\times\left(C_{12}(q,p;b_1,a_1)\left|f_1'(a_1)\right|^q+C_{10}(q,p;b_1,a_1)\left|f_1'(b_1)\right|^q+C_{14}(q,p;b_1,a_1)\mu\right)^{\frac{1}{q}}\bigg]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)} \times \left(\frac{\lambda^{r+1} + (1-\lambda)^{r+1}}{2(r+1)}\right)^{\frac{1}{r}} \Big[\big(C_9(q,p;a_1,b_1)\big|f_1'(a_1)\big|^q \\
&\quad + C_{11}(q,p;a_1,b_1)\big|f_1'(b_1)\big|^q + C_{13}(q,p;a_1,b_1)\mu\big)^{\frac{1}{q}} \\
&\quad + \big(C_{12}(q,p;b_1,a_1)\big|f_1'(a_1)\big|^q \\
&\quad + C_{10}(q,p;b_1,a_1)\big|f_1'(b_1)\big|^q + C_{14}(q,p;b_1,a_1)\mu\big)^{\frac{1}{q}}\Big],
\end{aligned} \tag{6.19}$$

which is the required result. □

Remark 6.7 Inserting $h(j) = j, \mu = 0$, and $l = 2$ with $\phi(j) = j(1-j)$ into Theorem 6.6, we obtain [36, Theorem 2.5].

For $\lambda = 0$, Theorem 6.6 reduces to the following result.

Corollary 6.8 *Let $M = [a_1, b_1] \subset \mathbb{R}\backslash\{0\}$ be a p-harmonic convex set, and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on the interior $\mathring{M}$ of M. If $f_1' \in L[a_1, b_1]$, $|f_1'|^q$ is in SR(ph) on M, $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&\left|f_1\left[\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{\frac{1}{p}}\right] - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1} \frac{f_1(x)}{x^{1+p}}\,dx\right| \\
&\quad \leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)} \times \left(\frac{1}{2(r+1)}\right)^{\frac{1}{r}} \Big[\big(C_9(q,p;a_1,b_1)\big|f_1'(a_1)\big|^q \\
&\quad + C_{11}(q,p;a_1,b_1)\big|f_1'(b_1)\big|^q + C_{14}(q,p;a_1,b_1)\mu\big)^{\frac{1}{q}} \\
&\quad + \big(C_{12}(q,p;b_1,a_1)\big|f_1'(a_1)\big|^q + C_{10}(q,p;b_1,a_1)\big|f_1'(b_1)\big|^q \\
&\quad + C_{14}(q,p;b_1,a_1)\mu\big)^{\frac{1}{q}}\Big],
\end{aligned} \tag{6.20}$$

where $C_9, C_{10}, C_{11}, C_{12}, C_{14}$, and C_{14} are given by (6.13)–(6.18).

Remark 6.9 Inserting $h(j) = j, \mu = 0$, and $l = 2$ with $\phi(j) = j(1-j)$ into Corollary 6.8, we obtain [36, Corollary 3.5].

For $\lambda = 1$, Theorem 6.6 reduces to the following result.

Corollary 6.10 *Let $M = [a_1, b_1] \subset \mathbb{R}\backslash\{0\}$ be a p-harmonic convex set, and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on the interior $\mathring{M}$ of M. If $f_1' \in L[a_1, b_1]$, $|f_1'|^q$ is a strongly reciprocally (p, h)-convex function of higher order on M, $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&\left|\frac{f_1(a_1) + f_1(b_1)}{2} - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1} \frac{f_1(x)}{x^{1+p}}\,dx\right| \\
&\quad \leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)} \times \left(\frac{1}{2(r+1)}\right)^{\frac{1}{r}} \Big[\big(C_9(q,p;a_1,b_1)\big|f_1'(a_1)\big|^q \\
&\quad + C_{11}(q,p;a_1,b_1)\big|f_1'(b_1)\big|^q + C_{14}(q,p;a_1,b_1)\mu\big)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(C_{12}(q,p;b_1,a_1)\left|f_1'(a_1)\right|^q + C_{10}(q,p;b_1,a_1)\left|f_1'(b_1)\right|^q + C_{14}(q,p;b_1,a_1)\mu\right)^{\frac{1}{q}}\Big],$$

where $C_9, C_{10}, C_{11}, C_{12}, C_{14}$, *and* C_{14} *are given by* (6.13)–(6.18).

Remark 6.11 Inserting $h(j) = j$, $\mu = 0$, and $l = 2$ with $\phi(j) = j(1-j)$ into Corollary 6.10, we obtain [36, Corollary 3.6].

For $\lambda = \frac{1}{3}$, Theorem 6.6 reduces to the following result.

Corollary 6.12 *Let* $M = [a_1, b_1] \subset \mathbb{R}\backslash\{0\}$ *be a p-harmonic convex set, and let* $f_1 : M = [a_1, b_1] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *be a differentiable function on the interior* $\mathring{M}$ *of* M. *If* $f_1' \in L[a_1, b_1]$, $|f_1'|^q$ *is a strongly reciprocally* (p,h)*-convex function of higher order on* M, $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, *and* $\lambda \in [0,1]$, *then*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f_1(a) + 4f_1\left[\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{\frac{1}{p}}\right] + f_1(b)\right] - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}\,dx\right| \\
&\quad\le \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)} \times \left(\frac{1 + 2^{r+1}}{6.3^r(r+1)}\right)^{\frac{1}{r}}\Big[\big(C_9(q,p;a_1,b_1)\left|f_1'(a_1)\right|^q \\
&\quad+ C_{11}(q,p;a_1,b_1)\left|f_1'(b_1)\right|^q + C_{14}(q,p;a_1,b_1)\mu\big)^{\frac{1}{q}} \\
&\quad+ \big(C_{12}(q,p;b_1,a_1)\left|f_1'(a_1)\right|^q + C_{10}(q,p;b_1,a_1)\left|f_1'(b_1)\right|^q + C_{14}(q,p;b_1,a_1)\mu\big)^{\frac{1}{q}}\Big],
\end{aligned}$$

where $C_9, C_{10}, C_{11}, C_{12}, C_{14}$, *and* C_{14} *are given by* (6.13)–(6.18).

Remark 6.13 Inserting $h(j) = j$, $\mu = 0$, and $l = 2$ with $\phi(j) = j(1-j)$ into Corollary 6.12, we obtain [36, Corollary 3.7].

For $\lambda = \frac{1}{2}$, Theorem 6.6) reduces to the following result.

Corollary 6.14 *Let* $M = [a_1, b_1] \subset \mathbb{R}\backslash\{0\}$ *be a p-harmonic convex set, and let* $f_1 : M = [a_1, b_1] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *be a differentiable function on the interior* $\mathring{M}$ *of* M. *If* $f_1' \in L[a_1, b_1]$, $|f_1'|^q$ *is a strongly reciprocally* (p,h)*-convex function of higher order on* M, $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, *and* $\lambda \in [0,1]$, *then*

$$\begin{aligned}
&\left|\frac{1}{4}\left[f_1(a) + 2f_1\left[\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{\frac{1}{p}}\right] + f_1(b)\right] - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}\,dx\right| \\
&\quad\le \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)} \times \left(\frac{2}{4.2^r(r+1)}\right)^{\frac{1}{r}}\Big[\big(C_9(q,p;a_1,b_1)\left|f_1'(a_1)\right|^q \\
&\quad+ C_{11}(q,p;a_1,b_1)\left|f_1'(b_1)\right|^q + C_{14}(q,p;a_1,b_1)\mu\big)^{\frac{1}{q}} \\
&\quad+ \big(C_{12}(q,p;b_1,a_1)\left|f_1'(a_1)\right|^q + C_{10}(q,p;b_1,a_1)\left|f_1'(b_1)\right|^q + C_{14}(q,p;b_1,a_1)\mu\big)^{\frac{1}{q}}\Big],
\end{aligned}$$

where $C_9, C_{10}, C_{11}, C_{12}, C_{14}$, *and* C_{14} *are given by* (6.13)–(6.18).

Remark 6.15 Inserting $h(j) = j$, $\mu = 0$, and $l = 2$ with $\phi(j) = j(1-j)$ into Corollary 6.14, we obtain [36, Corollary 3.8].

7 Conclusions

In this paper, we investigated the properties of a newly introduced class of functions called strongly reciprocally (p, h)-convex functions of higher order. Through the study, the authors have established Hermite–Hadamard-type and Fejér-type inequalities for this class of functions. These findings are significant, as they provide new insights into the behavior of these functions, which can be used to analyze a range of mathematical models and problems.

The paper also presents fractional integral inequalities applicable to strongly reciprocally (p, h)-convex functions of higher order. This result is particularly relevant in the field of fractional calculus, where the fractional derivatives and integrals are increasingly used to model various physical and biological phenomena.

Overall, the paper provides a comprehensive analysis of strongly reciprocally (p, h)-convex functions of higher order, and presents several important mathematical results that will be useful in various areas of mathematics and its applications. The findings of this paper may inspire further research on this class of functions and lead to the development of new techniques for analyzing and solving mathematical problems.

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Author contributions
H.L. analyzed the results, M.S.S. prepare the introduction section and verified the results, I.A. proved the main results and K.N.A. wrote the paper.

Author details
[1]School of Mathematics, ChengDu Normal University, ChengDu, 611130, China. [2]Department of Mathematics, University of Okara, Okara, Pakistan. [3]Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore, Pakistan.

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References

1. Definetti, B.: Sulla stratificazioni convesse. Ann. Math. Pures Appl. **30**, 173–183 (1949)
2. Manasarian, O.L.: Pseudo-convex functions. SIAM J. Control **3**, 281–290 (1965)
3. Páolyak, B.T.: Existence theorems and convergence of minimizing sequences in extremum problems with restrictions. Sov. Math. Dokl. **7**, 72–75 (1966)
4. Jensen, J.L.W.V.: On konvexe funktioner oguligheder mellem middlvaerdier. Nyt Tidsskr. Math. **16**, 49–69 (1905)
5. Abramovich, S.: Convexity, subadditivity and generalized Jensen's inequality. Ann. Funct. Anal. **4**(2), 183–194 (2013)
6. Anastassiou, G.A.: Basic and s-convexity Ostrowski and Grušs type inequalities involving several functions. Commun. Appl. Anal. **17**(2), 189–212 (2013)
7. Ozdemir, M.E., Avci, M., Kavrmaci, H.: Hermite–Hadamard-type inequalities via (α,m)-convexity. Comput. Math. Appl. **61**(9), 2614–2620 (2011)
8. Set, E., Ozdemir, M.E., Dragomir, S.S.: On Hadamard-type inequalities involving several kinds of convexity. J. Inequal. Appl. **2010**, Article ID 286845 (2010)
9. Bector, C.R., Singh, C.: B-Vex functions. J. Optim. Theory Appl. **71**(2), 237–253 (1991)
10. Hanson, M.A.: On sufficiency of the Kuhn–Tucker conditions. J. Math. Anal. Appl. **80**, 545–550 (1981)
11. Fejer, L.: Uber die Fourierreihen II. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **24**, 369–390 (1960). (in Hungarian)
12. Kizil, S., Ardic, M.A.: Inequalities for strongly convex functions via Atangana–Baleanu integral operators. Turk. J. Sci. **6**(2), 96–109 (2021)

13. Set, E., Akdemir, A.O., Özata, F.: Grüss type inequalities for fractional integral operator involving the extended generalized Mittag-Leffler function. Appl. Comput. Math. **19**(3), 402–414 (2020)
14. Özdemir, M.E.: New refinements of Hadamard integral inequality via k-fractional integrals for *p*-convex function. Turk. J. Sci. **6**(1), 1–5 (2019)
15. Avci Ardic, M., Akdemir, A., Set, E.: On new integral inequalities via geometric-arithmetic convex functions with applications. Sahand Commun. Math. Anal. (2022). Advance online publication. https://doi.org/10.22130/scma.2021.535821.971
16. Ekinci, A., Özdemir, M.E., Set, E.: New integral inequalities of Ostrowski type for quasi-convex functions with applications. Turk. J. Sci. **5**(3), 290–304 (2020)
17. Wang, S.-H., Shi, X.-T.: Hermite–Hadamard- type inequalities for *n*-time differentiable and GA-convex functions with applications to means. J. Anal. Number Theory **4**(1), 15–22 (2006)
18. Hamdan, M.H., Razali, N.A.M., Ali, A.I., Ahmad, S.: Numerical simulation of blood flow through stenotic artery. Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. **69**(2), 2313–2323 (2020). https://doi.org/10.31801/cfsuasmas.748013
19. Zhu, H., Yan, H.: Global dynamics for a Volterra difference equation with distributed delays. Mathematics **8**(11), 2015–2028 (2020). https://doi.org/10.3934/math.2020300
20. Wang, J., Jiao, J.: An improved Adomian decomposition method for nonlinear boundary value problems. Tbil. Math. J. **12**(3), 139–148 (2019). https://doi.org/10.32513/tbilisi/1593223217
21. Li, X., Li, X., Zeng, C.: Stability analysis of a class of delay differential equations. Math. Methods Appl. Sci. **44**(8), 5801–5810 (2021). https://doi.org/10.1002/mma.6712
22. Brodmann, M., Lanzilotta, M., Sather–Wagstaff, S.: Gröbner bases, determinantal ideals, and algebraic statistics. Results Math. **74**(1), 1–21 (2019). https://doi.org/10.1007/s00025-019-0960-8
23. Zhang, K.S.: *p*-convex functions and their properties. Pure Appl. Math. **1**(23), 130–133 (2007)
24. Maden, S., Turhan, S.: Hermite–Hadamard inequality for strongly *p*-convex functions (2018). https://doi.org/10.13140/RG.2.2.26582.14404
25. Shi, H.N., Zhang, J.: Some new judgement theorems of Schur geometric and Schur harmonic convexities for a class of symmetric functions. J. Inequal. Appl. **2013**, Article ID 527 (2013)
26. Noor, M.A., Noor, K.I., Iftikhar, S.: Hermite–Hadamard inequalities for harmonic nonconvex functions. MAGNT Res. Rep. **4**(1), 24–40 (2016)
27. Bracamonte, M., Medina, J., Vivas, M.: On some inequalities for strongly reciprocally convex functions. Extr. Math. **33**, 109–126 (2018)
28. Saleem, M.S., Hussain, I., Imran, M.: On strongly reciprocally p-convex functions
29. Varosanec, S.: On *h*-convexity. J. Math. Anal. Appl. **326**, 303–311 (2007). https://doi.org/10.1016/j.jmaa.2006.02.08
30. Fang, Z.B., Shi, R.: On the (*p*, *h*)-convex function and some integral inequalities. J. Inequal. Appl. **2014**, 45 (2014)
31. Lin, G.H., Fukushima, M.: Some exact penalty results for nonlinear programs and mathematical programs with equilibrium constraints. J. Optim. Theory Appl. **118**, 67–80 (2003)
32. Farid, G.T.: Some integral inequalities for m-convex functions via fractional integrals. J. Inequal. Spec. Funct. **8**(1), 170–185 (2017)
33. Farid, G.A.: Generalizations of some Hermite–Hadamard–Fejér type inequalities for *p*-convex functions via generalized fractional integrals. J. Fract. Calc. Appl. **9**(2), 56–76 (2018)
34. Iscan, I.: Hermite–Hadamard and Fejer type inequalities for convex functions via fractional integrals. Stud. Univ. Babeş–Bolyai, Math. **60**(3), 355–366 (2015)
35. Xi, B., Qi, Y.: Some integral inequalities of Hermite–Hadamard type for convex functions with applications to means. J. Funct. Spaces Appl. **2012**), Article ID 980438 (2012). https://doi.org/10.1155/2012/980438
36. Noor, M.A., Noor, K.I.: Integral inequalities for differentiable *p*-harmonic convex functions. Filomat **31**(20), 6575–6584 (2017)

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