Huang et al. *Journal of Inequalities and Applications* (2023) 2023:53
https://doi.org/10.1186/s13660-023-02959-5

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Approximate controllability of second-order impulsive neutral stochastic differential equations with state-dependent delay and Poisson jumps

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Abstract

We consider the approximate controllability for a class of second-order impulsive neutral stochastic differential equations with state-dependent delay and Poisson jumps in a real separable Hilbert space. Under the sufficient conditions, we obtain approximate controllability results by virtue of the theory of a strongly continuous cosine family of bounded linear operators combined with stochastic inequality technique and the Sadovskii fixed point theorem. Finally, we illustrate the main results by an example.

MSC: 34K50; 34K45; 93B05

Keywords: Approximate controllability; Strongly continuous cosine family; Second order; Impulsive neutral stochastic differential equations; State-dependent delay; Poisson jumps

1 Introduction

Controllability plays an important role in both deterministic and stochastic control systems throughout the history of modern control theory. It is well known that controllability of deterministic and stochastic equations has been frequently used in many fields such as physics, engineering, artificial intelligence, automatic control, biochemical, and so on (see [1–4] and the references therein). In the actual industrial process, the control usually does not affect the complete state of the dynamical systems, but only affects a part of it, and thus two basic concepts of exact controllability and approximate controllability are derived. Generally speaking, controllability means that it is possible to steer dynamical systems from an arbitrary initial state to the desired final state using the set of admissible controls.

The controllability of deterministic differential equations has been widely investigated by many authors in the past decades; see, for example, [5–7]. In fact, stochastic differential equations (SDEs) have attracted much attention for many inevitable random factors in real phenomena and played an important role in many branches of science and industry, such as physical, biological, medical, neural networks, financial, and engineering problems (see [8–11] and the references therein). In addition, impulsive effects exist in many dynamic systems, which describe abrupt changes of states at certain instant of time [12]. Therefore there is a real need to investigate the controllability of impulsive stochastic differential equations, and many results were obtained in recent years; see, for example, [13–15].

Differential equations with state-dependent delay (SDD) are well known in modeling many practical problems, and for this reason, the study of this type equations have gain more attention in recent years [16–20]. It should be pointed out that Muthukumar and Rajivganthi [17] investigated the following impulsive neutral stochastic functional differential system with SDD:

$$\begin{aligned} &d\big[x(t)+F(t,x_t)\big]=\big[Ax(t)+Bu(t)\big]\,dt+G(t,x_{\rho(t,x_t)})\,dw(t),\\ &t\in J=[0,b]\setminus\{t_1,t_2,\ldots,t_m\},\\ &x_0=\phi\in\mathcal{B},\\ &\Delta x|_{t=t_k}=x\big(t_k^+\big)-x\big(t_k^-\big)=I_k\big(x(t_k)\big),\quad k=1,2,\ldots,m. \end{aligned}\tag{1.1}$$

By means of the Sadovskii fixed point theorem and semigroup theory the approximate controllability results are obtained under some sufficient conditions. Note that the publications mentioned focus on the first-order SDEs.

Second-order abstract differential equations have gained much more attention due to their wide applications in physics and engineering. For example, the system of dynamical buckling of a hinged extensible beam can be modeled by second-order differential equations [21, 22]. Fitzgibbon [21] has discussed the extensible beam equation

$$\frac{\partial^2 z}{\partial t^2}+\frac{\alpha_1\partial^4 z}{\partial y^4}-\left(\alpha_2+\alpha_3\int_0^L\left|\frac{\partial z(y,t)}{\partial y}\right|^2dy\right)\frac{\partial^2 z}{\partial y^2}+f\left(\frac{\partial z}{\partial t}\right)=0\tag{1.2}$$

subject to the boundary conditions at the ends of the beam being hinged, namely,

$$z(0,t)=z(L,t)=z_{yy}(0,t)=z_{yy}(L,t)=0,$$

where $z(y,t)$ is the deflection of the beam at point y and time t, f is a nondecreasing numerical function, L is the length of the beam, and $\alpha_i>0$ $(i=1,2,3)$ are given parameters. The nonlinear friction force $f(\frac{\partial z}{\partial t})$ is the dissipative term. Under some reasonable assumptions, Eq. (1.2) can be rewritten as a second-order abstract differential equations as follows:

$$z''+A^2z+M\big(\|A^{\frac{1}{2}}z\|_H^2\big)Az+f\big(z'\big)=0,\tag{1.3}$$

where the A is an infinitesimal generator of a strongly continuous cosine family on a Hilbert space H, and M and f are real functions. By means of the theory of semigroup of linear operators and the theory of cosine operators, the global existence and boundedness of solutions of Eq. (1.2) are obtained under some suitable conditions.

In fact, the second-order SDEs are ideal models to describe the integrated process in continuous time, which can be made stationary. For instance, it is useful for engineers to model mechanical vibrations or charge on a capacitor or condenser subjected to white noise through second-order SDEs. Recently, Duan and Wang [9] have introduced the following stochastic wave equation:

$$
\begin{aligned}
&V_{tt} = c^2 V_{xx} + \epsilon \dot{W}, \quad 0 < x < l, \\
&V(0,t) = V(l,t) = 0, \\
&V(x,0) = f(x), \qquad V_t(x,0) = g(x),
\end{aligned}
\tag{1.4}
$$

where $V(x,t)$ is the displacement of a vibrating string at position x and time t, c is a positive constant (wave speed), ϵ is a positive real parameter modeling the noise intensity, $W(t)$ is a Brownian motion taking values in the Hilbert space $L^2(0,l)$, and the initial data f and g are deterministic for simplicity. Obviously, Eq. (1.4) can be abstracted as a kind of second-order SDEs, and many results on the controllability of second-order stochastic control systems are obtained in recent years [23–27].

Since, actually, in real life, we will always meet jump-type stochastic perturbations, SDEs with Poisson jumps have become very popular in describing many natural phenomena arising from fields such as economics, stochastic population control, stochastic pathwise control, engineering, and so on [28, 29]. It is natural and necessary to include a jump term in the SDEs. Moreover, many practical systems (such as sudden stock price variation resulting from market crashes, war, epidemics, and so on) may undergo some jump-type stochastic perturbations. The path continuity supposition does not seem plausible for these models. Therefore we should consider stochastic processes with jumps in modeling such systems. In general, these jump models are derived from Poisson random measure. For this reason, in recent years, many papers reported on the qualitative properties of SDEs with Poisson jumps, such as existence, stability, and controllability. Especially, we refer to [30–34]. As we know, most of the existing literature focuses on the first-order SDEs with Poisson jumps. It is worth mentioning that, very recently, the controllability of second-order SDEs with Poisson jumps has begun to attract the attention of researchers, but the relevant results remain limited. Huan and Gao [35] discussed the controllability of nonlocal second-order impulsive neutral stochastic functional integro-differential equations with delay and Poisson jumps by employing the theory of a strong continuous cosine family of bounded linear operators and the Banach fixed point theorem. However, Triggiani [36, 37] has proved that the notion of exact controllability is usually too strong and has limited applicability in infinite-dimensional spaces, whereas the approximate controllability is more suitable for describing the control systems. Muthukumar and Rajivganthi [38] investigated the following second-order neutral stochastic differential equations with infinite delay and Poisson jumps:

$$
\begin{aligned}
&d\big[x'(t) - f(t,x_t)\big] = \big[Ax(t) + Bu(t)\big]\,dt + g(t,x_t)\,dw(t) \\
&\qquad\qquad + \int_Z h(t,x_t,\eta)\widetilde{N}(dt,d\eta), \quad t \in J = [0,b], \\
&x_0 = \phi \in \mathcal{B}, \qquad x'(0) = \xi.
\end{aligned}
\tag{1.5}
$$

Under some suitable conditions, the approximate controllability results are obtained by using the theory of cosine family of operators and the successive approximation technique.

To the best of our knowledge, the approximate controllability of second-order impulsive neutral stochastic differential equations with SDD and Poisson jumps has not been investigated yet. To fill this gap, motivated by the work of [17, 35, 38], in this paper, we are concerned with second-order impulsive neutral stochastic differential equations with SDD and Poisson jumps of the form

$$\begin{aligned} d\big[x'(t) - F(t,x_t)\big] &= \big[Ax(t) + f(t,x_t) + Bu(t)\big]\,dt + \sigma(t, x_{\rho(t,x_t)})\,dw(t) \\ &\quad + \int_{\mathcal{U}} h\big(t, x(t-), \nu\big)\widetilde{N}(dt, d\nu), \quad t \in J = [0,T], t \neq t_k, \\ \Delta x(t_k) &= I_k(x_{t_k}), \qquad \Delta x'(t_k) = \tilde{I}_k(x_{t_k}), \quad k = 1,2,\ldots,n, \\ x_0 &= \phi \in \mathcal{B}, \qquad x'(0) = x_1 \in H, \end{aligned} \tag{1.6}$$

where the stochastic process x takes values in a real separable Hilbert space H, $A : D(A) \subseteq H \to H$ is the infinitesimal generator of a strongly continuous cosine family on H, K is another real separable Hilbert space, and $w(t)$ is a given K-valued Brownian motion or Wiener process. Let $\widetilde{N}(dt, d\nu)$ be the compensated Poisson measure that is independent of $w(t)$, which will be specified later.

Our contributions of this paper are as follows.

(i) From a practical viewpoint, Eq. (1.6) is in fact an abstract impulsive neutral second-order stochastic wave equation with SDD and Poisson jumps. Therefore the approximate controllability of a more realistic abstract model of Eq. (1.4) can be considered by introducing Poisson jumps, SDD, impulsive, neutral, and control terms as given in Eq. (1.6).

(ii) The approximate controllability of second-order impulsive neutral stochastic differential equations with SDD and Poisson jumps is an untreated topic in the literature, which is an additional motivation for writing this paper.

(iii) We obtain sufficient conditions ensuring the approximate controllability of Eq. (1.6) by using the theory of a strongly continuous cosine family of bounded linear operators, Sadovskii's fixed point theorem, and the stochastic inequality technique (such as the Doob martingale inequality and Burkholder-type inequality for stochastic integrals driven by Poisson processes).

(iv) It is worth mentioning that, compared with Eq. (1.5), Eq. (1.6) takes the impulsive effect into account, and the delay is also state-dependent. Thus Eq. (1.6) is more general and more difficult to handle technically. In addition, the main technique in [38] was based on successive approximation, which is different from that in this paper.

(v) In Sect. 4, an example about the approximate controllability of second-order stochastic wave equations is given to illustrate the obtained main results.

This paper is organized as follows. In the next section, we present some notation and preliminaries adopted from [2, 8, 9, 12, 16, 18, 29, 32, 39–46]. Section 3 is devoted to the approximate controllability of Eq. (1.6). In Sect. 4, an example is provided to illustrate the main results. Finally, conclusions are presented in the last section.

2 Notation and preliminaries

Let $(H, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and $(K, \|\cdot\|_K, \langle\cdot,\cdot\rangle_K)$ be two real separable Hilbert spaces, let $\{e_m\}_{m=1}^{\infty}$ be a complete orthonormal basis of K, and let $\{w(t) : t \geq 0\}$ be a cylindrical K-value Q-Wiener process, where Q is a finite nuclear covariance operator. Denote $\mathrm{Tr}(Q) = \sum_{m=1}^{\infty} \lambda_m < \infty$, wherer $Qe_m = \lambda_m e_m$ ($\lambda_m \geq 0$, $m = 1, 2, \ldots$). Set

$$w(t) = \sum_{m=1}^{\infty} \sqrt{\lambda_m}\alpha_m(t)e_m, \quad t \geq 0,$$

where $\{\alpha_m(t)\}_{m=1}^{\infty}$ is a sequence of real-valued independent one-dimensional standard Brownian motions over a complete probability space $(\Omega, \mathcal{F}, P)$.

We assume that $\mathcal{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by w and $\mathcal{F}_T = \{\mathcal{F}_t\}_{t\geq 0}$. Let $\xi \in \mathcal{L}(K, H)$ and define

$$\|\xi\|_Q^2 = \mathrm{Tr}\big(\xi Q \xi^*\big) = \sum_{m=1}^{\infty} \|\sqrt{\lambda_m}\xi e_m\|^2,$$

where ξ^* is the adjoint of the operator ξ, and $\mathcal{L}(K, H)$ is the space of all bounded linear operators from K into H endowed with the same norm $\|\cdot\|$. If $\|\xi\|_Q^2 < \infty$, then ξ is called a Q-Hilbert–Schmidt operator. The completion $\mathcal{L}_Q(K, H)$ of $\mathcal{L}(K, H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$, is a Hilbert space with the above norm topology, where $\|\xi\|_Q = \langle \xi, \xi\rangle^{\frac{1}{2}}$. The collection of all strongly measurable square-integrable H-valued random variables, denoted by $\mathcal{L}_2(\Omega, H)$, is a Banach space equipped with the norm $\|x\|_{\mathcal{L}_2} = (E\|x\|^2)^{\frac{1}{2}}$, where the expectation E is defined by $Ex = \int_\Omega x(w)\,dP$. Let $\mathcal{L}_2(\Omega, \mathcal{F}_T, H)$ be the Banach space of $\mathcal{F}_T$-measurable square-integrable random variables with values in H, and let $\mathcal{L}_2^{\mathcal{F}_t}(J, H)$ be the space of all square-integrable and $\mathcal{F}_t$-adapted processes.

Let $C(J, \mathcal{L}_2(\Omega, H))$ be the Banach space of all continuous maps from J into $\mathcal{L}_2(\Omega, H)$ satisfying the condition $\sup_{0\leq t\leq T} E\|x(t)\|^2 < \infty$. An important subspace of $\mathcal{L}_2(\Omega, H)$ is given by

$$\mathcal{L}_2^0(\Omega, H) = \big\{x \in \mathcal{L}_2(\Omega, H) : x \text{ is } \mathcal{F}_0\text{-measurable}\big\}.$$

Let $p = (p(t))$ $(t \in D_p)$ be a stationary $\mathcal{F}_t$-Poisson point process with characteristic measure $\lambda(d\nu)$, and let $N(dt, d\nu)$ be the Poisson counting measure associated with p. Then, $N(t, \mathcal{U}) = \sum_{s\in D_p, s\leq t} I_{\mathcal{U}}(p(s))$ with measurable set $\mathcal{U} \in \mathfrak{B}(K - \{0\})$, which is the Borel σ-field of $K - \{0\}$. Let $\widetilde{N}(dt, d\nu) = N(dt, d\nu) - dt\lambda(d\nu)$ be the compensated Poisson measure independent of $w(t)$, and let $\mathbb{P}_2(J \times \mathcal{U}; H)$ be the space of all predictable mappings $h : J \times \mathcal{U} \to H$ for which $\int_0^T \int_{\mathcal{U}} E\|h(t, \nu)\|_H^2\,dt\lambda(d\nu) < \infty$. Then we can define the H-valued stochastic integral $\int_0^T \int_{\mathcal{U}} h(t, \nu)\widetilde{N}(dt, d\nu)$, which is a centered square-integrable martingale.

For more detail, we refer the reader to Da Prato and Zabczyk [39], Gawarecki and Mandrekar [40], and Situ [29].

Consider the following second-order impulsive neutral stochastic differential equations with SDD and Poisson jumps:

$$\begin{aligned} &d\big[x'(t) - F(t, x_t)\big] = \big[Ax(t) + f(t, x_t) + Bu(t)\big]\,dt + \sigma(t, x_{\rho(t,x_t)})\,dw(t) \\ &\qquad\qquad + \int_{\mathcal{U}} h\big(t, x(t-), \nu\big)\widetilde{N}(dt, d\nu), \quad t \in J = [0, T], t \neq t_k, \\ &\Delta x(t_k) = I_k(x_{t_k}), \qquad \Delta x'(t_k) = \tilde{I}_k(x_{t_k}), \quad k = 1, 2, \ldots, n, \\ &x_0 = \phi \in \mathcal{B}, \qquad x'(0) = x_1 \in H, \end{aligned} \tag{2.1}$$

where the history $x_t : (-\infty, 0] \to H$, $x_t(\theta) = x(t+\theta)$, $t \geq 0$, belongs to the phase space $\mathcal{B}$, which will be described axiomatically later. Assume that the mappings $F : J \times \mathcal{B} \to H$, $f : J \times \mathcal{B} \to H$, $\sigma : J \times \mathcal{B} \to \mathcal{L}_Q(K,H)$, $h : J \times H \times \mathcal{U} \to H$, and $\rho : J \times \mathcal{B} \to (-\infty, T]$, $I_k, \tilde{I}_k : \mathcal{B} \to H$ $(k = 1, 2, \ldots, n)$ are appropriate functions that will be specified later. The control function u takes its values in $\mathcal{L}_2^{\mathcal{F}_t}(J, \mathcal{U})$ of admissible control functions for a separable Hilbert space $\mathcal{U}$, and B is a bounded linear operator from $\mathcal{U}$ into H. The initial data $\phi(t) \in \mathcal{L}_2(\Omega, \mathcal{B})$ and $x_1(t)$ are H-valued $\mathcal{F}_t$-measurable random variables independent of the Wiener process with finite second moment. Moreover, let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ be prefixed points, and let $\Delta x(t_k)$ represent the jump of the function x at t_k, which is defined by $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at $t = t_k$, respectively. Similarly, $x'(t_k^+)$ and $x'(t_k^-)$ denote, respectively, the right and left limits of $x'(t)$ at $t = t_k$.

We say that a function $x : [\alpha, \beta] \to H$ is a normalized piecewise continuous function on $[\alpha, \beta]$ if x is piecewise continuous and left continuous on $(\alpha, \beta]$. We denote by $\mathcal{PC}([\alpha, \beta], H)$ the space of the normalized piecewise continuous, $\mathcal{F}_t$-adapted measurable processes from $[\alpha, \beta]$ into H. Particularly, we introduce the space $\mathcal{PC}$ of all $\mathcal{F}_t$-adapted measurable H-valued stochastic processes $\{x(t) : t \in [0, T]\}$ such that x is continuous at $t \neq t_k$, $x(t_k^-) = x(t_k)$, and $x(t_k^+)$ exists for $k = 1, 2, \ldots, n$. Then $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space with norm

$$\|x\|_{\mathcal{PC}} = \sup_{s \in J} \bigl(E\|x(s)\|^2\bigr)^{\frac{1}{2}}.$$

For $x \in \mathcal{PC}$, we denote by $\tilde{x}_k$, $k = 1, 2, \ldots, n$, the function $\tilde{x}_k \in C([t_k, t_{k+1}]; \mathcal{L}_2(\Omega, H))$ given by

$$\tilde{x}_k(t) = \begin{cases} x(t) & \text{for } t \in (t_k, t_{k+1}], \\ x(t_k^+) & \text{for } t = t_k. \end{cases}$$

Moreover, for $\mathcal{D} \subseteq \mathcal{PC}$, we denote $\tilde{\mathcal{D}}_k = \{\tilde{x}_k : x \in \mathcal{D}\}$, $k = 0, 1, \ldots, n,$.

Lemma 2.1 ([16]) *A set $\mathcal{D} \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if and only if the set $\tilde{\mathcal{D}}_k$ is relatively compact in $C([t_k, t_{k+1}]; \mathcal{L}_2(\Omega, H))$ for every $k = 0, 1, \ldots, n$.*

Further in this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato [41]; see also [42] for details. The axioms of the space $\mathcal{B}$ are established for $\mathcal{F}_0$-measurable functions from $(-\infty, 0]$ into H, endowed with seminorm $\|\cdot\|_{\mathcal{B}}$ that satisfies the following axioms.

Axiom A *If $x : (-\infty, \eta + T] \to H$, $T > 0$, is such that $x_\eta \in \mathcal{B}$ and $x|_{[\eta, \eta+T]} \in \mathcal{PC}([\eta, \eta + T], H)$, then for every $t \in [\eta, \eta + T)$, the following conditions hold*:

(i) *x_t is in $\mathcal{B}$,*
(ii) *$E\|x(t)\| \leq L\|x_t\|_{\mathcal{B}}$,*
(iii) *$\|x_t\|_{\mathcal{B}} \leq N(t-\eta) \sup_{\eta \leq s \leq t} E\|x(s)\| + M(t-\eta)\|x_\eta\|_{\mathcal{B}}$, where $L > 0$ is a constant, $M, N : [0, +\infty) \to [1, +\infty)$, N is continuous, M is locally bounded, and L, M, and N are independent of x.*

Axiom B *The space $\mathcal{B}$ is complete.*

To prove the main results, we need the following essential properties.

Lemma 2.2 ([20]) *Let* $x : (-\infty, T] \to H$ *be an* $\mathcal{F}_t$*-adapted measurable process such that the* $\mathcal{F}_0$*-adapted process* $x_0 = \phi(t) \in \mathcal{L}_2^0(\Omega, \mathcal{B})$ *and* $x|_J \in \mathcal{PC}(J, H)$. *Then*

$$\|x_s\|_{\mathcal{B}} \leq M_T \|\phi\|_{\mathcal{B}} + N_T \sup_{0 \leq s \leq T} E\big\|x(s)\big\|,$$

where $N_T = \sup_{t\in J} N(t)$ *and* $M_T = \sup_{t\in J} M(t)$.

Next, we introduce the theory of cosine functions of operators and the second-order abstract Cauchy problem, which appeared in [43–45].

Definition 2.1 ([43, 45]) (1) A one-parameter family $\{C(t) : t \in R\}$ of operators in H is said to be a strongly continuous cosine family if the following conditions hold:

(i) $C(0) = I$, the identity operator in H;

(ii) $C(t)x$ is continuous in t on R for all $x \in H$;

(iii) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $t, s \in R$.

(2) The corresponding strong continuous sine family $\{S(t) : t \in R\} \subset \mathcal{L}(H)$, associated with the family $\{C(t) : t \in R\}$ is defined by

$$S(t)x = \int_0^t C(s)x\,ds, \quad t \in R, x \in H.$$

(3) The infinitesimal generator $A : H \to H$ of $\{C(t) : t \in R\}$ is given by $Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}$ for all $x \in D(A) = \{x \in H : C(\cdot)x \in C^2(R; H)\}$.

It is known that the infinitesimal generator A is a closed densely defined operator on H with the following properties.

Lemma 2.3 ([43, 45]) *Suppose that* A *is the infinitesimal generator of a cosine family of operators* $\{C(t) : t \in R\}$. *Then*:

(i) *there exist* $M_A \geq 1$ *and* $\omega \geq 0$ *such that* $\|C(t)\| \leq M_A e^{\omega|t|}$, *and hence* $\|S(t)\| \leq M_A e^{\omega|t|}$;

(ii) $A \int_s^r S(u)x\,du = [C(r) - C(s)]x$ *for all* $0 \leq s \leq r < \infty$;

(iii) *there exists* $N \geq 1$ *such that* $\|S(s) - S(r)\| \leq N|\int_s^r e^{\omega|s|}\,ds|$, $0 \leq s \leq r < \infty$;

(iv) $S(s + t) = S(s)C(t) + S(t)C(s)$ *for all* $s, t \in R$.

From the uniform bounded principle and Lemma 2.3 *we easily to see that* $\{C(t) : t \in J\}$ *and* $\{S(t) : t \in J\}$ *are uniformly bounded by* $\widetilde{M}_A = M_A e^{\omega|T|}$.

Consider the second-order linear abstract Cauchy problem

$$x''(t) = Ax(t) + g(t), \quad t \in J, x(0) = w_1, x'(0) = w_2, \tag{2.2}$$

where $g : J \to H$ is an integrable function. The mild solution $x : J \to H$ of the equation (2.2) is given by

$$x(t) = C(t)w_1 + S(t)w_2 + \int_0^t S(t - s)g(s)\,ds, \quad t \in J,$$

which is continuously differentiable:

$$x'(t) = AS(t)w_1 + C(t)w_2 + \int_0^t C(t-s)g(s)\,ds, \quad t \in J.$$

For more detail about cosine family theory, we refer to [43–45]. Now we state Sadovskii's fixed point theorem and the Burkholder–Davis–Gundy inequality, which are used in the proof of the main results.

Lemma 2.4 ([39]) *For any $r \geq 1$ and arbitrary $L_Q(K,H)$-valued predictable process Φ,*

$$\sup_{s\in[0,t]} E\left\| \int_0^s \Phi(u)\,dw(u) \right\|_H^{2r} \leq \bigl(r(2r-1)\bigr)^r \left(\int_0^t \bigl(E\|\Phi(s)\|_Q^{2r}\bigr)^{\frac{1}{r}}\,ds \right)^r. \tag{2.3}$$

Lemma 2.5 ([46]) *Let Ψ be a condensing operator on a Banach space X, i.e., Ψ is continuous and takes bounded sets into bounded sets, and $\mu(\Psi(B)) \leq \mu(B)$ for every bounded set B of X with $\mu(B) > 0$, where μ denotes the Kuratowskii measure of noncompactness. If $\Psi(N) \subset N$ for a convex closed bounded set N of X, then Ψ has a fixed point in X.*

Remark 2.1 Note that every map defined on a compact set is condensing. Further, the completely continuous operators, contractions, and also the sums of these two types are condensing operators. We refer the reader to [46] for more detail about the condensing operators.

Definition 2.2 An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, T] \to H$ is called a mild solution of Eq. (2.1) if $x_0 = \phi \in \mathcal{B}$, $x_{\rho(s,x_s)} \in \mathcal{B}$ satisfies $x_0 \in \mathcal{L}_2^0(\Omega, H)$, $x|_J \in \mathcal{PC}$, and $x'(0) = x_1 \in H$ satisfies $x_1 \in \mathcal{L}_2^0(\Omega, H)$. The functions $C(t-s)F(s,x_s)$ and $S(t-s)f(s,x_s)$ are integrable on $[0,T)$, and the following conditions hold:

(i) $\{x_t : t \in J\}$ is $\mathcal{B}$-valued, and the restriction of x to the interval $(t_k, t_{k+1}]$, $k = 1, 2, \ldots, n$, is continuous;

(ii) $\Delta x(t_k) = I_k(x_{t_k})$, $\Delta x'(t_k) = \tilde{I}_k(x_{t_k})$, $k = 1, 2, \ldots, n$;

(iii) for each $t \in J$, $x(t)$ satisfies the following integral equation

$$\begin{aligned} x(t) = {}& C(t)\phi(0) + S(t)\bigl[x_1 - F(0,\phi)\bigr] + \int_0^t C(t-s)F(s,x_s)\,ds \\ & + \int_0^t S(t-s)f(s,x_s)\,ds + \int_0^t S(t-s)\sigma(s, x_{\rho(s,x_s)})\,dw(s) \\ & + \int_0^t S(t-s)\int_{\mathcal{U}} h\bigl(s, x(s-), \nu\bigr)\widetilde{N}(ds,d\nu) + \int_0^t S(t-s)Bu(s)\,ds \\ & + \sum_{0<t_k<t} C(t-t_k)I_k(x_{t_k}) + \sum_{0<t_k<t} S(t-t_k)\tilde{I}_k(x_{t_k}). \end{aligned} \tag{2.4}$$

Definition 2.3 System (2.1) is said to be approximately controllable on J if $\overline{\mathbf{R}(T;\phi,u)} = \mathcal{L}_2(\Omega, \mathcal{F}_T, H)$, where the reachable set $\mathbf{R}(T;\phi,u)$ is defined as $\mathbf{R}(T;\phi,u) = \{x(T;\phi,u) : u \in \mathcal{L}_2^{\mathcal{F}_t}(J,U)\}$, and $\overline{\mathbf{R}(T;\phi,u)}$ is the closure of the reachable set of system (2.1).

Lemma 2.6 ([47]) *For any $x^T \in \mathcal{L}_2(\Omega, \mathcal{F}_T, H)$, there exists $\varphi \in \mathcal{L}_2^{\mathcal{F}_t}(\Omega, \mathcal{L}_2(J, \mathcal{L}_Q(K,H)))$ such that $x^T = E(x^T) + \int_0^T \varphi(s)\,dw(s)$.*

In what follows, we always assume that $\rho : J \times \mathcal{B} \to (-\infty, T]$ is continuous and $\phi \in \mathcal{B}$. In addition, we need the following hypotheses:

(H_1) Let $\mathcal{R}(\rho^-) = \{\rho(s,\psi) \le 0, \rho(s,\psi) : (s,\psi) \in J \times \mathcal{B}\}$. The function $t \to \phi_t$ is well defined from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$, and there exists a continuous bounded function $J^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathcal{B}} \le J^\phi(t)\|\phi\|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.

(H_2) The cosine family of operators $\{C(t) : t \in J\}$ and the corresponding sine family $\{S(t) : t \in J\}$ are compact for $t > 0$, and there exist positive constants M_C and M_S such that $\|C(t)\|^2 \le M_C$, $\|S(t)\|^2 \le M_S$.

(H_3) $B \in \mathcal{L}_2^{\mathcal{F}_t}(J, U)$, and there exists a positive constant M_B such that $\|B\|^2 \le M_B$.

(H_4) There exist constants $L_F > 0$ and $\widetilde{L}_F > 0$ such that

$$E\big\|F(t,x) - F(t,y)\big\|^2 \le L_F\|x-y\|_{\mathcal{B}}^2$$

for all $x, y \in \mathcal{B}$ and $t \in J$, and $\widetilde{L}_F = \sup_{0\le t\le T} E\|F(t,0)\|^2$.

(H_5) The function $f : J \times \mathcal{B} \to H$ satisfies the following properties:

(i) there exist an integrable function $m : J \to [0,\infty)$ and a nondecreasing function $\Omega_1 \in C([0,\infty);(0,\infty))$ such that, for every $(t,x) \in J \times \mathcal{B}$,

$$E\big\|f(t,x)\big\|^2 \le m(t)\Omega_1\big(\|x\|_{\mathcal{B}}^2\big), \qquad \liminf_{\xi\to\infty} \frac{\Omega_1(\xi)}{\xi} = \Lambda < \infty;$$

(ii) there exists a constant $L_f > 0$ such that

$$E\big\|f(t,x) - f(t,y)\big\|^2 \le L_f\|x-y\|_{\mathcal{B}}^2$$

for all $x, y \in \mathcal{B}$, $t \in J$.

(H_6) The function $\sigma : J \times \mathcal{B} \to \mathcal{L}_Q(K,H)$ satisfies the following properties:

(i) the function $\sigma(\cdot,x) : J \to \mathcal{L}_Q(K,H)$ is strongly measurable for every $x \in \mathcal{B}$;

(ii) the function $\sigma(t,\cdot) : \mathcal{B} \to \mathcal{L}_Q(K,H)$ is continuous on $\mathcal{R}(\rho^-) \cup J$;

(iii) there exist an integrable function $q : J \to [0,\infty)$ and a nondecreasing function $\Omega_2 \in C([0,\infty);(0,\infty))$ such that, for every $(t,x) \in J \times \mathcal{B}$,

$$E\big\|\sigma(t,x)\big\|^2 \le q(t)\Omega_2\big(\|x\|_{\mathcal{B}}^2\big), \qquad \liminf_{\zeta\to\infty} \frac{\Omega_2(\zeta)}{\zeta} = \Theta < \infty;$$

(iv) there exists a constant $L_\sigma > 0$ such that

$$E\big\|\sigma(t,x) - \sigma(t,y)\big\|^2 \le L_\sigma\|x-y\|_{\mathcal{B}}^2$$

for all $x, y \in \mathcal{B}$ and $t \in J$,

(H_7) The maps I_k and $\tilde{I}_k$ are completely continuous, and there exist positive constants c_k^j $(j = 1,2,3,4)$, L_k, and $\widetilde{L}_k$, $k = 1,2,\ldots,n$, such that

$$E\big\|I_k(x)\big\|^2 \le c_k^1\|x\|_{\mathcal{B}}^2 + c_k^2, \qquad E\big\|I_k(x) - I_k(y)\big\|^2 \le L_k\|x-y\|_{\mathcal{B}}^2$$

and

$$E\left\|\tilde{I}_k(x)\right\|^2 \le c_k^3\|x\|_{\mathcal{B}}^2 + c_k^4, \qquad E\left\|\tilde{I}_k(x) - \tilde{I}_k(y)\right\|^2 \le \widetilde{L}_k\|x-y\|_{\mathcal{B}}^2$$

for all $x, y \in \mathcal{B}$.

(H_8) The function $h : J \times H \times \mathcal{U} \to H$ is a Borel-measurable function, and there exist positive constants M_h and $\widetilde{M}_h$ such that for all $x, y \in \mathcal{L}_2^{\mathcal{F}_t}(J, H)$ and $t \in J$,

$$\begin{aligned}
&E\left(\int_0^t \int_{\mathcal{U}} \left\|h\left(s, x(s-), \nu\right)\right\|^2 \lambda(d\nu)\, ds\right) \vee E\left(\int_0^t \int_{\mathcal{U}} \left\|h\left(s, x(s-), \nu\right)\right\|^4 \lambda(d\nu)\, ds\right)^{\frac{1}{2}} \\
&\quad \le \widetilde{M}_h E\int_0^t \left(1 + \left\|x(s)\right\|^2\right) ds, \\
&E\left(\int_0^t \int_{\mathcal{U}} \|h\left(s, x(s-), \nu\right) - h\left(s, y(s-), \nu\right)\|^2 \lambda(d\nu)\, ds\right) \\
&\qquad \vee E\left(\int_0^t \int_{\mathcal{U}} \|h\left(s, x(s-), \nu\right) - h\left(s, y(s-), \nu\right)\|^4 \lambda(d\nu)\, ds\right)^{\frac{1}{2}} \\
&\quad \le M_h E\int_0^t \|x(s) - y(s)\|^2\, ds.
\end{aligned}$$

(H_9) For each $0 \le t < T$, the operator $\alpha R(\alpha, \Gamma_t^T) = \alpha(\alpha I + \Gamma_t^T)^{-1} \to 0$ in the strong operator topology as $\alpha \to 0^+$, where the controllability operator Γ_t^T associated with system (2.1) is defined as

$$\Gamma_t^T = \int_t^T S(T-s)BB^*S^*(T-s)\, ds,$$

where S^* denotes the adjoint operator of S.

The next lemma can be proved by using the phase spaces axioms, and its proof is omitted.

Lemma 2.7 *Let $x : (-\infty, T] \to H$ be a function such that $x_0 = \phi$, $x_0' = x_1 \in H$, and $x|_J \in \mathcal{PC}$. Then*

$$\|x_s\|_{\mathcal{B}} \le \left(M_T + J_0^{\phi}\right)\|\phi\|_{\mathcal{B}} + N_T \sup E\left\{\left\|x(\theta)\right\| : \theta \in \left[0, \max\{0, s\}\right]\right\}, \quad s \in \mathcal{R}\left(\rho^-\right) \cup J,$$

where $J_0^{\phi} = \sup\{J^{\phi}(t) : t \in \mathcal{R}(\rho^-)\}$.

Remark 2.2 Let $\phi \in \mathcal{B}$ and $t \le 0$. The notation ϕ_t represents the function defined by $\phi_t(\theta) = \phi(t + \theta)$. Consequently, if the function x in Axiom A is such that $x_0 = \phi$, then $x_t = \phi_t$. We observe that ϕ_t is well defined for $t < 0$, since the domain of ϕ is $(-\infty, 0]$.

3 Approximate controllability

In this section, we investigate the approximate controllability of system (2.1).

Let us introduce the space $\mathcal{BPC} = \{x : (-\infty, T] \to H; x_0 = \phi \in \mathcal{B}, x|_J \in \mathcal{PC}\}$, and let $\|\cdot\|_T$ be a seminorm in $\mathcal{BPC}$ defined by

$$\|x\|_T = \|x_0\|_{\mathcal{B}} + \sup_{t \in J}\left(E\left\|x(s)\right\|^2\right)^{\frac{1}{2}}.$$

For all $\alpha > 0$, define the control for system (2.1) as

$$\begin{aligned}
u^{\alpha}(t,x) &= B^{*}S^{*}(T-t)\Big[\big(\alpha I+\Gamma_{0}^{T}\big)^{-1}\big(Ex^{T}-C(T)\phi(0)-S(T)\big(x_{1}-F(0,\phi)\big)\big) \\
&\quad + \int_{0}^{t}\big(\alpha I+\Gamma_{s}^{T}\big)^{-1}\varphi(s)\,dw(s)\Big] \\
&\quad - B^{*}S^{*}(T-t)\int_{0}^{t}\big(\alpha I+\Gamma_{s}^{T}\big)^{-1}C(T-s)F(s,x_{s})\,ds \\
&\quad - B^{*}S^{*}(T-t)\int_{0}^{t}\big(\alpha I+\Gamma_{s}^{T}\big)^{-1}S(T-s)f(s,x_{s})\,ds \\
&\quad - B^{*}S^{*}(T-t)\int_{0}^{t}\big(\alpha I+\Gamma_{s}^{T}\big)^{-1}S(T-s)\sigma(s,x_{\rho(s,x_{s})})\,dw(s) \\
&\quad - B^{*}S^{*}(T-t)\int_{0}^{t}\big(\alpha I+\Gamma_{s}^{T}\big)^{-1}S(T-s)\int_{\mathcal{U}}h\big(s,x(s-),\nu\big)\widetilde{N}(ds,d\nu) \\
&\quad - B^{*}S^{*}(T-t)\big(\alpha I+\Gamma_{0}^{T}\big)^{-1}\sum_{0<t_{k}<T}C(T-t_{k})I_{k}(x_{t_{k}}) \\
&\quad - B^{*}S^{*}(T-t)\big(\alpha I+\Gamma_{0}^{T}\big)^{-1}\sum_{0<t_{k}<T}S(T-t_{k})\tilde{I}_{k}(x_{t_{k}}),
\end{aligned}$$

and define the operator Φ on $\mathcal{BPC}$ as follows:

$$\begin{aligned}
\Phi x(t) &= \phi(t), \quad t\in J_{0}=(-\infty,0], \\
\Phi x(t) &= C(t)\phi(0)+S(t)\big[x_{1}-F(0,\phi)\big]+\int_{0}^{t}C(t-s)F(s,x_{s})\,ds \\
&\quad + \int_{0}^{t}S(t-s)f(s,x_{s})\,ds+\int_{0}^{t}S(t-s)\sigma(s,x_{\rho(s,x_{s})})\,dw(s) \\
&\quad + \int_{0}^{t}S(t-s)\int_{\mathcal{U}}h\big(s,x(s-),\nu\big)\widetilde{N}(ds,d\nu)+\int_{0}^{t}S(t-s)Bu^{\alpha}(s,x)\,ds \\
&\quad + \sum_{0<t_{k}<t}C(t-t_{k})I_{k}(x_{t_{k}})+\sum_{0<t_{k}<t}S(t-t_{k})\tilde{I}_{k}(x_{t_{k}}), \quad t\in J.
\end{aligned}$$

To prove the approximate controllability of system (2.1), we will first show that there exists a fixed point of the operator Φ for all $\alpha > 0$.

Theorem 3.1 *Assume the (H_1)–(H_9) are satisfied. Then for each $0 < \alpha \leq 1$, the operator Φ has a fixed point, provided that*

$$8K_{0}\left(1+10T^{2}M_{S}^{2}M_{B}^{2}\frac{1}{\alpha^{2}}\right)\leq 1 \tag{3.1}$$

and

$$3T\big(TM_{C}L_{F}N_{T}^{2}+2CM_{h}\big)+18T^{2}M_{S}^{2}M_{B}^{2}\frac{1}{\alpha^{2}}K_{1}<1, \tag{3.2}$$

where

$$K_0 = 2N_T^2\Bigg(2T^2M_CL_F + TM_S\int_0^T m(s)\,ds\Lambda + \mathrm{Tr}(Q)M_S\Theta\int_0^T q(s)\,ds + 2TC\widetilde{M}_hL^2 + nM_C\sum_{k=1}^n c_k^1 + nM_S\sum_{k=1}^n c_k^3\Bigg), \tag{3.3}$$

$$K_1 = M_CT^2L_FN_T^2 + N_T^2M_SL_fT^2 + M_STTr(Q)L_\sigma N_T^2 + 2TCM_h + nM_CN_T^2\sum_{k=1}^n L_k + nM_SN_T^2\sum_{k=1}^n \widetilde{L}_k. \tag{3.4}$$

Proof For $\phi \in \mathcal{B}$, we define $\widetilde{\phi}$ by

$$\widetilde{\phi}(t) = \begin{cases} \phi(t) & \text{for } t \in J_0 = (-\infty, 0], \\ C(t)\phi(0) & \text{for } t \in J. \end{cases}$$

Then from the properties of $\mathcal{B}$ we infer $\widetilde{\phi} \in \mathcal{BPC}$. Let $x(t) = z(t) + \widetilde{\phi}(t)$, $t \in (-\infty, T]$. It is easy to see that x satisfies (2.4) if and only if z satisfies $z_0 = 0$, $x'(0) = x_1 = z'(0) = z_1$, and

$$\begin{aligned} z(t) &= S(t)\big[z_1 - F(0,\widetilde{\phi}_0)\big] + \int_0^t C(t-s)F(s,z_s+\widetilde{\phi}_s)\,ds \\ &\quad + \int_0^t S(t-s)f(s,z_s+\widetilde{\phi}_s)\,ds + \int_0^t S(t-s)\sigma(s,z_{\rho(s,z_s)}+\widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)})\,dw(s) \\ &\quad + \int_0^t S(t-s)\int_{\mathcal{U}} h\big(s,z(s-)+\widetilde{\phi}(s-),\nu\big)\widetilde{N}(ds,d\nu) + \int_0^t S(t-s)Bu^\alpha(s,z+\widetilde{\phi})\,ds \\ &\quad + \sum_{0<t_k<t} C(t-t_k)I_k(z_{t_k}+\widetilde{\phi}_{t_k}) + \sum_{0<t_k<t} S(t-t_k)\tilde{I}_k(z_{t_k}+\widetilde{\phi}_{t_k}), \quad t \in J. \end{aligned}$$

Define $\mathcal{B}^0\mathcal{PC} = \{y : (-\infty, T] \to H, y_0 = 0, y|_J \in \mathcal{PC}\}$. For any $y \in \mathcal{B}^0\mathcal{PC}$,

$$\|y\|_T = \|y_0\|_{\mathcal{B}} + \sup_{s\in J}\big(E\|y(s)\|^2\big)^{\frac{1}{2}} = \sup_{s\in J}\big(E\|y(s)\|^2\big)^{\frac{1}{2}} = \|y\|_{\mathcal{PC}},$$

and thus $(\mathcal{B}^0\mathcal{PC}, \|\cdot\|_T)$ is a Banach space. Let $B_r = \{y \in \mathcal{B}^0\mathcal{PC} : \|y\|_T^2 \le r\}$ for some $r \ge 0$. Then the family $B_r \subseteq \mathcal{B}^0\mathcal{PC}$ is uniformly bounded. For $z \in B_r$, we have

$$\begin{aligned} \|z_t + \widetilde{\phi}_t\|_{\mathcal{B}}^2 &\le 2\big(\|z_t\|_{\mathcal{B}}^2 + \|\widetilde{\phi}_t\|_{\mathcal{B}}^2\big) \\ &\le 2\Big[N_T^2 \sup_{s\in[0,t]}\big(E\|z(s)\|^2\big) + 2\big(M_T + J_0^\phi\big)^2\|\widetilde{\phi}_0\|_{\mathcal{B}}^2 + 2N_T^2 \sup_{s\in[0,t]}\big(E\|\widetilde{\phi}(s)\|^2\big)\Big] \\ &\le 2\big[N_T^2 r + 2\big(M_T + J_0^\phi\big)^2\|\widetilde{\phi}\|_{\mathcal{B}}^2 + 2N_T^2M_C^2E\|\phi(0)\|^2\big] \\ &= r^*. \end{aligned}$$

Define the operator $\bar{\Phi} : \mathcal{B}^0\mathcal{PC} \to \mathcal{B}^0\mathcal{PC}$ by

$$\bar{\Phi}z(t) = 0, \quad t \in J_0 = (-\infty, 0],$$

$$
\begin{aligned}
\bar{\Phi}z(t) &= S(t)\bigl[z_1 - F(0,\widetilde{\phi}_0)\bigr] + \int_0^t C(t-s)F(s,z_s+\widetilde{\phi}_s)\,ds \\
&\quad + \int_0^t S(t-s)f(s,z_s+\widetilde{\phi}_s)\,ds + \int_0^t S(t-s)\sigma(s,z_{\rho(s,z_s)}+\widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)})\,dw(s) \\
&\quad + \int_0^t S(t-s)\int_{\mathcal{U}} h\bigl(s,z(s-)+\widetilde{\phi}(s-),\nu\bigr)\widetilde{N}(ds,d\nu) + \int_0^t S(t-s)Bu^\alpha(s,z+\widetilde{\phi})\,ds \\
&\quad + \sum_{0<t_k<t} C(t-t_k)I_k(z_{t_k}+\widetilde{\phi}_{t_k}) + \sum_{0<t_k<t} S(t-t_k)\tilde{I}_k(z_{t_k}+\widetilde{\phi}_{t_k}), \quad t\in J.
\end{aligned}
$$

From assumptions (H_2), (H_4), (H_5), Hölder inequality, and Bochner theorem [48] we can easily obtain that $C(t-s)F(s,z_s+\widetilde{\phi}_s)$ and $S(t-s)f(s,z_s+\widetilde{\phi}_s)$ are integrable on $[0,t)$. In addition, by using the strong continuity of $C(t)$ and $S(t)$, combined with Lemma 2.3 and Lebesgue's dominated convergence theorem, we conclude that $\bar{\Phi}$ is continuous. Therefore $\bar{\Phi}$ is a well-defined operator from $\mathcal{B}^0\mathcal{PC}$ into $\mathcal{B}^0\mathcal{PC}$. Obviously, the operator $\bar{\Phi}$ has a fixed point, which means that Φ has a fixed point.

To prove that $\bar{\Phi}$ has a fixed point, we will first study the control $u^\alpha(s,z+\widetilde{\phi})$. Let $z,\bar{z}\in B_r$. From the assumptions, the Hölder inequality, the Doob martingale inequality, and a particular case of Burkholder-type inequality for stochastic integrals driven by Poisson jumps [32] we have

$$
\begin{aligned}
&E\bigl\|u^\alpha(s,z+\widetilde{\phi})\bigr\|^2 \\
&\quad\le 10M_BM_S\frac{1}{\alpha^2}\bigl[E\bigl\|x^T\bigr\|^2 + M_CL^2\|\phi\|_{\mathcal{B}}^2 + 2M_SE\|z_1\|^2 + 4M_S\bigl(L_F\|\phi\|_{\mathcal{B}}^2+\widetilde{L}_F\bigr)\bigr] \\
&\qquad + 10M_BM_S\frac{1}{\alpha^2}\operatorname{Tr}(Q)\int_0^T E\bigl\|\varphi(s)\bigr\|^2\,ds + 10M_BM_S\frac{1}{\alpha^2}M_CT^2\bigl(2L_Fr^* + 2\widetilde{L}_F\bigr) \\
&\qquad + 10M_BM_S^2\frac{1}{\alpha^2}T\int_0^T m(s)\Omega_1\bigl(r^*\bigr)\,ds + 10M_BM_S^2\frac{1}{\alpha^2}\operatorname{Tr}(Q)\int_0^T q(s)\Omega_2\bigl(r^*\bigr)\,ds \\
&\qquad + 20M_BM_S\frac{1}{\alpha^2}TC\widetilde{M}_h\bigl(1+L^2r^*\bigr) + 10M_BM_S\frac{1}{\alpha^2}nM_C\sum_{k=1}^n\bigl(c_k^1r^*+c_k^2\bigr) \\
&\qquad + 10M_BM_S^2\frac{1}{\alpha^2}n\sum_{k=1}^n\bigl(c_k^3r^*+c_k^4\bigr) \\
&\quad= 10M_BM_S\frac{1}{\alpha^2}\Biggl\{E\bigl\|x^T\bigr\|^2 + M_CL^2\|\phi\|_{\mathcal{B}}^2 + 2M_SE\|z_1\|^2 + 4M_S\bigl(L_F\|\phi\|_{\mathcal{B}}^2+\widetilde{L}_F\bigr) \\
&\qquad + \operatorname{Tr}(Q)\int_0^T E\bigl\|\varphi(s)\bigr\|^2\,ds + M_CT^2\bigl(2L_Fr^*+2\widetilde{L}_F\bigr) + TM_S\Omega_1\bigl(r^*\bigr)\int_0^T m(s)\,ds \\
&\qquad + \operatorname{Tr}(Q)M_S\Omega_2\bigl(r^*\bigr)\int_0^T q(s)\,ds + 2TC\widetilde{M}_h\bigl(1+L^2r^*\bigr) + nM_C\sum_{k=1}^n\bigl(c_k^1r^*+c_k^2\bigr) \\
&\qquad + nM_S\sum_{k=1}^n\bigl(c_k^3r^*+c_k^4\bigr)\Biggr\} = \mathcal{A}_0, \\
&E\bigl\|u^\alpha(s,z+\widetilde{\phi}) - u^\alpha(s,\bar{z}+\widetilde{\phi})\bigr\|^2 \\
&\quad\le 6M_BM_S\frac{1}{\alpha^2}\Bigl(M_CT^2L_FN_T^2 + M_ST^2L_fN_T^2 + M_STTr(Q)L_\sigma N_T^2
\end{aligned}
$$

$$+ 2CTM_h + nM_C N_T^2 \sum_{k=1}^{n} L_k + nM_S N_T^2 \sum_{k=1}^{n} \widetilde{L}_k \Bigg) \sup_{t\in J} E\big\|z(t) - \bar{z}(t)\big\|^2,$$

where C is a positive constant. Next, we show that the operator $\bar{\Phi}$ has a fixed point in the following steps.

Step 1: $\bar{\Phi}(B_r) \subseteq B_r$ for some $r > 0$.

We affirm that there exists a positive constant $r > 0$ such that $\bar{\Phi}(B_r) \subseteq B_r$. If this statement is false, then for each $r > 0$, there exists a function $z^r(t^r) \in B_r$, but $\bar{\Phi}(z^r) \notin B_r$, i.e., $r < E\|(\bar{\Phi}z^r)(t^r)\|^2$ for some $t^r \in J$. However, from the assumptions and Axiom we obtain

$$\begin{aligned}
& r < E\big\|\big(\bar{\Phi}z^r\big)\big(t^r\big)\big\|^2 \\
& \leq 8\Bigg\{2M_S\big[E\|z_1\|^2 + 2L_F\|\widetilde{\phi}\|_{\mathcal{B}}^2 + 2\widetilde{L}_F\big] + 2T^2M_C\big(L_F r^* + \widetilde{L}_F\big) \\
& \quad + TM_S\Omega_1\big(r^*\big)\int_0^T m(s)\,ds + \operatorname{Tr}(Q)M_S\Omega_2\big(r^*\big)\int_0^T q(s)\,ds + 2TC\widetilde{M}_h\big(1 + L^2 r^*\big) \\
& \quad + nM_C\sum_{k=1}^{n}\big(c_k^1 r^* + c_k^2\big) + nM_S\sum_{k=1}^{n}\big(c_k^3 r^* + c_k^4\big) + T^2M_SM_B\mathcal{A}_0\Bigg\} \\
& \leq 8\Bigg\{2M_S\big[E\|z_1\|^2 + 2L_F\|\widetilde{\phi}\|_{\mathcal{B}}^2 + 2\widetilde{L}_F\big] + 2T^2M_C\widetilde{L}_F + 2TC\widetilde{M}_h + nM_C\sum_{k=1}^{n} c_k^2 \\
& \quad + nM_S\sum_{k=1}^{n} c_k^4\Bigg\} + 80T^2M_B^2M_S^2\frac{1}{\alpha^2}\Bigg\{E\big\|x^T\big\|^2 + M_CL^2\|\phi\|_{\mathcal{B}}^2 + 2M_SE\|z_1\|^2 \\
& \quad + 4M_S\big(L_F\|\phi\|_{\mathcal{B}}^2 + \widetilde{L}_F\big) + \operatorname{Tr}(Q)\int_0^T E\big\|\varphi(s)\big\|^2\,ds + 2T^2M_C\widetilde{L}_F + 2TC\widetilde{M}_h \\
& \quad + nM_C\sum_{k=1}^{n} c_k^2 + nM_S\sum_{k=1}^{n} c_k^4\Bigg\} + 8\Bigg\{2T^2M_CL_Fr^* + TM_S\Omega_1\big(r^*\big)\int_0^T m(s)\,ds \\
& \quad + \operatorname{Tr}(Q)M_S\Omega_2\big(r^*\big)\int_0^T q(s)\,ds + 2TC\widetilde{M}_hL^2r^* + nM_C\sum_{k=1}^{n} c_k^1 r^* + nM_S\sum_{k=1}^{n} c_k^3 r^*\Bigg\} \\
& \quad + 80T^2M_B^2M_S^2\frac{1}{\alpha^2}\Bigg\{2T^2M_CL_Fr^* + TM_S\Omega_1\big(r^*\big)\int_0^T m(s)\,ds + 2TC\widetilde{M}_hL^2r^* \\
& \quad + \operatorname{Tr}(Q)M_S\Omega_2\big(r^*\big)\int_0^T q(s)\,ds + nM_C\sum_{k=1}^{n} c_k^1 r^* + nM_S\sum_{k=1}^{n} c_k^3 r^*\Bigg\}.
\end{aligned}$$

Dividing both sides by r and taking the limit as $r \to \infty$, we obtain

$$1 < 8K_0\left(1 + 10T^2M_S^2M_B^2\frac{1}{\alpha^2}\right),$$

where K_0 is defined by (3.3). From (3.1) we can see that $\bar{\Phi}(B_r) \subseteq B_r$ for some positive number r.

To prove that $\bar{\Phi}$ is condensing from B_r into B_r, we decompose $\bar{\Phi} = \bar{\Phi}_1 + \bar{\Phi}_2$ by

$$\bar{\Phi}_1 z(t) = S(t)\big[z_1 - F(0,\widetilde{\phi}_0)\big] + \int_0^t C(t-s)F(s,z_s + \widetilde{\phi}_s)\,ds$$

$$+\int_0^t S(t-s)\int_{\mathcal{U}} h\bigl(s,z(s-)+\widetilde{\phi}(s-),\nu\bigr)\widetilde{N}(ds,d\nu)$$
$$+\int_0^t S(t-s)Bu^{\alpha}(s,z+\widetilde{\phi})\,ds$$

and

$$\bar{\Phi}_2 z(t)=\int_0^t S(t-s)f(s,z_s+\widetilde{\phi}_s)\,ds+\int_0^t S(t-s)\sigma(s,z_{\rho(s,z_s)}+\widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)})\,dw(s)$$
$$+\sum_{0<t_k<t}C(t-t_k)I_k(z_{t_k}+\widetilde{\phi}_{t_k})+\sum_{0<t_k<t}S(t-t_k)\tilde{I}_k(z_{t_k}+\widetilde{\phi}_{t_k}).$$

We will verify that $\bar{\Phi}_1$ is a contraction operator, whereas $\bar{\Phi}_2$ is a completely continuous operator.

Step 2: $\bar{\Phi}_1$ is a contraction operator.

Take arbitrary $z,\bar{z}\in B_r$. Then for each $t\in J$, we have

$$\begin{aligned}
&E\bigl\|(\bar{\Phi}_1 z)(t)-(\bar{\Phi}_1\bar{z})(t)\bigr\|^2\\
&\quad\le 3T^2M_CL_FN_T^2\sup_{t\in J}E\bigl\|z(t)-\bar{z}(t)\bigr\|^2+6CTM_h\sup_{t\in J}E\bigl\|z(t)-\bar{z}(t)\bigr\|^2\\
&\qquad+18T^2M_S^2M_B^2\frac{1}{\alpha^2}\Biggl(M_CT^2L_FN_T^2+N_T^2M_SL_fT^2+M_STTr(Q)L_\sigma N_T^2\\
&\qquad+2TCM_h+nM_CN_T^2\sum_{k=1}^n L_k+nM_SN_T^2\sum_{k=1}^n\widetilde{L}_k\Biggr)\sup_{t\in J}E\bigl\|z(t)-\bar{z}(t)\bigr\|^2\\
&\quad\le\biggl[3T\bigl(TM_CL_FN_T^2+2CM_h\bigr)+18T^2M_S^2M_B^2\frac{1}{\alpha^2}K_1\biggr]\sup_{t\in J}E\bigl\|z(t)-\bar{z}(t)\bigr\|^2.
\end{aligned}$$

Therefore we get

$$\|\bar{\Phi}_1 z-\bar{\Phi}_1\bar{z}\|_T^2\le K_2\|z-\bar{z}\|_T^2,$$

where $K_2=3T(TM_CL_FN_T^2+2CM_h)+18T^2M_S^2M_B^2\frac{1}{\alpha^2}K_1$ with K_1 defined by (3.4). From (3.2) we know that $K_2<1$, and thus $\bar{\Phi}_1$ is a contraction operator.

Step 3: $\bar{\Phi}_2$ is a completely continuous operator on B_r. For better readability, we break the proof into five steps.

(i) $\bar{\Phi}_2$ is continuous on B_r.

Let $\{z^n\}\subseteq B_r$ with $z^n\to z$ $(n\to\infty)$ in $\mathcal{B}^0\mathcal{PC}$ for some $z\in B_r$. From Axiom A, Lemma 2.3, and Lemma 2.7 it is easy to see that $z^n_{\rho(s,z^n_s)}\to z_{\rho(s,z_s)}$ and $z^n_s\to z_s$ uniformly as $n\to\infty$ for $s\in(-\infty,T]$. By assumptions (H_5) and (H_6) we have

$$f\bigl(s,z_s^n+\widetilde{\phi}_s\bigr)\to f(s,z_s+\widetilde{\phi}_s)$$

and

$$\sigma\bigl(s,z^n_{\rho(s,z^n_s)}+\widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\bigr)\to\sigma\bigl(s,z_{\rho(s,z_s)}+\widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\bigr)$$

as $n \to \infty$ for each $s \in J$, and since

$$E\left\|f\left(s, z_s^n + \widetilde{\phi}_s\right) - f\left(s, z_s + \widetilde{\phi}_s\right)\right\|^2 \le 2m(t)\Omega_1\left(r^*\right),$$
$$E\left\|\sigma\left(s, z^n_{\rho(s,z_s^n)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\right) - \sigma\left(s, z_{\rho(s,z_s)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\right)\right\|^2 \le 2q(t)\Omega_2\left(r^*\right),$$

then by the complete continuity of I_k, $\tilde{I}_k$ $(k = 1, 2, \dots, n)$ and the dominated convergence theorem we have

$$\begin{aligned}
&\left\|\bar{\Phi}_2 z^n - \bar{\Phi}_2 z\right\|_T^2 \\
&\quad \le \sup_{t\in J} E\left\| \int_0^t S(t-s)\left[f\left(s, z_s^n + \widetilde{\phi}_s\right) - f\left(s, z_s + \widetilde{\phi}_s\right)\right] ds \right. \\
&\qquad + \int_0^t S(t-s)\left[\sigma\left(s, z^n_{\rho(s,z_s^n)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\right) - \sigma\left(s, z_{\rho(s,z_s)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\right)\right] dw(s) \\
&\qquad + \sum_{0<t_k<t} C(t-t_k)\left[I_k\left(z^n_{t_k} + \widetilde{\phi}_{t_k}\right) - I_k\left(z_{t_k} + \widetilde{\phi}_{t_k}\right)\right] \\
&\qquad \left. + \sum_{0<t_k<t} S(t-t_k)\left[\tilde{I}_k\left(z^n_{t_k} + \widetilde{\phi}_{t_k}\right) - \tilde{I}_k\left(z_{t_k} + \widetilde{\phi}_{t_k}\right)\right] \right\|^2 \\
&\quad \le 4TM_S \int_0^t E\left\|f\left(s, z_s^n + \widetilde{\phi}_s\right) - f\left(s, z_s + \widetilde{\phi}_s\right)\right\|^2 ds \\
&\qquad + 4M_S \operatorname{Tr}(Q) \int_0^t E\left\|\sigma\left(s, z^n_{\rho(s,z_s^n)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\right) - \sigma\left(s, z_{\rho(s,z_s)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\right)\right\|^2 ds \\
&\qquad + 4nM_C \sum_{0<t_k<t} \left\|I_k\left(z^n_{t_k} + \widetilde{\phi}_{t_k}\right) - I_k\left(z_{t_k} + \widetilde{\phi}_{t_k}\right)\right\|^2 \\
&\qquad + 4nM_S \sum_{0<t_k<t} \left\|\tilde{I}_k\left(z^n_{t_k} + \widetilde{\phi}_{t_k}\right) - \tilde{I}_k\left(z_{t_k} + \widetilde{\phi}_{t_k}\right)\right\|^2 \\
&\quad \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus $\bar{\Phi}_2$ is continuous on B_r.

(ii) $\bar{\Phi}_2$ maps bounded sets into bounded sets in $\mathcal{B}^0\mathcal{PC}$.

For each $z \in B_r$, from Lemma 2.7 and assumptions (H_5)–(H_7) we have

$$\begin{aligned}
E\left\|\bar{\Phi}_2 z(t)\right\|^2 &\le 4E\left\|\int_0^t S(t-s) f\left(s, z_s + \widetilde{\phi}_s\right) ds\right\|^2 \\
&\quad + 4E\left\|\int_0^t S(t-s)\sigma\left(s, z_{\rho(s,z_s)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\right) dw(s)\right\|^2 \\
&\quad + 4E\left\|\sum_{0<t_k<t} C(t-t_k) I_k\left(z_{t_k} + \widetilde{\phi}_{t_k}\right)\right\|^2 \\
&\quad + 4E\left\|\sum_{0<t_k<t} S(t-t_k)\tilde{I}_k\left(z_{t_k} + \widetilde{\phi}_{t_k}\right)\right\|^2 \\
&\le 4TM_S\Omega_1\left(r^*\right)\int_0^T m(s)\, ds + 4M_S\Omega_2\left(r^*\right)\operatorname{Tr}(Q)\int_0^T q(s)\, ds
\end{aligned}$$

$$+ 4nM_C \sum_{k=1}^{n} \bigl(c_k^1 r^* + c_k^2\bigr) + 4nM_S \sum_{k=1}^{n} \bigl(c_k^3 r^* + c_k^4\bigr) = K^*,$$

which shows the claim.

To show that $\bar{\Phi}_2(B_r)$ is equicontinuous and $\bar{\Phi}_2(B_r)(t)$ is precompact in $\mathcal{B}^0\mathcal{PC}$, we decompose $\bar{\Phi}_2$ as $\Pi_1 + \Pi_2$, where Π_1 and Π_2 are the operators on B_r defined respectively by

$$\Pi_1 z(t) = \int_0^t S(t-s) f(s, z_s + \widetilde{\phi}_s)\, ds + \int_0^t S(t-s)\sigma\bigl(s, z_{\rho(s,z_s)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\bigr)\, dw(s)$$

and

$$\Pi_2 z(t) = \sum_{0<t_k<t} C(t-t_k) I_k(z_{t_k} + \widetilde{\phi}_{t_k}) + \sum_{0<t_k<t} S(t-t_k)\tilde{I}_k(z_{t_k} + \widetilde{\phi}_{t_k}), \quad t \in J.$$

(iii) We first show that $\Pi_1(B_r)$ is equicontinuous.

Let $z \in B_r$, $0 < t_1 < t_2 \le T$. From assumptions (H_5) and (H_6) and Lemma 2.3(iii) we have

$$\begin{aligned}
&E\bigl\|(\Pi_1 z)(t_2) - (\Pi_1 z)(t_1)\bigr\|^2 \\
&\quad\le 2E\biggl\|\int_0^{t_2} S(t_2-s) f(s, z_s + \widetilde{\phi}_s)\, ds - \int_0^{t_1} S(t_1-s) f(s, z_s + \widetilde{\phi}_s)\, ds\biggr\|^2 \\
&\qquad + 2E\biggl\|\int_0^{t_2} S(t_2-s)\sigma\bigl(s, z_{\rho(s,z_s)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\bigr)\, dw(s) \\
&\qquad - \int_0^{t_1} S(t_1-s)\sigma\bigl(s, z_{\rho(s,z_s)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\bigr)\, dw(s)\biggr\|^2 \\
&\quad\le 4E\biggl\|\int_0^{t_1} \bigl[S(t_2-s) - S(t_1-s)\bigr] f(s, z_s + \widetilde{\phi}_s)\, ds\biggr\|^2 \\
&\qquad + 4E\biggl\|\int_{t_1}^{t_2} S(t_2-s) f(s, z_s + \widetilde{\phi}_s)\, ds\biggr\|^2 \\
&\qquad + 4E\biggl\|\int_0^{t_1} \bigl[S(t_2-s) - S(t_1-s)\bigr]\sigma\bigl(s, z_{\rho(s,z_s)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\bigr)\, dw(s)\biggr\|^2 \\
&\qquad + 4E\biggl\|\int_{t_1}^{t_2} S(t_2-s)\sigma\bigl(s, z_{\rho(s,z_s)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\bigr)\, dw(s)\biggr\|^2 \\
&\quad\le \widetilde{C}|t_2 - t_1|,
\end{aligned}$$

where the constant $\widetilde{C}$ does not depend on z, from which it follows that $E\|(\Pi_1 z)(t_2) - (\Pi_1 z)(t_1)\|^2 \to 0$ as $t_2 - t_1 \to 0$, and thus $\Pi_1(B_r)$ is equicontinuous. Here we consider only the case $0 < t_1 < t_2 \le T$, because the other cases $t_1 < t_2 \le 0$ and $t_1 < 0 \le t_2 \le T$ are very simple.

(iv) Π_1 maps B_r into a precompact set in H. That is, for every fixed $t \in J$, the set $V(t) = \{(\Pi_1 z)(t); z \in B_r\}$ is precompact in H. It is obvious that $V(0) = (\Pi_1 z)(0)$ is precompact. Let $0 < t \le T$ be fixed, and let $\varepsilon \in (0, t)$. For $z \in B_r$, we define

$$\bigl(\Pi_1^\varepsilon z\bigr)(t) = \int_0^{t-\varepsilon} S(t-s) f(s, z_s + \widetilde{\phi}_s)\, ds + \int_0^{t-\varepsilon} S(t-s)\sigma\bigl(s, z_{\rho(s,z_s)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)}\bigr)\, dw(s).$$

Since $S(t)$, $t > 0$, is compact, from Lemma 2.3(iv) we know that the set $V_\varepsilon(t) = \{(\Pi_1^\varepsilon z)(t) : z \in B_r\}$ is precompact in H for every $\varepsilon \in (0, t)$. Moreover, for every $z \in B_r$, we have

$$\begin{aligned}
&E\big\|(\Pi_1 z)(t) - \big(\Pi_1^\varepsilon z\big)(t)\big\|^2 \\
&\quad \le 2E\left\|\int_{t-\varepsilon}^t S(t-s)f(s, z_s + \widetilde{\phi}_s)\,ds\right\|^2 \\
&\qquad + 2E\left\|\int_{t-\varepsilon}^t S(t-s)\sigma(s, z_{\rho(s,z_s)} + \widetilde{\phi}_{\rho(s,\widetilde{\phi}_s)})\,dw(s)\right\|^2 \\
&\quad \le 2TM_S\Omega_1\big(r^*\big)\int_{t-\varepsilon}^t m(s)\,ds + 2M_S\Omega_2\big(r^*\big)\operatorname{Tr}(Q)\int_{t-\varepsilon}^t q(s)\,ds.
\end{aligned}$$

Therefore $E\|(\Pi_1 z)(t) - (\Pi_1^\varepsilon z)(t)\|^2 \to 0$ as $\varepsilon \to 0^+$, and there are precompact sets arbitrarily close to the set $V(t)$. Thus the set $V(t)$ is precompact in H.

(v) Π_2 is completely continuous. Now we decompose Π_2 as $\mathcal{G}_1 + \mathcal{G}_2$, where

$$(\mathcal{G}_1 z)(t) = \sum_{0<t_k<t} S(t-t_k)\tilde{I}_k(z_{t_k} + \widetilde{\phi}_{t_k})$$

and

$$(\mathcal{G}_2 z)(t) = \sum_{0<t_k<t} C(t-t_k)I_k(z_{t_k} + \widetilde{\phi}_{t_k}), \quad t \in J.$$

Firstly, we show that $\mathcal{G}_1$ is completely continuous. According to the proof in Step 3 (i) and (ii), we can see that $\mathcal{G}_1$ is continuous and $\mathcal{G}_1(B_r)$ is bounded in $\mathcal{B}^0\mathcal{PC}$. Next, we need to show that $\mathcal{G}_1(B_r)(t)$ is relatively compact and $\mathcal{G}_1(B_r)$ is equicontinuous. From the definition of $\mathcal{G}_1$, for $r > 0$, $t \in [t_k, t_{k+1}]$, $k = 1, 2, \dots, n$, and $z \in B_r$, we find that

$$(\widetilde{\mathcal{G}_1 z})(t) \in \begin{cases} \sum_{i=1}^k S(t-t_i)\tilde{I}_i(B_{r^*}(0,H)) & \text{if } t \in (t_k, t_{k+1}), \\ \sum_{i=1}^k S(t_{k+1}-t_i)\tilde{I}_i(B_{r^*}(0,H)) & \text{if } t = t_{k+1}, \\ \sum_{i=1}^{k-1} S(t_k-t_i)\tilde{I}_i(B_{r^*}(0,H)) + \tilde{I}_k(B_{r^*}(0,H)) & \text{if } t = t_k, \end{cases}$$

which proves that $[\widetilde{\mathcal{G}_1(B_r)}]_k(t)$ is relatively compact in H for every $t \in [t_k, t_{k+1}]$, since the maps $\tilde{I}_k$ are completely continuous for all $k = 1, 2, \dots, n$. Moreover, using the compactness of the operators $\tilde{I}_k$ and the strong continuity of $\{S(t) : t \in J\}$, we can prove that $[\widetilde{\mathcal{G}_1(B_r)}]_k$ is equicontinuous at every $t \in [t_k, t_{k+1}]$. Then from Lemma 2.1 we know that $\mathcal{G}_1$ is completely continuous. The proof of complete continuity for $\mathcal{G}_2$ is similar to that of $\mathcal{G}_1$, so we omit it. Therefore we obtain that Π_2 is completely continuous. These arguments prove that $\bar{\Phi}_2$ is completely continuous. Consequently, it follows from Sadovskii's fixed point theorem that the operator $\bar{\Phi}$ has a fixed point $z \in B_r$. Let $x(t) = z(t) + \widetilde{\phi}(t)$, $t \in (-\infty, T]$. Then x is a fixed point of the operator Φ. The proof of Theorem 3.1 is completed. □

Theorem 3.2 *Under hypotheses (H_1)–(H_9) and the assumptions of Theorem 3.1, suppose that also the functions F, f, h, and σ are uniformly bounded. Then system (2.1) is approximately controllable on J.*

Proof Let x^α be a fixed point of Φ in $\mathcal{BPC}$. Using the stochastic Fubini theorem [39], it is easy to see that

$$
\begin{aligned}
x^\alpha(T) &= C(T)\phi(0) + S(T)\bigl[x_1 - F(0,\phi)\bigr] + \int_0^T C(T-s)F\bigl(s,x_s^\alpha\bigr)\,ds \\
&\quad + \int_0^T S(T-s)f\bigl(s,x_s^\alpha\bigr)\,ds + \int_0^T S(T-s)\sigma\bigl(s,x^\alpha_{\rho(s,x_s^\alpha)}\bigr)\,dw(s) \\
&\quad + \int_0^T S(T-s)\int_{\mathcal{U}} h\bigl(s,x^\alpha(s-),\nu\bigr)\widetilde{N}(ds,d\nu) + \sum_{k=1}^n C(T-t_k)I_k\bigl(x^\alpha_{t_k}\bigr) \\
&\quad + \sum_{k=1}^n S(T-t_k)\tilde{I}_k\bigl(x^\alpha_{t_k}\bigr) + + \int_0^T S(T-s)B\biggl\{B^*S^*(T-s) \\
&\quad \times \biggl[\bigl(\alpha I + \Gamma_0^T\bigr)^{-1}\bigl(Ex^T - C(T)\phi(0) - S(T)\bigl(x_1 - F(0,\phi)\bigr)\bigr) \\
&\quad + \int_0^s \bigl(\alpha I + \Gamma_\tau^T\bigr)^{-1}\varphi(\tau)\,dw(\tau)\biggr] \\
&\quad - B^*S^*(T-s)\int_0^s \bigl(\alpha I + \Gamma_\tau^T\bigr)^{-1}C(T-\tau)F\bigl(\tau,x_\tau^\alpha\bigr)\,d\tau \\
&\quad - B^*S^*(T-s)\int_0^s \bigl(\alpha I + \Gamma_\tau^T\bigr)^{-1}S(T-\tau)f\bigl(\tau,x_\tau^\alpha\bigr)\,d\tau \\
&\quad - B^*S^*(T-s)\int_0^s \bigl(\alpha I + \Gamma_\tau^T\bigr)^{-1}S(T-\tau)\sigma\bigl(\tau,x^\alpha_{\rho(\tau,x_\tau^\alpha)}\bigr)\,dw(\tau) \\
&\quad - B^*S^*(T-s)\int_0^s \bigl(\alpha I + \Gamma_\tau^T\bigr)^{-1}S(T-\tau)\int_{\mathcal{U}} h\bigl(\tau,x^\alpha(\tau-),\nu\bigr)\widetilde{N}(d\tau,d\nu) \\
&\quad - B^*S^*(T-s)\bigl(\alpha I + \Gamma_0^T\bigr)^{-1}\sum_{k=1}^n C(T-t_k)I_k\bigl(x^\alpha_{t_k}\bigr) \\
&\quad - B^*S^*(T-s)\bigl(\alpha I + \Gamma_0^T\bigr)^{-1}\sum_{k=1}^n S(T-t_k)\tilde{I}_k\bigl(x^\alpha_{t_k}\bigr)\biggr\}\,ds \\
&= x^T - \alpha\bigl(\alpha I + \Gamma_0^T\bigr)^{-1}\bigl[Ex^T - C(T)\phi(0) - S(T)\bigl(x_1 - F(0,\phi)\bigr)\bigr] \\
&\quad + \alpha\int_0^T \bigl(\alpha I + \Gamma_s^T\bigr)^{-1}C(T-s)F\bigl(s,x_s^\alpha\bigr)\,ds \\
&\quad + \alpha\int_0^T \bigl(\alpha I + \Gamma_s^T\bigr)^{-1}S(T-s)f\bigl(s,x_s^\alpha\bigr)\,ds \\
&\quad + \alpha\int_0^T \bigl(\alpha I + \Gamma_s^T\bigr)^{-1}\bigl[S(T-s)\sigma\bigl(s,x^\alpha_{\rho(s,x_s^\alpha)}\bigr) - \varphi(s)\bigr]\,dw(s) \\
&\quad + \alpha\int_0^T \bigl(\alpha I + \Gamma_s^T\bigr)^{-1}S(T-s)\int_{\mathcal{U}} h\bigl(s,x^\alpha(s-),\nu\bigr)\widetilde{N}(ds,d\nu) \\
&\quad + \alpha\bigl(\alpha I + \Gamma_0^T\bigr)^{-1}\sum_{k=1}^n C(T-t_k)I_k\bigl(x^\alpha_{t_k}\bigr) \\
&\quad + \alpha\bigl(\alpha I + \Gamma_0^T\bigr)^{-1}\sum_{k=1}^n S(T-t_k)\tilde{I}_k\bigl(x^\alpha_{t_k}\bigr).
\end{aligned}
$$

Moreover, since F, f, h, and σ are uniformly bounded on J, there are subsequences, still denoted by $\{F(s,x_s^\alpha)\}$, $\{f(s,x_s^\alpha)\}$, $\{\sigma(s,x^\alpha_{\rho(s,x_s^\alpha)})\}$, and $\{h(s,x^\alpha(s-),\nu)\}$, that converge weakly to, say, $F(s), f(s), \sigma(s)$, and $h(s,\nu)$ in H, H, $\mathcal{L}_Q(K,H)$, and H, respectively. The compactness of $\{S(t): t\geq 0\}$ and $\{C(t): t\geq 0\}$ implies that $C(T-s)F(s,x_s^\alpha)\to C(T-s)F(s)$, $S(T-s)f(s,x_s^\alpha)\to S(T-s)f(s)$, $S(T-s)\sigma(s,x^\alpha_{\rho(s,x_s^\alpha)})\to S(T-s)\sigma(s)$, and $S(T-s)h(s,x^\alpha(s-),\nu)\to S(T-s)h(s,\nu)$. By (H_9) the operator $\alpha(\alpha I+\Gamma_s^T)^{-1}\to 0$ as $\alpha\to 0^+$, and $\|\alpha(\alpha I+\Gamma_s^T)^{-1}\|\leq 1$ for all $0\leq s<T$. Then by the Lebesgue dominated convergence theorem we have

$$
\begin{aligned}
&E\|x^\alpha(T)-x^T\|^2\\
&\quad\leq 12E\big\|\alpha\big(\alpha I+\Gamma_0^T\big)^{-1}\big[Ex^T-C(T)\phi(0)-S(T)\big(x_1-F(0,\phi)\big)\big]\big\|^2\\
&\qquad+12E\bigg(\int_0^T\big\|\alpha\big(\alpha I+\Gamma_s^T\big)^{-1}\big\|\big\|C(T-s)\big[F\big(s,x_s^\alpha\big)-F(s)\big]\big\|\,ds\bigg)^2\\
&\qquad+12E\bigg(\int_0^T\big\|\alpha\big(\alpha I+\Gamma_s^T\big)^{-1}C(T-s)F(s)\big\|\,ds\bigg)^2\\
&\qquad+12E\bigg(\int_0^T\big\|\alpha\big(\alpha I+\Gamma_s^T\big)^{-1}\big\|\big\|S(T-s)\big[f\big(s,x_s^\alpha\big)-f(s)\big]\big\|\,ds\bigg)^2\\
&\qquad+12E\bigg(\int_0^T\big\|\alpha\big(\alpha I+\Gamma_s^T\big)^{-1}S(T-s)f(s)\big\|\,ds\bigg)^2\\
&\qquad+24\widehat{C}\bigg\{E\bigg(\int_0^T\int_{\mathcal{U}}\big\|\alpha\big(\alpha I+\Gamma_s^T\big)^{-1}\big(h\big(s,x^\alpha(s-),\nu\big)-h(s,\nu)\big)\big\|^2\lambda(d\nu)\,ds\bigg)\\
&\qquad+E\bigg(\int_0^T\int_{\mathcal{U}}\big\|\alpha\big(\alpha I+\Gamma_s^T\big)^{-1}\big(h\big(s,x^\alpha(s-),\nu\big)-h(s,\nu)\big)\big\|^4\lambda(d\nu)\,ds\bigg)^{\frac{1}{2}}\bigg\}\\
&\qquad+24\widehat{C}\bigg\{E\bigg(\int_0^T\int_{\mathcal{U}}\big\|\alpha\big(\alpha I+\Gamma_s^T\big)^{-1}h(s,\nu)\big\|^2\lambda(d\nu)\,ds\bigg)\\
&\qquad+E\bigg(\int_0^T\int_{\mathcal{U}}\big\|\alpha\big(\alpha I+\Gamma_s^T\big)^{-1}h(s,\nu)\big\|^4\lambda(d\nu)\,ds\bigg)^{\frac{1}{2}}\bigg\}\\
&\qquad+12E\int_0^T\big\|\alpha\big(\alpha I+\Gamma_s^T\big)^{-1}\varphi(s)\big\|_Q^2\,ds\\
&\qquad+12E\int_0^T\big\|\alpha\big(\alpha I+\Gamma_s^T\big)^{-1}\big\|^2\big\|S(T-s)\big[\sigma\big(s,x^\alpha_{\rho(s,x_s^\alpha)}\big)-\sigma(s)\big]\big\|_Q^2\,ds\\
&\qquad+12E\int_0^T\big\|\alpha\big(\alpha I+\Gamma_s^T\big)^{-1}S(T-s)\sigma(s)\big\|_Q^2\,ds\\
&\qquad+12E\bigg\|\alpha\big(\alpha I+\Gamma_0^T\big)^{-1}\sum_{k=1}^n C(T-t_k)I_k\big(x_{t_k}^\alpha\big)\bigg\|^2\\
&\qquad+12E\bigg\|\alpha\big(\alpha I+\Gamma_0^T\big)^{-1}\sum_{k=1}^n S(T-t_k)\tilde{I}_k\big(x_{t_k}^\alpha\big)\bigg\|^2\longrightarrow 0\quad\text{as }\alpha\to 0^+,
\end{aligned}
$$

where $\widehat{C}$ is a positive constant. This gives the approximate controllability of system (2.1) and completes the proof. □

Now let us consider a particular case for system (2.1). If $h(t,x(t-),\nu)\equiv 0$, then system (2.1) becomes the impulsive neutral second-order stochastic differential equations with SDD of the form

$$\begin{aligned} &d\big[x'(t)-F(t,x_t)\big]=\big[Ax(t)+f(t,x_t)+Bu(t)\big]\,dt+\sigma(t,x_{\rho(t,x_t)})\,dw(t),\\ &t\in J=[0,T],\quad t\neq t_k,\\ &\Delta x(t_k)=I_k(x_{t_k}),\qquad \Delta x'(t_k)=\tilde{I}_k(x_{t_k}),\quad k=1,2,\dots,n,\\ &x_0=\phi\in\mathcal{B},\qquad x'(0)=x_1\in H. \end{aligned}\tag{3.5}$$

Corollary 3.1 *Assume that all assumptions of Theorem 3.1 hold, except (H_8) and that the functions F,f, and σ are uniformly bounded. If*

$$7K_3\left(1+9T^2M_S^2M_B^2\frac{1}{\alpha^2}\right)\leq 1$$

and

$$2T^2M_CL_FN_T^2+10T^2M_S^2M_B^2\frac{1}{\alpha^2}N_T^2K_4<1,$$

where

$$\begin{aligned} K_3={}&2N_T^2\left(2T^2M_CL_F+TM_S\int_0^T m(s)\,ds\Lambda+\operatorname{Tr}(Q)M_S\Theta\int_0^T q(s)\,ds\right.\\ &\left.+nM_C\sum_{k=1}^n c_k^1+nM_S\sum_{k=1}^n c_k^3\right) \end{aligned}$$

and

$$K_4=M_CT^2L_F+M_SL_fT^2+M_ST\operatorname{Tr}(Q)L_\sigma+nM_C\sum_{k=1}^n L_k+nM_S\sum_{k=1}^n\widetilde{L}_k,$$

then system (3.5) is approximately controllable on J.

4 An example

It is well known that wave equations with random disturbances have attracted more and more attention for their strong applications in physics, relativistic quantum mechanics, and oceanography ([9, 49] and references therein). In fact, stochastic wave equations are hyperbolic stochastic partial differential equations, and the well-posedness of the solutions is quite different from those of other stochastic partial differential equations. It is more realistic to take into account the impulsive effects, state-dependent delays, Poisson jumps, and neutral terms in Eq. (1.4), which is introduced in Sect. 1. Therefore, in this section, we give an example about the approximate controllability of stochastic wave equations to illustrate the obtained main results. Specifically, we discuss the following impulsive neutral stochastic wave equation with state-dependent delay and Poisson jumps:

$$\partial\left[\frac{\partial}{\partial t}z(t,y)-\int_{-\infty}^t\int_0^\pi q_1(t-s,\tau,y)z(s,\tau)\,d\tau\,ds\right]$$

$$
\begin{aligned}
&= \left[\frac{\partial^2}{\partial y^2} z(t,y) + Bu(t,y) + \int_{-\infty}^{t} q_2(s-t) z(s,y)\, ds\right] \partial t \\
&\quad + \left(\int_{-\infty}^{t} q_3(s-t) z\big(s-\rho_1(t)\rho_2\big(\|z(t,y)\|\big), y\big)\, ds\right) dw(t) \\
&\quad + \int_{\mathcal{U}} z(t-,y) \nu \widetilde{N}(dt,d\nu), \quad 0 \le y \le \pi, \tau > 0, t \in J = [0,T] \setminus \{t_1,\dots,t_n\},
\end{aligned}
$$

$$
\begin{aligned}
&z(t,0) = z(t,\pi) = 0, \quad t \in J, \qquad (4.1) \\
&\frac{\partial}{\partial t} z(0,y) = x_1(y), \quad 0 \le y \le \pi, \\
&z(t,y) = \phi(t,y), \quad t \in (-\infty,0], 0 \le y \le \pi, \\
&\Delta z(t_k)(y) = \int_{-\infty}^{t_k} \eta_k(t_k - s) z(s,y)\, ds, \quad k = 1,2,\dots,n, 0 \le y \le \pi, \\
&\Delta z'(t_k)(y) = \int_{-\infty}^{t_k} \xi_k(t_k - s) z(s,y)\, ds, \quad k = 1,2,\dots,n, 0 \le y \le \pi,
\end{aligned}
$$

where $0 < t_1 < \cdots < t_n < T$ are prefixed numbers, $\rho_1, \rho_2 : [0,\infty) \to [0,\infty)$ are continuous functions, q_i $(i = 1,2,3)$, η_k, ξ_k $(k = 1,2,\dots,n)$, ϕ, and x_1 are appropriate functions, which will be specified later. Let $w(t)$ denote a stochastic cylindrical Wiener process in a separable real Hilbert space $H = \mathcal{L}_2([0,\pi])$ defined on a stochastic space $(\Omega, \mathcal{F}, P)$. Let $\{p(t), t \in J\}$ be a K-valued Poisson point process (independent of $w(t)$) with characteristic measure $\lambda(d\nu)$ on $\mathcal{U} \in \mathfrak{B}(K - \{0\})$, where $\mathcal{U} = \{\nu \in R : 0 < \|\nu\|_R \le c, c > 0\}$ and $K = [0,\infty)$. We denote by $N(ds,d\nu)$ the Poisson counting measure induced by p, and the compensating martingale measure by $\widetilde{N}(ds,d\nu) = N(ds,d\nu) - \lambda(d\nu)\, ds$.

Define $A : H \to H$ by $A = \frac{\partial^2}{\partial y^2}$ with domain $D(A) = \{\xi \in H, \xi(0) = \xi(\pi)\}$. The operator A has a discrete spectrum with eigenvalues $-n^2$ for $n \in N$, and $e_n(y) = \sqrt{\frac{2}{\pi}} \sin(ny)$ $(n = 1,2,\dots)$ is an orthonormal basis of H. Then $Az = \sum_{n=1}^{\infty} -n^2 \langle z, e_n \rangle e_n$, $z \in D(A)$, and the operators $C(t)$ are defined by $C(t)z = \sum_{n=1}^{\infty} \cos(nt) \langle z, e_n \rangle e_n$ from a cosine function on H, with associated sine function $S(t)z = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle z, e_n \rangle e_n$, $t \in R$. It is clear that $C(\cdot)z$ and $S(\cdot)z$ are periodic functions with $\|C(t)\| \le 1$ and $\|S(t)\| \le 1$ for all $z \in H$ and $t \in R$ [44].

Let $\psi(\theta)y = \psi(\theta,y)$, $(\theta,y) \in (-\infty,0] \times [0,\pi]$, and $z(t)(y) = z(t,y)$. Let $g : (-\infty,0] \to (0,\infty)$ be a Lebesgue-integrable function with $l = \int_{-\infty}^{0} g(t)\, dt < \infty$. For any $b > 0$, define $\mathcal{B} = \{\psi : (-\infty,0] \to H | (E\|\psi(\theta)\|^2)^{\frac{1}{2}}$ is a bounded and measurable function on $[-b,0]$, and $\int_{-\infty}^{0} g(s)(E\|\psi(s)\|^2)^{\frac{1}{2}}\, ds < \infty\}$. Now we take $g(t) = e^{2t}$, $t < 0$. Then we get $l = \int_{-\infty}^{0} g(t)\, dt = \frac{1}{2}$ and define

$$
\|\psi\|_{\mathcal{B}} = \int_{-\infty}^{0} g(s) \sup_{s \le \theta \le 0} \left(E\|\psi(\theta)\|^2\right)^{\frac{1}{2}} ds.
$$

It is easy to verify that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space.

Define the infinite-dimensional space

$$
U = \left\{ u = \sum_{n=2}^{\infty} u_n e_n(y) : \sum_{n=2}^{\infty} u_n^2 < \infty \right\}
$$

with norm $\|u\|_U = (\sum_{n=2}^{\infty} u_n^2)^{\frac{1}{2}}$ and the linear continuous mapping $B : \mathcal{U} \to H$ by

$$Bu = 2u_2 e_1(y) + \sum_{n=2}^{\infty} u_n e_n(y).$$

It is easy to see that

$$Bu(t) = 2u_2(t)e_1(y) + \sum_{n=2}^{\infty} u_n(t)e_n(y) \in \mathcal{L}_2^{\mathcal{F}_t}(J,H)$$

for $u(t,y,w) = \sum_{n=2}^{\infty} u_n(t,w)e_n(y) \in \mathcal{L}_2^{\mathcal{F}_t}(J,H)$. Moreover,

$$B^* v = 2(v_1 + v_2)e_2(y) + \sum_{n=3}^{\infty} v_n e_n(y),$$

$$B^* S^*(t)z = \left(2z_1 e^{-t} + z_2 e^{-4t}\right)e_2(y) + \sum_{n=3}^{\infty} z_n e^{-n^2 t} e_n(y)$$

for $v = \sum_{n=1}^{\infty} v_n e_n(y)$ and $z = \sum_{n=1}^{\infty} z_n e_n(y)$. Let $\|B^* S^*(t)z\| = 0$, $t \in J$. It follows that

$$\begin{aligned} \left\|2z_1 e^{-t} + z_2 e^{-4t}\right\|^2 + \sum_{n=3}^{\infty} \left\|z_n e^{-n^2 t}\right\|^2 = 0 \quad &\Rightarrow \quad z_n = 0, \quad n = 1,2,\ldots \\ &\Rightarrow \quad z = 0, \quad t \in J. \end{aligned}$$

Thus by Theorem 4.1.7 in [50] the deterministic linear system corresponding to (4.1) is approximately controllable on J. Therefore hypothesis (H_9) is satisfied. Also, from the definition of B we know that B is bounded and satisfies hypothesis (H_3).

The functions $F : J \times \mathcal{B} \to H$, $\sigma : J \times \mathcal{B} \to \mathcal{L}_Q(K,H)$, $f : J \times \mathcal{B} \to H$, $\rho : J \times \mathcal{B} \to (-\infty, 0]$, $h : J \times H \times \mathcal{U} \to H$, and $I_k, \tilde{I}_k : \mathcal{B} \to H$ are defined respectively by

$$\begin{aligned} F(\psi)(y) &= \int_{-\infty}^{0} \int_{0}^{\pi} q_1(s,\tau,y)\psi(s,\tau)\,d\tau\,ds, \\ \sigma(\psi)(y) &= \int_{-\infty}^{0} q_3(s)\psi(s,y)\,ds, \\ \rho(t,\psi) &= t - \rho_1(t)\rho_2\left(\left\|\psi(0)\right\|\right), \\ f(\psi)(y) &= \int_{-\infty}^{0} q_2(s)\psi(s,y)\,ds, \\ I_k(\psi)(y) &= \int_{-\infty}^{0} \eta_k(-s)\psi(s,y)\,ds, \quad k = 1,2,\ldots,n, \\ \tilde{I}_k(\psi)(y) &= \int_{-\infty}^{0} \xi_k(-s)\psi(s,y)\,ds, \quad k = 1,2,\ldots,n, \\ h\left(\psi(y), v\right) &= \psi(y)v. \end{aligned}$$

We can represent system (4.1) by system (2.1). To get the result on the approximate controllability for system (4.1), we need the following conditions:

(i) The functions $\frac{\partial^i q_1(s,\tau,y)}{\partial y^i}$, $i = 0, 1$, are measurable, $q_1(s,\tau,0) = q_1(s,\tau,\pi) = 0$, and

$$L_F = \max\left\{\left\{\int_0^\pi \int_{-\infty}^0 \int_0^\pi \frac{1}{g(s)}\left(\frac{\partial^i q_1(s,\tau,y)}{\partial y^i}\right)^2 d\tau\, ds\, dy\right\}^{\frac{1}{2}}; i = 0, 1\right\} < \infty.$$

(ii) $q_2, q_3 : R \to R$ are continuous, and

$$L_f = \left(\int_{-\infty}^0 \frac{(q_2(s))^2}{g(s)}\, ds\right)^{\frac{1}{2}} < \infty, \qquad L_\sigma = \left(\int_{-\infty}^0 \frac{(q_3(s))^2}{g(s)}\, ds\right)^{\frac{1}{2}} < \infty.$$

(iii) The functions $\eta_k, \xi_k \in C(R,R)$, and

$$\beta_k = \left(\int_{-\infty}^0 \frac{\eta_k^2(-s)}{g(s)}\, ds\right)^{\frac{1}{2}} < \infty, \quad k = 1, 2, \ldots, n,$$

$$\tilde{\beta}_k = \left(\int_{-\infty}^0 \frac{\xi_k^2(-s)}{g(s)}\, ds\right)^{\frac{1}{2}} < \infty, \quad k = 1, 2, \ldots, n.$$

(iv) $\int_{\mathcal{U}} \nu^2 \lambda(d\nu) < \infty$ and $\int_{\mathcal{U}} \nu^4 \lambda(d\nu) < \infty$.

Under the above assumptions, we obtain that the mappings F, f, σ, I_k, and $\tilde{I}_k$ are bounded: $E\|F\|^2 \le L_F$, $E\|f\|^2 \le L_f$, $E\|\sigma\|^2 \le L_\sigma$, $E\|I_k\|^2 \le \beta_k$, and $E\|\tilde{I}_k\|^2 \le \tilde{\beta}_k$. Then all the conditions stated in Theorems 3.1 and 3.2 are satisfied. Hence by Theorem 3.2 system (4.1) is approximately controllable.

5 Conclusions

In this paper, we focus on a new kind of second-order impulsive neutral stochastic differential equations with state-dependent delay and Poisson jumps in a real separable Hilbert space, which are abstracted from stochastic wave equations. The results of approximate controllability were obtained by employing the Sadovskii fixed point theorem and the theory of a strongly continuous cosine family of bounded linear operators. Finally, an example illustrates the effectiveness of the main results. It should be emphasized that Eqs. (2.1) considered in this paper are more general than those in the existing literature, for example, [23, 24, 35, 38].

Second-order Volterra integro-differential equations were introduced by Hirokazu Oka [51]. As we know, up to now, there is no literature reported on the second-order stochastic Volterra integro-differential equations. Therefore we will try to study such equations in future work, which is a novel and interesting subject. In addition, we will also consider the controllability and stability of solutions for second-order impulsive neutral SDEs with jumps or driven by fractional Brownian motion.

Acknowledgements
Not applicable.

Funding
This work is supported by the Key Natural Science Research Project of Universities of Anhui Province (No. KJ2020A0121), the Support Program for Outstanding Young Talents of Universities in Anhui Province (No. gxyq2021204), and the National Natural Science Foundation of China (No. 11771001).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

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Received: 28 November 2022 Accepted: 29 March 2023 Published online: 07 April 2023

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