Zhang *Journal of Inequalities and Applications* (2023) 2023:50
https://doi.org/10.1186/s13660-023-02958-6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Global strong solutions of 3D compressible isothermal magnetohydrodynamics

Mingyu Zhang[1*]

*Correspondence:
wfumath@126.com
[1] School of Mathematics and Information Science, Weifang University, Weifang 261061, P.R. China

Abstract

This paper establishes the global existence of strong solutions for the three-dimensional compressible isothermal magnetohydrodynamic (MHD) flows. It is essentially shown that for the regular data with small energy but possible large oscillations, the global well-posedness of strong solutions is established.

MSC: 35B45; 35G55; 76N10

Keywords: Compressible isothermal magnetohydrodynamic equations; Global strong solutions; Large oscillations

1 Introduction

In this paper, we are concerned with the 3D compressible isothermal magnetohydrodynamic (MHD) equations, which are a combination of the compressible Navier–Stokes equations of fluid dynamics and Maxwells equations of electromagnetism (see [1, 13]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\mu + \lambda)\nabla \operatorname{div} \mathbf{u} + \nabla P(\rho) = (\operatorname{curl} \mathbf{B}) \times \mathbf{B}, \\ \mathbf{B}_t - \operatorname{curl}\,(\mathbf{u} \times \mathbf{B}) = \nu \Delta \mathbf{B}, \qquad \operatorname{div} \mathbf{B} = 0, \end{cases} \tag{1.1}$$

where $t \geq 0$ is the time and $x \in \mathbb{R}^3$ is the spatial coordinate. The unknown functions $\rho, \mathbf{u} = (u^1, u^2, u^3)^{\mathrm{tr}}$, $\mathbf{B} = (b^1, b^2, b^3)^{\mathrm{tr}}$, and P denote the fluid density, velocity, magnetic field, and pressure, respectively. The viscosity coefficients μ and λ satisfy the physical restrictions

$$\mu > 0, \qquad 2\mu + 3\lambda \geq 0. \tag{1.2}$$

Positive constant ν is the magnetic diffusion coefficient. The pressure $P(\rho)$ is determined through the equations of states. Here, we consider the isothermal gas dynamics

$$P(\rho) \triangleq a\rho \quad \text{with } a > 0. \tag{1.3}$$

The main aim of this paper is to study the Cauchy problem of (1.1)–(1.3) with the initial data

$$(\rho, \mathbf{u}, \mathbf{B})|_{t=0} = (\rho_0, \mathbf{u}_0, \mathbf{B}_0)(x), \quad x \in \mathbb{R}^3, \tag{1.4}$$

and the far field behavior

$$(\rho, \mathbf{u}, \mathbf{B})(x,t) \to (\tilde{\rho}, 0, 0) \quad \text{as } |x| \to \infty, t > 0, \tag{1.5}$$

where $\tilde{\rho}$ is a given nonnegative constant. Without loss of generality, we assume that $\tilde{\rho} = 1$.

The compressible MHD system (1.1) describes the relationship between the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism, and it has been studied by many articles [2, 3, 7, 9] and the references cited therein. For the one-dimensional isentropic compressible Navier–Stokes equations, Hoff [4], Kazhikhov and Shelukhin [12], and Serre [21, 22] have studied the isothermal problems respectively. For the multi-dimensional case, Itaya [11], Nash [19], Serrin [23], and Tani [24] investigated the local existence and uniqueness of classical solutions with vacuum, respectively. Matsumura and Nishida in [16–18] proved the global smooth solutions when the initial data are close to a non-vacuum equilibrium. Later, Huang, Li, and Xin [10] investigated the vacuum and non-vacuum state for the three-dimensional case, and they obtained the global existence and uniqueness of classical solutions. Yu in [25] studied the 3D compressible isothermal Navier–Stokes equations with a vacuum at infinity and proved the global existence of strong solutions. For the isentropic MHD system, Hu and Wang [8, 9] obtained the global existence of renormalized solutions with general large initial data. Li, Xu, and Zhang [15] considered the Cauchy problem of the 3D case and obtained the global well-posedness of classical solution with small energy.

For the isothermal Navier–Stokes system away from vacuum, Nishida in [20] proved the global existence of BV solutions for one-dimensional MHD equations. Hoff in [5, 6] obtained the global weak solutions for three-dimensional case. Matsumura and Nishida in [17] obtained the global smooth solutions. A natural question to ask is whether or not smooth solutions exist globally in three-dimensional MHD equations. Therefore, the main purpose of this paper is to investigate the global existence of strong solutions for the 3D compressible isothermal MHD system.

Before stating the main results, we explain the notation and conventions used throughout this paper. We denote

$$\int f(x)\,dx \triangleq \int_{\mathbb{R}^3} f(x)\,dx,$$

and the standard homogeneous and inhomogeneous Sobolev spaces as follows:

$$\begin{cases} L^r = L^r(\mathbb{R}^3), \qquad D^{k,r} = \{\mathbf{u} \in L^1_{\text{loc}} \,|\, \|\nabla^k \mathbf{u}\|_{L^r} < \infty\}, \qquad \|\mathbf{u}\|_{D^{k,r}} = \|\nabla^k \mathbf{u}\|_{L^r}, \\ W^{k,r} = L^r \cap D^{k,r}, \qquad H^k = W^{k,2}, \qquad D^k = D^{k,2}, \qquad D^1 = \{\mathbf{u} \in L^6 \,|\, \|\nabla \mathbf{u}\|_{L^2} < \infty\}, \end{cases}$$

for $1 < r < \infty$ and $k \in \mathbb{Z}$. The total energy is defined as

$$m_0 \triangleq \int \left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + G(\rho_0) + \frac{1}{2}|\mathbf{B}_0|^2\right)(x)\,dx,$$

where $G(\rho)$ denotes the potential energy density

$$G(\rho) \triangleq \rho \int_1^\rho \frac{P(s) - P(1)}{s^2}\, ds,$$

and it is clear that $G(\rho) \sim (\rho - 1)^2$.

We now state the definition of strong solution of (1.1)–(1.5) as follows.

Definition 1.1 A triple of functions $(\rho, \mathbf{u}, \mathbf{B})$ is said to be a strong solution of (1.1)–(1.5) provided that $(\rho, \mathbf{u}, \mathbf{B})$ satisfies equations (1.1)–(1.5) almost everywhere and belongs to the class of functions (1.8) in which the uniqueness can be shown to hold.

The main result of this paper is formulated as follows.

Theorem 1.1 *For any given numbers $M > 0$ (not necessary small), $\bar{\rho} \geq 2$ and $p \in [2, 6]$, assume that*

$$\begin{cases} \rho_0|\mathbf{u}_0|^2 + (\rho_0 - 1)^2 + |\mathbf{B}_0|^2 \in L^1, \quad (\mathbf{u}_0, \mathbf{B}_0) \in D^1 \cap D^2, \\ 0 \leq \inf \rho_0 \leq \rho_0(x) \leq \sup \rho_0 \leq \bar{\rho}, \quad (\rho_0 - 1) \in H^1 \cap W^{1,p}, \\ \|\nabla \mathbf{u}_0\|_{L^2}^2 + \|\nabla \mathbf{B}_0\|_{L^2}^2 \leq M, \end{cases} \tag{1.6}$$

and the compatibility condition holds

$$-\mu \Delta \mathbf{u}_0 - (\lambda + \mu)\nabla \operatorname{div} \mathbf{u}_0 + \nabla P(\rho_0) - (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 = \rho_0^{1/2} g \tag{1.7}$$

for some $g \in L^2$. Then there exists a positive constant ε depending only on μ, ν, λ, a, M, and $\bar{\rho}$ such that if

$$m_0 \leq \varepsilon,$$

then for any $0 < T < \infty$, there exists a unique global strong solution $(\rho, \mathbf{u}, \mathbf{B})$ of problem (1.1)–(1.5) on $\mathbb{R}^3 \times [0, T]$ satisfying

$$0 \leq \rho \leq 2\bar{\rho} \quad \text{for all } x \in \mathbb{R}^3, \quad t \geq 0,$$

and

$$\begin{cases} (\rho - 1) \in C([0, T]; H^1 \cap W^{1,p}), \quad \rho_t \in C([0, T]; L^p), \\ \mathbf{u} \in C([0, T]; D^1 \cap D^2) \cap L^2(0, T; D^{2,p}), \\ \sqrt{\rho}\mathbf{u}_t \in L^\infty(0, T; L^2), \quad \mathbf{u}_t \in L^2(0, T; D^1), \\ \mathbf{B} \in C([0, T]; H^2) \cap L^2(0, T; H^2), \quad \mathbf{B}_t \in L^2(0, T; H^1). \end{cases} \tag{1.8}$$

To obtain the strong solutions globally in time, we need global a priori estimates on smooth solutions for $(\rho, \mathbf{u}, \mathbf{B})$. The main difficulty is due to the appearance of the strong coupling between the velocity field and the magnetic field. Another difficulty is the weaker compatibility condition (1.7). Therefore, to overcome the two difficulties, we first give some known inequalities and facts in Sect. 2 and then establish the estimates of the global strong solutions that are independent of time t to problem (1.1)–(1.5) in Sect. 3. Finally, with the help of global (uniform) estimates at hand, in Sect. 4 we prove that Theorem 1.1 holds.

2 Preliminaries

In this section, we recall some known facts and elementary inequalities which will be used later. Firstly, we give the following local existence due to [17].

Proposition 2.1 *Assume that the initial data* $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$ *satisfies* (1.6) *and* (1.7). *Then there exist a small time* $T_* > 0$ *and a strong solution* $(\rho, \mathbf{u}, \mathbf{B})$ *to problem* (1.1)–(1.5) *on* $\mathbb{R}^3 \times (0, T_*]$.

Lemma 2.1 ([26]) *Let* $y \in W^{1,1}(0,T)$ *satisfy the ODE system*

$$y' = g(y) + b'(t) \quad \text{on } [0,T], y(0) = y_0,$$

where $b \in W^{1,1}(0,T)$, $g \in C(\mathbb{R})$, *and* $g(+\infty) = -\infty$. *Assume that there are two constants* $N_0 \geq 0$ *and* $N_1 \geq 0$ *such that for all* $0 \leq t_1 < t_2 \leq T$,

$$b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1). \tag{2.1}$$

Then

$$y(t) \leq \max\{y_0, \xi^*\} + N_0 < +\infty \quad \text{on } [0,T],$$

where $\xi^* \in \mathbb{R}$ *is a constant such that*

$$g(\xi) \leq -N_1 \quad \text{for } \xi \geq \xi^*. \tag{2.2}$$

The following well-known Gagliardo–Nirenberg inequality can be found in [14].

Lemma 2.2 *For* $p \in [2,6]$, $q \in (1,\infty)$, *and* $r \in (3,\infty)$, *assume that* $f \in H^1(\mathbb{R}^3)$ *and* $g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3)$. *Then there exists a generic constant* $C > 0$, *depending only on* q *and* r, *such that*

$$\|f\|_{L^p} \leq C\|f\|_{L^2}^{\frac{6-p}{2p}} \|\nabla f\|_{L^2}^{\frac{3p-6}{2p}}, \tag{2.3}$$

$$\|g\|_{L^\infty} \leq C\|g\|_{L^q}^{\frac{q(r-3)}{3r+q(r-3)}} \|\nabla g\|_{L^r}^{\frac{3r}{3r+q(r-3)}}. \tag{2.4}$$

Finally, we introduce the effective viscous flux F, the vorticity ω, and the material derivative "$\cdot$", which are defined as follows:

$$F \triangleq (2\mu + \lambda)\operatorname{div}\mathbf{u} - \big(P(\rho) - P(1)\big) - \frac{1}{2}|\mathbf{B}|^2, \quad \omega \triangleq \nabla \times \mathbf{u}, \dot{\mathbf{u}} \triangleq \mathbf{u}_t + \mathbf{u} \cdot \nabla \dot{\mathbf{u}},$$

then

$$\Delta F = \operatorname{div}(\rho \dot{\mathbf{u}}) - \operatorname{div}\operatorname{div}(\mathbf{B} \otimes \mathbf{B}) \quad \text{and} \quad \mu \Delta \omega = \nabla \times \big(\rho \dot{\mathbf{u}} - \operatorname{div}(\mathbf{B} \otimes \mathbf{B})\big).$$

Thus, it follows from Lemma 2.2 and the standard L^p-estimates of elliptic equations that we have the following lemma.

Lemma 2.3 ([15]) *Let* $(\rho, \mathbf{u}, \mathbf{B})$ *be a smooth solution of* (1.1)–(1.5). *Then there exists a generic constant* $C > 0$ *such that for any* $p \in [2, 6]$,

$$\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} \le C\big(\|\rho \dot{\mathbf{u}}\|_{L^p} + \|\nabla \mathbf{B} \cdot \mathbf{B}\|_{L^p}\big), \tag{2.5}$$

$$\begin{aligned} \|F\|_{L^p} + \|\omega\|_{L^p} \le{}& C\big(\|\nabla \mathbf{u}\|_{L^2} + \big\|P(\rho) - P(1)\big\|_{L^2} + \big\||\mathbf{B}|^2\big\|_{L^2}\big)^{(6-p)/2p} \\ &\times \big(\|\rho \dot{\mathbf{u}}\|_{L^2} + \|\nabla \mathbf{B} \cdot \mathbf{B}\|_{L^2}\big)^{(3p-6)/2p}, \end{aligned} \tag{2.6}$$

$$\|\nabla \mathbf{u}\|_{L^p} \le C\big(\|F\|_{L^p} + \big\|P(\rho) - P(1)\big\|_{L^p} + \big\||\mathbf{B}|^2\big\|_{L^p} + \|\omega\|_{L^p}\big). \tag{2.7}$$

3 A priori estimates

In this section, we establish the uniform a priori estimates of solutions to problem (1.1)–(1.5) to extend the local strong solution guaranteed by Proposition 2.1. Assume that $(\rho, \mathbf{u}, \mathbf{B})$ is a smooth solution to (1.1)–(1.5) on $\mathbb{R}^3 \times (0, T)$ for some positive time $T > 0$ with smooth initial data $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$ satisfying (1.6) and (1.7). Set $\sigma(t) \triangleq \min\{1, t\}$ and define

$$A_1(T) \triangleq \sup_{0\le t\le T} \sigma\big(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2\big),$$

$$A_2(T) \triangleq \sup_{0\le t\le T} \big(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2\big).$$

We have the following key a priori estimates on $(\rho, \mathbf{u}, \mathbf{B})$.

Proposition 3.1 *For given constant* $\bar{\rho} > 0$ *and* M (*not necessarily small*), *assume that* $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$ *satisfies* (1.6) *and* (1.7). *Then there exist positive constants* K *and* ε_0, *depending only on* μ, ν, λ, a, $\bar{\rho}$, *and* M, *such that if* $(\rho, \mathbf{u}, \mathbf{B})$ *is a smooth solution of* (1.1)–(1.5) *on* $\mathbb{R}^3 \times (0, T]$ *satisfying*

$$\begin{cases} 0 \le \rho(x,t) \le 2\bar{\rho} & \text{for all } (x,t) \in \mathbb{R}^3 \times [0,T], \\ A_1(T) \le 2m_0^{1/2}, & A_2(T) \le 3K, \end{cases} \tag{3.1}$$

then

$$\begin{cases} 0 \le \rho(x,t) \le \frac{7}{4}\bar{\rho} & \text{for all } (x,t) \in \mathbb{R}^3 \times [0,T], \\ A_1(T) \le m_0^{1/2}, & A_2(T) \le 2K, \end{cases} \tag{3.2}$$

provided $m_0 \le \varepsilon_0$.

Proof The proof of Proposition 3.1 will be done by a series of lemmas below. □

Throughout this paper, we denote by C, C_i $(i = 1, 2, \ldots)$ the generic positive constants that may depend on μ, ν, λ, a, $\bar{\rho}$, and M, but are independent of time $T > 0$. We also use $C(\alpha)$ to emphasize the dependence on α.

We first begin with the following standard energy estimates, which can be easily deduced from (1.1)–(1.5).

Lemma 3.1 *Let* $(\rho, \mathbf{u}, \mathbf{B})$ *be a smooth solution of* (1.1)–(1.5) *on* $\mathbb{R}^3 \times [0,T]$. *Then*

$$\sup_{t\in[0,T]} \int \left((\rho-1)^2 + \rho|\mathbf{u}|^2 + |\mathbf{B}|^2\right) dx + \int_0^T \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2\right) dt \le Cm_0. \tag{3.3}$$

Proof Multiplying $(1.1)_1$, $(1.1)_2$, and $(1.1)_3$ by $G'(\rho)$, $\mathbf{u}$, and $\mathbf{B}$, respectively, integrating the resulting equations by parts over $\mathbb{R}^3$, and adding them together, one has

$$\frac{d}{dt}\int \left(G(\rho) + \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{B}|^2\right) dx + \mu\|\nabla \mathbf{u}\|_{L^2}^2 + (\mu+\lambda)\|\operatorname{div}\mathbf{u}\|_{L^2}^2 + \nu\|\nabla \mathbf{B}\|_{L^2}^2 = 0,$$

which, integrated over $(0,T)$, immediately leads to (3.3). □

Lemma 3.2 *Under the conditions of Proposition* 3.1, *one has*

$$\sup_{t\in[0,T]} \|\mathbf{B}\|_{L^3}^3 + \int_0^T \|\mathbf{B}\|_{L^9}^3 \, dt \le C\|\mathbf{B}_0\|_{L^3}^3. \tag{3.4}$$

Proof By virtue of (3.1) and (3.3), we infer from Lemma 2.2 ($p = 6$ in (2.3))that

$$\begin{aligned}
&\int_0^T \left(\|\mathbf{u}\|_{L^6}^4 + \|\nabla \mathbf{u}\|_{L^2}^4 + \|\nabla \mathbf{B}\|_{L^2}^4\right) dt \\
&\quad\le C\int_0^{\sigma(T)} \left(\|\nabla \mathbf{u}\|_{L^2}^4 + \|\nabla \mathbf{B}\|_{L^2}^4\right) dt + C\int_{\sigma(T)}^T \sigma\left(\|\nabla \mathbf{u}\|_{L^2}^4 + \|\nabla \mathbf{B}\|_{L^2}^4\right) dt \\
&\quad\le C\sup_{t\in[0,\sigma(T)]}\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2\right)\int_0^{\sigma(T)} \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2\right) dt \\
&\qquad + C\sup_{t\in[\sigma(T),T]} \sigma\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2\right)\int_{\sigma(T)}^T \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2\right) dt \\
&\quad\le Cm_0.
\end{aligned} \tag{3.5}$$

Multiplying $(1.1)_3$ by $3|\mathbf{B}|\mathbf{B}$ and integrating by parts over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\mathbf{B}\|_{L^3}^3 + 3\nu\int \left(|\mathbf{B}||\nabla \mathbf{B}|^2 + |\mathbf{B}|\left|\nabla\left(|\mathbf{B}|\right)\right|^2\right) dx \\
&\quad\le \nu\int |\mathbf{B}||\nabla \mathbf{B}|^2 \, dx + C\|\nabla \mathbf{u}\|_{L^2}^2 \|\mathbf{B}\|_{L^{9/2}}^3,
\end{aligned} \tag{3.6}$$

where the last term on the right-hand side in (3.6) comes from the following inequality:

$$\begin{aligned}
\int |\nabla \mathbf{u}||\mathbf{B}|^3 \, dx &\le C\|\nabla \mathbf{u}\|_{L^2}\|\mathbf{B}\|_{L^{9/2}}^{3/2}\|\mathbf{B}\|_{L^9}^{3/2} \\
&\le C\|\nabla \mathbf{u}\|_{L^2}\|\mathbf{B}\|_{L^{9/2}}^{3/2}\left\||\mathbf{B}|^{3/2}\right\|_{L^6} \\
&\le C\left(\|\nabla \mathbf{u}\|_{L^2}^2\|\mathbf{B}\|_{L^{9/2}}^3\right)^{1/2}\left\||\nabla \mathbf{B}||\mathbf{B}|^{1/2}\right\|_{L^2}.
\end{aligned}$$

To deal with the right-hand side of (3.6), we notice that

$$\|\mathbf{B}\|_{L^9}^3 \le C\left\||\mathbf{B}|^{3/2}\right\|_{L^6}^2 \le C\left\||\nabla \mathbf{B}||\mathbf{B}|^{1/2}\right\|_{L^2}^2,$$

then

$$\|\mathbf{B}\|_{L^{9/2}} \le C\|\mathbf{B}\|_{L^3}^{1/2}\|\mathbf{B}\|_{L^9}^{1/2} \le C\|\mathbf{B}\|_{L^3}^{1/2}\left\|\nabla|\mathbf{B}|^{3/2}\right\|_{L^2}^{1/3},$$

which together with (3.6) yields

$$\frac{d}{dt}\|\mathbf{B}\|_{L^3}^3 + \|\mathbf{B}\|_{L^9}^3 \le C\|\nabla\mathbf{u}\|_{L^2}^4\|\mathbf{B}\|_{L^3}^3,$$

which together with (3.5) and Gronwall's inequality yields the desired estimate (3.4). □

Lemma 3.3 *Under the conditions of Proposition 3.1,*

$$\begin{aligned}
&\left(\mu\beta\|\nabla\mathbf{u}\|_{L^2}^2 + (\mu+\lambda)\beta\|\operatorname{div}\mathbf{u}\|_{L^2}^2 + \nu\beta\|\nabla\mathbf{B}\|_{L^2}^2\right)_t + \beta\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^2 + \beta\|\mathbf{B}_t\|_{L^2}^2 + \beta\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2 \\
&\quad \le \left(\beta\int a(\rho-1)\operatorname{div}\mathbf{u}\,dx + \beta\int\left(\mathbf{B}\cdot\nabla\mathbf{B} - \frac{1}{2}\nabla|\mathbf{B}|^2\right)\cdot\mathbf{u}\,dx\right)_t \\
&\qquad + C\left(\beta + \left|\beta'\right|\right)\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{B}\|_{L^2}^2\right) \\
&\qquad + C\left(\beta + \left|\beta'\right|\right)\left(\|\nabla\mathbf{u}\|_{L^2}^6 + \|\nabla\mathbf{B}\|_{L^2}^6\right) + Cm_0,
\end{aligned} \tag{3.7}$$

where $\beta = \beta(t) \ge 0$ is a piecewise smooth function.

Proof Multiplying $(1.1)_2$ by $\beta(t)\dot{\mathbf{u}}$ and integrating by parts over $\mathbb{R}^3$, one has

$$\begin{aligned}
\int\beta\rho|\dot{\mathbf{u}}|^2\,dx &= -\int\beta\nabla\left(P(\rho)-P(1)\right)\cdot\dot{\mathbf{u}}\,dx + \mu\beta\int\Delta\mathbf{u}\cdot\dot{\mathbf{u}}\,dx + \beta(\mu+\lambda)\int\nabla\operatorname{div}\mathbf{u}\cdot\dot{\mathbf{u}} \\
&\quad + \int\beta\left(\mathbf{B}\cdot\nabla\mathbf{B} - \frac{1}{2}\nabla|\mathbf{B}|^2\right)\cdot\dot{\mathbf{u}}\,dx \triangleq \sum_{i=1}^4 I_i.
\end{aligned} \tag{3.8}$$

The right-hand side terms of (3.8) can be estimated as follows. First, noting that

$$\left(P(\rho)-P(1)\right)_t + \mathbf{u}\cdot\nabla\left(P(\rho)-P(1)\right) + P(\rho)\operatorname{div}\mathbf{u} = 0, \tag{3.9}$$

we obtain from (3.3), (3.9) and Cauchy–Schwarz's inequality that

$$\begin{aligned}
I_1 &= \left(\int\beta\left(P(\rho)-P(1)\right)\operatorname{div}\mathbf{u}\,dx\right)_t - \beta'\int\left(P(\rho)-P(1)\right)\operatorname{div}\mathbf{u}\,dx \\
&\quad - \int\beta\left(P(\rho)-P(1)\right)_t\operatorname{div}\mathbf{u}\,dx + \int\beta\left(P(\rho)-P(1)\right)\partial_i\left(u^j\partial_j u^i\right)dx \\
&\le \left(\int\beta\left(P(\rho)-P(1)\right)\operatorname{div}\mathbf{u}\,dx\right)_t + C\beta'\|\nabla\mathbf{u}\|_{L^2}\left\|P(\rho)-P(1)\right\|_{L^2} \\
&\quad + \int\beta P(1)(\operatorname{div}\mathbf{u})^2\,dx + \beta\int\left(P(\rho)-P(1)\right)\partial_i u^i\partial_j u^j\,dx \\
&\le a\left(\int\beta(\rho-1)\operatorname{div}\mathbf{u}\,dx\right)_t + C\left(\beta + \left|\beta'\right|\right)\|\nabla\mathbf{u}\|_{L^2}^2 + C|\beta|'m_0.
\end{aligned}$$

By virtue of Cauchy-Schwarz's inequality, one has

$$
\begin{aligned}
I_2 &= -\mu \int \beta \partial_i u^j \partial_i \left(u_t^j + u^k \partial_k u^j\right) dx \\
&= -\frac{\mu}{2}\left(\int \beta |\nabla \mathbf{u}|^2 \, dx\right)_t + \frac{\mu}{2}\beta' \|\nabla \mathbf{u}\|_{L^2}^2 - \mu\beta \int \left(\partial_i u^j \partial_i u^k \partial_k u^j - \frac{1}{2}|\nabla \mathbf{u}|^2 \operatorname{div} \mathbf{u}\right) dx \\
&\le -\frac{\mu}{2}\left(\int \beta |\nabla \mathbf{u}|^2 \, dx\right)_t + C\left|\beta'\right| \|\nabla \mathbf{u}\|_{L^2}^2 + C_1 \beta \|\nabla \mathbf{u}\|_{L^3}^3.
\end{aligned}
$$

Similarly,

$$
I_3 \le -\frac{\mu+\lambda}{2}\left(\int \beta |\operatorname{div} \mathbf{u}|^2 \, dx\right)_t + C\left|\beta'\right| \|\nabla \mathbf{u}\|_{L^2}^2 + C_1 \beta \|\nabla \mathbf{u}\|_{L^3}^3.
$$

For I_4, it follows from (2.3) and (2.4) that

$$
\begin{aligned}
I_4 &= \left(\int \beta \left(\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2}\nabla |\mathbf{B}|^2\right) \cdot \mathbf{u} \, dx\right)_t - \beta' \int \left(\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2}\nabla |\mathbf{B}|^2\right) \cdot \mathbf{u} \, dx \\
&\quad - \beta \int \left(\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2}\nabla |\mathbf{B}|^2\right)_t \cdot \mathbf{u} \, dx + \beta \int \left(\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2}\nabla |\mathbf{B}|^2\right) \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, dx \\
&\le \left(\int \beta \left(\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2}\nabla |\mathbf{B}|^2\right) \cdot \mathbf{u} \, dx\right)_t + C\beta' \|\mathbf{B}\|_{L^2}^{1/2} \|\nabla \mathbf{B}\|_{L^2}^{3/2} \|\nabla \mathbf{u}\|_{L^2} \\
&\quad + C\beta \|\nabla \mathbf{B}\|_{L^2}^{1/2} \left\|\nabla^2 \mathbf{B}\right\|_{L^2}^{1/2} \|\mathbf{B}_t\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} + C\beta \|\nabla \mathbf{B}\|_{L^2} \left\|\nabla^2 \mathbf{B}\right\|_{L^2} \|\nabla \mathbf{u}\|_{L^2}^2 \\
&\le \left(\int \beta \left(\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2}\nabla |\mathbf{B}|^2\right) \cdot \mathbf{u} \, dx\right)_t + \frac{\beta}{4} \|\mathbf{B}_t\|_{L^2}^2 + \frac{\nu\beta}{8} \left\|\nabla^2 \mathbf{B}\right\|_{L^2}^2 \\
&\quad + C\left(\beta + \left|\beta'\right|\right) \|\nabla \mathbf{u}\|_{L^2}^6 + C\left(\beta + \left|\beta'\right|\right) \|\nabla \mathbf{B}\|_{L^2}^6 + C\left|\beta'\right| \|\nabla \mathbf{B}\|_{L^2}^2 + C\left|\beta'\right| m_0^6.
\end{aligned}
$$

On the other hand, it follows from (2.7) and Cauchy-Schwarz's inequality that

$$
\begin{aligned}
\|\nabla \mathbf{u}\|_{L^3}^3 &\le \left(\left\|\rho^{1/2} \dot{\mathbf{u}}\right\|_{L^2} + \|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^2}\right)^{3/2} \left(\|\nabla \mathbf{u}\|_{L^2} + \left\|P(\rho) - P(1)\right\|_{L^2} + \left\||\mathbf{B}|^2\right\|_{L^2}\right)^{3/2} \\
&\quad + \left\|P(\rho) - P(1)\right\|_{L^3}^3 + \left\||\mathbf{B}|^2\right\|_{L^3}^3 \\
&\le \frac{1}{2C_1} \left\|\rho^{1/2} \dot{\mathbf{u}}\right\|_{L^2}^2 + \frac{\nu}{8C_1} \left\|\nabla^2 \mathbf{B}\right\|_{L^2}^2 + C\left(\|\nabla \mathbf{u}\|_{L^2}^6 + \|\nabla \mathbf{B}\|_{L^2}^6\right) + Cm_0.
\end{aligned} \tag{3.10}
$$

Substituting I_1, I_1, I_3, I_4 into (3.8) and using (3.10), we immediately obtain

$$
\begin{aligned}
&\left(\mu\beta \|\nabla \mathbf{u}\|_{L^2}^2 + (\mu+\lambda)\beta \|\operatorname{div} \mathbf{u}\|_{L^2}^2\right)_t + \beta \left\|\rho^{1/2} \dot{\mathbf{u}}\right\|_{L^2}^2 \\
&\quad \le \left(\beta \int a(\rho - 1) \operatorname{div} \mathbf{u} \, dx + \beta \int \left(\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2}\nabla |\mathbf{B}|^2\right) \cdot \mathbf{u} \, dx\right)_t + Cm_0 \\
&\qquad + C\left(\beta + \left|\beta'\right|\right)\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2\right) + C\left(\beta + \left|\beta'\right|\right)\left(\|\nabla \mathbf{u}\|_{L^2}^6 + \|\nabla \mathbf{B}\|_{L^2}^6\right).
\end{aligned} \tag{3.11}
$$

We infer from (1.1)$_3$ and (2.3) that

$$
\begin{aligned}
&\left(\nu\beta\|\nabla\mathbf{B}\|_{L^2}^2\right)_t + \nu^2\beta\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2 + \beta\|\mathbf{B}_t\|_{L^2}^2 = \beta\int(\mathbf{B}_t - \nu\Delta\mathbf{B})^2\,dx - \nu\beta'\|\nabla\mathbf{B}\|_{L^2}^2 \\
&= \beta\int|\mathbf{B}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{B} - \mathbf{B}\operatorname{div}\mathbf{u}|^2\,dx - \nu\beta'\|\nabla\mathbf{B}\|_{L^2}^2 \\
&\le C\beta\|\nabla\mathbf{B}\|_{L^2}^2 + C\beta\|\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{B}\|_{L^3}^2 \\
&\le \frac{\beta\nu}{2}\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2 + C\left|\beta'\right|\|\nabla\mathbf{B}\|_{L^2}^2 + C\beta\left(\|\nabla\mathbf{u}\|_{L^2}^6 + \|\nabla\mathbf{B}\|_{L^2}^6\right),
\end{aligned}
$$

thus

$$
\left(\nu\beta\|\nabla\mathbf{B}\|_{L^2}^2\right)_t + \nu^2\beta\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2 + \beta\|\mathbf{B}_t\|_{L^2}^2 \le C\left|\beta'\right|\|\nabla\mathbf{B}\|_{L^2}^2 + C\beta\left(\|\nabla\mathbf{u}\|_{L^2}^6 + \|\nabla\mathbf{B}\|_{L^2}^6\right),
$$

which, together with (3.11), yields (3.7). □

Lemma 3.4 *Under the conditions of Proposition 3.1, there exist positive constants $K \ge M + 1$ and $\varepsilon_1 < 1$, depending only on μ, ν, λ, a, $\bar{\rho}$, and M, such that*

$$
A_2\big(\sigma(T)\big) + \int_0^{\sigma(T)}\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \left\|\nabla^2\mathbf{B}\right\|_{L^2}^2\right)dt \le 2K, \tag{3.12}
$$

provided $m_0 \le \varepsilon_1$.

Proof Taking $\beta = 1$ in (3.7) and integrating it over $(0,\sigma(T))$, we deduce from (1.6), (1.7), (3.1), (3.3), (2.7), (3.3), and (3.4) that

$$
\begin{aligned}
&A_2\big(\sigma(T)\big) + \int_0^{\sigma(T)}\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \left\|\nabla^2\mathbf{B}\right\|_{L^2}^2\right)dt \\
&\le C(M+1) + a\int(\rho-1)\operatorname{div}\mathbf{u}\,dx\Big|_0^{\sigma(T)} + \int\left(\mathbf{B}\cdot\nabla\mathbf{B} - \frac{1}{2}\nabla|\mathbf{B}|^2\right)dx\Big|_0^{\sigma(T)} \\
&\quad + C\int_0^{\sigma(T)}\left(\|\nabla\mathbf{u}\|_{L^2}^6 + \|\nabla\mathbf{B}\|_{L^2}^6\right)dt \\
&\le \frac{1}{2}A_2\big(\sigma(T)\big) + Cm_0^{1/2}A_2^{3/2}\big(\sigma(T)\big) + C(M+1)^2,
\end{aligned}
$$

thus

$$
\begin{aligned}
&A_2\big(\sigma(T)\big) + \int_0^{\sigma(T)}\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \left\|\nabla^2\mathbf{B}\right\|_{L^2}^2\right)dt \\
&\le C_2m_0^{1/2}K^{3/2} + C_3(M+1)^2 \le 2K,
\end{aligned}
$$

provided

$$
m_0 \le \varepsilon_1 \triangleq \min\left\{1, \frac{K}{C_2^2}\right\} \quad \text{with } K \triangleq C_3(M+1)^2.
$$

Thus, we immediately obtain the desired estimate (3.12). The proof of Lemma 3.4 is therefore completed. □

Lemma 3.5 *Under the conditions of Proposition 3.1, there exists a positive constant ε_2, depending only on μ, ν, λ, a, $\bar{\rho}$, and M, such that*

$$A_1(T) + \int_{i-1}^{i+1} \sigma_i \big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 \big)\, dt \le m_0^{1/2}, \tag{3.13}$$

provided $m_0 \le \varepsilon_2$, where $\sigma_i \triangleq \sigma(t+1-i)$ and i is an integer satisfying $1 \le i \le [T]-1$ with $[T]$ denoting the largest integer less than or equal to T.

Proof Without loss of generality, assume that $T \ge 2$. Otherwise, things can be done by choosing a suitable small step size. For integer $i(1 \le i \le [T]-1)$, taking $\beta = \sigma_i(t)$ in (3.7) and integrating the results over $(i-1, i+1]$, one deduces from (3.1), (3.3), and (3.5) that

$$\begin{aligned}
&\sup_{t\in[i-1,i+1]} \big(\sigma_i \|\nabla \mathbf{u}\|_{L^2}^2 + \sigma_i \|\nabla \mathbf{B}\|_{L^2}^2\big) + \int_{i-1}^{i+1} \sigma_i \big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 \big)\, dt \\
&\quad\le Cm_0 + \sigma_i \int \big(P(\rho) - P(1)\big) \operatorname{div} \mathbf{u}\, dx + \sigma_i \int \left(\mathbf{B}\cdot\nabla\mathbf{B} - \frac{1}{2}\nabla|\mathbf{B}|^2 \right)\cdot \mathbf{u}\, dx \\
&\qquad + C\int_{i-1}^{i+1} \sigma_i \big(\|\nabla \mathbf{u}\|_{L^2}^6 + \|\nabla \mathbf{B}\|_{L^2}^6 \big)\, dt + C\int_{i-1}^{i+1} \sigma_i \big(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2 \big)\, dt \\
&\quad\le \frac{\sigma_i}{2}\|\nabla \mathbf{u}\|_{L^2}^2 + C_4 m_0^{1/2} M^{1/2} \big(\sigma_i \|\nabla \mathbf{B}\|_{L^2}^2\big) + Cm_0,
\end{aligned}$$

thus

$$\begin{aligned}
&\sup_{t\in[i-1,i+1]} \big(\sigma_i \|\nabla \mathbf{u}\|_{L^2}^2 + \sigma_i \|\nabla \mathbf{B}\|_{L^2}^2\big) + \int_{i-1}^{i+1} \sigma_i \big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 \big)\, dt \\
&\quad\le C_5 m_0 \le C m_0^{1/2},
\end{aligned}$$

provided

$$m_0 \le \varepsilon_2 \triangleq \min\left\{1, \frac{1}{2C_4^2 M}, \frac{1}{C_5^2}\right\}. \qquad \square$$

Lemma 3.6 *Under the conditions of Proposition 3.1, it holds that*

$$\begin{aligned}
&\big(\xi \|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + \xi \|\mathbf{B}_t\|_{L^2}^2\big)_t + \xi \big(\|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2\big) \\
&\quad\le |\xi'| \big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 \big) + C\xi \big(\|\nabla \mathbf{u}\|_{L^2}^4 + \|\nabla \mathbf{B}\|_{L^2}^4 \big) \big(\|\mathbf{B}_t\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 \big) \\
&\qquad + C\xi \|\nabla \mathbf{u}\|_{L^2}^2 + C\xi \|\nabla \mathbf{u}\|_{L^2}^4 \|\nabla \mathbf{B}\|_{L^2}^{3/2} \|\nabla^2 \mathbf{B}\|_{L^2}^{3/2} + C\xi m_0 \|\nabla \mathbf{B}\|_{L^2}^3 \|\nabla^2 \mathbf{B}\|_{L^2}^3 \\
&\qquad + C\xi \big(\|\nabla \mathbf{u}\|_{L^2} + m_0^{1/2} + m_0^{1/4} \|\nabla \mathbf{B}\|_{L^2}^{3/2} \big) \big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^3 + \|\nabla^2 \mathbf{B}\|_{L^2}^3 \big) + C\xi m_0,
\end{aligned} \tag{3.14}$$

where $\xi = \xi(t) \ge 0$ is a piecewise smooth function.

Proof Operating $\xi \dot{u}^j[\partial_t + \operatorname{div}(\mathbf{u}\cdot)]$ to $(1.1)_2^j$, summing with respect to j, and integrating the resulting equation over $\mathbb{R}^3$, one has after integration by parts

$$\begin{aligned}
&\left(\frac{\xi}{2}\int \rho|\dot{\mathbf{u}}|^2\,dx\right)_t - \frac{\xi'}{2}\int \rho|\dot{\mathbf{u}}|^2\,dx \\
&\quad = -\xi\int \dot{u}^j\big[\partial_j P_t + \operatorname{div}(\mathbf{u}\partial_j P)\big]\,dx + \xi\mu\int \dot{u}^j\big[\Delta u_t^j + \operatorname{div}\big(\mathbf{u}\Delta u^j\big)\big]\,dx \\
&\qquad + \xi(\lambda+\mu)\int \dot{u}^j\big[\partial_j\partial_t(\operatorname{div}\mathbf{u}) + \operatorname{div}\big(\mathbf{u}\partial_j(\operatorname{div}\mathbf{u})\big)\big]\,dx \\
&\qquad + \xi\int \dot{u}^j\big[\partial_t\big(\mathbf{B}\cdot\nabla B^j\big) + \operatorname{div}\big(\mathbf{u}\mathbf{B}\cdot\nabla B^j\big)\big]\,dx \\
&\qquad - \frac{\xi}{2}\int \dot{u}^j\big[\partial_t\partial_j\big(|\mathbf{B}|^2\big) + \operatorname{div}\big(\mathbf{u}\partial_j\big(|\mathbf{B}|^2\big)\big)\big]\,dx \triangleq \sum_{i=1}^5 J_i,
\end{aligned} \tag{3.15}$$

where the first term on the right-hand side of (3.15) can be estimated as follows. Based on integrating by parts and (3.1), we obtain

$$\begin{aligned}
J_1 &= \xi\int \big(\partial_j\dot{u}^j P(\rho)_t + \partial_k\dot{u}^j u^k\partial_j P(\rho)\big)\,dx \\
&= \xi\int \big(-a\operatorname{div}\mathbf{u}\partial_j\dot{u}^j - a\operatorname{div}\big((\rho-1)\mathbf{u}\big)\partial_j\dot{u}^j - \partial_j\big(\partial_k\dot{u}^j u^k\big)\big(P(\rho)-P(1)\big)\big)\,dx \\
&= \xi\int \big(-a\operatorname{div}\mathbf{u}\partial_j\dot{u}^j - \partial_k\dot{u}^j\partial_j u^k\big(P(\rho)-P(1)\big)\big)\,dx \\
&\le \frac{\mu}{8}\xi\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C(\bar{\rho})\xi(\|\nabla\mathbf{u}\|_{L^2}^2.
\end{aligned}$$

Similarly,

$$J_2 = \mu\xi\int \dot{u}^j\big[\Delta u_t^j + \operatorname{div}\big(\mathbf{u}\Delta u^j\big)\big]\,dx \le -\frac{3\mu}{4}\xi\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\xi\|\nabla\mathbf{u}\|_{L^4}^4$$

and

$$J_3 \le -\frac{\lambda+\mu}{2}\xi\|\operatorname{div}\dot{\mathbf{u}}\|_{L^2}^2 + \frac{\mu}{4}\xi\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\xi\|\nabla\mathbf{u}\|_{L^4}^4.$$

Keeping in mind that $\operatorname{div}\mathbf{B} = 0$ and integrating by parts over $\mathbb{R}^3$, one has

$$\begin{aligned}
J_4 &= \xi\int \big(\dot{u}^j\big(B_t^i\partial_i B^j + B^i\partial_i B_t^j\big) - \partial_k\dot{u}^j u^k B^i\partial_i B^j\big)\,dx \\
&\le -\xi\int \big(B^j\partial_i\dot{u}^j B_t^i + B_t^j\partial_i\dot{u}^j B^i\,dx + \partial_k\dot{u}^j u^k B^i\partial_i B^j\big)\,dx \\
&\le \frac{\mu}{8}\xi\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\xi\|\mathbf{B}\|_{L^3}^2\|\nabla\mathbf{B}_t\|_{L^2}^2 + C\xi\|\nabla\mathbf{B}\|_{L^2}^2\big\|\nabla^2\mathbf{B}\big\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2 \\
&\le \frac{\mu}{8}\xi\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\xi m_0^{1/2}\|\nabla\mathbf{B}_t\|_{L^2}^2 + C\xi\big(\|\nabla\mathbf{B}\|_{L^2}^4 + \|\nabla\mathbf{u}\|_{L^2}^4\big)\big\|\nabla^2\mathbf{B}\big\|_{L^2}^2.
\end{aligned}$$

Similarly,

$$J_5 \le \frac{\mu}{8}\xi\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\xi m_0^{1/2}\|\nabla\mathbf{B}_t\|_{L^2}^2 + C\xi\big(\|\nabla\mathbf{B}\|_{L^2}^4 + \|\nabla\mathbf{u}\|_{L^2}^4\big)\big\|\nabla^2\mathbf{B}\big\|_{L^2}^2.$$

It follows from (2.6), (2.7), (3.4) and Cauchy–Schwarz's inequality that

$$\begin{aligned}
\|\nabla \mathbf{u}\|_{L^4}^4 &\le \big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^3 + \|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^2}^3\big)\big(\|\nabla \mathbf{u}\|_{L^2} + \|P(\rho)-P(1)\|_{L^2} + \||\mathbf{B}|^2\|_{L^2}\big) \\
&\quad + \|P(\rho)-P(1)\|_{L^4}^4 + \||\mathbf{B}|^2\|_{L^4}^4 \\
&\le \big(\|\nabla \mathbf{u}\|_{L^2} + m_0^{1/2} + \|\mathbf{B}\|_{L^2}^{1/2}\|\nabla\mathbf{B}\|_{L^2}^{3/2}\big)\big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^3 + \|\mathbf{B}\|_{L^3}^3\|\nabla^2\mathbf{B}\|_{L^2}^3\big) \\
&\quad + C(\bar\rho)m_0 + \|\mathbf{B}\|_{L^2}^2\|\mathbf{B}\|_{L^\infty}^6 \\
&\le \big(\|\nabla \mathbf{u}\|_{L^2} + m_0^{1/2} + m_0^{1/4}\|\nabla\mathbf{B}\|_{L^2}^{3/2}\big)\big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^3 + \|\nabla^2\mathbf{B}\|_{L^2}^3\big) \\
&\quad + C(\bar\rho)m_0 + \|\nabla\mathbf{B}\|_{L^2}^3\|\nabla^2\mathbf{B}\|_{L^2}^3. \qquad (3.16)
\end{aligned}$$

Substituting $J_1, J_2, \ldots, J_5$ into (3.15) and using (3.16), we obtain

$$\begin{aligned}
&\big(\xi\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2\big)_t + \xi\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 \\
&\le \frac{1}{2}\xi\|\nabla\mathbf{B}_t\|_{L^2}^2 + |\xi'|\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + C\xi\big(\|\nabla\mathbf{u}\|_{L^2}^4 + \|\nabla\mathbf{B}\|_{L^2}^4\big)\|\nabla^2\mathbf{B}\|_{L^2}^2 \\
&\quad + C\xi\|\nabla\mathbf{u}\|_{L^2}^2 + C\xi m_0\|\nabla\mathbf{B}\|_{L^2}^3\|\nabla^2\mathbf{B}\|_{L^2}^3 + C\xi m_0 \\
&\quad + C\xi\big(\|\nabla \mathbf{u}\|_{L^2} + m_0^{1/2} + m_0^{1/4}\|\nabla\mathbf{B}\|_{L^2}^{3/2}\big)\big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^3 + \|\nabla^2\mathbf{B}\|_{L^2}^3\big). \qquad (3.17)
\end{aligned}$$

On the other hand, it follows from $(1.1)_3$ that

$$\mathbf{B}_{tt} - \nu\Delta\mathbf{B}_t = (\mathbf{B}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{B} - \mathbf{B}\operatorname{div}\mathbf{u})_t. \qquad (3.18)$$

Multiplying (3.18) by $\xi\mathbf{B}_t$ and integrating over $\mathbb{R}^3$ yield

$$\begin{aligned}
&\left(\frac{\xi}{2}\|\mathbf{B}_t\|_{L^2}^2\right)_t - \frac{\xi'}{2}\|\mathbf{B}_t\|_{L^2}^2 + \nu\xi\|\nabla\mathbf{B}_t\|_{L^2}^2 \\
&= \xi\int(\mathbf{B}_t\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{B}_t - \mathbf{B}_t\cdot\operatorname{div}\mathbf{u})\cdot\mathbf{B}_t\,dx \\
&\quad + \xi\int\big(-\mathbf{B}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) + (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\mathbf{B} + \mathbf{B}\operatorname{div}(\mathbf{u}\cdot\nabla\mathbf{u})\big)\cdot\mathbf{B}_t\,dx \\
&\quad + \xi\int(\mathbf{B}\cdot\nabla\dot{\mathbf{u}} - \dot{\mathbf{u}}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\operatorname{div}\dot{\mathbf{u}})\cdot\mathbf{B}_t\,dx \triangleq \sum_{i=1}^{3}N_i. \qquad (3.19)
\end{aligned}$$

Now, we estimate N_i $(i = 1, 2, 3)$ as follows. By using (2.1), (2.2) and integrating by parts, we have

$$N_1 \le C\xi\|\mathbf{B}_t\|_{L^3}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{B}_t\|_{L^2} \le \frac{\nu}{6}\xi\|\nabla\mathbf{B}_t\|_{L^2}^2 + C\|\mathbf{B}_t\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^4.$$

Due to (2.3), (2.4) and Cauchy–Schwarz's inequality, we have

$$\begin{aligned}
N_2 &= \xi\int\big(u^k\partial_k u^j B^i\partial_i B_t^j + u^k\partial_k u^i\partial_i B^j B_t^j - u^k\partial_k u^i\partial_i B^j B_t^j - u^k\partial_k u^i B^j\partial_i B_t^j\big)\,dx \\
&\le C\xi\|\mathbf{B}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^3}\|\nabla\mathbf{B}_t\|_{L^2} + C\xi\|\nabla\mathbf{B}\|_{L^3}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^3}\|\mathbf{B}_t\|_{L^6} \\
&\le \frac{\nu}{6}\xi\|\nabla\mathbf{B}_t\|_{L^2}^2 + C\xi\|\nabla\mathbf{B}\|_{L^2}^{3/2}\|\nabla^2\mathbf{B}\|_{L^2}^{3/2}\|\nabla\mathbf{u}\|_{L^2}^4 + C\xi\|\nabla\mathbf{u}\|_{L^4}^4,
\end{aligned}$$

and inequality (3.4) gives

$$N_3 \le C\xi \|\mathbf{B}\|_{L^3}\|\nabla \mathbf{B}_t\|_{L^2}\|\nabla \dot{\mathbf{u}}\|_{L^2} \le C\xi m_0^{1/2}\|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + \frac{\nu}{6}\xi \|\nabla \mathbf{B}_t\|_{L^2}^2.$$

Thus, substituting N_1, N_2, N_3 into (3.19), we infer from (3.16) that

$$\begin{aligned}
&\left(\xi \|\mathbf{B}_t\|_{L^2}^2\right)_t + \xi \|\nabla \mathbf{B}_t\|_{L^2}^2 \\
&\quad \le \frac{1}{2}\xi \|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + \left|\xi'\right|\|\mathbf{B}_t\|_{L^2}^2 + C\xi \|\nabla \mathbf{u}\|_{L^2}^4 \|\mathbf{B}_t\|_{L^2}^2 \\
&\qquad + C\xi \|\nabla \mathbf{u}\|_{L^2}^4 \|\nabla \mathbf{B}\|_{L^2}^{3/2}\left\|\nabla^2 \mathbf{B}\right\|_{L^2}^{3/2} + C\xi m_0 \|\nabla \mathbf{B}\|_{L^2}^3 \left\|\nabla^2 \mathbf{B}\right\|_{L^2}^3 \\
&\qquad + C\xi\left(\|\nabla \mathbf{u}\|_{L^2} + m_0^{1/2} + m_0^{1/4}\|\nabla \mathbf{B}\|_{L^2}^{3/2}\right)\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^3 + \left\|\nabla^2 \mathbf{B}\right\|_{L^2}^3\right) + C\xi m_0,
\end{aligned}$$

which together with (3.19) gives (3.14). □

Lemma 3.7 *Under the conditions of Proposition 3.1, it holds that*

$$\sup_{t\in[0,T]} \sigma^2\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \left\|\nabla^2 \mathbf{B}\right\|_{L^2}^2\right) \le C m_0^{1/2}. \tag{3.20}$$

Moreover, for any $0 \le t_1 < t_2 \le T$, *it holds that*

$$\int_{t_1}^{t_2} \sigma^2\left(\|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2\right) dt \le C m_0^{1/2} + C m_0 (t_2 - t_1). \tag{3.21}$$

Proof For any integer $1 \le i \le [T] - 1$, taking $\xi = \sigma_i^2$ with $\sigma_i(t) \triangleq \sigma(t+1-i)$ in (3.14) and integrating it over $(i-1, i+1]$, we obtain from (3.1), (3.3), and (3.13) that

$$\begin{aligned}
&\sup_{t\in[i-1,i+1]} \sigma_i^2\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2\right) + \int_{i-1}^{i+1} \sigma_i^2\left(\|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2\right) dt \\
&\quad \le C m_0 + C\int_{i-1}^{i+1} \sigma_i \sigma_i'\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2\right) dt + C\int_{i-1}^{i+1} \sigma_i^2 \|\nabla \mathbf{u}\|_{L^2}^2 \,dt \\
&\qquad + C\int_{i-1}^{i+1} \sigma_i^2\left(\|\nabla \mathbf{u}\|_{L^2}^4 + \|\nabla \mathbf{B}\|_{L^2}^4\right)\left(\|\mathbf{B}_t\|_{L^2}^2 + \left\|\nabla^2 \mathbf{B}\right\|_{L^2}^2\right) dt \\
&\qquad + C\int_{i-1}^{i+1} \sigma_i^2 \|\nabla \mathbf{u}\|_{L^2}^4 \|\nabla \mathbf{B}\|_{L^2}^{3/2}\left\|\nabla^2 \mathbf{B}\right\|_{L^2}^{3/2} dt + C m_0 \int_{i-1}^{i+1} \sigma_i^2 \|\nabla \mathbf{B}\|_{L^2}^3 \left\|\nabla^2 \mathbf{B}\right\|_{L^2}^3 dt \\
&\qquad + C\int_{i-1}^{i+1} \sigma_i^2\left(\|\nabla \mathbf{u}\|_{L^2} + m_0^{1/2} + m_0^{1/4}\|\nabla \mathbf{B}\|_{L^2}^{3/2}\right)\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^3 + \left\|\nabla^2 \mathbf{B}\right\|_{L^2}^3\right) dt \\
&\quad \le \sup_{t\in[i-1,i+1]}\left(\|\nabla \mathbf{u}\|_{L^2} + m_0^{1/2} + m_0^{1/4}\|\nabla \mathbf{B}\|_{L^2}^{3/2}\right)\int_{i-1}^{i+1} \sigma_i^2\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^3 + \left\|\nabla^2 \mathbf{B}\right\|_{L^2}^3\right) dt \\
&\qquad + C m_0\left(\sup_{t\in[i-1,i+1]} \|\nabla \mathbf{B}\|_{L^2}^3\right)\int_{i-1}^{i+1} \sigma_i^2 \left\|\nabla^2 \mathbf{B}\right\|_{L^2}^3 dt + C m_0^{1/2} \\
&\quad \le C m_0^{1/2} \sup_{t\in[i-1,i+1]} \sigma_i^2\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2} + \left\|\nabla^2 \mathbf{B}\right\|_{L^2}\right) + C m_0^{1/2}.
\end{aligned} \tag{3.22}$$

On the other hand, we obtain from $(1.1)_3$ that

$$\left\|\nabla^2 \mathbf{B}\right\|_{L^2} \le C\|\mathbf{B}_t\|_{L^2} + C\|\nabla \mathbf{u}\|_{L^2}^2 \|\nabla \mathbf{B}\|_{L^2}, \tag{3.23}$$

which together with (3.22) yields (3.20).

Next, to prove (3.21), taking $\beta = \sigma$ in (3.7) and integrating the results over $[t_1, t_2] \subseteq [0, T]$, we deduce from (3.1) and (3.3) that

$$\begin{aligned}
&\int_{t_1}^{t_2} \sigma\big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \|\nabla^2\mathbf{B}\|_{L^2}^2\big)\,dt \\
&\le C \sup_{t\in[t_1,t_2]} \sigma\big(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{B}\|_{L^2}^2\big)^2 \int_{t_1}^{t_2} \sigma\big(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{B}\|_{L^2}^2\big)\,dt \\
&\quad + C\big(m_0 + A_1(T)\big) + \|\mathbf{B}\|_{L^3} A_1(T) + Cm_0 + Cm_0(t_2 - t_1) \\
&\le Cm_0^{1/2} + Cm_0(t_2 - t_1).
\end{aligned} \tag{3.24}$$

Taking $\xi = \sigma^2$ in (3.14) and integrating it over $[t_1, t_2] \subseteq [0, T]$, we deduce from (3.1), (3.3), (3.20), and (3.24) that

$$\begin{aligned}
&\int_{t_1}^{t_2} \sigma^2\big(\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla\mathbf{B}_t\|_{L^2}^2\big)\,dt \\
&\le C \sup_{t\in[t_1,t_2]} \sigma\big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla^2\mathbf{B}\|_{L^2}^2\big)^2 \int_{t_1}^{t_2} \sigma\big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla^2\mathbf{B}\|_{L^2}^2\big)\,dt \\
&\quad + Cm_0^{1/2} + Cm_0(t_2 - t_1) \le Cm_0^{1/2} + Cm_0(t_2 - t_1).
\end{aligned}$$

The proof of Lemma 3.7 is therefore completed. □

Lemma 3.8 *Under the conditions of Proposition 3.1, it holds that*

$$\begin{aligned}
&\sup_{t\in[0,\sigma(T)]} \sigma\big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \|\nabla^2\mathbf{B}\|_{L^2}^2\big) \\
&\quad + \int_0^{\sigma(T)} \sigma\big(\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla\mathbf{B}_t\|_{L^2}^2\big)\,dt \le C.
\end{aligned} \tag{3.25}$$

Proof Taking $\xi = \sigma$ in (3.14) and integrating it over $[0, \sigma(T)]$, we deduce from (3.1), (3.3), and (3.20) that

$$\begin{aligned}
&\sup_{t\in[0,\sigma(T)]} \sigma\big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2\big) + \int_0^{\sigma(T)} \sigma\big(\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla\mathbf{B}_t\|_{L^2}^2\big)\,dt \\
&\le Cm_0 + C\int_0^{\sigma(T)} \sigma'\big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2\big)\,dt + C\int_0^{\sigma(T)} \sigma\|\nabla\mathbf{u}\|_{L^2}^2\,dt \\
&\quad + C\int_0^{\sigma(T)} \sigma\big(\|\nabla\mathbf{u}\|_{L^2}^4 + \|\nabla\mathbf{B}\|_{L^2}^4\big)\big(\|\mathbf{B}_t\|_{L^2}^2 + \|\nabla^2\mathbf{B}\|_{L^2}^2\big)\,dt \\
&\quad + C\int_0^{\sigma(T)} \sigma\|\nabla\mathbf{u}\|_{L^2}^4\|\nabla\mathbf{B}\|_{L^2}^{3/2}\|\nabla^2\mathbf{B}\|_{L^2}^{3/2}\,dt + \int_0^{\sigma(T)} \sigma\|\nabla\mathbf{B}\|_{L^2}^3\|\nabla^2\mathbf{B}\|_{L^2}^3\,dt \\
&\quad + C\int_0^{\sigma(T)} \sigma\big(\|\nabla\mathbf{u}\|_{L^2} + m_0^{1/2} + m_0^{1/4}\|\nabla\mathbf{B}\|_{L^2}^{3/2}\big)\big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^3 + \|\nabla^2\mathbf{B}\|_{L^2}^3\big)\,dt \\
&\le \sup_{t\in[0,\sigma(T)]} \big(\|\nabla\mathbf{u}\|_{L^2} + m_0^{1/2} + m_0^{1/4}\|\nabla\mathbf{B}\|_{L^2}^{3/2}\big) \int_0^{\sigma(T)} \sigma\big(\|\rho^{1/2}\dot{\mathbf{u}}\|_{L^2}^3 + \|\nabla^2\mathbf{B}\|_{L^2}^3\big)\,dt
\end{aligned}$$

$$
\begin{aligned}
&+ Cm_0\Big(\sup_{t\in[0,\sigma(T)]}\|\nabla\mathbf{B}\|_{L^2}^3\Big)\int_0^{\sigma(T)}\sigma\big\|\nabla^2\mathbf{B}\big\|_{L^2}^3\,dt + C\\
&\le \frac{1}{2}\sup_{t\in[0,\sigma(T)]}\sigma\big(\big\|\rho^{1/2}\dot{\mathbf{u}}\big\|_{L^2} + \big\|\nabla^2\mathbf{B}\big\|_{L^2}\big) + C,
\end{aligned}
$$

which together with (3.23) yields (3.24). □

Lemma 3.9 *Let $(\rho,\mathbf{u},\mathbf{B})$ be a smooth solution of (1.1)–(1.5) on $\mathbb{R}^3\times(0,T]$ satisfying (3.1). Then there exists a positive constant $\varepsilon>0$, depending only on μ, λ, ν, a, $\bar\rho$, and M, such that*

$$
0\le\rho(x,t)\le\frac{7}{4}\bar\rho,\quad \forall\,(x,t)\in\mathbb{R}^3\times[0,T), \tag{3.26}
$$

provided $m_0\le\varepsilon$.

Let $D_t\triangleq\partial_t+\mathbf{u}\cdot\nabla$ denote the material derivation operator. Then we can rewrite $(3.1)_1$ as follows:

$$
D_t\rho = g(\rho) + b'(t),
$$

where

$$
g(\rho)\triangleq -\frac{a\rho}{2\mu+\lambda}(\rho-1),\qquad b(t)\triangleq -\frac{1}{2\mu+\lambda}\int_0^t\Big(\rho F+\frac{1}{2}\rho|\mathbf{B}|^2\Big)\,ds.
$$

Obviously, it holds that $g(\infty)=-\infty$. So, to apply Lemma 2.1, we still need to deal with $b(t)$. To do this, we first use (2.4)–(2.6), (3.3), (3.4), (3.12), (3.13), and (3.25) to deduce that for any $0\le t_1\le t_2\le\sigma(T)\le1$,

$$
\begin{aligned}
&\big|b(t_2)-b(t_1)\big|\\
&\le C(\bar\rho)\int_0^{\sigma(T)}\big(\|F\|_{L^\infty}+\|\mathbf{B}\|_{L^\infty}^2\big)\,dt\\
&\le C\int_0^{\sigma(T)}\big(\|F\|_{L^2}^{1/4}\|\nabla F\|_{L^6}^{3/4}+\|\mathbf{B}\|_{L^6}\|\nabla\mathbf{B}\|_{L^6}\big)\,dt\\
&\le C\int_0^{\sigma(T)}\big(\|\nabla\mathbf{u}\|_{L^2}+m_0^{1/2}+\|\mathbf{B}\|_{L^2}^{2/3}\|\mathbf{B}\|_{L^3}^{1/2}\big)^{1/4}\big(\|\nabla\dot{\mathbf{u}}\|_{L^2}+\|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^2}\big)^{3/4}\,dt\\
&\quad + C\int_0^{\sigma(T)}\|\nabla\mathbf{B}\|_{L^2}\big\|\nabla^2\mathbf{B}\big\|_{L^2}\,dt\\
&\le Cm_0^{1/16}\Big(\int_0^{\sigma(T)}\sigma^{-4/5}\,dt\Big)^{5/8}\Big(\int_0^{\sigma(T)}\sigma\|\nabla\dot{\mathbf{u}}\|_{L^2}^2\,dt\Big)^{3/8}\\
&\quad + Cm_0^{1/12}\Big(\int_0^{\sigma(T)}\sigma^{-3/5}\,dt\Big)^{5/8}\Big(\int_0^{\sigma(T)}\sigma\|\nabla\dot{\mathbf{u}}\|_{L^2}^2\,dt\Big)^{3/8}\\
&\quad + Cm_0^{7/64}+Cm_0^{1/2}\Big(\int_0^{\sigma(T)}\big\|\nabla^2\mathbf{B}\big\|_{L^2}^2\,dt\Big)^{1/2}\le C_6m_0^{1/16}.
\end{aligned}
$$

Therefore, for $t \in [0, \sigma(T)]$, we can choose N_0 and N_1 in (2.1) as

$$N_1 = 0, \qquad N_0 = C_6 m_0^{1/16},$$

and $\xi^* = 1$ in (2.2), then we obtain from Lemma 2.1 that

$$\sup_{t\in[0,\sigma(T)]} \|\rho\|_{L^\infty} \le \max\{\bar{\rho}, \xi^*\} + N_0 \le \bar{\rho} + C_6 m_0^{1/16} \le \frac{3\bar{\rho}}{2}, \tag{3.27}$$

provided that

$$m_0 \le \varepsilon_3 \triangleq \left\{\varepsilon_1, \varepsilon_2, \left(\frac{\bar{\rho}}{2C_6}\right)^{16}\right\}.$$

For $t \in [\sigma(T), T]$, we obtain from (2.4)–(2.6), (3.3), (3.4), (3.12), (3.13), and (3.25) that, for all $\sigma(T) \le t_1 < t_2 \le T$,

$$\begin{aligned}
&\left|b(t_2) - b(t_1)\right| \\
&\le C(\bar{\rho}) \int_{t_1}^{t_2} \left(\|F\|_{L^\infty} + \|\mathbf{B}\|_{L^\infty}^2\right) dt \\
&\le \frac{a}{4\mu + 2\lambda}(t_2 - t_1) + C \int_{t_1}^{t_2} \left(\|F\|_{L^2}^{2/3} \|\nabla F\|_{L^6}^2 + \|\mathbf{B}\|_{L^6} \|\nabla \mathbf{B}\|_{L^6}\right) dt \\
&\le \frac{a}{4\mu + 2\lambda}(t_2 - t_1) + C \int_{t_1}^{t_2} \left(\|\nabla \mathbf{u}\|_{L^2} + m_0^{1/3}\right)^{2/3} \left(\|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^2}^2\right) dt \\
&\le \left(\frac{a}{4\mu + 2\lambda} + C_7 m_0^{7/6}\right)(t_2 - t_1) + C_8 m_0^{2/3} \\
&\le \frac{a}{2\mu + \lambda}(t_2 - t_1) + C_8 m_0^{2/3},
\end{aligned}$$

provided that

$$m_0 \le \varepsilon_4 \triangleq \left\{\varepsilon_3, \left(\frac{a}{C_7(4\mu + 2\lambda)}\right)^{6/7}\right\}.$$

Therefore, for $t \in [\sigma(T), T]$, we can choose N_0 and N_1 in (2.1) as

$$N_1 = \frac{a}{2\mu + \lambda}(t_2 - t_1), \qquad N_0 = C_8 m_0^{2/3},$$

and $\xi^* = 2$ in (2.2), then we obtain from Lemma 2.1 that

$$\sup_{t\in[\sigma(T),T]} \|\rho\|_{L^\infty} \le \max\left\{\frac{3}{2}\bar{\rho}, \xi^*\right\} + N_0 \le \frac{3}{2}\bar{\rho} + C_8 m_0^{2/3} \le \frac{7\bar{\rho}}{4}, \tag{3.28}$$

provided that

$$m_0 \le \varepsilon \triangleq \left\{\varepsilon_4, \left(\frac{\bar{\rho}}{4C_8}\right)^{3/2}\right\}. \tag{3.29}$$

The combination of (3.27) with (3.28) yields (3.26).

4 Proof of Theorem 1.1

In this section, we always assume that the initial energy m_0 satisfies (3.29) and the positive constant C may depend on T and g besides μ, λ, ν, a, $\bar{\rho}$, and M.

Lemma 4.1 *Under the conditions of Theorem 1.1, the following estimates hold*:

$$\sup_{t\in[0,\sigma(T)]}\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^2+\|\mathbf{B}_t\|_{L^2}^2+\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2\right)+\int_0^{\sigma(T)}\left(\|\nabla\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla\mathbf{B}_t\|_{L^2}^2\right)\le C. \tag{4.1}$$

Proof Taking $\xi=1$ in (3.14) and integrating the resulting equation over $(0,\sigma(T)]$, we deduce from (3.23) that

$$\begin{aligned}&\sup_{t\in[0,\sigma(T)]}\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^2+\|\mathbf{B}_t\|_{L^2}^2+\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2\right)+\int_0^{\sigma(T)}\left(\|\nabla\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla\mathbf{B}_t\|_{L^2}^2\right)\\&\quad\le\left\|\rho_0^{1/2}g\right\|_{L^2}^2+Cm_0+C\int_0^{\sigma(T)}\left(\|\nabla\mathbf{u}\|_{L^2}^4+\|\nabla\mathbf{B}\|_{L^2}^4\right)\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^2+\|\mathbf{B}_t\|_{L^2}^2\right)dt+C\\&\quad\le C,\end{aligned}$$

which leads to (4.1). □

Lemma 4.2 *Under the conditions of Theorem 1.1, the following estimates hold*:

$$\sup_{t\in[0,T]}\left(\left\|\rho^{1/2}\dot{\mathbf{u}}\right\|_{L^2}^2+\|\mathbf{B}_t\|_{L^2}^2+\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2\right)+\int_0^{T}\left(\|\nabla\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla\mathbf{B}_t\|_{L^2}^2\right)\le C, \tag{4.2}$$

and for any $p\in[2,6]$,

$$\sup_{0\le t\le T}\left(\|\nabla\rho\|_{L^2\cap L^p}+\|\nabla\mathbf{u}\|_{H^1}\right)+\int_0^T\|\nabla\mathbf{u}\|_{L^\infty}\,dt\le C(T). \tag{4.3}$$

Proof Inequality (4.2) can be obtained directly from Lemma 3.7 and Lemma 4.1. Similar to the proof of [10], we can obtain (4.3). □

Proof of Theorem 1.1 To prove Theorem 1.1, by virtue of Proposition 2.1, we know that there exists a positive time $T_*>0$ such that problem (1.1)–(1.5) possesses a strong solution $(\rho,\mathbf{u},\mathbf{B})$ in $\mathbb{R}^3\times(0,T_*]$. Next, with all the a priori estimates established in Sect. 3, we will extend the local strong solution to all time.

First, in view of the definitions of $A_1(T)$ and $A_2(T)$, it is easily deduced from (3.1) that

$$A_1(0)\le m_0^{1/2},\qquad A_2(0)\le 2K,\quad 0\le\rho\le\frac{7}{4}\bar{\rho},$$

due to $m_0\le\varepsilon$. Thus, there exists a time $T_1\in(0,T_*]$ such that (3.1) holds for $T=T_1$.

Set

$$T^*\triangleq\sup\left\{T|(\rho,\mathbf{u},\mathbf{B})\text{ is a strong solution on }[0,T]\right\}$$

and

$$T_1^*\triangleq\sup\left\{T|(\rho,\mathbf{u},\mathbf{B})\text{ is a strong solution on }[0,T]\text{ satisfying (3.1)}\right\}.$$

Thus, $T_1^* \geq T_1 > 0$. By virtue of Proposition 3.1, we know

$$T^* = T_1^*,$$

provided $m_0 \leq \varepsilon$.

Next, similar to the proof of [10, Sect. 4], we can claim that $T^* = \infty$. Thus, the proof of Theorem 1.1 is complete. □

Acknowledgements
The author would like to thank the anonymous referee for his/her helpful comments, which improved the presentation of the paper.

Funding
This research is partially supported by the Natural Science Foundation of Shandong Province of China (Grant No. ZR2021QA049) and the Science and Technology Project of Weifang (2022GX006).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Mingyu Zhang wrote the full manuscript text and reviewed the manuscript.

Received: 21 December 2022 Accepted: 28 March 2023 Published online: 05 April 2023

References

1. Cabannes, H.: Theoretical Magnetofluiddynamics. Academic Press, New York (1970)
2. Ducomet, B., Feireisl, E.: The equations of magnetohydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars. Commun. Math. Phys. **226**, 595–629 (2006)
3. Fan, J., Jiang, S., Nakamura, G.: Vanishing shear viscosity limit in the magnetohydrodynamic equations. Commun. Math. Phys. **270**, 691–708 (2007)
4. Hoff, D.: Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data. Trans. Am. Math. Soc. **303**(1), 169–181 (1987)
5. Hoff, D.: Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data. Trans. Am. Math. Soc. **303**(1), 169–181 (1987)
6. Hoff, D.: Spherically symmetric solutions of the Navier-Stokes equations for compressible, isothermal flow with large, discontinuous initial data. Indiana Univ. Math. J. **41**(4), 1225–1302 (1992)
7. Hoff, D., Tsyganov, E.: Uniqueness and continuous dependence of weak solutions in compressible magnetohydrodynamics. Z. Angew. Math. Phys. **56**, 791–804 (2005)
8. Hu, X., Wang, D.: Global solutions to the three-dimensional full compressible magnetohydrodynamic flows. Commun. Math. Phys. **283**, 255–284 (2008)
9. Hu, X., Wang, D.: Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows. Arch. Ration. Mech. Anal. **197**, 203–238 (2010)
10. Huang, X.D., Li, J., Xin, Z.P.: Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations. Commun. Pure Appl. Math. **65**, 549–585 (2012)
11. Itaya, N.: On the Cauchy problem for the system of fundamental equations describing the movement of compressible viscous fluid. Kodai Math. J. **23**(1), 60–120 (1971)
12. Kazhikhov, A.V., Shelukhin, V.V.: Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. Prikl. Mat. Meh. **41**, 282–291 (1977)
13. Kulikovskiy, A.G., Lyubimov, G.A.: Magnetohydrodynamics. Addison-Wesley, Reading (1965)
14. Ladyzhenskaya, O.A.: The Mathematical Theory of Viscous Incompressible Flow. Mathematics and Its Applications, vol. 2, p. xviii + 224 pp. Gordon and Breach, New York (1969). Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu
15. Li, H.L., Xu, X.Y., Zhang, J.W.: Global classical solutions to 3D compressible magnetohydrodynamic equations with large oscillations and vacuum. SIAM J. Math. Anal. **45**(3), 1356–1387 (2013)
16. Matsumura, A., Nishida, T.: The initial boundary value problem for the equations of motion of compressible viscous and heat conductive fluids. Proc. Jpn. Acad., Ser. A, Math. Sci. **55**, 337–342 (1979)
17. Matsumura, A., Nishida, T.: The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. **20**(1), 67–104 (1980)
18. Matsumura, A., Nishida, T.: Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. Commun. Math. Phys. **89**, 445–464 (1983)

19. Nash, J.: Le problème de Cauchy pour les équations différentielles d'un fluide général. Bull. Soc. Math. Fr. **90**, 487–497 (1962)
20. Nishida, T.: Global solution for an initial-boundary value problem of a quasilinear hyperbolic systems. Proc. Jpn. Acad. **44**, 642–646 (1968)
21. Serre, D.: Solutions faibles globales des équations de Navier-Stokes pour un fluide compressible. C. R. Acad. Sci., Sér. 1 Math. **303**, 639–642 (1986)
22. Serre, D.: Sur l'équation monodimensionnelle d'un fluide visqueux, compressible et conducteur de chaleur. C. R. Acad. Sci., Sér. 1 Math. **303**, 703–706 (1986)
23. Serrin, J.: On the uniqueness of compressible fluid motion. Arch. Ration. Mech. Anal. **3**, 271–288 (1959)
24. Tani, A.: On the first initial-boundary value problem of compressible viscous fluid motion. Publ. Res. Inst. Math. Sci. **13**(1), 193–253 (1977)
25. Yu, H.B.: The global existence of strong solutions to the 3D compressible isothermal Navier-Stokes equations. Acta Math. Sci. **42B**(1), 233–246 (2022)
26. Zlotnik, A.A.: Uniform estimates and stabilization of symmetric solutions of a system of quasilinear equations. Differ. Equ. **36**, 701–716 (2000)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.