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Hermite–Hadamard-type inequalities via different convexities with applications

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Abstract

In this paper, we explore a class of Hermite–Hadamard integral inequalities for convex and *m*-convex functions. The Hölder inequality is used to create this class, which has a wide range of applications in optimization theory. Some trapezoid-type inequalities and midpoint error estimates are investigated. Inequalities for several *q*-special functions are highlighted. As particular cases, we have included several previous results.

MSC: Primary 26D15; secondary 26A51; 26A33; 05A30

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1 Introduction

In the theory of inequalities, convex functions are very important. The concept of convexity has been expanded and summarized in various ways. For example, Wu et al. [1] used the convexity to find estimates of upper bounds, Hu et al. [2] applied the concept of convexity via local fractional integral, Awan et al. [3] introduced a class of M-convex functions and discussed its properties, Samraiz et al. [4] explored mean type inequalities via different convexities. In continuation, we found plenty of papers [5–10] that have strong applications of theory of convexity. The convexity is also important to deal with nonlinear problems [11]. A generalization of convex functions to a real-valued functions defined on any real linear space is fairly natural [12]. In pure and applied mathematics, convex functions emerge in numerous problems. The idea of convexity is essential in studying both linear and nonlinear programming issues. Convexity has endless uses in industry, business, medicine, and art, which have a significant impact on our daily lives [13, 14]. Convex functions can be applied to solving problems in management, economics, and, in fact, our everyday lives. For further literature review, we refer the reader to [15–17].

Definition 1.1 A function $\psi : J \to \mathbb{R}$, where J is an interval on the real line, is said to be convex if for any two points ς and τ in J and for any $0 \le s \le 1$,

$$\psi\big(s\varsigma + (1-s)\tau\big) \le s\psi(\varsigma) + (1-s)\psi(\tau).$$

In nonlinear analysis the Hermite–Hadamard inequality is significant. This idea of inequalities has been applied in a variety of ways [18–20].

Definition 1.2 Let $\psi : J \to \mathbb{R}$ be a convex mapping, where $J \subseteq \mathbb{R}$. For any two points ς and τ in J with $\varsigma < \tau$, we have

$$\psi\left(\frac{\varsigma+\tau}{2}\right) \le \frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\,dz \le \frac{\psi(\varsigma)+\psi(\tau)}{2}.$$

If ψ is concave, then the inequalities are reversed.

Toader [21] (also see [22]) introduced the concept of an m-convex function as follows.

Definition 1.3 Let $\psi : [0,a] \to \mathbb{R}$ be an m-convex mapping, where $m \in (0,1]$. For any two points ς and τ in $[0,a]$ with $s \in [0,1]$, we have

$$\psi\big(s\varsigma + m(1-s)\tau\big) \le s\psi(\varsigma) + m(1-s)\psi(\tau).$$

Definition 1.4 ([23]) The beta function is a special function closely related to the gamma function and binomial coefficients. It is defined by the integral

$$B(y,z) = \int_0^1 s^{y-1}(1-s)^{z-1}\,ds, \quad \Re(z) > 0, \Re(y) > 0.$$

To establish the main results, we need some lemmas. The first following lemma is given in [13].

Lemma 1.5 *Let $J \subseteq \mathbb{R}$, and let $\psi : J \to \mathbb{R}$ be a differentiable function on J^0. If ς and τ are any two points in J with $\varsigma < \tau$, then we have*

$$\frac{\psi(\varsigma)+\psi(\tau)}{2} - \frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\,dz = \frac{(\tau-\varsigma)^2}{2}\int_0^1 s(1-s)\psi''\big(s\varsigma+(1-s)\tau\big)\,ds.$$

Kirmaci et al. [24] proved the following lemma.

Lemma 1.6 *Let $\psi : J \to \mathbb{R}$ be a differentiable function on J^0, and let $J \subseteq \mathbb{R}$. If ς and τ are any two points in J^0 with $\varsigma < \tau$, then we have*

$$\begin{aligned}&\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\,dz - \psi\left(\frac{\varsigma+\tau}{2}\right)\\&\quad= (\tau-\varsigma)\left[\int_0^{\frac{1}{2}} s\psi'\big(\tau+(\varsigma-\tau)s\big)\,ds + \int_{\frac{1}{2}}^1 (s-1)\psi'\big(\tau+(\varsigma-\tau)s\big)\,ds\right].\end{aligned}$$

The proof of the next lemma can be found in [13].

Lemma 1.7 *Let $\psi : J \to \mathbb{R}$ be a differentiable mapping on J^0, where $J \subseteq \mathbb{R}$ and $\varsigma < \tau$. If ψ is convex, then we have the following inequalities*:

$$\psi\left(\frac{\varsigma+\tau}{2}\right) \le \frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\,dz \le \frac{1}{4}\left[2\psi\left(\frac{\varsigma+\tau}{2}\right) + \psi\left(\frac{3\tau-\varsigma}{2}\right) + \psi\left(\frac{3\varsigma-\tau}{2}\right)\right]$$

and

$$\left|\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\,dz-\frac{\psi(\frac{\varsigma+\tau}{2})}{2}\right|\leq\left|\frac{1}{4}\left[\psi\left(\frac{3\tau-\varsigma}{2}\right)+\psi\left(\frac{3\varsigma-\tau}{2}\right)\right]\right|.$$

This paper is organized as follows. In Sect. 2, we establish Hermite–Hadamard-type inequalities via m-convex functions fruitfully applying the Hölder inequality. Some applications to special means of real numbers are discussed in Sect. 3. In the last section, we conclude about the findings of all previous sections.

2 Main results

This section is devoted to Hermite–Hadamard-type inequalities via m-convex functions fruitfully applying the Hölder inequality. To derive the results of this section, we use the definition and properties of the beta function. The first main result of this section is as follows.

Theorem 2.1 *Let $\psi : J \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on J^0, let $\varsigma, \tau \in J^0$ with $\varsigma < \tau$, and let $\varpi > 1$. If $|\psi'|^{\varrho}$ is m-convex on $[\varsigma, \tau]$ and $\frac{1}{\varpi} + \frac{1}{\varrho} = 1$, then*

$$\begin{aligned}&\left|\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\,dz-\psi\left(\frac{\varsigma+\tau}{2}\right)\right|\\&\quad\leq\frac{\tau-\varsigma}{16}\left(\frac{4}{\varpi+1}\right)^{\frac{1}{\varpi}}\left[\left(\left|\psi'(\varsigma)\right|^{\varrho}+3m\left|\psi'\left(\frac{\tau}{m}\right)\right|^{\varrho}\right)^{\frac{1}{\varrho}}\right.\\&\qquad\left.+\left(3\left|\psi'(\varsigma)\right|^{\varrho}+m\left|\psi'\left(\frac{\tau}{m}\right)\right|^{\varrho}\right)^{\frac{1}{\varrho}}\right].\end{aligned}\tag{2.1}$$

Proof Using Lemma 1.6 and the Hölder inequality, we deduce

$$\begin{aligned}&\left|\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\,dz-\psi\left(\frac{\varsigma+\tau}{2}\right)\right|\\&\quad\leq(\tau-\varsigma)\left[\left(\int_0^{\frac{1}{2}}s^{\varpi}\,ds\right)^{\frac{1}{\varpi}}\left(\int_0^{\frac{1}{2}}\left|\psi'\big(s\varsigma+(1-s)\tau\big)\right|^{\varrho}\,ds\right)^{\frac{1}{\varrho}}\right.\\&\qquad\left.+\left(\int_{\frac{1}{2}}^{1}|s-1|^{\varpi}\right)^{\frac{1}{\varpi}}\left(\int_{\frac{1}{2}}^{1}\left|\psi'\big(s\varsigma+(1-s)\tau\big)\right|^{\varrho}\,ds\right)^{\frac{1}{\varrho}}\right].\end{aligned}\tag{2.2}$$

From the m-convexity of $|\psi'|^{\varrho}$ we get

$$\int_0^{\frac{1}{2}}s\left|\psi'\big(s\varsigma+(1-s)\tau\big)\right|^{\varrho}\,ds\leq\frac{1}{8}\left[\left|\psi'(\varsigma)\right|^{\varrho}+3m\left|\psi'\left(\frac{\tau}{m}\right)\right|^{\varrho}\right].\tag{2.3}$$

Similarly,

$$\int_{\frac{1}{2}}^{1}\left|\psi'\big(s\varsigma+(1-s)\tau\big)\right|^{\varrho}\,ds\leq\frac{1}{8}\left[3\left|\psi'(\varsigma)\right|^{\varrho}+m\left|\psi'\left(\frac{\tau}{m}\right)\right|^{\varrho}\right].\tag{2.4}$$

Using relations (2.3) and (2.4) in (2.2), by simple calculations we obtain the desired result. □

Remark 2.2 If we set $m = 1$ in Theorem 2.1, then we get [14, Theorem 2.3].

Theorem 2.3 *Let $\psi : J \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on J^0, let $\varsigma, \tau \in J^0$ with $\varsigma < \tau$, and let $\varpi > 1$. If the mapping $|\psi'|^{\varrho}$ is m-convex on $[\varsigma, \tau]$ and $\frac{1}{\varpi} + \frac{1}{\varrho} = 1$, then*

$$\begin{aligned} &\left|\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\,dz - \psi\left(\frac{\varsigma+\tau}{2}\right)\right| \\ &\quad \le \frac{\tau-\varsigma}{4}\left(\frac{4}{\varpi+1}\right)^{\frac{1}{\varpi}}\left(\left|\psi'(\varsigma)\right| + m\left|\psi'\left(\frac{\tau}{m}\right)\right|\right). \end{aligned} \tag{2.5}$$

Proof By using Theorem 2.1 with substitutions

$$\begin{aligned} &\varsigma_1 = \left|\psi'(\varsigma)\right|^{\varrho}, \qquad \tau_1 = 3m\left|\psi'\left(\frac{\tau}{m}\right)\right|^{\varrho}, \\ &\varsigma_2 = 3\left|\psi'(\varsigma)\right|^{\varrho}, \qquad \tau_2 = m\left|\psi'\left(\frac{\tau}{m}\right)\right|^{\varrho}, \end{aligned}$$

where $0 \le \frac{1}{\varrho} < 1$ for $\varpi > 1$, and using the inequality

$$\sum_{p=1}^{n}(\varsigma_p + \tau_p)^s \le \sum_{p=1}^{n}\varsigma_p^s + \sum_{p=1}^{n}\tau_p^s$$

for $\varsigma_i, \tau_i \ge 0$ $(i = 1, 2, \dots, n)$ and $0 \le s < 1$, we obtain inequality (2.5). □

Remark 2.4 If we set $m = 1$ in Theorem 2.3, then we get [14, Theorem 2.4].

Theorem 2.5 *Let $\psi : J \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on J^0, let $\varsigma, \tau \in J^0$ with $\varsigma < \tau$, and let $m\tau \neq \varsigma$. If ψ'' is an m-convex function, then*

$$\begin{aligned} &\frac{\psi(\varsigma) + m\psi(\tau)}{2} - \frac{1}{(m\tau-\varsigma)}\int_{\varsigma}^{m\tau}\psi(z)\,dz \\ &\quad = \frac{(m\tau-\varsigma)^2}{2}\int_0^1 s(1-s)\psi''\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds \\ &\quad \le \frac{(m\tau-\varsigma)^2}{24}\bigl[\psi''(\varsigma) + m\psi''(\tau)\bigr]. \end{aligned} \tag{2.6}$$

Proof Consider the middle part of (2.6), i.e.,

$$I := \frac{(m\tau-\varsigma)^2}{2}\int_0^1 s(1-s)\psi''\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds.$$

Integrating by parts, we get

$$I = -\frac{(m\tau-\varsigma)^2}{2}\int_0^1 \frac{(1-2s)\psi'(s\varsigma + m(1-s)\tau)}{\varsigma - m\tau}\,ds.$$

Again integrating by parts, we obtain

$$I = \frac{\psi(\varsigma) + m\psi(\tau)}{2} - \int_0^1 \psi\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds.$$

Using the change of variable $z = s\varsigma + m(1-s)\tau$, where $s \in [0,1]$, we get

$$\frac{\psi(\varsigma) + m\psi(\tau)}{2} - \frac{1}{(m\tau - \varsigma)} \int_{\varsigma}^{m\tau} \psi(z)\,dz = \frac{(m\tau - \varsigma)^2}{2} \int_0^1 s(1-s)\psi''\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds. \tag{2.7}$$

Again considering the middle part of (2.6) and using the m-convexity of ψ'', we get

$$\frac{(m\tau - \varsigma)^2}{2} \int_0^1 s(1-s)\psi''\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds \le \frac{(\tau m - \varsigma)^2}{24}\bigl[\psi''(\varsigma) + m\psi''(\tau)\bigr]. \tag{2.8}$$

Combining (2.7) and (2.8), we achieve (2.6). □

Theorem 2.6 *Let $\psi : J \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on J^0, and let $\varsigma, \tau \in J^0$ with $\varsigma < \tau$. If ψ' is an m-convex function, then*

$$\begin{aligned}
&\frac{1}{m\tau - \varsigma} \int_{\varsigma}^{m\tau} \psi(z)\,dz - \psi\left(\frac{\varsigma + m\tau}{2}\right) \\
&\quad = (m\tau - \varsigma)\left[\int_0^{\frac{1}{2}} s\psi'\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds + \int_{\frac{1}{2}}^1 (s-1)\psi'\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds\right] \\
&\quad \le \frac{(\tau m - \varsigma)^2}{8}\bigl[\psi'(\varsigma) + m\psi'(\tau)\bigr].
\end{aligned} \tag{2.9}$$

Proof Consider the middle part of (2.9), i.e.,

$$I := (m\tau - \varsigma)\left[\int_0^{\frac{1}{2}} s\psi'\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds + \int_{\frac{1}{2}}^1 (s-1)\psi'\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds\right].$$

Integrating by parts, we get

$$I = -\psi\left(\frac{\varsigma + m\tau}{2}\right) + \int_0^{\frac{1}{2}} \psi\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds + \int_{\frac{1}{2}}^1 \psi\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds.$$

Using the change of variable $z = s\varsigma + m(1-s)\tau$, where $s \in [0,1]$, we get

$$\begin{aligned}
&\frac{1}{m\tau - \varsigma} \int_{\varsigma}^{m\tau} \psi(z)\,dz - \psi\left(\frac{\varsigma + m\tau}{2}\right) \\
&\quad = (m\tau - \varsigma)\left[\int_0^{\frac{1}{2}} s\psi'\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds + \int_{\frac{1}{2}}^1 (s-1)\psi'\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds\right].
\end{aligned} \tag{2.10}$$

Again considering the middle part of (2.9) and using the m-convexity of ψ', we get

$$\begin{aligned}
&(m\tau - \varsigma)\left[\int_0^{\frac{1}{2}} s\psi'\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds + \int_{\frac{1}{2}}^1 (s-1)\psi'\bigl(s\varsigma + m(1-s)\tau\bigr)\,ds\right] \\
&\quad \le \frac{(\tau m - \varsigma)^2}{8}\bigl[\psi'(\varsigma) + m\psi'(\tau)\bigr].
\end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11), we achieve (2.9). □

Theorem 2.7 *Let $\psi : J \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on J^0, let $\varsigma, \tau \in J^0$ with $\varsigma < \tau$, and $\psi' \in C[\frac{3\varsigma-\tau}{2}, \frac{3\tau-\varsigma}{2}]$ be such that $\psi'(z) \in \mathbb{R}$ for all $z \in (\frac{3\varsigma-\tau}{2}, \frac{3\tau-\varsigma}{2})$. If $\varrho \geq 1$ and $|\psi'|^{\varrho}$ is an m-convex mapping on $[\frac{3\varsigma-\tau}{2}, \frac{3\tau-\varsigma}{2}]$, then we have the following inequality*:

$$\begin{aligned}
&\left|\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\,dz-\psi\left(\frac{\varsigma+\tau}{2}\right)\right| \\
&\quad \leq \frac{\tau-\varsigma}{8}\left(\left|\psi'\left(\frac{3\varsigma-\tau}{2}\right)\right|^{\varrho}+m\left|\psi'\left(\frac{3\tau-\varsigma}{2m}\right)\right|^{\varrho}\right)^{\frac{1}{\varrho}}.
\end{aligned}$$

Proof By Lemma 1.6 we have

$$\begin{aligned}
&\frac{1}{2(\tau-\varsigma)}\int_{\frac{3\varsigma-\tau}{2}}^{\frac{3\tau-\varsigma}{2}}\psi(z)\,dz-\psi\left(\frac{\varsigma+\tau}{2}\right) \\
&\quad = 2(\tau-\varsigma)\left(\int_0^{\frac{1}{2}} s\psi'\left(\frac{3\tau-\varsigma}{2}+2(\varsigma-\tau)s\right)ds\right. \\
&\qquad \left.+\int_{\frac{1}{2}}^{1}(s-1)\psi'\left(\frac{3\tau-\varsigma}{2}+2(\varsigma-\tau)s\right)ds\right).
\end{aligned}$$

By Lemma 1.7 we obtain

$$\begin{aligned}
&\left|\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\,dz-\psi\left(\frac{\varsigma+\tau}{2}\right)\right| \\
&\quad \leq (\tau-\varsigma)\left(\int_0^{\frac{1}{2}} s\left|\psi'\left(\frac{3\tau-\varsigma}{2}+2(\tau-\varsigma)s\right)\right|ds\right. \\
&\qquad \left.+\int_{\frac{1}{2}}^{1}(1-s)\left|\psi'\left(\frac{3\tau-\varsigma}{2}+2(\tau-\varsigma)s\right)\right|ds\right).
\end{aligned} \tag{2.12}$$

We consider the following two cases.

(i) For $\varrho = 1$, using the m-convexity of $|\psi'|$ on $[\frac{3\varsigma-\tau}{2}, \frac{3\tau-\varsigma}{2}]$ with $s \in [0,1]$, we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{2}} s\left|\psi'\left(\frac{3\tau-\varsigma}{2}+2(\varsigma-\tau)s\right)\right|ds \\
&\quad = \int_0^{\frac{1}{2}} s\left|\psi'\left(s\left(\frac{3\varsigma-\tau}{2}\right)+m(1-s)\left(\frac{3\tau-\varsigma}{2m}\right)\right)\right|ds \\
&\quad \leq \frac{|\psi'(\frac{3\varsigma-\tau}{2})|+2m|\psi'(\frac{3\tau-\varsigma}{2m})|}{24}.
\end{aligned} \tag{2.13}$$

Similarly, we get

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}(1-s)\left|\psi'\left(\frac{3\tau-\varsigma}{2}+2(\tau-\varsigma)s\right)\right|ds \\
&\quad = \int_{\frac{1}{2}}^{1}(1-s)\left|\psi'\left(s\left(\frac{3\varsigma-\tau}{2}\right)+m(1-s)\left(\frac{3\tau-\varsigma}{2m}\right)\right)\right|ds \\
&\quad \leq \frac{2|\psi'(\frac{3\varsigma-\tau}{2})|+m|\psi'(\frac{3\tau-\varsigma}{2m})|}{24}.
\end{aligned} \tag{2.14}$$

Substituting inequalities (2.13) and (2.14) into (2.12), we get

$$\begin{aligned}&\left|\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\,dz-\psi\left(\frac{\varsigma+\tau}{2}\right)\right|\\&\quad\leq\frac{\tau-\varsigma}{8}\left(\left|\psi'\left(\frac{3\varsigma-\tau}{2}\right)\right|+m\left|\psi'\left(\frac{3\tau-\varsigma}{2m}\right)\right|\right).\end{aligned}$$

(ii) Now suppose that $\varrho>1$. Using the Hölder inequality for $\varrho>1$ and $\varpi=\frac{\varrho}{\varrho-1}$, we get

$$\begin{aligned}&\int_0^{\frac{1}{2}}s\left|\psi'\left(\frac{3\tau-\varsigma}{2}+2(\varsigma-\tau)s\right)\right|ds\\&\quad=\int_0^{\frac{1}{2}}s\left|\psi'\left(s\left(\frac{3\varsigma-\tau}{2}\right)+(1-s)\left(\frac{3\tau-\varsigma}{2}\right)\right)\right|ds\\&\quad=\int_0^{\frac{1}{2}}s^{1-\frac{1}{\varrho}}\left(s^{\frac{1}{\varrho}}\left|\psi'\left(s\left(\frac{3\varsigma-\tau}{2}\right)+(1-s)\left(\frac{3\tau-\varsigma}{2}\right)\right)\right|\right)ds\\&\quad\leq\left(\int_0^{\frac{1}{2}}s\,ds\right)^{1-\frac{1}{\varrho}}\left(\int_0^{\frac{1}{2}}s\left|\psi'\left(s\left(\frac{3\varsigma-\tau}{2}\right)+(1-s)\left(\frac{3\tau-\varsigma}{2}\right)\right)\right|^{\varrho}ds\right)^{\frac{1}{\varrho}}\\&\quad\leq\left(\frac{1}{8}\right)^{1-\frac{1}{\varrho}}\left(\int_0^{\frac{1}{2}}s\left|\psi'\left(s\left(\frac{3\varsigma-\tau}{2}\right)+m(1-s)\left(\frac{3\tau-\varsigma}{2m}\right)\right)\right|^{\varrho}ds\right)^{\frac{1}{\varrho}}\\&\quad\leq\left(\frac{1}{8}\right)^{1-\frac{1}{\varrho}}\left(\frac{|\psi'(\frac{3\varsigma-\tau}{2})|^{\varrho}+2m|\psi'(\frac{3\tau-\varsigma}{2m})|^{\varrho}}{24}\right)^{\frac{1}{\varrho}}.\end{aligned}\tag{2.15}$$

In the same way, we get

$$\begin{aligned}&\int_{\frac{1}{2}}^{1}(1-s)\left|\psi'\left(\frac{3\tau-\varsigma}{2}+2(\tau-\varsigma)s\right)\right|ds\\&\quad\leq\left(\frac{1}{8}\right)^{1-\frac{1}{\varrho}}\left(\frac{2|\psi'(\frac{3\varsigma-\tau}{2})|^{\varrho}+m|\psi'(\frac{3\tau-\varsigma}{2m})|^{\varrho}}{24}\right)^{\frac{1}{\varrho}}.\end{aligned}\tag{2.16}$$

So inequalities (2.12), (2.15), and (2.16) prove the theorem. □

Remark 2.8 If we set $m=1$ in Theorem 2.7, then we get [13, Theorem 1].

Theorem 2.9 *Let $\psi:J\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on J^0, let $\varsigma,\tau\in J^0$ with $\varsigma<\tau$, and let $\psi'\in C[\frac{3\varsigma-\tau}{2},\frac{3\tau-\varsigma}{2}]$ be such that $\psi'(z)\in\mathbb{R}$ for all $z\in(\frac{3\varsigma-\tau}{2},\frac{3\tau-\varsigma}{2})$. If $\varrho>1$ and $|\psi'|^{\varrho}$ is an m-convex mapping on $[\frac{3\varsigma-\tau}{2},\frac{3\tau-\varsigma}{2}]$, then we have the following inequality*:

$$\begin{aligned}&\left|\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\,dz-\psi\left(\frac{\varsigma+\tau}{2}\right)\right|\\&\quad\leq(\tau-\varsigma)\left(\frac{1}{(\varpi+1)2^{\varpi+1}}\right)^{\frac{1}{\varpi}}\left(\frac{|\psi'(\frac{3\varsigma-\tau}{2})|^{\varrho}+m|\psi'(\frac{3\tau-\varsigma}{2m})|^{\varrho}}{2}\right)^{\frac{1}{\varrho}}\end{aligned}\tag{2.17}$$

with $\frac{1}{\varpi}+\frac{1}{\varrho}=1$.

Proof By the Hölder inequality we have

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} s\left|\psi'\left(s\left(\frac{3\varsigma-\tau}{2}\right)+(1-s)\left(\frac{3\tau-\varsigma}{2}\right)\right)\right| ds \\
&\leq \left(\int_0^{\frac{1}{2}} s^{\varpi}\, ds\right)^{\frac{1}{\varpi}}\left(\int_0^{\frac{1}{2}}\left|\psi'\left(s\left(\frac{3\varsigma-\tau}{2}\right)+(1-s)\left(\frac{3\tau-\varsigma}{2}\right)\right)\right|^{\varrho} ds\right)^{\frac{1}{\varrho}} \\
&= \left(\int_0^{\frac{1}{2}} s^{\varpi}\, ds\right)^{\frac{1}{\varpi}}\left(\int_0^{\frac{1}{2}}\left|\psi'\left(s\left(\frac{3\varsigma-\tau}{2}\right)+m(1-s)\left(\frac{3\varsigma-\tau}{2m}\right)\right)\right|^{\varrho} ds\right)^{\frac{1}{\varrho}} \\
&\leq \left(\frac{1}{(\varpi+1)2^{\varpi+1}}\right)^{\frac{1}{\varpi}} \\
&\quad\times\left(\left|\psi'\left(\frac{3\varsigma-\tau}{2}\right)\right|^{\varrho}\int_0^{\frac{1}{2}} s\, ds+m\left|\psi'\left(\frac{3\tau-\varsigma}{2m}\right)\right|^{\varrho}\int_0^{\frac{1}{2}}(1-s)\, ds\right)^{\frac{1}{\varrho}} \\
&\leq \left(\frac{1}{2^{\varpi+1}(\varpi+1)}\right)^{\frac{1}{\varpi}}\left(\frac{|\psi'(\frac{3\varsigma-\tau}{2})|^{\varrho}+3m|\psi'(\frac{3\tau-\varsigma}{2m})|^{\varrho}}{8}\right)^{\frac{1}{\varrho}}.
\end{aligned}
\tag{2.18}
$$

Similarly, we get

$$
\begin{aligned}
&\int_{\frac{1}{2}}^{1}(1-s)\left|\psi'\left(s\left(\frac{3\tau-\varsigma}{2}\right)+(1-s)\left(\frac{3\tau-\varsigma}{2}\right)\right)\right| ds \\
&\leq \left(\frac{1}{(\varpi+1)2^{\varpi+1}}\right)^{\frac{1}{\varpi}}\left(\frac{3|\psi'(\frac{3\varsigma-\tau}{2})|^{\varrho}+m|\psi'(\frac{3\tau-\varsigma}{2m})|^{\varrho}}{8}\right)^{\frac{1}{\varrho}}.
\end{aligned}
\tag{2.19}
$$

Thus by combining inequalities (2.18) and (2.19) we get the required result. □

Remark 2.10 If we set $m = 1$ in Theorem 2.9, then we get [13, Theorem 2].

Corollary 2.11 *Under the assumption? of Theorems 2.7 and 2.9, we obtain the following inequality for $\varrho > 1$:*

$$
\begin{aligned}
&\left|\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\, dz-\psi\left(\frac{\varsigma+\tau}{2}\right)\right| \\
&\leq \min\{K_1,K_2\}(\tau-\varsigma)\left(\left|\psi'\left(\frac{3\varsigma-\tau}{2}\right)\right|^{\varrho}+m\left|\psi'\left(\frac{3\tau-\varsigma}{2m}\right)\right|^{\varrho}\right)^{\frac{1}{\varrho}},
\end{aligned}
\tag{2.20}
$$

where $K_1=\frac{1}{8}$, $K_2=(\frac{1}{(\varpi+1)2^{\varpi+1+\frac{1}{\varpi\varrho}}})^{\frac{1}{\varpi}}$, *and* $\frac{1}{\varpi}+\frac{1}{\varrho}=1$.

Theorem 2.12 *Let $\psi: J\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function such that ψ'' exists on J^0, let $\varsigma,\tau\in J^0$ with $\varsigma<\tau$, and let $\psi'':[\frac{3\varsigma-\tau}{2},\frac{3\tau-\varsigma}{2}]\to\mathbb{R}$ be a continuous function. If $\varrho\geq 1$ and $|\psi''|^{\varrho}$ is an m-convex function on $[\frac{3\varsigma-\tau}{2},\frac{3\tau-\varsigma}{2}]$, then we have the following inequality:*

$$
\begin{aligned}
&\left|\frac{1}{(\tau-\varsigma)}\int_{\varsigma}^{\tau}\psi(z)\, dz-\frac{\psi(\frac{3\varsigma-\tau}{2})+\psi(\frac{3\tau-\varsigma}{2})+2\psi(\frac{\varsigma+\tau}{2})}{4}\right| \\
&\leq \frac{(\tau-\varsigma)^2}{3}\left(\frac{|\psi''(\frac{3\varsigma-\tau}{2})|^{\varrho}+m|\psi''(\frac{3\tau-\varsigma}{2m})|^{\varrho}}{2}\right)^{\frac{1}{\varrho}}.
\end{aligned}
\tag{2.21}
$$

Proof By Lemma 1.5 we have

$$\begin{aligned}
&\frac{1}{2(\tau-\varsigma)}\int_{\frac{3\varsigma-\tau}{2}}^{\frac{3\tau-\varsigma}{2}}\psi(z)\,dz \\
&\quad=\frac{\psi(\frac{3\tau-\varsigma}{2})+\psi(\frac{3\varsigma-\tau}{2})}{2} \\
&\qquad-2(\tau-\varsigma)^2\int_0^1 s(1-s)\psi''\left(s\left(\frac{3\varsigma-\tau}{2}\right)+(1-s)\left(\frac{3\tau-\varsigma}{2}\right)\right)ds. \qquad (2.22)
\end{aligned}$$

Thus by applying Lemma 1.7 to (2.22), we obtain

$$\begin{aligned}
&\left|\frac{1}{\tau-\varsigma}\int_\varsigma^\tau\psi(z)\,dz-\frac{\psi(\frac{3\varsigma-\tau}{2})+\psi(\frac{3\tau-\varsigma}{2})+2\psi(\frac{\varsigma+\tau}{2})}{4}\right| \\
&\quad\le 2(\tau-\varsigma)^2\int_0^1 s(1-s)\left|\psi''\left(s\left(\frac{3\varsigma-\tau}{2}\right)+(1-s)\left(\frac{3\tau-\varsigma}{2}\right)\right)\right|ds. \qquad (2.23)
\end{aligned}$$

In the case $\varrho=1$ the function $|\psi''|$ is m-convex on $[\frac{3\varsigma-\tau}{2},\frac{3\tau-\varsigma}{2}]$, and we get

$$\begin{aligned}
&\int_0^1 s(1-s)\left|\psi''\left(s\left(\frac{3\varsigma-\tau}{2}\right)+(1-s)\left(\frac{3\tau-\varsigma}{2}\right)\right)\right|ds \\
&\quad\le\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|\int_0^1 s^2(1-s)\,ds+m\left|\psi''\left(\frac{3\tau-\varsigma}{2m}\right)\right|\int_0^1 s(1-s)^2\,ds \\
&\quad=\frac{1}{12}\left(\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|+m\left|\psi''\left(\frac{3\tau-\varsigma}{2m}\right)\right|\right). \qquad (2.24)
\end{aligned}$$

By using this value in (2.23) we deduce that inequality (2.23) holds for $\varrho=1$.

Now assume that $\varrho>1$. Using the Hölder inequality for $\frac{1}{\varrho}+\frac{1}{\varpi}=1$, we get

$$\begin{aligned}
&\int_0^1\left(s-s^2\right)\left|\psi''\left(s\left(\frac{3\varsigma-\tau}{2}\right)+(1-s)\left(\frac{3\tau-\varsigma}{2}\right)\right)\right|ds \\
&\quad=\int_0^1\left(\left(s-s^2\right)^{1-\frac{1}{\varrho}}\left(s-s^2\right)^{\frac{1}{\varrho}}\right)\left|\psi''\left(s\left(\frac{3\varsigma-\tau}{2}\right)+(1-s)\left(\frac{3\tau-\varsigma}{2}\right)\right)\right|ds \\
&\quad\le\left(\int_0^1\left(s-s^2\right)\right)^{1-\frac{1}{\varrho}}\left(\left(s-s^2\right)\left|\psi''\left(s\left(\frac{3\varsigma-\tau}{2}\right)+m(1-s)\left(\frac{3\tau-\varsigma}{2m}\right)\right)\right|^{\varrho}ds\right)^{\frac{1}{\varrho}} \\
&\quad\le\left(\frac{1}{6}\right)^{1-\frac{1}{\varrho}}\left(\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|^{\varrho}\int_0^1\left(s^2-s^3\right)ds\right. \\
&\qquad\left.+m\left|\psi''\left(\frac{3\tau-\varsigma}{2m}\right)\right|^{\varrho}\int_0^1\left(s^3-2s^2+s\right)ds\right)^{\frac{1}{\varrho}} \\
&\quad\le\left(\frac{1}{6}\right)^{1-\frac{1}{\varrho}}\left(\frac{|\psi''(\frac{3\varsigma-\tau}{2})|^{\varrho}+m|\psi''(\frac{3\tau-\varsigma}{2m})|^{\varrho}}{12}\right)^{\frac{1}{\varrho}}.
\end{aligned}$$

This completes the proof. □

Remark 2.13
(i) If we set $m = 1$ in Theorem 2.12, then we get [13, Theorem 3].
(ii) If $|\psi''(z)| \leq K$ on $[\frac{3\varsigma-\tau}{2}, \frac{3\tau-\varsigma}{2}]$ in Theorem 2.12, then we get

$$\left|\frac{1}{(\tau-\varsigma)}\int_{\varsigma}^{\tau}\psi(z)\,dz - \frac{1}{4}\left[\psi\left(\frac{3\varsigma-\tau}{2}\right)+\psi\left(\frac{3\tau-\varsigma}{2}\right)+2\psi\left(\frac{\varsigma+\tau}{2}\right)\right]\right|
\leq \frac{K(\tau-\varsigma)^2}{3}\left(\frac{m+1}{2}\right)^{\frac{1}{\varrho}}.$$

Theorem 2.14 *Let $\psi : J^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on J^0, let $\varsigma, \tau \in J^0$ with $\varsigma < \tau$, and let $\psi'' \in C[\frac{3\varsigma-\tau}{2} \frac{3\tau-\varsigma}{2}]$ be such that $\psi''(z) \in \mathbb{R}$ for all $z \in (\frac{3\varsigma-\tau}{2}, \frac{3\tau-\varsigma}{2})$. If $\varrho > 1$ and $|\psi''|^{\varrho}$ is an m-convex function on $[\frac{3\varsigma-\tau}{2}, \frac{3\tau-\varsigma}{2}]$, then we have the following inequality*:

$$\left|\frac{1}{\tau-\varsigma}\int_{\varsigma}^{\tau}\psi(z)\,dz - \frac{1}{4}\left[\psi\left(\frac{3\varsigma-\tau}{2}\right)+\psi\left(\frac{3\tau-\varsigma}{2}\right)+2\psi\left(\frac{\varsigma+\tau}{2}\right)\right]\right|
\leq \frac{(\tau-\varsigma)^2}{2}\left(\frac{\sqrt{\pi}\,\Gamma(\varpi+1)}{2\Gamma(\varpi+\frac{3}{2})}\right)^{\frac{1}{\varpi}}\left(\frac{|\psi''(\frac{3\varsigma-\tau}{2})|^{\varrho}+m|\psi''(\frac{3\tau-\varsigma}{2m})|^{\varrho}}{2}\right),$$

where $\frac{1}{\varpi}+\frac{1}{\varrho}=1$.

Proof Using first the Hölder inequality and then the m-convexity of the function $|\psi''|^{\varrho}$, we have

$$\begin{aligned}
I &:= \int_0^1 (s-s^2)\left|\psi''\left(s\left(\frac{3\varsigma-\tau}{2}\right)+(1-s)\left(\frac{3\tau-\varsigma}{2}\right)\right)\right|ds \\
&\leq \left(\int_0^1 (s-s^2)^{\varpi}\,ds\right)^{\frac{1}{\varpi}}\left(\int_0^1\left|\psi''\left(s\left(\frac{3\varsigma-\tau}{2}\right)+m(1-s)\left(\frac{3\tau-\varsigma}{2m}\right)\right)\right|^{\varrho}ds\right)^{\frac{1}{\varrho}} \\
&\leq \left(\int_0^1 (s-s^2)^{\varpi}\,ds\right)^{\frac{1}{\varpi}} \\
&\quad\times\left(\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|^{\varrho}\int_0^1 s\,ds + m\left|\psi''\left(\frac{3\tau-\varsigma}{2m}\right)\right|^{\varrho}\int_0^1(1-s)\,ds\right)^{\frac{1}{\varrho}}.
\end{aligned}$$

By the definition of the beta function we get

$$\begin{aligned}
I &\leq \left[B(\varpi+1,\varpi+1)\right]^{\frac{1}{\varpi}} \\
&\quad\times\left(\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|^{\varrho}\int_0^1 s\,ds + m\left|\psi''\left(\frac{3\tau-\varsigma}{2m}\right)\right|^{\varrho}\int_0^1(1-s)\,ds\right)^{\frac{1}{\varrho}}.
\end{aligned}$$

Using the equalities $B(z,z) = 2^{1-2z}B(\frac{1}{2},z)$ and $B(y,z) = \frac{\Gamma(y)\Gamma(z)}{\Gamma(y+z)}$, we get

$$I = \left(\frac{\sqrt{\pi}\,\Gamma(\varpi+1)}{\Gamma(\varpi+\frac{3}{2})2^{1+2\varpi}}\right)^{\frac{1}{\varpi}}\left(\frac{|\psi''(\frac{3\varsigma-\tau}{2})|^{\varrho}+m|\psi''(\frac{3\tau-\varsigma}{2m})|^{\varrho}}{2}\right)^{\frac{1}{\varrho}}. \tag{2.25}$$

Finally, from (2.23) and (2.25) we obtain the desired result. □

Remark 2.15

(i) If we set $m = 1$ in Theorem 2.14, then we get [13, Theorem 4];

(ii) Using the assumptions of Theorem 2.14 with $\psi''(z) \le K$ on $[\frac{3\varsigma-\tau}{2}, \frac{3\tau-\varsigma}{2}]$, we get

$$\begin{aligned}
&\left|\frac{1}{\tau-\varsigma}\int_\varsigma^\tau \psi(z)\,dz - \frac{\psi(\frac{3\varsigma-\tau}{2}) + (\frac{3\tau-\varsigma}{2}) + 2\psi(\frac{\varsigma+\tau}{2})}{4}\right| \\
&\quad \le K\frac{(\tau-\varsigma)^2}{2}\left(\frac{m+1}{2}\right)^{\frac{1}{\varrho}}\left(\frac{\sqrt{\pi}\,\Gamma(\varpi+1)}{2\Gamma(\varpi+\frac{3}{2})}\right)^{\frac{1}{\varpi}}. \qquad (2.26)
\end{aligned}$$

Theorem 2.16 *Under the assumptions of Theorem* 2.14, *we have the following inequality*:

$$\begin{aligned}
&\left|\frac{1}{\tau-\varsigma}\int_\varsigma^\tau \psi(z)\,dz - \frac{\psi(\frac{3\varsigma-\tau}{2}) + (\frac{3\tau-\varsigma}{2}) + 2\psi(\frac{\varsigma+\tau}{2})}{4}\right| \\
&\quad \le (\tau-\varsigma)^2 K(\varpi,\varrho)\left(\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|^\varrho + m(\varrho+1)\left|\psi''\left(\frac{3\tau-\varsigma}{2m}\right)\right|^\varrho\right)^{\frac{1}{\varrho}}, \qquad (2.27)
\end{aligned}$$

where

$$K(\varpi,\varrho) = 2\left(\frac{1}{\varpi+1}\right)^{\frac{1}{\varpi}}\left(\frac{1}{(\varrho+1)(\varrho+2)}\right)^{\frac{1}{\varrho}}.$$

Proof Using first the Hölder inequality and then the m-convexity, we get

$$\begin{aligned}
&\int_0^1 (s-s^2)\left|\psi''\left(s\left(\frac{3\varsigma-\tau}{2}\right) + m(1-s)\left(\frac{3\tau-\varsigma}{2m}\right)\right)\right| ds \\
&\quad \le \left(\int_0^1 s^\varpi\,ds\right)^{\frac{1}{\varpi}}\left(\int_0^1 (1-s)^\varrho\left|\psi''\left(s\left(\frac{3\varsigma-\tau}{2}\right) + m(1-s)\left(\frac{3\tau-\varsigma}{2m}\right)\right)\right|^\varrho ds\right)^{\frac{1}{\varrho}} \\
&\quad \le \left(\int_0^1 s^\varpi\,ds\right)^{\frac{1}{\varrho}} \\
&\qquad \times \left(\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|^\varrho \int_0^1 s(1-s)^\varrho\,ds + m\left|\psi''\left(\frac{3\tau-\varsigma}{2}\right)\right|^\varrho \int_0^1 (1-s)^{\varrho+1}\,ds\right)^{\frac{1}{\varrho}} \\
&\quad = \left(\frac{1}{\varpi+1}\right)^{\frac{1}{\varpi}}\left(B(2,\varrho+1)\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|^\varrho + \frac{m|\psi''(\frac{3\tau-\varsigma}{2m})|^\varrho}{\varrho+2}\right)^{\frac{1}{\varrho}} \\
&\quad = \left(\frac{1}{\varpi+1}\right)^{\frac{1}{\varpi}}\left(\frac{\Gamma(2)\Gamma(\varrho+1)}{\Gamma(\varrho+3)}\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|^\varrho + \frac{m|\psi''(\frac{3\tau-\varsigma}{2m})|^\varrho}{\varrho+2}\right)^{\frac{1}{\varrho}} \\
&\quad = \left(\frac{1}{\varpi+1}\right)^{\frac{1}{\varpi}}\left(\frac{1}{(\varrho+1)(\varrho+2)}\right)^{\frac{1}{\varrho}} \\
&\qquad \times \left(\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|^\varrho + m(\varrho+1)\left|\psi''\left(\frac{3\tau-\varsigma}{2m}\right)\right|^\varrho\right)^{\frac{1}{\varrho}}. \qquad (2.28)
\end{aligned}$$

Keeping in mind (2.23) and (2.28), we obtain (2.27). □

Remark 2.17 If we set $m = 1$ in Theorem 2.16, then we get [13, Theorem 5].

Theorem 2.18 *Let $\psi : J^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on J^0, let $\varsigma, \tau \in J$ with $\varsigma < \tau$, and let $\psi'' \in C[\frac{3\varsigma-\tau}{2}, \frac{3\tau-\varsigma}{2}]$ be such that $\psi''(z) \in \mathbb{R}$ for all $z \in (\frac{3\varsigma-\tau}{2}, \frac{3\tau-\varsigma}{2})$. If $\varrho \geq 1$ and $|\psi''|^\varrho$ is an m-convex mapping on $[\frac{3\varsigma-\tau}{2}, \frac{3\tau-\varsigma}{2}]$, then we have the following inequality*:

$$\begin{aligned} &\left|\frac{1}{\tau-\varsigma}\int_\varsigma^\tau \psi(z)\,dz - \frac{\psi(\frac{3\varsigma-\tau}{2}) + \psi(\frac{3\tau-\varsigma}{2}) + 2\psi(\frac{\varsigma+\tau}{2})}{4}\right| \\ &\quad\leq (\tau-\varsigma)^2 K_2(\varrho)\left(2\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|^\varrho + m(q+1)\left|\psi''\left(\frac{3\tau-\varsigma}{2m}\right)\right|^\varrho\right)^{\frac{1}{\varrho}}, \end{aligned} \tag{2.29}$$

where

$$K_2(\varrho) = \left(\frac{2}{(\varrho+1)(\varrho+2)(\varrho+3)}\right)^{\frac{1}{\varrho}}.$$

Proof Let $\varrho > 1$. Using first the Hölder inequality and then the m-convexity, we get

$$\begin{aligned} &\int_0^1 (s-s^2)\left|\psi''\left(s\left(\frac{3\varsigma-\tau}{2}\right) + m(1-s)\left(\frac{3\tau-\varsigma}{2m}\right)\right)\right| ds \\ &\quad= \int_0^1 s^{1-\frac{1}{\varrho}}\left(s^{\frac{1}{\varrho}}(1-s)\left|\psi''\left(s\left(\frac{3\varsigma-\tau}{2}\right) + m(1-s)\left(\frac{3\tau-\varsigma}{2m}\right)\right)\right|\right) ds \\ &\quad\leq \left(\int_0^1 s\,ds\right)^{1-\frac{1}{\varrho}}\left(\int_0^1 s(1-s)^\varrho\left|\psi''\left(s\left(\frac{3\varsigma-\tau}{2}\right) + m(1-s)\left(\frac{3\tau-\varsigma}{2m}\right)\right)\right|^\varrho ds\right)^{\frac{1}{\varrho}} \\ &\quad\leq \left(\int_0^1 s\,ds\right)^{1-\frac{1}{\varrho}}\left(\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|^\varrho \int_0^1 s^2(1-s)^\varrho\,ds \right. \\ &\qquad\left. + m\left|\psi''\left(\frac{3\tau-\varsigma}{2m}\right)\right|^\varrho \int_0^1 s(1-s)^{\varrho+1}\,ds\right)^{\frac{1}{\varrho}} \\ &\quad= \left(\frac{1}{2}\right)^{1-\frac{1}{\varrho}}\left(B(3,\varrho+1)\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|^\varrho + mB(2,\varrho+2)\left|\psi''\left(\frac{3\tau-\varsigma}{2m}\right)\right|^\varrho\right)^{\frac{1}{\varrho}} \\ &\quad= \left(\frac{1}{2}\right)^{1-\frac{1}{\varrho}}\left(\frac{1}{(\varrho+1)(\varrho+2)(\varrho+3)}\right)^{\frac{1}{\varrho}} \\ &\qquad\times\left(2\left|\psi''\left(\frac{3\varsigma-\tau}{2}\right)\right|^\varrho + m(\varrho+1)\left|\psi''\left(\frac{3\tau-\varsigma}{2m}\right)\right|^\varrho\right)^{\frac{1}{\varrho}}. \end{aligned} \tag{2.30}$$

In view of (2.23) and (2.30), we deduce that (2.29) holds when $\varrho > 1$. From (2.24) we deduce that (2.29) is true when $\varrho = 1$. This completes the proof of Theorem 2.18. □

Remark 2.19 If we set $m = 1$ in Theorem 2.18, then we get [13, Theorem 6].

3 Applications to special means

In this section, we skilfully use the main results of Sect. 2 to give some applications to special means of positive real numbers. We first need to recall the following basic definitions of different means and techniques of numerical integration. For arbitrary positive numbers ς, τ such that $\varsigma \neq \tau$, we consider the following means.

(i) The arithmetic mean:

$$\mathcal{A} := \mathcal{A}(\varsigma,\tau) = \frac{\varsigma+\tau}{2}.$$

(ii) The geometric mean:

$$\mathcal{G} := \mathcal{G}(\varsigma,\tau) = \sqrt{\varsigma\tau}.$$

(iii) The logarithmic mean:

$$\mathcal{L}(\varsigma,\tau) := \frac{\tau-\varsigma}{\log(\tau)-\log(\varsigma)}.$$

(iv) The generalized logarithmic mean:

$$\mathcal{L}_n(\varsigma,\tau) := \left[\frac{\tau^{n+1}-\varsigma^{n+1}}{(\tau-\varsigma)(n+1)}\right]^{\frac{1}{n}},$$

where $n \in \mathbb{Z}\setminus\{-1,0\}$;

(v) The midpoint formula: Let d be a partition with points $\varsigma = z_0 < z_1 < \cdots < z_{m-1} < z_m = \tau$ of the interval $[\varsigma,\tau]$ and consider the quadrature formula

$$\int_\varsigma^\tau \psi(z)\,dz = T_j(\psi,d) + E_j(\psi,d), \quad j = 1,2,$$

where

$$T_1(\psi,d) = \sum_{j=0}^{m-1} \frac{\psi(z_j)+\psi(z_{j+1})}{2}(z_{j+1}-z_j)$$

for the trapezoidal version, and

$$T_2(\psi,d) = \sum_{j=0}^{m-1} \psi\left(\frac{z_j+z_{j+1}}{2}\right)(z_{j+1}-z_j)$$

for the midpoint version, whereas $E_j(f,d)$ represents the approximation error.

Proposition 3.1 *If $n \in \mathbb{Z}\setminus\{-1,0\}$ and $\varsigma,\tau \in \mathbb{R}$ with $0 < \varsigma < \tau$, then we have the following inequality*:

$$\begin{aligned} &\left|\mathcal{A}^n(\varsigma,\tau) - \mathcal{L}_n^n(\varsigma,\tau)\right| \\ &\quad \le \min\{K_1,K_2\}\left(2^{\frac{1}{\varrho}}|n|(\tau-\varsigma)\right)\left[\mathcal{A}\left(\left|\frac{3\varsigma-\tau}{2}\right|^{(n-1)\varrho}, m\left|\frac{3\tau-\varsigma}{2m}\right|^{(n-1)\varrho}\right)\right]^{\frac{1}{\varrho}}. \end{aligned} \tag{3.1}$$

Proof Using Corollary 2.11 with substitution $\psi(z) = z^n$, by simple mathematical calculations we get (3.1). □

Proposition 3.2 *If $\varsigma,\tau\in\mathbb{R}$ with $0<\varsigma<\tau$, then we have the following inequality*:

$$\begin{aligned}&\left|\mathcal{G}^{-2}(\varsigma,\tau)-\mathcal{A}^{-2}(\varsigma,\tau)\right|\\&\quad\le\min\{K_1,K_2\}\left(4^{\frac{1}{\varrho}}(\tau-\varsigma)\right)\left[\mathcal{A}\left(\left|\frac{3\varsigma-\tau}{2}\right|^{-3\varrho},m\left|\frac{3\tau-\varsigma}{2m}\right|^{-3\varrho}\right)\right]^{\frac{1}{\varrho}}.\end{aligned}\tag{3.2}$$

Proof Using Corollary 2.11 with substitution $\psi(z)=\frac{1}{z^2}$, by simple mathematical calculations we get (3.2). □

Proposition 3.3 *If $\varrho\ge1$ and $\varsigma,\tau\in\mathbb{R}$ with $0<\varsigma<\tau$, then we have the following inequality*:

$$\begin{aligned}&\left|\mathcal{A}^{-1}(\varsigma,\tau)-\mathcal{L}^{-1}(\varsigma,\tau)\right|\\&\quad\le\min\{K_1,K_2\}\left(2^{\frac{1}{\varrho}}(\tau-\varsigma)\right)\left[\mathcal{A}\left(\left|\frac{3\varsigma-\tau}{2}\right|^{-2\varrho},m\left|\frac{3\tau-\varsigma}{2m}\right|^{-2\varrho}\right)\right]^{\frac{1}{\varrho}}.\end{aligned}\tag{3.3}$$

Proof Using Corollary 2.11 with substitution $\psi(z)=\frac{1}{z}$, by simple mathematical calculations we get (3.3). □

Proposition 3.4 *If $\varrho\ge1$ and $|\psi'|^{\varrho}$ is an m-convex function, then for every partition of $[\frac{3\varsigma-\tau}{2},\frac{3\tau-\varsigma}{2}]$, the midpoint error satisfies*

$$\begin{aligned}\left|E_2(\psi;d)\right|&\le\min(K_1,K_2)\sum_{j=0}^{m-1}(z_{j+1}-z_j)^2\left[\left|\psi'\left(\frac{3z_j-z_{j+1}}{2}\right)\right|^{\varrho}+m\left|\psi'\left(\frac{3z_{j+1}-z_j}{2m}\right)\right|^{\varrho}\right]^{\frac{1}{\varrho}}\\&\le2\min(K_1,K_2)\sum_{j=0}^{m-1}(z_{j+1}-z_j)^2\max\left[\left|\psi'\left(\frac{3z_j-z_{j+1}}{2}\right)\right|,m\left|\psi'\left(\frac{3z_{j+1}-z_j}{2m}\right)\right|\right].\end{aligned}$$

Proof From Corollary 2.11 we obtain

$$\begin{aligned}&\left|\int_{z_j}^{z_{j+1}}\psi(z)\,dz(z_{j+1}-z_j)\psi\left(\frac{z_j+z_{j+1}}{2}\right)\right|\\&\quad\le\min(K_1,K_2)(z_{j+1}-z_j)^2\left[\left|\psi'\left(\frac{3z_j-z_{j+1}}{2}\right)\right|^{\varrho}+m\left|\psi'\left(\frac{3z_{j+1}-z_j}{2m}\right)\right|^{\varrho}\right]^{\frac{1}{\varrho}}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\left|\left\{\int_{\varsigma}^{\tau}\psi(z)\,dz-T_2(\psi,d)\right\}\right|\\&\quad=\left|\sum_{j=0}^{m-1}\left\{\int_{z_j}^{z_{j+1}}\psi(z)\,dz-(z_{j+1}-z_j)\psi\left(\frac{z_j+z_{j+1}}{2}\right)\right\}\right|\\&\quad\le\min(K_1,K_2)\sum_{j=0}^{m-1}(z_{j+1}-z_j)^2\left[\left|\psi'\left(\frac{3z_j-z_{j+1}}{2}\right)\right|^{\varrho}+m\left|\psi'\left(\frac{3z_{j+1}-z_j}{2m}\right)\right|^{\varrho}\right]^{\frac{1}{\varrho}}\\&\quad\le2\min(K_1,K_2)\sum_{j=0}^{m-1}(z_{j+1}-z_j)^2\max\left[\left|\psi'\left(\frac{3z_j-z_{j+1}}{2}\right)\right|,m\left|\psi'\left(\frac{3z_{j+1}-z_j}{2m}\right)\right|\right].\end{aligned}$$

□

4 Conclusions

Inequalities and convexity are correlated and are used to obtain optimal results. In this paper, we have used the m-convexity and Hölder inequality, which play an important role in optimization theory. A class of Hermite–Hadamard-type inequalities is developed using the traditional convex and m-convex maps. Furthermore, the main results are used to establish certain means, and quadrature formulae are used to calculate the error estimates. In future studies, the researchers can design similar forms of inequalities using various convexities.

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