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On Toeplitz operators between different Fock–Sobolev-type spaces

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Abstract
In this paper, we characterize those positive Borel measurable symbols μ on $\mathbb{C}^n$ that induce the Toeplitz operators T_μ^α to be bounded or compact between different Fock–Sobolev–type spaces F_α^p and F_α^∞ with $0 < p \le \infty$.

MSC: Primary 47B38; secondary 47B35

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1 Introduction

Let $\mathbb{C}^n$ be the complex n-space and dv be the ordinary volume measure on $\mathbb{C}^n$ that is normalized so that $\int_{\mathbb{C}^n} e^{-|z|^2}\,dv(z) = 1$. For points $z = (z_1, z_2, \dots, z_n)$ and $w = (w_1, w_2, \dots, w_n)$ in $\mathbb{C}^n$, we write

$$z\overline{w} = \sum_{j=1}^{n} z_j\overline{w_j}, \qquad |z| = \sqrt{z\overline{z}}.$$

For every $0 < p < \infty$, $\alpha \in \mathbb{R}$, we denote by $L_\alpha^p(\mathbb{C}^n)$ the space of measurable functions f such that

$$\|f\|_{L_\alpha^p} = \left(\int_{\mathbb{C}^n} \left|f(z)e^{-\frac{1}{2}|z|^2}\right|^p \frac{dv(z)}{(1+|z|)^\alpha}\right)^{\frac{1}{p}} < \infty.$$

For $p = \infty$, we denote by $L_\alpha^\infty(\mathbb{C}^n)$ the spaces of the Lebesgue measurable function f on $\mathbb{C}^n$ such that

$$\|f\|_{L_\alpha^\infty} = \operatorname{esssup}\left\{\frac{|f(z)|e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} : z \in \mathbb{C}^n\right\} < \infty.$$

Let $H(\mathbb{C}^n)$ be the set of entire functions on $\mathbb{C}^n$. Then, for a given $0 < p < \infty$, the Fock–Sobolev-type space F_α^p with the norm $\|\cdot\|_{F_\alpha^p} = \|\cdot\|_{L_\alpha^p}$ is defined as

$$F_\alpha^p = L_\alpha^p\left(\mathbb{C}^n\right) \cap H\left(\mathbb{C}^n\right).$$

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Obviously, the Fock–Sobolev-type space F^2_α equipped with the natural inner product defined by

$$\langle f,g\rangle_{L^2_\alpha} = \int_{\mathbb{C}^n} f(z)\overline{g(z)}e^{-|z|^2}\frac{dv(z)}{(1+|z|)^\alpha}$$

is a reproducing kernel Hilbert space for every real α. As stated in [4], with respect to the above inner product, it is difficult to compute the reproducing kernel of F^2_α explicitly. Hence, we use the equivalent norm with respect to a new measure $|z|^{-\alpha}\,dv(z)$. In detail, for $\alpha \le 0$, we will let

$$\langle f,g\rangle_\alpha = \int_{\mathbb{C}^n} f(z)\overline{g(z)}e^{-|z|^2}\frac{dv(z)}{|z|^\alpha}$$

and for $\alpha > 0$ we let

$$\langle f,g\rangle_\alpha = \int_{\mathbb{C}^n} f^-_{\frac{\alpha}{2}}(z)\overline{g^-_{\frac{\alpha}{2}}(z)}e^{-|z|^2}\,dv(z) + \int_{\mathbb{C}^n} f^+_{\frac{\alpha}{2}}(z)\overline{g^+_{\frac{\alpha}{2}}(z)}e^{-|z|^2}\frac{dv(z)}{|z|^\alpha},$$

where $f^-_{\frac{\alpha}{2}}$ is the Taylor expansion of f up to order $\frac{\alpha}{2}$ and $f^+_{\frac{\alpha}{2}} = f - f^-_{\frac{\alpha}{2}}$. Now, we can make sure that the inner product $\langle\cdot,\cdot\rangle_\alpha$ generates a new Hilbert space norm on F^p_α that is equivalent to the F^p_α norm $\|\cdot\|_{F^p_\alpha}$. In particular, if we define the norm $\|\cdot\|_{\widetilde{F^p_\alpha}}$ on F^p_α, when $\alpha \le 0$, by

$$\|f\|_{\widetilde{F^p_\alpha}} = \left(\int_{\mathbb{C}^n} \left|f(z)\right|^p e^{-\frac{p}{2}|z|^2}\frac{dv(z)}{|z|^\alpha}\right)^{\frac{1}{p}}$$

and when $\alpha > 0$,

$$\|f\|_{\widetilde{F^p_\alpha}} = \left(\int_{\mathbb{C}^n} \left|f^-_{\frac{\alpha}{p}}(z)\right|^p e^{-\frac{p}{2}|z|^2}\,dv(z)\right)^{\frac{1}{p}} + \left(\int_{\mathbb{C}^n} \left|f^+_{\frac{\alpha}{p}}(z)\right|^p e^{-\frac{p}{2}|z|^2}\frac{dv(z)}{|z|^\alpha}\right)^{\frac{1}{p}},$$

then we have that both $\|\cdot\|_{\widetilde{F^p_\alpha}}$ and $\|\cdot\|_{F^p_\alpha}$ are equivalent norms.

As is well known, F^2_α is indeed a reproducing kernel Hilbert space (see [4, Lemma 2.1] for more details). Therefore, we have

$$K^\alpha_z(w) = \sum_\beta \phi_\beta(w)\overline{\phi_\beta(z)},$$

where $\{\phi_\beta\}$ is any orthonormal basis for F^2_α with respect to $\langle\cdot,\cdot\rangle_\alpha$. Note that polynomials form a dense subset of F^p_α (see [3, Proposition 2.3]). Also, note that monomials are mutually orthogonal, which means that $\{\frac{z^\beta}{\sqrt{\langle z^\beta,z^\beta\rangle_\alpha}}\}$ is an orthonormal basis for F^2_α. The arguments that are identical to those in the proof of [3, Theorem 4.5] then give us that

$$K^\alpha_z(w) = \begin{cases} \mathcal{I}^{-\frac{\alpha}{2}}K_z(w), & \text{if } \alpha \le 0; \\ \mathcal{I}^{-\frac{\alpha}{2}}K_z(w) + (K_z)^-_{\frac{\alpha}{2}}(w), & \text{if } \alpha > 0. \end{cases}$$

Here, $K_z(w) = e^{\bar{z}w}$ and $\mathcal{I}^s$ is the fractional integration operator defined as

$$\mathcal{I}^s f(z) = \begin{cases} \sum_{k=0}^\infty \frac{\Gamma(n+k)}{\Gamma(n+s+k)} f_k(z), & \text{if } s \ge 0; \\ \sum_{k>|s|}^\infty \frac{\Gamma(n+k)}{\Gamma(n+s+k)} f_k(z), & \text{if } s < 0. \end{cases}$$

Now, it is easy to see that if $\alpha \le 0$, $(F_\alpha^2, \|\cdot\|_{\widetilde{F_\alpha^2}})$ is a closed subspace of L_α^2 with respect to $\langle\cdot,\cdot\rangle_\alpha$. In this case, let P_α denote the orthogonal projection such that

$$P_\alpha f(z) = \langle f, K_z^\alpha\rangle_\alpha$$

for any $f \in L_\alpha^2$. Unfortunately, the inner product $\langle\cdot,\cdot\rangle_\alpha$ does not make sense on L_α^2 when $\alpha > 0$. That means we cannot define the Toeplitz operator on F_α^2 in the usual way in terms of this inner product. However, according to the ideas of [4], it makes sense to define the Toeplitz operator with the complex Borel measure on $\mathbb{C}^n$ by the formula

$$T_\mu^\alpha f(z) = \int_{\mathbb{C}^n} f(w)\overline{K_z^\alpha(w)}e^{-|w|^2}|w|^{-\alpha}\,d\mu(w)$$

if $\alpha \le 0$ and

$$\begin{aligned} T_\mu^\alpha f(z) = &\int_{\mathbb{C}^n} f_{\frac{\alpha}{2}}^-(w)\overline{\left(K_z^\alpha\right)_{\frac{\alpha}{2}}^-(w)}e^{-|w|^2}\,d\mu(w) \\ &+ \int_{\mathbb{C}^n} f_{\frac{\alpha}{2}}^+(w)\overline{\left(K_z^\alpha\right)_{\frac{\alpha}{2}}^+(w)}e^{-|w|^2}\frac{d\mu(w)}{|w|^\alpha} \end{aligned}$$

if $\alpha > 0$. Note that if μ satisfies

$$\int_{\mathbb{C}^n}\left|K_z^\alpha(w)\right|^2 e^{-|w|^2}\frac{d|\mu|(w)}{(1+|w|)^\alpha} < \infty, \quad z \in \mathbb{C}^n,$$

then the Toeplitz operator T_μ^α is densely defined on F_α^2. Seqentially, we let

$$B(z,r) = \left\{w \in \mathbb{C}^n : |w-z| < r\right\}$$

for $z \in \mathbb{C}^n$ and $r > 0$. Given a Borel measure μ on $\mathbb{C}^n$, the average of μ on $B(z,r)$ is $\frac{\mu(B(z,r))}{v(B(z,r))}$. Since the Lebesgue volume $v(B(z,r)) \simeq r^{2n}$, we simply set the average function of μ as

$$\widehat{\mu}_r(z) = \mu\left(B(z,r)\right).$$

For Borel measure μ on $\mathbb{C}^n$, $0 < t < \infty$, the t-Berezin transform of a Toeplitz operator on $\mathbb{C}^n$ is defined by

$$\widetilde{\mu}_t^\alpha(z) = \frac{1}{(1+|z|)^{(\frac{t}{2}-1)\alpha}}\int_{\mathbb{C}^n}\left|k_{2,z}^\alpha(w)e^{-\frac{1}{2}|w|^2}\right|^t\frac{d\mu(w)}{|w|^\alpha}$$

if $\alpha \le 0$, and

$$\begin{aligned} \widetilde{\mu}_t^\alpha(z) = &\frac{1}{(1+|z|)^{(\frac{t}{2}-1)\alpha}}\int_{\mathbb{C}^n}\left|\left(k_{2,z}^\alpha\right)_{\frac{\alpha}{t}}^-(w)e^{-\frac{1}{2}|w|^2}\right|^t d\mu(w) \\ &+ \frac{1}{(1+|z|)^{(\frac{t}{2}-1)\alpha}}\int_{\mathbb{C}^n}\left|\left(k_{2,z}^\alpha\right)_{\frac{\alpha}{t}}^+(w)e^{-\frac{1}{2}|w|^2}\right|^t\frac{d\mu(w)}{|w|^\alpha} \end{aligned}$$

if $\alpha > 0$, where $k_{2,z}^\alpha(w)$ is the normalization of the kernel $K_z^\alpha(w)$, and, in general, we denote by $k_{p,z}^\alpha = K_z^\alpha\|K_z^\alpha\|_{F_\alpha^p}^{-1}$, for $0 < p \le \infty$, $z \in \mathbb{C}^n$.

Although there has been much more research on Toeplitz operators on Bergman or Hardy spaces of various domains than on Fock spaces, Toeplitz operators on Fock spaces have been studied for many years, see [10] and references therein for examples. In 2014, Mengestie had characterized the Toeplitz operators with positive measure symbols between classic Fock spaces F_α^p and F_α^∞. In more detail, the Toeplitz operator $T_\mu : F_\alpha^p \to F_\alpha^\infty$ is bounded (compact) if and only if μ is a (vanishing) $(1,q)$ Fock–Carleson measure, where $1 \le p \le q < \infty$, if and only if $\widehat{\mu}_\delta, \widetilde{\mu}_t \in L^\infty$ for some or any $t, \delta > 0$. The Toepitz operator $T_\mu : F_\alpha^\infty \to F_\alpha^p$ is bounded or compact if and only μ is a vanishing (∞, q) Fock–Carleson measure, where $1 \le q < \infty$, if and only if $\widehat{\mu}_\delta, \widetilde{\mu}_t \in L^1$ for some or any $t, \delta > 0$. Those achievements would be based on some his own results in [7]. In his other paper [8], he had researched the basic properties of Carleson-type measures of Fock–Sobolev spaces. It is worth noting that the positive measure μ is a (∞, q) Fock–Carleson measure for $0 < q < \infty$ if and only if it is also vanishing, if and only $\widehat{\mu}_{mq,r}, \widetilde{\mu}_{t,mq} \in L^1$ for some or any $t, r > 0$.

In 2019, Lv had, in [6], successfully obtained the boundedness and compactness of Toeplitz operators with positive measure symbols on doubling the Fock space between F_φ^p and F_φ^∞ for $0 < p \le \infty$. She pointed out that Toeplitz operator $T_\mu : F_\varphi^p \to F_\varphi^\infty$ is bounded (compact) if and only if μ is a (vanishing) $\frac{p}{p+1}$ Fock–Carleson measure. Conversely, the Toeplitz operator $T_\mu : F_\varphi^\infty \to F_\varphi^p$ is bounded or compact if and only if $\widehat{\mu}_\delta, \widetilde{\mu}_t \in L^p$ for some or any $t, \delta > 0$. Something interesting attracts our attention, that is the so-called $\frac{p}{p+1}$ Fock–Carleson measure. As stated in [6], μ is a (p,q) Fock–Carleson measure if and only if it is a (tp, tq) Fock–Carleson measure for any $t > 0$, because it is also equivalent to both $\widehat{\mu}_\delta \rho^{2(1-\frac{q}{p})}$ and $\widetilde{\mu}_t \rho^{2(1-\frac{q}{p})}$ that are bounded. That is why we call it a $\frac{p}{p+1}$ Fock–Carleson measure and we write it as $\|\mu\|_{\frac{p}{q}} = \|i\|_{F_\varphi^{p/q} \to L_\varphi^1(\mu)}$. Luckily, these good achievements keep being correct in the case of large Fock spaces, for which we can refer to our latest paper [1].

Unfortunately, this phenomenon will not occur in the case of Fock–Sobolev-type spaces, because the definition of the largest Fock–Soboblev–type space F_α^∞ differs, totally, from doubling Fock spaces F_φ^∞. In other cases, including the classic Fock space, doubling Fock space, and large Fock space, the infinite index would be thought of as the limit of the finite index. However, This is not the case here. Moreover, $F_\alpha^\infty \ne \lim_{p\to\infty} F_\alpha^p$. In this paper, some terminologies and symbols are still similar to those in [6]. We are going to achieve some characterizations on those $\mu \ge 0$ such that Toeplitz operator T_μ is bounded or compact from F_α^p to F_α^∞ and the converse case, respectively, for $0 < p \le \infty$.

We will end this introduction with a comment on some notations.

Firstly, for conciseness, we will denote by $I_{(\star)}$ the integral in Line ($\star$). Take, for example, that $I_{(1)}$ refers to the integral in Line (1).

Secondly, for positive quantities A and B (which may depend on a variety of parameters or variables), we will use the notation $A \lesssim B$ if there exists an unimportant constant C such that $A \le CB$. The notation $A \gtrsim B$ will have a similar meaning. We write $A \simeq B$ if $A \lesssim B$ and $A \gtrsim B$ at the same time.

2 Carleosn measures and related results

In this section, we are going to characterize Fock–Carleson measures. For this purpose, we need some conclusions about the reproducing kernel K_z^α, which are partly from [3, 4].

Lemma 2.1 *The upper pointwise estimate and the properties of the Bergman kernel* K_z^α *are as follows*:

(1) *Suppose that f belongs to the Fock–Sobolev-type space F^p_α for any real α. Then, for any $z, w \in \mathbb{C}^n$, $p, a, t > 0$, $\alpha \in \mathbb{R}$, we have*

$$\frac{|f(z)|^p e^{-a|z|^2}}{(1+|z|)^\alpha} \lesssim \int_{|w-z|<t} \left|f(w)\right|^p e^{-a|w|^2} \frac{dv(w)}{(1+|w|)^\alpha}.$$

(2) *The Bergman kernel satisfies*

$$\left|K^\alpha_z(w)\right| \lesssim \begin{cases} (1+|z||w|)^{\frac{\alpha}{2}} \exp(\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2 - \frac{1}{8}|z-w|^2), & \text{if } \alpha \le 0; \\ (1+|w\bar{z}|)^{\frac{\alpha}{2}} \exp(\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2 - \frac{1}{8}|z-w|^2), & \text{if } \alpha > 0. \end{cases}$$

More specifically,

$$\left|K^\alpha_z(z)\right| \simeq \left(1+|z|\right)^\alpha e^{|z|^2}$$

for any $z \in \mathbb{C}^n$, and there is a small enough $r_0 > 0$ such that

$$\left|K^\alpha_z(w)\right| \gtrsim \left(1+|z|\right)^\alpha \exp\left(\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2\right), \quad \forall z \in B(w, r_0).$$

(3) *Together with the above, the estimate of the norm of the reproducing kernel is*

$$\left\|K^\alpha_z\right\|_{F^p_\alpha} \lesssim \begin{cases} (1+|z|)^{\alpha-\frac{\alpha}{p}} e^{\frac{|z|^2}{2}}, & 0 < p < \infty; \\ e^{\frac{|z|^2}{2}}, & p = \infty. \end{cases}$$

(4) *For $0 < p \le \infty$, the normalization of the kernel $k^\alpha_{p,z} \to 0$ uniformly on compact subsets of $\mathbb{C}^n$ as $|z| \to \infty$.*

Proof We only discuss the statement (4) in the case of $p = \infty$, as the others are the basic properties in [3, 4] like [4, Lemmas 2.1, 2.3] and [3, Proposition 4.8]. By the statement (3), to be continued, we can easily see that

$$\frac{|K^\alpha_z(w)|}{\|K^\alpha_z\|_{F^\infty_\alpha}} \lesssim \left|K^\alpha_z(w)\right| e^{-\frac{|z|^2}{2}}.$$

Moreover, in terms of the statement (2),

$$\left|K^\alpha_z(w)\right| e^{-\frac{|z|^2}{2}} \lesssim \begin{cases} (1+|z||w|)^{\frac{\alpha}{2}} \exp(\frac{1}{2}|w|^2 - \frac{1}{8}|z-w|^2), & \text{if } \alpha \le 0; \\ (1+|w\bar{z}|)^{\frac{\alpha}{2}} \exp(\frac{1}{2}|w|^2 - \frac{1}{8}|z-w|^2), & \text{if } \alpha > 0. \end{cases}$$

This tells us the statement (4) is true when w is fixed and $|z| \to \infty$. □

Sequentially, we will study Carleson measures on F^p_α. If $\mu \ge 0$ is a Borel measure, then we define the norm $\|\cdot\|_{L^p_{\alpha,\mu} \cap H(\mathbb{C}^n)}$ on $L^p_{\alpha,\mu} \cap H(\mathbb{C}^n)$ by

$$\|f\|^p_{L^p_{\alpha,\mu} \cap H(\mathbb{C}^n)} = \int_{\mathbb{C}^n} \left|f^-_{\frac{\alpha}{p}}(z) e^{-\frac{1}{2}|z|^2}\right|^p d\mu(z) + \int_{\mathbb{C}^n} \left|f^+_{\frac{\alpha}{p}}(z) e^{-\frac{1}{2}|z|^2}\right|^p \frac{d\mu(z)}{|z|^\alpha},$$

when $\alpha > 0$ and

$$\|f\|^p_{L^p_{\alpha,\mu}\cap H(\mathbb{C}^n)} = \int_{\mathbb{C}^n} \left|f(z)e^{-\frac{1}{2}|z|^2}\right|^p \frac{d\mu(z)}{|z|^\alpha},$$

when $\alpha \leq 0$. If $0 < p, q < \infty$ and μ is a weighted (p,q)-Fock–Carleson measure, then the inclusion

$$\iota_p : F^p_\alpha \to L^q_{\alpha,\mu} \cap H(\mathbb{C}^n)$$

is bounded. We call μ a vanishing weighted (p,q)-Fock–Carleson measure for F^p_α if

$$\lim_{j\to\infty} \int_{\mathbb{C}^n} \left|(f_j)^-_{\frac{\alpha}{q}}(z)e^{-\frac{1}{2}|z|^2}\right|^q d\mu(z) + \int_{\mathbb{C}^n} \left|(f_j)^+_{\frac{\alpha}{q}}(z)e^{-\frac{1}{2}|z|^2}\right|^q \frac{d\mu(z)}{|z|^\alpha} = 0,$$

when $\alpha > 0$ and

$$\lim_{j\to\infty} \int_{\mathbb{C}^n} \left|f_j(z)e^{-\frac{1}{2}|z|^2}\right|^q \frac{d\mu(z)}{|z|^\alpha} = 0,$$

when $\alpha \leq 0$, whenever $\{f_j\}$ is bounded in F^p_α and converges to 0 uniformly on a compact subset of $\mathbb{C}^n$ as $j \to \infty$.

The following three theorems characterize (vanishing) (p,q) Fock–Carleson measures for all possible $0 < p, q < \infty$. All of them have been proven in [2].

Theorem 2.2 *Let $0 < p \leq q < \infty$ and $\mu \geq 0$. Then, the following statements are equivalent*:

(1) *μ is a (p,q)-Fock–Carleson measure*;
(2) $(1+|z|)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widetilde{\mu}^\alpha_t(z) \in L^\infty(dv)$ *for some or any $t > 0$*;
(3) $(1+|z|)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widehat{\mu}_\delta(z) \in L^\infty(dv)$ *for some or any $\delta > 0$*;
(4) *The sequence*

$$\left\{\left(1+|a_k|\right)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widehat{\mu}_r(a_k)\right\}^\infty_{k=1} \in l^\infty$$

for some or any r-lattice $\{a_k\}^\infty_{k=1}$.

Furthermore,

$$\begin{aligned}\|\iota_p\|^q_{F^p_\alpha\to L^q_{\alpha,\mu}\cap H(\mathbb{C}^n)} &\simeq \left\|\left(1+|\cdot|\right)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widetilde{\mu}^\alpha_t\right\|_{L^\infty(dv)} \\ &\simeq \left\|\left(1+|\cdot|\right)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widehat{\mu}_\delta\right\|_{L^\infty(dv)} \\ &\simeq \left\|\left\{\left(1+|a_k|\right)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widehat{\mu}_r(a_k)\right\}^\infty_k\right\|_{l^\infty}.\end{aligned}$$

Theorem 2.3 *Let $0 < p \leq q < \infty$ and $\mu \geq 0$. Then, the following statements are equivalent*:

(1) *μ is a vanishing (p,q)-Fock–Carleson measure*;
(2) $(1+|z|)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widetilde{\mu}^\alpha_t(z) \to 0$ *as $|z| \to \infty$ for some or any $t > 0$*;
(3) $(1+|z|)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widehat{\mu}_\delta(z) \to 0$ *as $|z| \to \infty$ for some or any $\delta > 0$*;
(4) *The sequence*

$$\lim_{k\to\infty}\left(1+|a_k|\right)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widehat{\mu}_r(a_k) = 0$$

for some or any r-lattice $\{a_k\}^\infty_{k=1}$.

Theorem 2.4 *Let $0 < q < p < \infty$ and $\mu \geq 0$. Then, the following statements are equivalent:*

(1) *μ is a (p,q)-Fock–Carleson measure;*
(2) *μ is a vanishing (p,q)-Fock–Carleson measure;*
(3) $(1+|z|)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widetilde{\mu}_t^{\alpha}(z) \in L^{\frac{p}{p-q}}(dv)$ *for some or any $t > 0$;*
(4) $(1+|z|)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widehat{\mu}_{\delta}(z) \in L^{\frac{p}{p-q}}(dv)$ *for some or any $\delta > 0$;*
(5) *The sequence*

$$\left\{\left(1+|a_k|\right)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widehat{\mu}_r(a_k)\right\}_{k=1}^{\infty} \in l^{\frac{p}{p-q}}$$

for some or any r-lattice $\{a_k\}_{k=1}^{\infty}$.

Furthermore,

$$\begin{aligned}\|\iota_p\|^q_{F^p_\alpha \to L^q_{\alpha,\mu}\cap H(\mathbb{C}^n)} &\simeq \left\|\left(1+|\cdot|\right)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widetilde{\mu}_t^{\alpha}\right\|_{L^{\frac{p}{p-q}}} \\ &\simeq \left\|\left(1+|\cdot|\right)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widehat{\mu}_{\delta}\right\|_{L^{\frac{p}{p-q}}} \\ &\simeq \left\|\left\{\left(1+|a_k|\right)^{(\frac{1}{p}-\frac{1}{q})q\alpha}\widehat{\mu}_r(a_k)\right\}_k^{\infty}\right\|_{l^{\frac{p}{p-q}}}.\end{aligned}$$

The well-informed reader will note that the descriptions in Theorem 2.2 are a little different from those in [5, Theorem 3.4] and [9, Theorem 25]. The reason why this situation occurred is mainly the measurement of the unit ball. For example, in the case of Fock-type spaces,

$$C_1\left[\Phi'\left(|z|^2\right)\right]^{-1}\left[\Psi'\left(|z|^2\right)\right]^{-(n-1)} \leq \left|B(z,r)\right| \leq C_2\left[\Phi'\left(|z|^2\right)\right]^{-1}\left[\Psi'\left(|z|^2\right)\right]^{-(n-1)}$$

for some constants C_1 and C_2. See [9] for more information.

3 Bounded Toeplitz operators

In this section, we are going to achieve some characterizations on those $\mu \geq 0$ such that the Toeplitz operator T_μ is bounded or compact from F^p_α to F^∞_α and the converse, respectively, for $0 < p \leq \infty$. To study the compactness, we need the following lemma, part of which can been found in [2, Lemma 3.2].

Lemma 3.1 *Let $0 < p,q \leq \infty$ and suppose μ is a t-Fock–Carleson measure. The Toeplitz operator T^α_μ is well defined on F^p_α. Moreover, for $R > 0$, the Toeplitz operator $T^\alpha_{\mu_R}$ is compact from F^p_α to F^q_α, where $\mu_R(V) = \int_{V\cap\{z:|z|\leq R\}} d\mu$ for $V \subset \mathbb{C}^n$ measurable.*

Proof It suffices to discuss the case of an infinite index, because by [2, Lemmas 3.1, 3.2], we can conclude that T^α_μ is well defined on F^p_α and $T^\alpha_{\mu_R}$ is compact from F^p_α to F^q_α for $0 < p,q < \infty$, respectively.

For any $f \in F^\infty_\alpha$, by the definition of a Toeplitz operator, it suffices to prove that, for any $z \in \mathbb{C}^n$, when $\alpha > 0$,

$$\begin{aligned}&\int_{\mathbb{C}^n}\left|f^+_{\frac{\alpha}{2}}(w)\right|\left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right|e^{-|w|^2}\frac{d\mu(w)}{|w|^\alpha} \\ &\quad = \int_{|w|\geq 1}\left|f^+_{\frac{\alpha}{2}}(w)\right|\left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right|e^{-|w|^2}\frac{d\mu(w)}{|w|^\alpha} \qquad (1)\end{aligned}$$

$$+\int_{|w|<1}\left|f^{+}_{\frac{\alpha}{2}}(w)\right|\left|\left(K^{\alpha}_{z}\right)^{+}_{\frac{\alpha}{2}}(w)\right|e^{-|w|^2}\frac{d\mu(w)}{|w|^{\alpha}}<\infty. \tag{2}$$

Note that for a fixed small $0<\delta<R$, Theorems 2.2 and 2.3 show us that

$$\left(1+|\cdot|\right)^{(t-1)\alpha}\widehat{\mu}_{\delta}(\cdot)\in L^{\infty}(dv), \tag{3}$$

for some t, maybe different from the condition. Now, we divide the integral above into two cases. When $|w|\geq 1$, [2, Lemma 2.3] tells us that

$$\begin{aligned}
\mathrm{I}_{(1)} &\lesssim \int_{|w|\geq 1}\left(\frac{|f^{+}_{\frac{\alpha}{2}}(w)|}{(1+|w|)^{\alpha}}e^{-\frac{1}{2}|w|^2}\right)\left|\left(K^{\alpha}_{z}\right)^{+}_{\frac{\alpha}{2}}(w)\right|e^{-\frac{1}{2}|w|^2}\widehat{\mu}_{\delta}(w)\,dv(w)\\
&\lesssim \|f\|_{F^{\infty}_{\alpha}}\left\|\frac{\widehat{\mu}_{\delta}(\cdot)}{(1+|\cdot|)^{(1-t)\alpha}}\right\|_{L^{\infty}}\underbrace{\int_{\mathbb{C}^n}\left|\left(K^{\alpha}_{z}\right)^{+}_{\frac{\alpha}{2}}(w)\right|e^{-\frac{1}{2}|w|^2}\frac{dv(w)}{|w|^{(t-1)\alpha}}}_{\mathrm{II}}.
\end{aligned}$$

According to the proof in [4, Lemma 3.4], for any $z\in\mathbb{C}^n$,

$$\begin{aligned}
\mathrm{II} &\lesssim \sum_{|m|>\frac{\alpha}{2}}\frac{|z|^{|m|}\Gamma(n+|m|)}{m!\Gamma(n-\frac{\alpha}{2}+|m|)}\int_{\mathbb{C}^n}|w|^{|m|+|t-1|\alpha}e^{-\frac{1}{2}|w|^2}\,dv(w)\\
&\lesssim \sum_{|m|>\frac{\alpha}{2}}|z|^{|m|}\left(|m!|\right)^{-1}n^{|m|}|m|^{\frac{\alpha}{2}}\Gamma\left(n+\frac{|m|+|t-1|\alpha}{2}\right)<\infty.
\end{aligned}$$

On the other hand, the Cauchy integral formula over the unit polydisk and the maximum modulus theorem show us that

$$\begin{aligned}
\sup_{|w|<1}\left|f^{+}_{\frac{\alpha}{2}}(w)\right||w|^{-\frac{\alpha}{2}} &\lesssim \sup_{|w|<1}\sum_{|\gamma|>\frac{\alpha}{2}}\left|\frac{\partial^{\gamma}f(0)}{\gamma!}\right||w|^{|\gamma|-\frac{\alpha}{2}}\\
&\lesssim \sup_{|w|<2}\left|f(w)\right|\sum_{k>\frac{\alpha}{2}}\frac{|w|^{k-\frac{\alpha}{2}}}{2^k}\\
&\lesssim \sup_{|w|<2}\left|f(w)\right|\left(1+|w|\right)^{-\alpha}.
\end{aligned} \tag{4}$$

Also, when $|w|<1$, by Stirling's formula,

$$\left|\left(K^{\alpha}_{z}\right)^{+}_{\frac{\alpha}{2}}(w)\right||w|^{-\frac{\alpha}{2}}\lesssim |w|^{-\frac{\alpha}{2}}\sum_{k>\frac{\alpha}{2}}\frac{k^{\frac{\alpha}{2}}}{k!}|z\bar{w}|^k\lesssim\sum_{k>\frac{\alpha}{2}}\frac{k^{\frac{\alpha}{2}}}{k!}|z|^k. \tag{5}$$

Therefore, we can see that

$$\begin{aligned}
I_{(2)} &\lesssim \int_{|w|<1}\left(\sup_{|w|<2}\frac{|f(w)|}{(1+|w|)^{\alpha}}\right)\left(\left|\left(K^{\alpha}_{z}\right)^{+}_{\frac{\alpha}{2}}(w)\right||w|^{-\frac{\alpha}{2}}\right)e^{-\frac{1}{2}|w|^2}\,d\mu(w)\\
&\lesssim \|f\|_{F^{\infty}_{\alpha}}\int_{|w|<1}\left(\left|\left(K^{\alpha}_{z}\right)^{+}_{\frac{\alpha}{2}}(w)\right||w|^{-\frac{\alpha}{2}}\right)e^{-\frac{1}{2}|w|^2}\,d\mu(w)\\
&\lesssim \|f\|_{F^{\infty}_{\alpha}}\left\|\frac{\widehat{\mu}_{\delta}(\cdot)}{(1+|\cdot|)^{(1-t)\alpha}}\right\|_{L^{\infty}}\sum_{k>\frac{\alpha}{2}}\frac{k^{\frac{\alpha}{2}}}{k!}|z|^k.
\end{aligned}$$

Sequentially, in order to prove the compactness of $T^{\alpha}_{\mu_R}$ for $q = \infty$, we suppose that $\{f_k\}_{k\geq 1} \subset F^p_\alpha(0 < p \leq \infty)$ is a bounded sequence and f_k uniformly converges to 0 on compact subset of $\mathbb{C}^n$ as $k \to \infty$. By Montel's theorem, we will show that

$$\lim_{k\to\infty} \left\| T^{\alpha}_{\mu_R} f_k \right\|_{F^\infty_\alpha} = 0.$$

By the definition of a Topelitz operator, we will omit the details about the case $\alpha \leq 0$ and the other comes into play:

$$\begin{aligned}
&\left|T^{\alpha}_{\mu_R} f_k(z)\right| \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \\
&\quad\lesssim \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \int_{\mathbb{C}^n} \left|(f_k)^-_{\frac{\alpha}{2}}(w)\right| \left|\left(K^\alpha_z\right)^-_{\frac{\alpha}{2}}(w)\right| e^{-|w|^2}\, d\mu_R(w) \\
&\qquad + \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \int_{\mathbb{C}^n} \left|(f_k)^+_{\frac{\alpha}{2}}(w)\right| \left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right| \frac{e^{-|w|^2}}{|w|^\alpha}\, d\mu_R(w).
\end{aligned} \tag{6}$$

Now, we pay attention to the integral $I_{(6)}$, because the first will be discussed in a similar way. For some suitable $r_0 > 0$, we divide the integral $I_{(6)}$ into

$$I_{(6)} = \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \int_{|w|\geq r_0} \left|(f_k)^+_{\frac{\alpha}{2}}(w)\right| \left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right| \frac{e^{-|w|^2}}{|w|^\alpha}\, d\mu_R(w) \tag{7}$$

$$\quad + \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \int_{|w|<r_0} \left|(f_k)^+_{\frac{\alpha}{2}}(w)\right| \left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right| \frac{e^{-|w|^2}}{|w|^\alpha}\, d\mu_R(w). \tag{8}$$

According to the proof in [4, Proposition 3.2],

$$\int_{\mathbb{C}^n} \left|K^\alpha_w(z)\right|^p e^{-\frac{p}{2}|z|^2} \frac{dv(z)}{(1+|z|)^\alpha} \lesssim \left(1+|w|\right)^{p\alpha-\alpha} e^{\frac{p}{2}|w|^2}, \quad 0 < p < \infty. \tag{9}$$

Therefore, if we combine [2, Lemma 2.3] and the condition (3), the integral $I_{(7)}$ comes into play, when $k \to \infty$,

$$\begin{aligned}
I_{(7)} &\lesssim \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \int_{r_0\leq|w|\leq R} \left|(f_k)^+_{\frac{\alpha}{2}}(w)\right| \frac{|(K^\alpha_z)^+_{\frac{\alpha}{2}}(w)|}{(1+|w|)^{(t-1)\alpha}} \frac{e^{-|w|^2}}{|w|^\alpha} \frac{\widehat{\mu}_\delta(w)\, dv(w)}{(1+|w|)^{(1-t)\alpha}} \\
&\lesssim \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \left\| \frac{\widehat{\mu}_\delta(\cdot)}{(1+|\cdot|)^{(1-t)\alpha}} \right\|_{L^\infty} \sup_{r_0\leq|w|\leq R} \left|(f_k)^+_{\frac{\alpha}{2}}(w)\right| \frac{|(K^\alpha_z)^+_{\frac{\alpha}{2}}(w)|}{e^{|w|^2}(1+|w|)^{t\alpha}} \\
&\lesssim \sup_{r_0\leq|w|\leq R} \frac{|f_k(w)|e^{-\frac{1}{2}|w|^2}}{(1+|w|)^{t\alpha}} \int_{\mathbb{C}^n} \frac{|K^\alpha_w(z)|}{e^{\frac{1}{2}|z|^2} e^{\frac{1}{2}|w|^2}} \frac{dv(z)}{(1+|z|)^\alpha} \to 0.
\end{aligned}$$

On the other hand, in view of the estimates (5) and (4),

$$\begin{aligned}
I_{(8)} &\lesssim \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \left(\sup_{|w|\leq 2r_0} \frac{|f_k(w)|e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha} \right) \int_{|w|\leq r_0} \left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right| e^{-\frac{1}{2}|w|^2} \frac{d\mu(w)}{|w|^{\frac{\alpha}{2}}} \\
&\lesssim \left(\frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \sum_{s>\frac{\alpha}{2}} \frac{s^{\frac{\alpha}{2}}}{s!} |z|^s \right) \left(\sup_{|w|\leq 2r_0} \frac{|f_k(w)|e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha} \right) \left\| \frac{\widehat{\mu}_\delta(\cdot)}{(1+|\cdot|)^{(1-t)\alpha}} \right\|_{L^\infty}
\end{aligned}$$

$$\lesssim \frac{|z|e^{|z|}e^{-\frac{1}{2}|z|^2}}{(1+|z|)^{-|\alpha|}}\left(\sup_{|w|\le 2r_0}\frac{|f_k(w)|e^{-\frac{1}{2}|w|^2}}{(1+|w|)^{\alpha}}\right)\left\|\frac{\widehat{\mu}_\delta(\cdot)}{(1+|\cdot|)^{(1-t)\alpha}}\right\|_{L^\infty}.$$

Thus, when $k\to\infty$, the integral $I_{(8)}$ will tend to zero. Now, the proof is finished. □

In the following, we will characterize the boundedness and compactness of positive Toeplitz operators from F^p_α to F^∞_α and from F^∞_α to F^p_α with $0<p<\infty$. Now, we state the main results as follows.

Theorem 3.2 *Let $0<p<\infty$, and $\mu\ge 0$. Then,*

(1) *$T^\alpha_\mu: F^p_\alpha\to F^\infty_\alpha$ is bounded if and only if μ is a p-Fock–Carleson measure. Furthermore,*

$$\|T^\alpha_\mu\|_{F^p_\alpha\to F^\infty_\alpha}\simeq\|\mu\|_p.$$

(2) *$T^\alpha_\mu: F^p_\alpha\to F^\infty_\alpha$ is compact if and only if μ is a vanishing p-Fock–Carleson measure.*

Proof (1) First, we assume that $T^\alpha_\mu: F^p_\alpha\to F^\infty_\alpha$ is bounded. From the proof in [2, Theorem 3.3], it follows that

$$\begin{aligned}\left|\widetilde{\mu}^\alpha_2(z)\right| &\lesssim \left(1+|z|\right)^{\alpha-\frac{\alpha}{p}}\left|T^\alpha_\mu\left(\frac{k^\alpha_{2,z}}{\|k^\alpha_{2,z}\|_{F^p_\alpha}}\right)(z)\right|\frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha}\\ &\lesssim \left(1+|z|\right)^{\alpha-\frac{\alpha}{p}}\|T^\alpha_\mu\|_{F^p_\alpha\to F^\infty_\alpha}\left\|\frac{k^\alpha_{2,z}}{\|k^\alpha_{2,z}\|_{F^p_\alpha}}\right\|_{F^p_\alpha}. \end{aligned}\tag{10}$$

Together with Theorem 2.2, μ is a p-Fock–Carleson measure. Furthermore,

$$\|\mu\|_p\simeq\sup_{z\in\mathbb{C}^n}\left|\widetilde{\mu}^\alpha_2(z)\right|\left(1+|z|\right)^{\frac{\alpha}{p}-\alpha}\lesssim\|T^\alpha_\mu\|_{F^p_\alpha\to F^\infty_\alpha}.$$

On the other hand, if μ is a p-Fock–Carleson measure, then $\widehat{\mu}_\delta(z)(1+|z|)^{\frac{\alpha}{p}-\alpha}$ is bounded for $\delta>0$. Given any $f\in F^p_\alpha$, we only want to obtain that, when $\alpha>0$,

$$\begin{aligned}&\sup_{z\in\mathbb{C}^n}\left|T^\alpha_\mu f(z)\right|\frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha}\\ &\quad\lesssim\sup_{z\in\mathbb{C}^n}\frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha}\int_{\mathbb{C}^n}\left|f^-_{\frac{\alpha}{2}}(w)\right|\left|\left(K^\alpha_z\right)^-_{\frac{\alpha}{2}}(w)\right|e^{-|w|^2}\,d\mu(w)\\ &\qquad+\sup_{z\in\mathbb{C}^n}\frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha}\int_{\mathbb{C}^n}\left|f^+_{\frac{\alpha}{2}}(w)\right|\left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right|\frac{e^{-|w|^2}}{|w|^\alpha}\,d\mu(w)<\infty.\end{aligned}\tag{11}$$

Now, we pay attention to the integral $I_{(11)}$, because the first will be discussed in a similar way. For some suitable $r_0>0$, we divide the integral $I_{(11)}$ into

$$I_{(11)}=\frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha}\int_{|w|\ge r_0}\left|f^+_{\frac{\alpha}{2}}(w)\right|\left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right|\frac{e^{-|w|^2}}{|w|^\alpha}\,d\mu(w)\tag{12}$$

$$+\frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha}\int_{|w|<r_0}\left|f^+_{\frac{\alpha}{2}}(w)\right|\left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right|\frac{e^{-|w|^2}}{|w|^\alpha}\,d\mu(w).\tag{13}$$

If choosing an r-lattice $\{a_k\}_{k\geq 1}$, then we can see that, by [2, Lemma 2.3] and the upper pointwise estimate in Lemma 2.1,

$$\begin{aligned}
I_{(12)} &\lesssim \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha}\int_{|w|\geq r_0}\left|(K_z^\alpha)^+_{\frac{\alpha}{2}}(w)\right|\frac{|f^+_{\frac{\alpha}{2}}(w)|e^{-\frac{1}{2}|w|^2}}{(1+|w|)^{\frac{\alpha}{p}}}\frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\frac{\widehat{\mu}^\alpha_\delta(w)\,dv(w)}{(1+|w|)^{-\frac{\alpha}{p}}} \\
&\lesssim \|f\|_{F^p_\alpha}\sum_{k:|a_k|\geq r+r_0}\int_{B(a_k,r)}\frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha}\left|(K_z^\alpha)^+_{\frac{\alpha}{2}}(w)\right|e^{-\frac{1}{2}|w|^2}\frac{\widehat{\mu}^\alpha_\delta(w)\,dv(w)}{(1+|w|)^{\alpha-\frac{\alpha}{p}}} \\
&\lesssim \|f\|_{F^p_\alpha}\left\|\frac{\widehat{\mu}_\delta(\cdot)}{(1+|\cdot|)^{\alpha-\frac{\alpha}{p}}}\right\|_{L^\infty}\sum_{k:|a_k|\geq r+r_0}\sup_{w\in B(a_k,r)}\frac{|(K_z^\alpha)^+_{\frac{\alpha}{2}}(w)|}{(1+|z|)^\alpha}\frac{e^{-\frac{1}{2}|w|^2}}{e^{\frac{1}{2}|z|^2}} \\
&\lesssim \|f\|_{F^p_\alpha}\left\|\frac{\widehat{\mu}_\delta(\cdot)}{(1+|\cdot|)^{\alpha-\frac{\alpha}{p}}}\right\|_{L^\infty}\sup_{|w|\geq r_0}\int_{\mathbb{C}^n}e^{-\frac{1}{2}|z|^2}\left|K_w^\alpha(z)\right|e^{-\frac{1}{2}|w|^2}\frac{dv(z)}{(1+|z|)^\alpha}.
\end{aligned}$$

Next, the integral $I_{(13)}$ comes into play in a similar way as in the estimate $I_{(8)}$,

$$\begin{aligned}
I_{(13)} &\lesssim \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha}\int_{|w|<r_0}\frac{|(K_z^\alpha)^+_{\frac{\alpha}{2}}(w)|}{|w|^{\frac{\alpha}{2}}e^{\frac{1}{2}|w|^2}}\left(\sup_{|w|<2r_0}\frac{|f(w)|e^{-\frac{1}{2}|w|^2}}{(1+|w|)^{\frac{\alpha}{p}}}\right)\frac{d\mu(w)}{(1+|w|)^{\alpha-\frac{\alpha}{p}}} \\
&\lesssim \|f\|_{F^p_\alpha}\left\|\frac{\widehat{\mu}_\delta(\cdot)}{(1+|\cdot|)^{\alpha-\frac{\alpha}{p}}}\right\|_{L^\infty}\left(\sum_{k>\frac{\alpha}{2}}\frac{k^{\frac{\alpha}{2}}}{k!}|z|^k\right)\frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha}.
\end{aligned}$$

All in all, T^α_μ is bounded from F^p_α to F^∞_α and $\|T^\alpha_\mu\|_{F^p_\alpha\to F^\infty_\alpha}\simeq\|\mu\|_p$.

(2) We suppose μ is a vanishing p-Fock–Carleson measure. By Theorem 2.3,

$$\widehat{\mu}_\delta(z)\big(1+|z|\big)^{\frac{\alpha}{p}-\alpha}\to 0,\quad \text{as } |z|\to\infty.$$

As $T^\alpha_{\mu_R}$ is compact from F^p_α to F^∞_α by Lemma 3.1, if we note that $\mu-\mu_R\geq 0$, $T^\alpha_{\mu-\mu_R}$ is also bounded from F^p_α to F^∞_α. For $r>0$,

$$\lim_{R\to\infty}\sup_{z\in\mathbb{C}^n}(\widehat{\mu-\mu_R})_r(z)\big(1+|z|\big)^{\frac{\alpha}{p}-\alpha}=0.$$

Therefore, the condition (1) tells us that, when $R\to\infty$,

$$\begin{aligned}
\left\|T^\alpha_\mu-T^\alpha_{\mu_R}\right\|_{F^p_\varphi\to F^\infty_\varphi} &= \left\|T^\alpha_{\mu-\mu_R}\right\|_{F^p_\varphi\to F^\infty_\varphi}\simeq\|\mu-\mu_R\|_p \\
&\simeq \sup_{z\in\mathbb{C}^n}(\widehat{\mu-\mu_R})_r(z)\big(1+|z|\big)^{\frac{\alpha}{p}-\alpha}\to 0.
\end{aligned}$$

Hence, we can see that $T^\alpha_\mu:F^p_\alpha\to F^\infty_\alpha$ is compact.

Conversely, we assume that $T^\alpha_\mu:F^p_\alpha\to F^\infty_\alpha$ is compact. Then, $\widehat{\mu}_\delta(z)(1+|z|)^{\frac{\alpha}{p}-\alpha}$ is bounded for $\delta>0$. Obviously, $\{\frac{k^\alpha_{2,z}}{\|k^\alpha_{2,z}\|_{F^p_\alpha}}:z\in\mathbb{C}^n\}$ is bounded in F^p_α. Therefore, $\{T^\alpha_\mu(\frac{k^\alpha_{2,z}}{\|k^\alpha_{2,z}\|_{F^p_\alpha}}):z\in\mathbb{C}^n\}$ is relatively compact in F^∞_α. For any sequence $\{z_j\}_{j\geq 1}$ with $|z_j|\to\infty$, there exists a subsequence of $\{T^\alpha_\mu(\frac{k^\alpha_{2,z_j}}{\|k^\alpha_{2,z_j}\|_{F^p_\alpha}})\}_{j\geq 1}$ converging to some h in F^∞_α. Without loss of generality, we may assume that

$$\lim_{|z_j|\to\infty}\left\|T^\alpha_\mu\left(\frac{k^\alpha_{2,z_j}}{\|k^\alpha_{2,z_j}\|_{F^p_\alpha}}\right)-h\right\|_{F^\infty_\alpha}=0.$$

Our goal is to show $h = 0$. We will only prove the case $\alpha > 0$. For any $w \in \mathbb{C}^n$, the definition of a Toeplitz operator implies that

$$\begin{aligned}
&\frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\left|T_\mu^\alpha\left(\frac{k_{2,z_j}^\alpha}{\|k_{2,z_j}^\alpha\|_{F_\alpha^p}}\right)(w)\right| \\
&\quad\lesssim \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\int_{\mathbb{C}^n}\left|\left(K_w^\alpha\right)_{\frac{\alpha}{2}}^-(\xi)\right|\frac{|(k_{2,z_j}^\alpha)_{\frac{\alpha}{2}}^-(\xi)|}{\|k_{2,z_j}^\alpha\|_{F_\alpha^p}}e^{-|\xi|^2}\,d\mu(\xi) \\
&\qquad+\frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\int_{\mathbb{C}^n}\left|\left(K_w^\alpha\right)_{\frac{\alpha}{2}}^+(\xi)\right|\frac{|(k_{2,z_j}^\alpha)_{\frac{\alpha}{2}}^+(\xi)|}{\|k_{2,z_j}^\alpha\|_{F_\alpha^p}}\frac{e^{-|\xi|^2}}{|\xi|^\alpha}\,d\mu(\xi). \qquad (14)
\end{aligned}$$

Now, we pay attention to the integral $I_{(14)}$ and we divide it into two integrals for some suitable $R > 0$

$$\begin{aligned}
I_{(14)} &\lesssim \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\int_{|\xi|>R}\left|\left(K_w^\alpha\right)_{\frac{\alpha}{2}}^+(\xi)\right|\frac{|(k_{2,z_j}^\alpha)_{\frac{\alpha}{2}}^+(\xi)|}{\|k_{2,z_j}^\alpha\|_{F_\alpha^p}}\frac{e^{-|\xi|^2}}{|\xi|^\alpha}\,d\mu(\xi) \qquad (15)\\
&\quad+\frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\int_{|\xi|\le R}\left|\left(K_w^\alpha\right)_{\frac{\alpha}{2}}^+(\xi)\right|\frac{|(k_{2,z_j}^\alpha)_{\frac{\alpha}{2}}^+(\xi)|}{\|k_{2,z_j}^\alpha\|_{F_\alpha^p}}\frac{e^{-|\xi|^2}}{|\xi|^\alpha}\,d\mu(\xi). \qquad (16)
\end{aligned}$$

First, the integral $I_{(15)}$ is estimated by, after denoting by $\{a_k\}_{k\ge1}$ an r-lattice and combing with [4, Proposition 3.2],

$$\begin{aligned}
I_{(15)} &\lesssim \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\int_{|\xi|>R}\left|\left(K_w^\alpha\right)_{\frac{\alpha}{2}}^+(\xi)\right|\frac{|(k_{2,z_j}^\alpha)_{\frac{\alpha}{2}}^+(\xi)|}{\|k_{2,z_j}^\alpha\|_{F_\alpha^p}}\frac{e^{-|\xi|^2}}{(1+|\xi|)^{\frac{\alpha}{p}}}\frac{\widehat{\mu}_\delta(\xi)\,d\nu(\xi)}{(1+|\xi|)^{\alpha-\frac{\alpha}{p}}} \\
&\lesssim \left\|\frac{\widehat{\mu}_\delta(\cdot)}{(1+|\cdot|)^{\alpha-\frac{\alpha}{p}}}\right\|_{L^\infty}\sum_{k:|a_k|>R+r}\left[\left(\sup_{\xi\in B(a_k,r)}\left|\left(K_w^\alpha\right)_{\frac{\alpha}{2}}^+(\xi)\right|e^{-\frac{1}{2}|\xi|^2}\frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\right)\right. \\
&\quad\left.\times\int_{B(a_k,r)}\frac{(1+|\xi|)^{(\frac{1}{2}-\frac{1}{p})\alpha}}{(1+|z_j|)^{(\frac{1}{2}-\frac{1}{p})\alpha}}\left|k_{2,z_j}^\alpha(\xi)\right|\frac{e^{-\frac{1}{2}|\xi|^2}}{|\xi|^{\frac{\alpha}{2}}}\,d\nu(\xi)\right] \\
&\lesssim \left\|\frac{\widehat{\mu}_\delta(\cdot)}{(1+|\cdot|)^{\alpha-\frac{\alpha}{p}}}\right\|_{L^\infty}\left(\sup_{|\xi|>R}\int_{\mathbb{C}^n}e^{-\frac{1}{2}|w|^2}\left|\left(K_\xi^\alpha\right)_{\frac{\alpha}{2}}^+(w)\right|e^{-\frac{1}{2}|\xi|^2}\frac{d\nu(w)}{(1+|w|)^\alpha}\right) \\
&\quad\times\sum_{k:|a_k|>R+r}\int_{B(a_k,r)}\frac{(1+|\xi|)^{(\frac{1}{2}-\frac{1}{p})\alpha}}{(1+|z_j|)^{(\frac{1}{2}-\frac{1}{p})\alpha}}\left|k_{2,z_j}^\alpha(\xi)\right|\frac{e^{-\frac{1}{2}|\xi|^2}}{|\xi|^{\frac{\alpha}{2}}}\,d\nu(\xi) \\
&\lesssim \left\|\frac{\widehat{\mu}_\delta(\cdot)}{(1+|\cdot|)^{\alpha-\frac{\alpha}{p}}}\right\|_{L^\infty}\int_{|\xi|>R}\left(1+|z_j-\xi|\right)^{|\frac{1}{2}-\frac{1}{p}||\alpha|}e^{-\frac{1}{8}|z_j-\xi|^2}\,d\nu(\xi).
\end{aligned}$$

This implies the integral $I_{(15)}$ will tend to zero as $R \to \infty$. Next, the integral $I_{(16)}$ will be divided into two subcases.

$$\begin{aligned}
I_{(16)} &\lesssim \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\int_{|\xi|<r_0}\left|\left(K_w^\alpha\right)_{\frac{\alpha}{2}}^+(\xi)\right|\frac{|(k_{2,z_j}^\alpha)_{\frac{\alpha}{2}}^+(\xi)|}{\|k_{2,z_j}^\alpha\|_{F_\alpha^p}}\frac{e^{-|\xi|^2}}{|\xi|^\alpha}\,d\mu(\xi) \\
&\quad+\frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\int_{r_0\le|\xi|\le R}\left|\left(K_w^\alpha\right)_{\frac{\alpha}{2}}^+(\xi)\right|\frac{|(k_{2,z_j}^\alpha)_{\frac{\alpha}{2}}^+(\xi)|}{\|k_{2,z_j}^\alpha\|_{F_\alpha^p}}\frac{e^{-|\xi|^2}}{|\xi|^\alpha}\,d\mu(\xi), \qquad (17)
\end{aligned}$$

for a small enough $0 < r_0 < R$. Since $\frac{k^\alpha_{2,z}}{\|k^\alpha_{2,z}\|_{F^p_\alpha}} \to 0$ uniformly on a compact subset of $\mathbb{C}^n$ when $|z| \to \infty$, it is easy to see that

$$\begin{aligned}&\lim_{|z_j|\to\infty}\int_{r_0\le|\xi|\le R}\left|\left(K^\alpha_w\right)^+_{\frac{\alpha}{2}}(\xi)\right|\frac{|(k^\alpha_{2,z_j})^+_{\frac{\alpha}{2}}(\xi)|}{\|k^\alpha_{2,z_j}\|_{F^p_\alpha}}\frac{e^{-|\xi|^2}}{|\xi|^\alpha}\,d\mu(\xi)\\&\quad\lesssim\left\|\frac{\widehat{\mu}_\delta(\cdot)}{(1+|\cdot|)^{\alpha-\frac{\alpha}{p}}}\right\|_{L^\infty}\lim_{|z_j|\to\infty}\int_{r_0\le|\xi|\le R}\left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(\xi)\right|\frac{|(k^\alpha_{2,z_j})^+_{\frac{\alpha}{2}}(\xi)|}{\|k^\alpha_{2,z_j}\|_{F^p_\alpha}}\frac{e^{-|\xi|^2}\,dv(\xi)}{(1+|\xi|)^{\frac{\alpha}{p}}}=0.\end{aligned}$$

The other is, as follows, if we use a similar way as in the estimate $I_{(8)}$,

$$\begin{aligned}I_{(17)}&\lesssim\frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\int_{|\xi|<r_0}\frac{|(K^\alpha_w)^+_{\frac{\alpha}{2}}(\xi)|}{|w|^{\frac{1}{2}\alpha}\|k^\alpha_{2,z_j}\|_{F^p_\alpha}}\left(\sup_{|\xi|\le2r_0}\frac{|(k^\alpha_{2,z_j})^+_{\frac{\alpha}{2}}(\xi)|}{(1+|\xi|)^{\frac{\alpha}{p}}}\right)\frac{e^{-|\xi|^2}\,d\mu(\xi)}{(1+|\xi|)^{\alpha-\frac{\alpha}{p}}}\\&\lesssim\frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\int_{|\xi|<r_0}\left(\left|\left(K^\alpha_w\right)^+_{\frac{\alpha}{2}}(\xi)\right||\xi|^{-\frac{\alpha}{2}}\right)\frac{e^{-\frac{1}{2}|\xi|^2}}{(1+|\xi|)^{\alpha-\frac{\alpha}{p}}}\,d\mu(\xi)\\&\lesssim\left\|\frac{\widehat{\mu}_\delta(\cdot)}{(1+|\cdot|)^{\alpha-\frac{\alpha}{p}}}\right\|_{L^\infty}\left(\sum_{k>\frac{\alpha}{2}}\frac{k^{\frac{\alpha}{2}}}{k!}|w|^k\right)\frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\\&\lesssim\left\|\frac{\widehat{\mu}_\delta(\cdot)}{(1+|\cdot|)^{\alpha-\frac{\alpha}{p}}}\right\|_{L^\infty}\frac{|w|e^{|w|}e^{-\frac{1}{2}|w|^2}}{(1+|w|)^{-|\alpha|}}.\end{aligned}$$

This implies that the integral $I_{(17)}$ tends to zero if we let r_0 go to zero simultaneously. All in all, we know that $\lim_{|z_j|\to\infty}T^\alpha_\mu(\frac{k^\alpha_{2,z_j}}{\|k^\alpha_{2,z_j}\|_{F^p_\alpha}})(w)=0$ for any $w\in\mathbb{C}^n$. Do not forget that $\lim_{|z_j|\to\infty}T^\alpha_\mu(\frac{k^\alpha_{2,z_j}}{\|k^\alpha_{2,z_j}\|_{F^p_\alpha}})(w)=h(w)$ for any $w\in\mathbb{C}^n$ under the assumption. Hence, $h=0$, that is

$$\lim_{|z_j|\to\infty}\left\|T^\alpha_\mu\left(\frac{k^\alpha_{2,z_j}}{\|k^\alpha_{2,z_j}\|_{F^p_\alpha}}\right)\right\|_{F^\infty_\varphi}=0.$$

This, together with the estimate (10), yields that, when $|z_j|\to\infty$,

$$\left|\widetilde{\mu}^\alpha_2(z)\right|\lesssim\left(1+|z|\right)^{\alpha-\frac{\alpha}{p}}\left\|T^\alpha_\mu\left(\frac{k^\alpha_{2,z_j}}{\|k^\alpha_{2,z_j}\|_{F^p_\alpha}}\right)\right\|_{F^\infty_\varphi}\to0.$$

Hence, we can conclude that μ is a vanishing p-Fock–Carleson measure. □

Theorem 3.3 *Let $0<p<\infty$ and $\mu\ge0$. Then, the following statements are equivalent*:

1. $T^\alpha_\mu: F^\infty_\alpha\to F^p_\alpha$ *is bounded*;
2. $T^\alpha_\mu: F^\infty_\alpha\to F^p_\alpha$ *is compact*;
3. $(1+|z|)^{\alpha-\frac{\alpha}{p}}\widetilde{\mu}^\alpha_t(z)\in L^p$ *for some or any* $t>0$;
4. $(1+|z|)^{\alpha-\frac{\alpha}{p}}\widehat{\mu}_\delta(z)\in L^p$ *for some or any* $\delta>0$;
5. *The sequence* $\{(1+|a_k|)^{\alpha-\frac{\alpha}{p}}\widehat{\mu}_r(a_k)\}^\infty_{k=1}\in l^p$ *for some or any r-lattice* $\{a_k\}^\infty_{k=1}$.

Furthermore,

$$\begin{aligned}\|T^\alpha_\mu\|_{F^p_\alpha\to F^q_\alpha}&\simeq\left\|\left(1+|\cdot|\right)^{\alpha-\frac{\alpha}{p}}\widetilde{\mu}^\alpha_t\right\|_{L^p}\simeq\left\|\left(1+|\cdot|\right)^{\alpha-\frac{\alpha}{p}}\widehat{\mu}_\delta\right\|_{L^p}\\&\simeq\left\|\left\{\left(1+|a_k|\right)^{\alpha-\frac{\alpha}{p}}\widehat{\mu}_r(a_k)\right\}^\infty_k\right\|_{l^p}.\end{aligned}$$

Proof In view of [2, Lemma 2.4], the statements (3), (4), and (5) are equivalent.

(1) ⇒ (5): Suppose that T_μ^α is bounded from F_α^∞ to F_α^p. Let $\{a_k\}_{k=1}^\infty$ be an r-lattice and $\{\lambda_k\}_{k\geq 1} \in l^\infty$, and set

$$f(z) = \sum_{k=1}^\infty \lambda_k \frac{k_{2,a_k}^\alpha(z)}{\|k_{2,a_k}^\alpha\|_{F_\alpha^\infty}} = \sum_{k=1}^\infty \lambda_k \frac{k_{2,a_k}^\alpha(z)}{(1+|a_k|)^{-\frac{\alpha}{2}}}, \quad z \in \mathbb{C}^n.$$

We can claim that

$$f \in F_\alpha^\infty \quad \text{and} \quad \|f\|_{F_\alpha^\infty} \lesssim \sup_k\{\lambda_k\}.$$

To that end, by the estimate (3) in Lemma 2.1, we can follow the norm $\|k_{2,a_k}^\alpha\|_{F_\alpha^\infty}$ from that

$$\frac{|k_{2,a_k}^\alpha(z)|e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} = \frac{|K_{a_k}^\alpha(z)|}{\sqrt{K_{a_k}^\alpha(a_k)}} \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \lesssim \frac{\|K_{a_k}^\alpha\|_{F_\alpha^\infty} e^{-\frac{1}{2}|a_k|^2}}{(1+|a_k|)^{\frac{\alpha}{2}}}.$$

Hence, we have $T_\mu^\alpha f \in F_\alpha^p$ by the condition. Khinchine's inequality and Fubini's Theorem show that

$$\begin{aligned}
&\int_{\mathbb{C}^n} \left(\sum_{k=1}^\infty \left| \lambda_k \frac{T_\mu^\alpha(k_{2,a_k}^\alpha)(z)}{(1+|a_k|)^{-\frac{\alpha}{2}}} \right|^2 \right)^{\frac{p}{2}} \frac{e^{-\frac{p}{2}|z|^2}}{(1+|z|)^\alpha} \, dv(z) \\
&\quad \lesssim \int_{\mathbb{C}^n} \int_0^1 \left| \sum_{k=1}^\infty \Psi_k(t) \lambda_j \frac{T_\mu^\alpha(k_{2,a_k}^\alpha)(z)}{(1+|a_k|)^{-\frac{\alpha}{2}}} \right|^p dt \frac{e^{-\frac{p}{2}|z|^2}}{(1+|z|)^\alpha} \, dv(z) \\
&\quad \lesssim \int_0^1 \left\| T_\mu^\alpha \left(\sum_{k=1}^\infty \Psi_k(t) \lambda_j \frac{k_{2,a_k}^\alpha(z)}{(1+|a_k|)^{-\frac{\alpha}{2}}} \right) \right\|_{F_\alpha^p}^p dt,
\end{aligned}$$

where Ψ_k is the kth Rademacher function on $[0,1]$. By the bounedness of T_μ^α,

$$\int_0^1 \left\| T_\mu^\alpha \left(\sum_{k=1}^\infty \Psi_k(t) \lambda_j \frac{k_{2,a_k}^\alpha(z)}{(1+|a_k|)^{-\frac{\alpha}{2}}} \right) \right\|_{F_\alpha^p}^p dt \lesssim \| T_\mu^\alpha \|_{F_\alpha^\infty \to F_\alpha^p}^p \sup_{k\geq 1} |\lambda_k|^p.$$

Using the property of an r-lattice, we will have the estimate in other directions,

$$\begin{aligned}
&\int_{\mathbb{C}^n} \left(\sum_{k=1}^\infty \left| \lambda_k \frac{T_\mu^\alpha(k_{2,a_k}^\alpha)(z)}{(1+|a_k|)^{-\frac{\alpha}{2}}} \right|^2 \right)^{\frac{p}{2}} \frac{e^{-\frac{p}{2}|z|^2}}{(1+|z|)^\alpha} \, dv(z) \\
&\quad \gtrsim \sum_{j=1}^\infty \int_{B(a_j,r)} \left(\sum_{k=1}^\infty \left| \lambda_k \frac{T_\mu^\alpha(k_{2,a_k}^\alpha)(z)}{(1+|a_k|)^{-\frac{\alpha}{2}}} \right|^2 \right)^{\frac{p}{2}} \frac{e^{-\frac{p}{2}|z|^2}}{(1+|z|)^\alpha} \, dv(z) \\
&\quad \gtrsim \sum_{j=1}^\infty \int_{B(a_j,r)} \left| \lambda_j \frac{T_\mu^\alpha(k_{2,a_j}^\alpha)(z)}{(1+|a_j|)^{-\frac{\alpha}{2}}} \right|^p \frac{e^{-\frac{p}{2}|z|^2}}{(1+|z|)^\alpha} \, dv(z).
\end{aligned}$$

By the upper pointwise estimate in Lemma 2.1, it is easy to see that

$$\left| \frac{T_\mu^\alpha(k_{2,a_j}^\alpha)(a_j)}{(1+|a_j|)^{-\frac{\alpha}{2}}} \right|^p \frac{e^{-\frac{p}{2}|a_j|^2}}{(1+|a_j|)^\alpha} \lesssim \int_{B(a_j,r)} \left| \lambda_j \frac{T_\mu^\alpha(k_{2,a_j}^\alpha)(z)}{(1+|a_j|)^{-\frac{\alpha}{2}}} \right|^p \frac{e^{-\frac{p}{2}|z|^2}}{(1+|z|)^\alpha} \, dv(z).$$

The above analysis implies that

$$\int_{\mathbb{C}^n}\left(\sum_{k=1}^{\infty}\left|\lambda_k\frac{T_\mu^\alpha(k_{2,a_k}^\alpha)(z)}{(1+|a_k|)^{-\frac{\alpha}{2}}}\right|^2\right)^{\frac{p}{2}}\frac{e^{-\frac{p}{2}|z|^2}\,d\nu(z)}{(1+|z|)^\alpha}\gtrsim\sum_{j=1}^{\infty}\left|\lambda_j\frac{T_\mu^\alpha(k_{2,a_j}^\alpha)(a_j)}{(1+|a_j|)^{-\frac{\alpha}{2}}}\right|^p\frac{e^{-\frac{p}{2}|a_j|^2}}{(1+|a_j|)^\alpha}.$$

Using the definition of the Toeplitz operator T_μ^α, and similar calculations as in [2, Page 42], we can have the details as follows, when $\alpha\le 0$,

$$\begin{aligned}&\sum_{j=1}^{\infty}\left|\lambda_j\frac{T_\mu^\alpha(k_{2,a_j}^\alpha)(a_j)}{(1+|a_j|)^{-\frac{\alpha}{2}}}\right|^p\frac{e^{-\frac{p}{2}|a_j|^2}}{(1+|a_j|)^\alpha}\\
&\quad\gtrsim\sum_{j=1}^{\infty}\frac{|\lambda_j|^p}{(1+|a_j|)^{\alpha-p\alpha}}\left(\int_{B(a_j,r)}\left|k_{2,a_j}^\alpha(w)\right|^2e^{-|w|^2}\frac{d\mu(w)}{|w|^\alpha}\right)^p\\
&\quad\gtrsim\sum_{j=1}^{\infty}\frac{|\lambda_j|^p}{(1+|a_j|)^{\alpha-p\alpha}}\left(\int_{B(a_j,r)}\frac{|w|^{-\alpha}}{(1+|a_j|)^{-\alpha}}\,d\mu(w)\right)^p\\
&\quad\gtrsim\sum_{j=1}^{\infty}|\lambda_j|^p\frac{\mu(B(a_j,r))^p}{(1+|a_j|)^{\alpha-p\alpha}}.\end{aligned}$$

In the case $\alpha>0$, because $\max_{z\in\mathbb{C}^n}\{1,|z|^\alpha\}\lesssim(1+|z|)^\alpha$ for any $z\in\mathbb{C}^n$,

$$\begin{aligned}&\sum_{j=1}^{\infty}\left|\lambda_j\frac{T_\mu^\alpha(k_{2,a_j}^\alpha)(a_j)}{(1+|a_j|)^{-\frac{\alpha}{2}}}\right|^p\frac{e^{-\frac{p}{2}|a_j|^2}}{(1+|a_j|)^\alpha}\\
&\quad\gtrsim\sum_{j=1}^{\infty}\frac{|\lambda_j|^p}{(1+|a_j|)^{\alpha-p\alpha}}\left(\int_{B(a_j,r)}\left|\left(k_{2,a_j}^\alpha\right)_{\frac{\alpha}{2}}^{-}(w)\right|^2e^{-|w|^2}\,d\mu(w)\right.\\
&\qquad\left.+\int_{B(a_j,r)}\left|\left(k_{2,a_j}^\alpha\right)_{\frac{\alpha}{2}}^{+}(w)\right|^2e^{-|w|^2}\frac{d\mu(w)}{|w|^\alpha}\right)^p\\
&\quad\gtrsim\sum_{j=1}^{\infty}\frac{|\lambda_j|^p}{(1+|a_j|)^{\alpha-p\alpha}}\left(\int_{B(a_j,r)}\left|k_{2,a_j}^\alpha(w)\right|^2e^{-|w|^2}\frac{d\mu(w)}{(1+|w|)^\alpha}\right)^p\\
&\quad\gtrsim\sum_{j=1}^{\infty}\frac{|\lambda_j|^p}{(1+|a_j|)^{\alpha-p\alpha}}\left(\int_{B(a_j,r)}\frac{(1+|a_j|)^\alpha}{(1+|w|)^\alpha}\,d\mu(w)\right)^p\\
&\quad\gtrsim\sum_{j=1}^{\infty}|\lambda_j|^p\frac{\mu(B(a_j,r))^p}{(1+|a_j|)^{\alpha-p\alpha}}.\end{aligned}$$

If we set $\beta_j=|\lambda_j|^p$, we then know that $\{\beta_j\}_{j=1}^\infty\in l^\infty$. Thus, we further have

$$\sum_{j=1}^{\infty}\beta_j\frac{\mu(B(a_j,r))^p}{(1+|a_j|)^{\alpha-p\alpha}}\lesssim\|T_\mu^\alpha\|_{F_\alpha^\infty\to F_\alpha^p}^p\sup_{k\ge1}|\lambda_k|^p.$$

This implies that

$$\left\|\left\{\frac{\mu(B(a_j,r))}{(1+|a_j|)^{\frac{\alpha}{p}-\alpha}}\right\}_{j\ge1}\right\|_{l^p}\lesssim\|T_\mu^\alpha\|_{F_\alpha^\infty\to F_\alpha^p}.$$

In order to finish the proof, we only need to prove (4) ⇒ (2) when $\alpha > 0$. The basic inequality $(a+b)^p \le 2^{p-1}(a^p+b^p)$ when $p > 1$, $(a+b)^p \le a^p + b^p$ when $0 < p \le 1$ for arbitrary positive numbers a, b tells us that, for any $f \in F_\alpha^\infty$,

$$\begin{aligned}
&\int_{\mathbb{C}^n} \left|T_\mu^\alpha f(z)\right|^p e^{-\frac{p}{2}|z|^2} \frac{dv(z)}{(1+|z|)^\alpha} \\
&\lesssim \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} \left|f_{\frac{\alpha}{2}}^-(w)\right| \left|(K_z^\alpha)_{\frac{\alpha}{2}}^-(w)\right| e^{-|w|^2} e^{-\frac{1}{2}|z|^2}\, d\mu(w) \right)^p \frac{dv(z)}{(1+|z|)^\alpha} \\
&\quad + \int_{\mathbb{C}^n} \underbrace{\left(\int_{\mathbb{C}^n} \left|f_{\frac{\alpha}{2}}^+(w)\right| \left|(K_z^\alpha)_{\frac{\alpha}{2}}^+(w)\right| e^{-|w|^2} e^{-\frac{1}{2}|z|^2} \frac{d\mu(w)}{|w|^\alpha} \right)^p}_{\text{III}} \frac{dv(z)}{(1+|z|)^\alpha}.
\end{aligned} \tag{18}$$

Now, we pay attention to the integral $I_{(18)}$, because the first will be discussed in a similar way. For some suitable $r_0 > 0$, we use that basic inequality again to divide the integral III into

$$\text{III} \lesssim \left(\int_{|w|\ge r_0} \left|(K_z^\alpha)_{\frac{\alpha}{2}}^+(w)\right| \left|f_{\frac{\alpha}{2}}^+(w)\right| e^{-|w|^2} e^{-\frac{1}{2}|z|^2} \frac{d\mu(w)}{|w|^\alpha} \right)^p \tag{19}$$

$$+ \left(\int_{|w|< r_0} \left|(K_z^\alpha)_{\frac{\alpha}{2}}^+(w)\right| \left|f_{\frac{\alpha}{2}}^+(w)\right| e^{-|w|^2} e^{-\frac{1}{2}|z|^2} \frac{d\mu(w)}{|w|^\alpha} \right)^p. \tag{20}$$

After we put the integrals $I_{(19)}$ and $I_{(20)}$ into the integral $I_{(18)}$, we find that

$$I_{(18)} \lesssim \int_{\mathbb{C}^n} \left(\int_{|w|\ge r_0} \left|f_{\frac{\alpha}{2}}^+(w)\right| \frac{|(K_z^\alpha)_{\frac{\alpha}{2}}^+(w)|}{e^{|w|^2} e^{\frac{1}{2}|z|^2}} \frac{d\mu(w)}{|w|^\alpha} \right)^p \frac{dv(z)}{(1+|z|)^\alpha} \tag{21}$$

$$+ \int_{\mathbb{C}^n} \left(\int_{|w|< r_0} \left|f_{\frac{\alpha}{2}}^+(w)\right| \frac{|(K_z^\alpha)_{\frac{\alpha}{2}}^+(w)|}{e^{|w|^2} e^{\frac{1}{2}|z|^2}} \frac{d\mu(w)}{|w|^\alpha} \right)^p \frac{dv(z)}{(1+|z|)^\alpha}. \tag{22}$$

If choosing an r-lattice $\{a_k\}_{k\ge 1}$, we can see that, when $0 < p \le 1$,

$$\begin{aligned}
I_{(19)} &\lesssim \left[\sum_{k:|a_k|\ge r_0+r} \int_{B(a_k,r)} \left|(K_z^\alpha)_{\frac{\alpha}{2}}^+(w)\right| \left|f_{\frac{\alpha}{2}}^+(w)\right| e^{-\frac{1}{2}|z|^2} \frac{e^{-|w|^2} \widehat{\mu}_\delta(w)}{(1+|w|)^\alpha}\, dv(w) \right]^p \\
&\lesssim \sum_{k:|a_k|\ge r_0+r} \sup_{w\in B(a_k,r)} \left|(K_z^\alpha)_{\frac{\alpha}{2}}^+(w)\right|^p \left|f_{\frac{\alpha}{2}}^+(w)\right|^p e^{-\frac{p}{2}|z|^2} e^{-p|w|^2} \frac{\widehat{\mu}_\delta^p(w)}{(1+|w|)^{p\alpha}} \\
&\lesssim \|f\|_{F_\alpha^\infty}^p \sum_{k:|a_k|\ge r_0+r} \frac{\widehat{\mu}_\delta^p(a_k)}{(1+|a_k|)^{-\alpha}} \sup_{w\in B(a_k,r)} \left|(K_z^\alpha)_{\frac{\alpha}{2}}^+(w)\right|^p \frac{e^{-\frac{p}{2}|z|^2} e^{-\frac{p}{2}|w|^2}}{(1+|w|)^\alpha} \\
&\lesssim \|f\|_{F_\alpha^\infty}^p \sum_{k:|a_k|\ge r_0+r} \frac{\widehat{\mu}_\delta^p(a_k)}{(1+|a_k|)^{-\alpha}} \int_{B(a_k,r)} \frac{|(K_z^\alpha)_{\frac{\alpha}{2}}^+(w)|^p}{e^{\frac{p}{2}|z|^2} e^{\frac{p}{2}|w|^2}} \frac{dv(w)}{(1+|w|)^\alpha} \\
&\lesssim \|f\|_{F_\alpha^\infty}^p \int_{\mathbb{C}^n} e^{-\frac{p}{2}|z|^2} \left|(K_z^\alpha)_{\frac{\alpha}{2}}^+(w)\right|^p e^{-\frac{p}{2}|w|^2} \widehat{\mu}_\delta^p(w)\, dv(w).
\end{aligned}$$

Next, Fubini's theorem and the estimate (9) show us, in detail, that

$$\begin{aligned}I_{(21)} &\lesssim \|f\|^p_{F^\infty_\alpha}\int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n} e^{-\frac{p}{2}|z|^2}\left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right|^p e^{-\frac{p}{2}|w|^2}\widehat{\mu}^p_\delta(w)\,dv(w)\right)\frac{dv(z)}{(1+|z|)^\alpha}\\ &\lesssim \|f\|^p_{F^\infty_\alpha}\int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n} e^{-\frac{p}{2}|z|^2}\left|\left(K^\alpha_w\right)^+_{\frac{\alpha}{2}}(z)\right|^p e^{-\frac{p}{2}|w|^2}\frac{dv(z)}{(1+|z|)^\alpha}\right)\widehat{\mu}^p_\delta(w)\,dv(w)\\ &\lesssim \|f\|^p_{F^\infty_\alpha}\int_{\mathbb{C}^n}\frac{\widehat{\mu}^p_\delta(w)}{(1+|w|)^{\alpha-p\alpha}}\,dv(w).\end{aligned}$$

Combining [2, Lemma 2.3], the estimate (9), and Hölder's inequality, we continue to estimate the case $p>1$ as follows,

$$\begin{aligned}I_{(19)} &\lesssim \left(\int_{|w|\ge r_0}\left|f^+_{\frac{\alpha}{2}}(w)\right|^p\left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right|e^{-\frac{p+1}{2}|w|^2}e^{-\frac{1}{2}|z|^2}\widehat{\mu}^p_\delta(w)\frac{dv(w)}{(1+|w|)^\alpha}\right)\\ &\quad\times\left(\int_{\mathbb{C}^n}\left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right|e^{-\frac{1}{2}|w|^2}e^{-\frac{1}{2}|z|^2}\frac{dv(w)}{(1+|w|)^\alpha}\right)^{p-1}\\ &\lesssim \int_{|w|\ge r_0}\frac{|f^+_{\frac{\alpha}{2}}(w)|^p e^{-\frac{p}{2}|w|^2}}{(1+|w|)^{p\alpha}}\left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right|e^{-\frac{1}{2}|w|^2}e^{-\frac{1}{2}|z|^2}\frac{\widehat{\mu}^p_\delta(w)\,dv(w)}{(1+|w|)^{\alpha-p\alpha}}\\ &\lesssim \|f\|^p_{F^\infty_\alpha}\int_{\mathbb{C}^n}e^{-\frac{1}{2}|z|^2}\left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right|e^{-\frac{1}{2}|w|^2}\frac{\widehat{\mu}^p_\delta(w)}{(1+|w|)^{\alpha-p\alpha}}\,dv(w).\end{aligned}$$

Again, Fubini's theorem and the estimate (9) show us, in detail, that

$$\begin{aligned}I_{(21)} &\lesssim \|f\|^p_{F^\infty_\alpha}\int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n}\frac{|(K^\alpha_z)^+_{\frac{\alpha}{2}}(w)|}{e^{\frac{1}{2}|z|^2}e^{\frac{1}{2}|w|^2}}\frac{\widehat{\mu}^p_\delta(w)\,dv(w)}{(1+|w|)^{\alpha-p\alpha}}\right)\frac{dv(z)}{(1+|z|)^\alpha}\\ &\lesssim \|f\|^p_{F^\infty_\alpha}\int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n}\frac{|(K^\alpha_z)^+_{\frac{\alpha}{2}}(w)|}{e^{\frac{1}{2}|z|^2}e^{\frac{1}{2}|w|^2}}\frac{dv(z)}{(1+|z|)^\alpha}\right)\frac{\widehat{\mu}^p_\delta(w)}{(1+|w|)^{\alpha-p\alpha}}\,dv(w)\\ &\lesssim \|f\|^p_{F^\infty_\alpha}\int_{\mathbb{C}^n}\frac{\widehat{\mu}^p_\delta(w)}{(1+|w|)^{\alpha-p\alpha}}\,dv(w).\end{aligned}$$

Then, the integral $I_{(20)}$ comes into that, after we use the estimate (4) twice,

$$\begin{aligned}I_{(20)} &\lesssim \left[\int_{|w|<r_0}\frac{|(K^\alpha_z)^+_{\frac{\alpha}{2}}(w)|}{e^{\frac{1}{2}|z|^2}e^{\frac{1}{2}|w|^2}}\left(\sup_{|w|<2r_0}\frac{|f(w)|e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\right)\frac{d\mu(w)}{|w|^{\frac{\alpha}{2}}}\right]^p\\ &\lesssim \|f\|^p_{F^\infty_\alpha}\mu^p\big(B(0,r_0)\big)\sup_{|w|<2r_0}\left|K^\alpha_z(w)\right|^p\frac{e^{-\frac{p}{2}|z|^2}e^{-\frac{p}{2}|w|^2}}{(1+|w|)^{p\alpha}}\\ &\lesssim \|f\|^p_{F^\infty_\alpha}\frac{\mu^p(B(0,r_0))}{(1+|w|)^{-\alpha}}\sup_{|w|<2r_0}\left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right|^p\frac{e^{-\frac{p}{2}|z|^2}e^{-\frac{p}{2}|w|^2}}{(1+|w|)^\alpha}\\ &\lesssim \|f\|^p_{F^\infty_\alpha}\int_{\mathbb{C}^n}e^{-\frac{p}{2}|z|^2}\left|\left(K^\alpha_z\right)^+_{\frac{\alpha}{2}}(w)\right|^p e^{-\frac{p}{2}|w|^2}\widehat{\mu}^p_\delta(w)\,dv(w).\end{aligned}$$

We can estimate the integral $I_{(22)}$ successfully using Fubini's theorem like the integral $I_{(21)}$.

All in all, the analysis above tells us that $\|T^\alpha_\mu\|_{F^\infty_\alpha\to F^p_\alpha}\lesssim\|(1+|\cdot|)^{\alpha-\frac{\alpha}{p}}\widehat{\mu}_\delta\|_{L^p}$. Finally, taking μ_R as in Lemma 3.1, then we have $\mu-\mu_R\ge 0$ and, moreover,

$$\left\|T^\alpha_\mu - T^\alpha_{\mu_R}\right\|_{F^\infty_\alpha\to F^p_\alpha}\lesssim\left\|\left(1+|\cdot|\right)^{\alpha-\frac{\alpha}{p}}(\widehat{\mu-\mu_R})_\delta\right\|_{L^p}\to 0,$$

when $R \to \infty$. Lemma 3.1 shows us that T_{μ_R} is compact from F_α^∞ to F_α^p, and so is T_μ. The condition (2) holds. □

The following theorem about the infinity case is not merged into Theorem 3.2, and the reasons for this can be found in the Introduction section. However, its proof is similar to that in Theorem 3.2. We can, however, give the details for completeness.

Theorem 3.4 *Let $\mu \ge 0$. Then,*

(1) $T_\mu^\alpha : F_\alpha^\infty \to F_\alpha^\infty$ *is bounded if and only if μ is a 1-Fock–Carleson measure. Furthermore,*

$$\left\| T_\mu^\alpha \right\|_{F_\alpha^\infty \to F_\alpha^\infty} \simeq \|\mu\|_1.$$

(2) $T_\mu^\alpha : F_\alpha^\infty \to F_\alpha^\infty$ *is compact if and only if μ is a vanishing 1-Fock–Carleson measure.*

Proof (1) Suppose μ is a 1-Fock–Carleson measure. It suffices to discuss

$$\sup_{z\in\mathbb{C}^n} \underbrace{\frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \int_{\mathbb{C}^n} \left|\left(K_z^\alpha\right)^+_{\frac{\alpha}{2}}(w)\right| \left|f^+_{\frac{\alpha}{2}}(w)\right| \frac{e^{-|w|^2}}{|w|^\alpha}\, d\mu(w)}_{\text{IV}} \lesssim \|f\|_{F_\alpha^\infty} \|\widehat{\mu}_\delta\|_{L^\infty},$$

when $\alpha > 0$, for any $f \in F_\alpha^\infty$. Therefore, fix a small enough $r_0 > 0$, and we will divide the integral IV into two cases.

$$\text{IV} = \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \int_{|w|\ge r_0} \left|\left(K_z^\alpha\right)^+_{\frac{\alpha}{2}}(w)\right| \left|f^+_{\frac{\alpha}{2}}(w)\right| \frac{e^{-|w|^2}}{|w|^\alpha}\, d\mu(w) \tag{23}$$

$$+ \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \int_{|w|<r_0} \left|\left(K_z^\alpha\right)^+_{\frac{\alpha}{2}}(w)\right| \left|f^+_{\frac{\alpha}{2}}(w)\right| \frac{e^{-|w|^2}}{|w|^\alpha}\, d\mu(w). \tag{24}$$

First, by [2, Lemma 2.3] and the estimate (9), the integral $I_{(23)}$ is estimated by

$$\begin{aligned}
I_{(23)} &\lesssim \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \int_{|w|\ge r_0} \left|\left(K_z^\alpha\right)^+_{\frac{\alpha}{2}}(w)\right| \frac{|f^+_{\frac{\alpha}{2}}(w)| e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha} e^{-\frac{1}{2}|w|^2} \widehat{\mu}_\delta(w)\, dv(w) \\
&\lesssim \|f\|_{F_\alpha^\infty} \|\widehat{\mu}_\delta\|_{L^\infty} \sum_{k:|a_k|>r+r_0} \sup_{w\in B(a_k,r)} \left|\left(K_z^\alpha\right)^+_{\frac{\alpha}{2}}(w)\right| e^{-\frac{1}{2}|w|^2} \frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \\
&\lesssim \|f\|_{F_\alpha^\infty} \|\widehat{\mu}_\delta\|_{L^\infty} \sup_{w\in\mathbb{C}^n} \int_{\mathbb{C}^n} e^{-\frac{1}{2}|z|^2} \left|K_w^\alpha(z)\right| e^{-\frac{1}{2}|w|^2} \frac{dv(z)}{(1+|z|)^\alpha}.
\end{aligned}$$

Using a similar way as in the estimate $I_{(8)}$, the integral $I_{(24)}$ is estimated by

$$\begin{aligned}
I_{(24)} &\lesssim \|\widehat{\mu}_\delta\|_{L^\infty} \left(\frac{e^{-\frac{1}{2}|z|^2}}{(1+|z|)^\alpha} \sum_{k>\frac{\alpha}{2}} \frac{k^{\frac{\alpha}{2}}}{k!} |z|^k \right) \left(\sup_{|w|\le 2r_0} \frac{|f(w)| e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha} \right) \\
&\lesssim \|\widehat{\mu}_\delta\|_{L^\infty} \frac{|z| e^{|z|} e^{-\frac{1}{2}|z|^2}}{(1+|z|)^{-|\alpha|}} \left(\sup_{|w|\le 2r_0} \frac{|f_k(w)| e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha} \right).
\end{aligned}$$

Therefore, our desired goal is obtained.

On the other hand, we assume that $T_\mu^\alpha : F_\alpha^\infty \to F_\alpha^\infty$ is bounded. Similar to the proof in [2, Theorem 3.3], we can calculate that, by the definitions of a Toeplitz operator and the Berezin transform, that

$$\begin{aligned} \left|\widetilde{\mu}_2^\alpha(z)\right| &\lesssim \left(1+|z|\right)^{-\alpha}\left|T_\mu^\alpha\left(\frac{k_{2,z}^\alpha}{(1+|z|)^{-\frac{\alpha}{2}}}\right)(z)\right|e^{-\frac{1}{2}|z|^2} \\ &\lesssim \left\|T_\mu^\alpha\right\|_{F_\alpha^\infty\to F_\alpha^\infty}\left\|\frac{k_{2,z}^\alpha}{(1+|z|)^{-\frac{\alpha}{2}}}\right\|_{F_\alpha^\infty}. \end{aligned} \tag{25}$$

Together with Theorem 2.2, μ is a 1-Fock–Carleson measure. Furthermore,

$$\|\mu\|_1 \simeq \sup_{z\in\mathbb{C}^n}\left|\widetilde{\mu}_2(z)\right| \lesssim \left\|T_\mu^\alpha\right\|_{F_\alpha^\infty\to F_\alpha^\infty}.$$

(2) Suppose that μ is a vanishing 1-Fock–Carleson measure. By Theorem 2.3,

$$\widehat{\mu}_\delta(z)\to 0, \quad \text{as } |z|\to\infty.$$

As $T_{\mu_R}^\alpha$ is compact from F_α^∞ to F_α^∞ by Lemma 3.1, if we note that $\mu-\mu_R\geq 0$, $T_{\mu-\mu_R}^\alpha$ is also bounded from F_α^∞ to F_α^∞. For $r>0$,

$$\lim_{R\to\infty}\sup_{z\in\mathbb{C}^n}\widehat{(\mu-\mu_R)}_r(z)=0.$$

Therefore, the condition (1) tells us that, when $R\to\infty$,

$$\left\|T_\mu^\alpha - T_{\mu_R}^\alpha\right\|_{F_\varphi^\infty\to F_\varphi^\infty} = \left\|T_{\mu-\mu_R}^\alpha\right\|_{F_\varphi^\infty\to F_\varphi^\infty} \simeq \sup_{z\in\mathbb{C}^n}\widehat{(\mu-\mu_R)}_r(z)\to 0.$$

Hence, we can see that $T_\mu^\alpha : F_\alpha^\infty \to F_\alpha^\infty$ is compact.

On the other hand, we know that $\widehat{\mu}_\delta(z)$ is bounded for $\delta>0$. Obviously, $\{\frac{k_{2,z}^\alpha}{(1+|z|)^{-\frac{\alpha}{2}}} : z\in\mathbb{C}^n\}$ is bounded in F_α^∞. Therefore, $\{T_\mu^\alpha(\frac{k_{2,z}^\alpha}{(1+|z|)^{-\frac{\alpha}{2}}}) : z\in\mathbb{C}^n\}$ is relatively compact in F_α^∞. Next, our goal will be to obtain that, as in Theorem 3.2,

$$\lim_{j\to\infty} T_\mu^\alpha\left(\frac{k_{2,z_j}^\alpha}{(1+|z_j|)^{-\frac{\alpha}{2}}}\right)(w)=0,$$

for any $w\in\mathbb{C}^n$. By the definition of a Topelitz operator, we will omit the details about the case $\alpha\leq 0$ and the other comes into play,

$$\begin{aligned} &T_\mu^\alpha\left(\frac{k_{2,z_j}^\alpha}{(1+|z_j|)^{-\frac{\alpha}{2}}}\right)(w)\frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha} \\ &\quad\lesssim \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\int_{\mathbb{C}^n}\left|\left(K_w^\alpha\right)_{\frac{\alpha}{2}}^-(\xi)\right|\frac{|(k_{2,z_j}^\alpha)_{\frac{\alpha}{2}}^-(\xi)|}{(1+|z_j|)^{-\frac{\alpha}{2}}}e^{-|\xi|^2}\,d\mu(\xi) \\ &\qquad + \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}\int_{\mathbb{C}^n}\left|\left(K_w^\alpha\right)_{\frac{\alpha}{2}}^+(\xi)\right|\frac{|(k_{2,z_j}^\alpha)_{\frac{\alpha}{2}}^+(\xi)|}{(1+|z_j|)^{-\frac{\alpha}{2}}}\frac{e^{-|\xi|^2}}{|\xi|^\alpha}\,d\mu(\xi). \end{aligned} \tag{26}$$

Now, we pay attention to the integral $I_{(26)}$ and we divide it into two integrals for suitable $0 < r_0 < R < \infty$,

$$I_{(26)} = \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha} \int_{|\xi|>R} \left|\left(K_w^\alpha\right)^+_{\frac{\alpha}{2}}(\xi)\right| \frac{|(k_{2,z_j}^\alpha)^+_{\frac{\alpha}{2}}(\xi)|}{(1+|z_j|)^{-\frac{\alpha}{2}}} \frac{e^{-|\xi|^2}}{|\xi|^\alpha}\, d\mu(\xi) \tag{27}$$

$$+ \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha} \int_{r_0\le|\xi|\le R} \left|\left(K_w^\alpha\right)^+_{\frac{\alpha}{2}}(\xi)\right| \frac{|(k_{2,z_j}^\alpha)^+_{\frac{\alpha}{2}}(\xi)|}{(1+|z_j|)^{-\frac{\alpha}{2}}} \frac{e^{-|\xi|^2}}{|\xi|^\alpha}\, d\mu(\xi)$$

$$+ \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha} \int_{|\xi|<r_0} \left|\left(K_w^\alpha\right)^+_{\frac{\alpha}{2}}(\xi)\right| \frac{|(k_{2,z_j}^\alpha)^+_{\frac{\alpha}{2}}(\xi)|}{(1+|z_j|)^{-\frac{\alpha}{2}}} \frac{e^{-|\xi|^2}}{|\xi|^\alpha}\, d\mu(\xi). \tag{28}$$

In view of [4, Proposition 3.2], the integral $I_{(27)}$ is estimated by

$$\begin{aligned}
I_{(27)} &\lesssim \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha} \int_{|\xi|>R} \left|\left(K_w^\alpha\right)^+_{\frac{\alpha}{2}}(\xi)\right| \frac{|(k_{2,z_j}^\alpha)^+_{\frac{\alpha}{2}}(\xi)|}{(1+|z_j|)^{-\frac{\alpha}{2}}} \frac{e^{-|\xi|^2}\widehat{\mu}_\delta(\xi)}{(1+|\xi|)^\alpha}\, dv(\xi)\\
&\lesssim \|\widehat{\mu}_\delta\|_{L^\infty} \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha} \int_{|\xi|>R} \left|\left(K_w^\alpha\right)^+_{\frac{\alpha}{2}}(\xi)\right| \left|\left(K_{z_j}^\alpha\right)^+_{\frac{\alpha}{2}}(\xi)\right| e^{-|\xi|^2} \frac{e^{-\frac{1}{2}|z_j|^2}\, dv(\xi)}{(1+|\xi|)^\alpha}\\
&\lesssim \|\widehat{\mu}_\delta\|_{L^\infty} \left(\sup_{|\xi|>R} \int_{\mathbb{C}^n} e^{-\frac{1}{2}|w|^2} \left|\left(K_\xi^\alpha\right)^+_{\frac{\alpha}{2}}(w)\right| e^{-\frac{1}{2}|\xi|^2} \frac{dv(w)}{(1+|w|)^\alpha}\right)\\
&\quad \times \sum_{k:|a_k|>R+r} \int_{B(a_k,r)} e^{-\frac{1}{2}|z_j|^2} \left|\left(K_{z_j}^\alpha\right)^+_{\frac{\alpha}{2}}(\xi)\right| e^{-\frac{1}{2}|\xi|^2} \frac{dv(\xi)}{(1+|\xi|)^\alpha}\\
&\lesssim \|\widehat{\mu}_\delta\|_{L^\infty} \int_{|\xi|>R} e^{-\frac{1}{2}|z_j|^2} \left|\left(K_{z_j}^\alpha\right)^+_{\frac{\alpha}{2}}(\xi)\right| e^{-\frac{1}{2}|\xi|^2} \frac{dv(\xi)}{(1+|\xi|)^\alpha}.
\end{aligned}$$

This implies the integral $I_{(27)}$ will tend to zero as $R \to \infty$. Since $\frac{|(k_{2,z_j}^\alpha)^+_{\frac{\alpha}{2}}(\xi)|}{(1+|z_j|)^{-\frac{\alpha}{2}}} \to 0$ uniformly on a compact subset of $\mathbb{C}^n$ when $|z_j| \to \infty$, it is easy to see that

$$\lim_{|z_j|\to\infty} \int_{r_0\le|\xi|\le R} \left|\left(K_w^\alpha\right)^+_{\frac{\alpha}{2}}(\xi)\right| \frac{|(k_{2,z_j}^\alpha)^+_{\frac{\alpha}{2}}(\xi)|}{(1+|z_j|)^{-\frac{\alpha}{2}}} \frac{e^{-|\xi|^2}}{|\xi|^\alpha}\, d\mu(\xi) = 0.$$

Next, the integral $I_{(28)}$ will be estimated by, in view of estimate (5),

$$\begin{aligned}
I_{(28)} &\lesssim \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha} \int_{|\xi|<r_0} \frac{|(K_w^\alpha)^+_{\frac{\alpha}{2}}(\xi)|}{|\xi|^{\frac{\alpha}{2}} e^{\frac{1}{2}|\xi|^2}} \frac{\|k_{2,z_j}^\alpha\|_{F_\alpha^\infty}}{(1+|z_j|)^{-\frac{\alpha}{2}}}\, d\mu(\xi)\\
&\lesssim \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha} \int_{|\xi|<r_0} \left(\left|\left(K_w^\alpha\right)^+_{\frac{\alpha}{2}}(\xi)\right| |\xi|^{-\frac{\alpha}{2}}\right) e^{-\frac{1}{2}|\xi|^2}\, d\mu(\xi)\\
&\lesssim \|\widehat{\mu}_\delta\|_{L^\infty} \sum_{k>\frac{\alpha}{2}} \frac{k^{\frac{\alpha}{2}}}{k!} |w|^k \frac{e^{-\frac{1}{2}|w|^2}}{(1+|w|)^\alpha}.
\end{aligned}$$

This implies the integral $I_{(28)}$ tends to zero if r_0 goes to zero simultaneously. Thus, our aims have been achieved. This, together with the estimate (25), yields that

$$\left|\widetilde{\mu}_2^\alpha(z)\right| \lesssim \left\|T_\mu^\alpha\right\|_{F_\alpha^\infty\to F_\alpha^\infty} \left\| \frac{k_{2,z_j}^\alpha}{(1+|z|)^{-\frac{\alpha}{2}}} \right\|_{F_\alpha^\infty} \to 0, \quad |z_j| \to \infty.$$

Hence, we can conclude that μ is a vanishing 1-Fock–Carleson measure. □

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
J.J. Chen developed the theoretical part and wrote this paper by himself. G.X. Xu helped perform the analysis with constructive discuss and read the final manuscript.

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