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Some new parameterized Newton-type inequalities for differentiable functions via fractional integrals

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Abstract

The main goal of the current study is to establish some new parameterized Newton-type inequalities for differentiable convex functions in the setting of fractional calculus. For this, first we prove a parameterized integral identity involving fractional integrals and then prove Newton-type inequalities for differentiable convex functions. It is also shown that the newly established parameterized inequalities are refinements of the already proved inequalities in the literature for different choices of parameters. Finally, we discuss a mathematical example along with a plot to show the validity of the newly established inequalities.

MSC: 34A08; 26A51; 26D15

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1 Introduction

The Hermite–Hadamard inequality was the first result given between convex functions and integrals. This inequality was introduced by Hermite [1] in 1883 and was later proved by Hadamard [2] in 1893. This inequality has the following mathematical form:

$$\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right) \le \frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\mathfrak{G}(\varkappa)\,d\varkappa \le \frac{\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)}{2}, \tag{1.1}$$

where $\mathfrak{G}$ is a convex function. This inequality also holds in the reverse direction for concave functions.

This inequality has many advantages, especially in approximation theory, and is widely used. Due to its wide applications, mathematicians started working on it and came up with many new results. For example, Dragomir and Agarwal [3] found the boundaries of the trapezoidal formula by taking the difference of the middle part and the right part of this inequality and used differentiable convexity in the whole process. Later, Kirmaci [4] gave the boundaries of the midpoint formula, which were formed from the same inequality, he took the difference of the middle part from the left part and he also derived his results by using differentiable convexity. Qi and Xi [5] took the difference of the middle part of this inequality with the average of the left and right parts to establish a new inequality that is known as Bullen's inequality.

However, because of their significance, researchers have used fractional calculus to create a variety of fractional integral inequalities that are useful in approximation theory. The bounds of mathematical integration formulas can be determined using inequalities such as Hermite–Hadamard, Simpson's, midpoint, Ostrowski's, and trapezoidal inequalities. In [6], the Hermite–Hadamard-type inequality and the bounds for the trapezoidal formula were established. Differentiable convexity was used in Set [7] to establish fractional Ostrowski-type inequalities. Through the use of Riemann–Liouville fractional integrals (RLFIs), İşcan and Wu [8] established certain bounds for numerical integration as well as an inequality of the Hermite–Hadamard type for reciprocal convex functions. Sarikaya and Yildirim established the midpoint bounds and a new version of the fractional inequality of the Hermite–Hadamard type in [9]. Sarikaya et al. [10] used the general convexity and RLFIs to obtain the bounds for Simpson's 1/3 formula. In [11], the authors used the RLFIs to discover some new boundaries for Simpson's 1/3 formula. The *s*-convexity was utilized by the authors of [12] to analyze different Simpson's 1/3 formula bounds. Generalized RLFIs were introduced as a new class of fractional integrals in 2020 by Sarikaya and Ertugral [13]; they also established Hermite–Hadamard-type inequalities related to the newly defined class of integrals. The ability to be transformed into the classical integral, RLFIs, *k*-RLFIs, Hadamard fractional integrals, etc. is the main benefit of the newly defined class of fractional integral operators. Zhao et al. used generalized RLFIs and reciprocal convex functions in [14] to obtain some bounds for a trapezoidal formula. Using the generalized RLFIs, Budak et al. [15] found certain approximations for Simpson's 1/3 formula for differentiable convex functions.

Recently, Sitthiwirattham et al. [16] found some bounds for Simpson's 3/8 formula using the RLFIs. For further inequalities that can be addressed using fractional and quantum integrals, see [17–27] and the references therein.

Motivated by the ongoing studies, we obtain some new parameterized inequalities of Simpson's 3/8 formula type using the convexity and RLFIs. The main benefit of the newly established inequalities is that these can be converted into classical and fractional inequalities of Newton type, trapezoidal type, and many others for different choices of the parameters and $\alpha = 1$ without being establishing one by one.

The following is a description of the paper: The basics of fractional calculus and more significant research in this area are briefly reviewed in Sect. 2. In Sect. 3, we establish an essential identity that is key in pinpointing the paper's major findings. In Sect. 4, we construct some new parameterized Newton-type inequalities for differentiable convex functions using RLFIs. In Section 5, we find numerous inequalities for $\alpha = 1$ and various parameter selections. A few suggestions for further research are included in Sect. 6.

2 Fractional integrals and related inequalities

In this section, some inequalities and basics of fractional calculus are recalled.

Definition 1 ([28, 29]) Let $\mathfrak{G} \in L_1[\theta_1, \theta_2]$. The RLFIs of order $\alpha > 0$ with $\theta_1 \geq 0$ are stated as follows:

$$J^{\alpha}_{\theta_1+}\mathfrak{G}(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\theta_1}^{\varkappa} (\varkappa - \rho)^{\alpha-1} \mathfrak{G}(\rho)\, d\rho, \quad \varkappa > \theta_1$$

and

$$J^{\alpha}_{\theta_2-}\mathfrak{G}(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\theta_2}(\rho-\varkappa)^{\alpha-1}\mathfrak{G}(\rho)\,d\rho, \quad \varkappa < \theta_2,$$

respectively, where Γ is used for the notation of the Gamma function.

The following fractional Hermite–Hadamard-type inequality was first demonstrated in 2013 by Sarikaya et al.

Theorem 1 ([6]) *For a positive and convex mapping* $\mathfrak{G}: I \subset \mathbb{R} \to \mathbb{R}$ *with* $\mathfrak{G} \in L_1[\theta_1,\theta_2]$ *and* $0 \leq \theta_1 < \theta_2$, *the following inequality holds*:

$$\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^{\alpha}}\left[J^{\alpha}_{\theta_1+}\mathfrak{G}(\theta_2) + J^{\alpha}_{\theta_2-}\mathfrak{G}(\theta_1)\right] \leq \frac{\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)}{2}.$$

Following that, Sarikaya and Yildrim demonstrated the new fractional Hermite–Hadamard inequality as follows:

Theorem 2 ([9]) *For a positive and convex mapping* $\mathfrak{G}: I \subset \mathbb{R} \to \mathbb{R}$ *with* $\mathfrak{G} \in L_1[\theta_1,\theta_2]$, $0 \leq \theta_1 < \theta_2$ *and* $\theta_1,\theta_2 \in I$, *the following inequality holds*:

$$\mathfrak{G}\left(\frac{\theta_1+\theta_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\theta_2-\theta_1)^{\alpha}}\left[J^{\alpha}_{(\frac{\theta_1+\theta_2}{2})+}\mathfrak{G}(\theta_2) + J^{\alpha}_{(\frac{\theta_1+\theta_2}{2})-}\mathfrak{G}(\theta_1)\right] \leq \frac{\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)}{2}.$$

3 A new and crucial identity

In this paper, we prove a RLFIs identity involving a three-step kernel and differentiable functions.

Lemma 1 *For a differentiable function* $\mathfrak{G}: [\theta_1,\theta_2] \to \mathbb{R}$ *over* (θ_1,θ_2) *with* $\mathfrak{G} \in L[\theta_1,\theta_2]$, *the following equality holds*:

$$\begin{aligned}&(1+\lambda-\nu)\left[\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)\right] + (\nu-\lambda)\left[\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right) + \mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right] \\ &\quad - \frac{\Gamma(\alpha+1)}{(\theta_2-\theta_1)^{\alpha}}\left[J^{\alpha}_{\theta_1+}\mathfrak{G}(\theta_2) + J^{\alpha}_{\theta_2-}\mathfrak{G}(\theta_1)\right] \\ &\quad = (\theta_2-\theta_1)\int_0^1 \Delta(\rho)\left[\mathfrak{G}'\left(\rho\theta_2+(1-\rho)\theta_1\right) - \mathfrak{G}'\left(\rho\theta_1+(1-\rho)\theta_2\right)\right]d\rho,\end{aligned} \tag{3.1}$$

where $\lambda,\mu,\nu \geq 0$ *and*

$$\Delta(\rho) = \begin{cases}\rho^{\alpha}-\lambda, & \rho \in [0,\frac{1}{3}),\\ \rho^{\alpha}-\mu, & \rho \in [\frac{1}{3},\frac{2}{3}),\\ \rho^{\alpha}-\nu, & \rho \in [\frac{2}{3},1].\end{cases}$$

Proof From the definition of $\Delta(\rho)$, we have

$$
\begin{aligned}
&\int_0^1 \Delta(\rho)\left[\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)-\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right]d\rho \\
&= \int_0^{\frac{1}{3}} \big(\rho^\alpha-\lambda\big)\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\,d\rho+\int_{\frac{1}{3}}^{\frac{2}{3}} \big(\rho^\alpha-\mu\big)\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\,d\rho \\
&\quad+\int_{\frac{2}{3}}^{1} \big(\rho^\alpha-\nu\big)\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\,d\rho \\
&\quad+\int_0^{\frac{1}{3}} \big(\lambda-\rho^\alpha\big)\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\,d\rho+\int_{\frac{1}{3}}^{\frac{2}{3}} \big(\mu-\rho^\alpha\big)\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\,d\rho \\
&\quad+\int_{\frac{2}{3}}^{1} \big(\nu-\rho^\alpha\big)\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\,d\rho \\
&= I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
\tag{3.2}
$$

From integration by parts, we have

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{3}} \big(\rho^\alpha-\lambda\big)\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\,d\rho \\
&= \big(\rho^\alpha-\lambda\big)\frac{\mathfrak{G}(\rho\theta_2+(1-\rho)\theta_1)}{\theta_2-\theta_1}\bigg|_0^{\frac{1}{3}}-\frac{\alpha}{\theta_2-\theta_1}\int_0^{\frac{1}{3}}\rho^{\alpha-1}\mathfrak{G}\big(\rho\theta_2+(1-\rho)\theta_1\big)\,d\rho \\
&= \frac{1}{\theta_2-\theta_1}\left[\left(\left(\frac{1}{3}\right)^\alpha-\lambda\right)\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+\lambda\mathfrak{G}(\theta_1)\right] \\
&\quad-\frac{\alpha}{\theta_2-\theta_1}\int_0^{\frac{1}{3}}\rho^{\alpha-1}\mathfrak{G}\big(\rho\theta_2+(1-\rho)\theta_1\big)\,d\rho,
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
I_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}} \big(\rho^\alpha-\mu\big)\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\,d\rho \\
&= \frac{1}{\theta_2-\theta_1}\left[\left(\left(\frac{2}{3}\right)^\alpha-\mu\right)\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)-\left(\left(\frac{1}{3}\right)^\alpha-\mu\right)\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\right] \\
&\quad-\frac{\alpha}{\theta_2-\theta_1}\int_{\frac{1}{3}}^{\frac{2}{3}}\rho^{\alpha-1}\mathfrak{G}\big(\rho\theta_2+(1-\rho)\theta_1\big)\,d\rho,
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
I_3 &= \int_{\frac{2}{3}}^{1} \big(\rho^\alpha-\nu\big)\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\,d\rho \\
&= \frac{1}{\theta_2-\theta_1}\left[(1-\nu)\mathfrak{G}(\theta_2)-\left(\left(\frac{2}{3}\right)^\alpha-\nu\right)\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right] \\
&\quad-\frac{\alpha}{\theta_2-\theta_1}\int_{\frac{2}{3}}^{1}\rho^{\alpha-1}\mathfrak{G}\big(\rho\theta_2+(1-\rho)\theta_1\big)\,d\rho.
\end{aligned}
\tag{3.5}
$$

By adding the equalities (3.3)–(3.5) and from the definition of the right RLFI, we have the following relation

$$\begin{aligned}
&(\theta_2-\theta_1)[I_1+I_2+I_3] \qquad (3.6)\\
&\quad=\lambda\mathfrak{G}(\theta_1)+(\mu-\lambda)\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+(\nu-\mu)\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\\
&\qquad+(1-\nu)\mathfrak{G}(\theta_2)-\frac{\Gamma(\alpha+1)}{(\theta_2-\theta_1)^\alpha}J^\alpha_{\theta_2-}\mathfrak{G}(\theta_1).
\end{aligned}$$

Similarly, from the definition of the left RLFI, we have

$$\begin{aligned}
&(\theta_2-\theta_1)[I_4+I_5+I_6] \qquad (3.7)\\
&\quad=\lambda\mathfrak{G}(\theta_2)+(\mu-\lambda)\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+(\nu-\mu)\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)\\
&\qquad+(1-\nu)\mathfrak{G}(\theta_1)-\frac{\Gamma(\alpha+1)}{(\theta_2-\theta_1)^\alpha}J^\alpha_{\theta_1+}\mathfrak{G}(\theta_2).
\end{aligned}$$

Thus, we obtain the desired equality by summing (3.6) and (3.7). □

4 Fractional Newton inequalities

In this section, some inequalities of Newton type are established using the RLFIs.

Theorem 3 *Let $\mathfrak{G}$ as in Lemma 1 hold. If $|\mathfrak{G}'|$ contains the convexity property, then we have*:

$$\begin{aligned}
&\left|(1+\lambda-\nu)\bigl[\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)\bigr]+(\nu-\lambda)\left[\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right]\right. \qquad (4.1)\\
&\qquad\left.-\frac{\Gamma(\alpha+1)}{(\theta_2-\theta_1)^\alpha}\bigl[J^\alpha_{\theta_1+}\mathfrak{G}(\theta_2)+J^\alpha_{\theta_2-}\mathfrak{G}(\theta_1)\bigr]\right|\\
&\quad\le(\theta_2-\theta_1)\bigl[A_1(\lambda,\alpha)+A_2(\mu,\alpha)+A_3(\nu,\alpha)\bigr]\bigl(\bigl|\mathfrak{G}'(\theta_1)\bigr|+\bigl|\mathfrak{G}'(\theta_2)\bigr|\bigr),
\end{aligned}$$

where

$$A_1(\lambda,\alpha)=\int_0^{\frac{1}{3}}\bigl|\rho^\alpha-\lambda\bigr|\,d\rho=\begin{cases}\frac{2\alpha}{1+\alpha}\lambda^{1+\frac{1}{\alpha}}+\frac{1}{3^{\alpha+1}(\alpha+1)}-\frac{\lambda}{3}, & 0<\lambda\le(\frac{1}{3})^\alpha,\\ \frac{\lambda}{3}-\frac{1}{3^{\alpha+1}(\alpha+1)}, & \lambda>(\frac{1}{3})^\alpha,\end{cases}$$

$$A_2(\mu,\alpha)=\int_{\frac{1}{3}}^{\frac{2}{3}}\bigl|\rho^\alpha-\mu\bigr|\,d\rho=\begin{cases}\frac{2^{1+\alpha}-1}{3^{\alpha+1}(\alpha+1)}-\frac{\mu}{3}, & 0<\mu\le(\frac{1}{3})^\alpha,\\ \frac{2\alpha}{1+\alpha}\mu^{1+\frac{1}{\alpha}}+\frac{1+2^{1+\alpha}}{3^{\alpha+1}(\alpha+1)}-\mu, & (\frac{1}{3})^\alpha<\mu\le(\frac{2}{3})^\alpha,\\ \frac{\mu}{3}-\frac{2^{1+\alpha}-1}{3^{\alpha+1}(\alpha+1)}, & \mu>(\frac{2}{3})^\alpha\end{cases}$$

and

$$A_3(\nu,\alpha)=\int_{\frac{2}{3}}^{1}\bigl|\rho^\alpha-\nu\bigr|\,d\rho=\begin{cases}\frac{3^{1+\alpha}-2^{1+\alpha}}{3^{\alpha+1}(\alpha+1)}-\frac{\nu}{3}, & 0<\nu\le(\frac{2}{3})^\alpha,\\ \frac{2\alpha}{1+\alpha}\nu^{1+\frac{1}{\alpha}}+\frac{2^{1+\alpha}+3^{1+\alpha}}{3^{\alpha+1}(\alpha+1)}-\frac{5\nu}{3}, & (\frac{2}{3})^\alpha<\nu\le1,\\ \frac{\nu}{3}-\frac{3^{1+\alpha}-2^{1+\alpha}}{3^{\alpha+1}(\alpha+1)}, & \nu>1.\end{cases}$$

Proof Taking the modulus in (3.1) and using the convexity of $|\mathfrak{G}'|$, we have

$$\begin{aligned}
&\bigg|(1+\lambda-\nu)\big[\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)\big]+(\nu-\lambda)\bigg[\mathfrak{G}\bigg(\frac{2\theta_1+\theta_2}{3}\bigg)+\mathfrak{G}\bigg(\frac{\theta_1+2\theta_2}{3}\bigg)\bigg]\\
&\quad-\frac{\Gamma(\alpha+1)}{(\theta_2-\theta_1)^\alpha}\big[J^\alpha_{\theta_1+}\mathfrak{G}(\theta_2)+J^\alpha_{\theta_2-}\mathfrak{G}(\theta_1)\big]\bigg|\\
&\leq(\theta_2-\theta_1)\int_0^1\big|\Delta(\rho)\big|\big[\big|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\big|+\big|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\big|\big]\,d\rho\\
&=(\theta_2-\theta_1)\bigg[\int_0^{\frac{1}{3}}\big|\rho^\alpha-\lambda\big|\big[\big|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\big|+\big|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\big|\big]\,d\rho\\
&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\big|\rho^\alpha-\mu\big|\big[\big|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\big|+\big|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\big|\big]\,d\rho\\
&\quad+\int_{\frac{2}{3}}^{1}\big|\rho^\alpha-\nu\big|\big[\big|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\big|+\big|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\big|\big]\,d\rho\bigg]\\
&\leq(\theta_2-\theta_1)\bigg[\big(\big|\mathfrak{G}'(\theta_1)\big|+\big|\mathfrak{G}'(\theta_2)\big|\big)\\
&\quad\times\bigg(\int_0^{\frac{1}{3}}\big|\rho^\alpha-\lambda\big|\,d\rho+\int_{\frac{1}{3}}^{\frac{2}{3}}\big|\rho^\alpha-\mu\big|\,d\rho+\int_{\frac{2}{3}}^{1}\big|\rho^\alpha-\nu\big|\,d\rho\bigg)\bigg]\\
&=(\theta_2-\theta_1)\big[A_1(\lambda,\alpha)+A_2(\mu,\alpha)+A_3(\nu,\alpha)\big]\big(\big|\mathfrak{G}'(\theta_1)\big|+\big|\mathfrak{G}'(\theta_2)\big|\big).
\end{aligned}$$

Thus, the proof is completed. □

Theorem 4 *Let $\mathfrak{G}$ as in Lemma 1 hold. If $|\mathfrak{G}'|^q$, $q\geq 1$ contains the convexity property, then we have*

$$\begin{aligned}
&\bigg|(1+\lambda-\nu)\big[\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)\big]+(\nu-\lambda)\bigg[\mathfrak{G}\bigg(\frac{2\theta_1+\theta_2}{3}\bigg)+\mathfrak{G}\bigg(\frac{\theta_1+2\theta_2}{3}\bigg)\bigg]\\
&\quad-\frac{\Gamma(\alpha+1)}{(\theta_2-\theta_1)^\alpha}\big[J^\alpha_{\theta_1+}\mathfrak{G}(\theta_2)+J^\alpha_{\theta_2-}\mathfrak{G}(\theta_1)\big]\bigg|\\
&\leq(\theta_2-\theta_1)\big[A_1^{1-\frac{1}{q}}(\lambda,\alpha)\big\{\big(A_4(\lambda,\alpha)\big|\mathfrak{G}'(\theta_2)\big|^q+\big(A_1(\lambda,\alpha)-A_4(\lambda,\alpha)\big)\big|\mathfrak{G}'(\theta_1)\big|^q\big)^{\frac{1}{q}}\\
&\quad+\big(A_4(\lambda,\alpha)\big|\mathfrak{G}'(\theta_1)\big|^q+\big(A_1(\lambda,\alpha)-A_4(\lambda,\alpha)\big)\big|\mathfrak{G}'(\theta_2)\big|^q\big)^{\frac{1}{q}}\big\}\\
&\quad+A_2^{1-\frac{1}{q}}(\mu,\alpha)\big\{\big(A_5(\mu,\alpha)\big|\mathfrak{G}'(\theta_2)\big|^q+\big(A_2(\mu,\alpha)-A_5(\mu,\alpha)\big)\big|\mathfrak{G}'(\theta_1)\big|^q\big)^{\frac{1}{q}}\\
&\quad+\big(A_5(\mu,\alpha)\big|\mathfrak{G}'(\theta_1)\big|^q+\big(A_2(\mu,\alpha)-A_5(\mu,\alpha)\big)\big|\mathfrak{G}'(\theta_2)\big|^q\big)^{\frac{1}{q}}\big\}\\
&\quad+A_3^{1-\frac{1}{q}}(\nu,\alpha)\big\{\big(A_6(\nu,\alpha)\big|\mathfrak{G}'(\theta_2)\big|^q+\big(A_3(\nu,\alpha)-A_6(\nu,\alpha)\big)\big|\mathfrak{G}'(\theta_1)\big|^q\big)^{\frac{1}{q}}\\
&\quad+\big(A_6(\nu,\alpha)\big|\mathfrak{G}'(\theta_1)\big|^q+\big(A_3(\nu,\alpha)-A_6(\nu,\alpha)\big)\big|\mathfrak{G}'(\theta_2)\big|^q\big)^{\frac{1}{q}}\big\}\big],
\end{aligned}$$

where

$$A_4(\lambda,\alpha)=\int_0^{\frac{1}{3}}\rho\big|\rho^\alpha-\lambda\big|\,d\rho=\begin{cases}\frac{\alpha}{2+\alpha}\lambda^{1+\frac{2}{\alpha}}+\frac{1}{3^{\alpha+2}(\alpha+2)}-\frac{\lambda}{18}, & 0<\lambda\leq(\frac{1}{3})^\alpha,\\ \frac{\lambda}{18}-\frac{1}{3^{\alpha+2}(\alpha+2)}, & \lambda>(\frac{1}{3})^\alpha,\end{cases}$$

$$A_5(\mu,\alpha)=\int_{\frac{1}{3}}^{\frac{2}{3}}\rho\left|\rho^\alpha-\mu\right|d\rho=\begin{cases}\frac{2^{2+\alpha}-1}{3^{\alpha+2}(\alpha+2)}-\frac{\mu}{6}, & 0<\mu\leq(\frac{1}{3})^\alpha,\\ \frac{\alpha}{2+\alpha}\mu^{1+\frac{2}{\alpha}}+\frac{1+2^{2+\alpha}}{3^{\alpha+2}(\alpha+2)}-\frac{5\mu}{18}, & (\frac{1}{3})^\alpha<\mu\leq(\frac{2}{3})^\alpha,\\ \frac{\mu}{6}-\frac{2^{2+\alpha}-1}{3^{\alpha+2}(\alpha+2)}, & \mu>(\frac{2}{3})^\alpha\end{cases}$$

and

$$A_6(\nu,\alpha)=\int_{\frac{2}{3}}^{1}\rho\left|\rho^\alpha-\nu\right|d\rho=\begin{cases}\frac{3^{2+\alpha}-2^{2+\alpha}}{3^{\alpha+2}(\alpha+2)}-\frac{5\nu}{18}, & 0<\nu\leq(\frac{2}{3})^\alpha,\\ \frac{\alpha}{2+\alpha}\nu^{1+\frac{2}{\alpha}}+\frac{2^{2+\alpha}+3^{2+\alpha}}{3^{\alpha+2}(\alpha+2)}-\frac{13\nu}{18}, & (\frac{2}{3})^\alpha<\nu\leq 1,\\ \frac{5\nu}{18}-\frac{3^{2+\alpha}-2^{2+\alpha}}{3^{\alpha+2}(\alpha+2)}, & \nu>1.\end{cases}$$

Proof Using the power mean inequality in (3.1) after taking the modulus and using the properties of the modulus, we have

$$\begin{aligned}
&\bigg|(1+\lambda-\nu)\big[\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)\big]+(\nu-\lambda)\bigg[\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\bigg]\\
&\quad-\frac{\Gamma(\alpha+1)}{(\theta_2-\theta_1)^\alpha}\big[J_{\theta_1+}^{\alpha}\mathfrak{G}(\theta_2)+J_{\theta_2-}^{\alpha}\mathfrak{G}(\theta_1)\big]\bigg|\\
&\leq(\theta_2-\theta_1)\int_0^1\left|\Delta(\rho)\right|\big[\left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right|+\left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|\big]\,d\rho\\
&=(\theta_2-\theta_1)\bigg[\int_0^{\frac{1}{3}}\left|\rho^\alpha-\lambda\right|\big[\left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right|+\left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|\big]\,d\rho\\
&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\rho^\alpha-\mu\right|\big[\left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right|+\left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|\big]\,d\rho\\
&\quad+\int_{\frac{2}{3}}^{1}\left|\rho^\alpha-\nu\right|\big[\left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right|+\left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|\big]\,d\rho\bigg]\\
&\leq(\theta_2-\theta_1)\bigg[\left(\int_0^{\frac{1}{3}}\left|\rho^\alpha-\lambda\right|d\rho\right)^{1-\frac{1}{q}}\bigg\{\left(\int_0^{\frac{1}{3}}\left|\rho^\alpha-\lambda\right|\left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right|^q d\rho\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^{\frac{1}{3}}\left|\rho^\alpha-\lambda\right|\left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|^q d\rho\right)^{\frac{1}{q}}\bigg\}\\
&\quad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\rho^\alpha-\mu\right|d\rho\right)^{1-\frac{1}{q}}\bigg\{\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\rho^\alpha-\mu\right|\left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right|^q d\rho\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\rho^\alpha-\mu\right|\left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|^q d\rho\right)^{\frac{1}{q}}\bigg\}\\
&\quad+\left(\int_{\frac{2}{3}}^{1}\left|\rho^\alpha-\nu\right|d\rho\right)^{1-\frac{1}{q}}\bigg\{\left(\int_{\frac{2}{3}}^{1}\left|\rho^\alpha-\nu\right|\left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right|^q d\rho\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{2}{3}}^{1}\left|\rho^\alpha-\nu\right|\left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|^q d\rho\right)^{\frac{1}{q}}\bigg\}\bigg].
\end{aligned}$$

Now, using the convexity of $|\mathfrak{G}'|^q$, we have

$$\begin{aligned}
&\bigg|(1+\lambda-\nu)\big[\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)\big]+(\nu-\lambda)\bigg[\mathfrak{G}\bigg(\frac{2\theta_1+\theta_2}{3}\bigg)+\mathfrak{G}\bigg(\frac{\theta_1+2\theta_2}{3}\bigg)\bigg] \\
&\quad -\frac{\Gamma(\alpha+1)}{(\theta_2-\theta_1)^\alpha}\big[J^\alpha_{\theta_1+}\mathfrak{G}(\theta_2)+J^\alpha_{\theta_2-}\mathfrak{G}(\theta_1)\big]\bigg| \\
&\leq (\theta_2-\theta_1) \\
&\quad \times\bigg[A_1^{1-\frac{1}{q}}(\lambda,\alpha)\bigg\{\bigg(\big|\mathfrak{G}'(\theta_2)\big|^q\int_0^{\frac{1}{3}}\rho\big|\rho^\alpha-\lambda\big|\,d\rho+\big|\mathfrak{G}'(\theta_1)\big|^q\int_0^{\frac{1}{3}}(1-\rho)\big|\rho^\alpha-\lambda\big|\,d\rho\bigg)^{\frac{1}{q}} \\
&\quad +\bigg(\big|\mathfrak{G}'(\theta_1)\big|^q\int_0^{\frac{1}{3}}\rho\big|\rho^\alpha-\lambda\big|\,d\rho+\big|\mathfrak{G}'(\theta_2)\big|^q\int_0^{\frac{1}{3}}(1-\rho)\big|\rho^\alpha-\lambda\big|\,d\rho\bigg)^{\frac{1}{q}}\bigg\} \\
&\quad +A_2^{1-\frac{1}{q}}(\mu,\alpha)\bigg\{\bigg(\big|\mathfrak{G}'(\theta_2)\big|^q\int_{\frac{1}{3}}^{\frac{2}{3}}\rho\big|\rho^\alpha-\mu\big|\,d\rho+\big|\mathfrak{G}'(\theta_1)\big|^q\int_{\frac{1}{3}}^{\frac{2}{3}}(1-\rho)\big|\rho^\alpha-\mu\big|\,d\rho\bigg)^{\frac{1}{q}} \\
&\quad +\bigg(\big|\mathfrak{G}'(\theta_1)\big|^q\int_{\frac{1}{3}}^{\frac{2}{3}}\rho\big|\rho^\alpha-\mu\big|\,d\rho+\big|\mathfrak{G}'(\theta_2)\big|^q\int_{\frac{1}{3}}^{\frac{2}{3}}(1-\rho)\big|\rho^\alpha-\mu\big|\,d\rho\bigg)^{\frac{1}{q}}\bigg\} \\
&\quad +A_3^{1-\frac{1}{q}}(\nu,\alpha)\bigg\{\bigg(\big|\mathfrak{G}'(\theta_2)\big|^q\int_{\frac{2}{3}}^{1}\rho\big|\rho^\alpha-\nu\big|\,d\rho+\big|\mathfrak{G}'(\theta_1)\big|^q\int_{\frac{2}{3}}^{1}(1-\rho)\big|\rho^\alpha-\nu\big|\,d\rho\bigg)^{\frac{1}{q}} \\
&\quad +\bigg(\big|\mathfrak{G}'(\theta_1)\big|^q\int_{\frac{2}{3}}^{1}\rho\big|\rho^\alpha-\nu\big|\,d\rho+\big|\mathfrak{G}'(\theta_2)\big|^q\int_{\frac{2}{3}}^{1}(1-\rho)\big|\rho^\alpha-\nu\big|\,d\rho\bigg)^{\frac{1}{q}}\bigg\}\bigg] \\
&=(\theta_2-\theta_1)\big[A_1^{1-\frac{1}{q}}(\lambda,\alpha)\big\{\big(A_4(\lambda,\alpha)\big|\mathfrak{G}'(\theta_2)\big|^q+\big(A_1(\lambda,\alpha)-A_4(\lambda,\alpha)\big)\big|\mathfrak{G}'(\theta_1)\big|^q\big)^{\frac{1}{q}} \\
&\quad +\big(A_4(\lambda,\alpha)\big|\mathfrak{G}'(\theta_1)\big|^q+\big(A_1(\lambda,\alpha)-A_4(\lambda,\alpha)\big)\big|\mathfrak{G}'(\theta_2)\big|^q\big)^{\frac{1}{q}}\big\} \\
&\quad +A_2^{1-\frac{1}{q}}(\mu,\alpha)\big\{\big(A_5(\mu,\alpha)\big|\mathfrak{G}'(\theta_2)\big|^q+\big(A_2(\mu,\alpha)-A_5(\mu,\alpha)\big)\big|\mathfrak{G}'(\theta_1)\big|^q\big)^{\frac{1}{q}} \\
&\quad +\big(A_5(\mu,\alpha)\big|\mathfrak{G}'(\theta_1)\big|^q+\big(A_2(\mu,\alpha)-A_5(\mu,\alpha)\big)\big|\mathfrak{G}'(\theta_2)\big|^q\big)^{\frac{1}{q}}\big\} \\
&\quad +A_3^{1-\frac{1}{q}}(\nu,\alpha)\big\{\big(A_6(\nu,\alpha)\big|\mathfrak{G}'(\theta_2)\big|^q+\big(A_3(\nu,\alpha)-A_6(\nu,\alpha)\big)\big|\mathfrak{G}'(\theta_1)\big|^q\big)^{\frac{1}{q}} \\
&\quad +\big(A_6(\nu,\alpha)\big|\mathfrak{G}'(\theta_1)\big|^q+\big(A_3(\nu,\alpha)-A_6(\nu,\alpha)\big)\big|\mathfrak{G}'(\theta_2)\big|^q\big)^{\frac{1}{q}}\big\}\big].
\end{aligned}$$

Thus, the proof is completed. □

Theorem 5 *Let $\mathfrak{G}$ as in Lemma 1 hold. If $|\mathfrak{G}'|^q$, $q>1$ contains the convexity property, then we have*

$$\begin{aligned}
&\bigg|(1+\lambda-\nu)\big[\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)\big]+(\nu-\lambda)\bigg[\mathfrak{G}\bigg(\frac{2\theta_1+\theta_2}{3}\bigg)+\mathfrak{G}\bigg(\frac{\theta_1+2\theta_2}{3}\bigg)\bigg] \\
&\quad -\frac{\Gamma(\alpha+1)}{(\theta_2-\theta_1)^\alpha}\big[J^\alpha_{\theta_1+}\mathfrak{G}(\theta_2)+J^\alpha_{\theta_2-}\mathfrak{G}(\theta_1)\big]\bigg| \\
&\leq(\theta_2-\theta_1)\bigg[\big(A_7(\lambda,\alpha,p)+A_8(\upsilon,\alpha,p)\big) \\
&\quad \times\bigg\{\bigg(\frac{5|\mathfrak{G}'(\theta_1)|^q+|\mathfrak{G}'(\theta_2)|^q}{18}\bigg)^{\frac{1}{q}}+\bigg(\frac{|\mathfrak{G}'(\theta_1)|^q+5|\mathfrak{G}'(\theta_2)|^q}{18}\bigg)^{\frac{1}{q}}\bigg\}
\end{aligned}$$

$$+ 2A_9(\mu,\alpha,p)\left(\frac{|\mathfrak{G}'(\theta_1)|^q + |\mathfrak{G}'(\theta_2)|^q}{6}\right)^{\frac{1}{q}}\Bigg],$$

where

$$A_7(\lambda,\alpha,p) = \left(\int_0^{\frac{1}{3}} \left|\rho^\alpha - \lambda\right|^p d\rho\right)^{\frac{1}{p}},$$

$$A_8(\upsilon,\alpha,p) = \left(\int_{\frac{2}{3}}^{1} \left|\rho^\alpha - \nu\right|^p d\rho\right)^{\frac{1}{p}},$$

$$A_9(\mu,\alpha,p) = \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left|\rho^\alpha - \mu\right|^p d\rho\right)^{\frac{1}{p}}$$

and $q^{-1} + p^{-1} = 1$.

Proof Using the Hölder inequality in (3.1) after taking the modulus and using the properties of the modulus, we have

$$\begin{aligned}
&\Bigg|(1+\lambda-\nu)\big[\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)\big] + (\nu-\lambda)\left[\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right) + \mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right] \\
&\quad - \frac{\Gamma(\alpha+1)}{(\theta_2-\theta_1)^\alpha}\big[J^\alpha_{\theta_1+}\mathfrak{G}(\theta_2) + J^\alpha_{\theta_2-}\mathfrak{G}(\theta_1)\big]\Bigg| \\
&\leq (\theta_2-\theta_1)\int_0^1 \left|\Delta(\rho)\right|\big[\left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right| + \left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|\big]\,d\rho \\
&= (\theta_2-\theta_1)\left[\int_0^{\frac{1}{3}} \left|\rho^\alpha-\lambda\right|\big[\left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right| + \left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|\big]\,d\rho\right. \\
&\quad + \int_{\frac{1}{3}}^{\frac{2}{3}} \left|\rho^\alpha-\mu\right|\big[\left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right| + \left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|\big]\,d\rho \\
&\quad \left.+ \int_{\frac{2}{3}}^{1} \left|\rho^\alpha-\nu\right|\big[\left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right| + \left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|\big]\,d\rho\right] \\
&\leq (\theta_2-\theta_1)\left[\left(\int_0^{\frac{1}{3}} \left|\rho^\alpha-\lambda\right|^p d\rho\right)^{\frac{1}{p}} \left\{\left(\int_0^{\frac{1}{3}} \left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right|^q d\rho\right)^{\frac{1}{q}}\right.\right. \\
&\quad \left. + \left(\int_0^{\frac{1}{3}} \left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|^q d\rho\right)^{\frac{1}{q}}\right\} \\
&\quad + \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left|\rho^\alpha-\mu\right|^p d\rho\right)^{\frac{1}{p}} \left\{\left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right|^q d\rho\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|^q d\rho\right)^{\frac{1}{q}}\right\} \\
&\quad + \left(\int_{\frac{2}{3}}^{1} \left|\rho^\alpha-\nu\right|^p d\rho\right)^{\frac{1}{p}} \left\{\left(\int_{\frac{2}{3}}^{1} \left|\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)\right|^q d\rho\right)^{\frac{1}{q}}\right. \\
&\quad \left.\left. + \left(\int_{\frac{2}{3}}^{1} \left|\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\right|^q d\rho\right)^{\frac{1}{q}}\right\}\right].
\end{aligned}$$

Now, using convexity we have,

$$\begin{aligned}
&\Bigg|(1+\lambda-\nu)\big[\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)\big]+(\nu-\lambda)\bigg[\mathfrak{G}\bigg(\frac{2\theta_1+\theta_2}{3}\bigg)+\mathfrak{G}\bigg(\frac{\theta_1+2\theta_2}{3}\bigg)\bigg]\\
&\quad-\frac{\Gamma(\alpha+1)}{(\theta_2-\theta_1)^\alpha}\big[J^\alpha_{\theta_1+}\mathfrak{G}(\theta_2)+J^\alpha_{\theta_2-}\mathfrak{G}(\theta_1)\big]\Bigg|\\
&\leq(\theta_2-\theta_1)\Bigg[A_7(\lambda,\alpha,p)\Bigg\{\bigg(\int_0^{\frac{1}{3}}\big(\rho\big|\mathfrak{G}'(\theta_2)\big|^q+(1-\rho)\big|\mathfrak{G}'(\theta_1)\big|^q\big)\,d\rho\bigg)^{\frac{1}{q}}\\
&\quad+\bigg(\int_0^{\frac{1}{3}}\big(\rho\big|\mathfrak{G}'(\theta_1)\big|^q+(1-\rho)\big|\mathfrak{G}'(\theta_2)\big|^q\big)\,d\rho\bigg)^{\frac{1}{q}}\Bigg\}\\
&\quad+A_9(\mu,\alpha,p)\Bigg\{\bigg(\int_{\frac{1}{3}}^{\frac{2}{3}}\big(\rho\big|\mathfrak{G}'(\theta_2)\big|^q+(1-\rho)\big|\mathfrak{G}'(\theta_1)\big|^q\big)\,d\rho\bigg)^{\frac{1}{q}}\\
&\quad+\bigg(\int_{\frac{1}{3}}^{\frac{2}{3}}\big(\rho\big|\mathfrak{G}'(\theta_1)\big|^q+(1-\rho)\big|\mathfrak{G}'(\theta_2)\big|^q\big)\,d\rho\bigg)^{\frac{1}{q}}\Bigg\}\\
&\quad+A_8(\nu,\alpha,p)\Bigg\{\bigg(\int_{\frac{2}{3}}^{1}\big(\rho\big|\mathfrak{G}'(\theta_2)\big|^q+(1-\rho)\big|\mathfrak{G}'(\theta_1)\big|^q\big)\,d\rho\bigg)^{\frac{1}{q}}\\
&\quad+\bigg(\int_{\frac{2}{3}}^{1}\big(\rho\big|\mathfrak{G}'(\theta_1)\big|^q+(1-\rho)\big|\mathfrak{G}'(\theta_2)\big|^q\big)\,d\rho\bigg)^{\frac{1}{q}}\Bigg\}\Bigg].\\
&=(\theta_2-\theta_1)\Bigg[\big(A_7(\lambda,\alpha,p)+A_8(\nu,\alpha,p)\big)\\
&\quad\times\Bigg\{\bigg(\frac{5|\mathfrak{G}'(\theta_1)|^q+|\mathfrak{G}'(\theta_2)|^q}{18}\bigg)^{\frac{1}{q}}+\bigg(\frac{|\mathfrak{G}'(\theta_1)|^q+5|\mathfrak{G}'(\theta_2)|^q}{18}\bigg)^{\frac{1}{q}}\Bigg\}\\
&\quad+2A_9(\mu,\alpha,p)\bigg(\frac{|\mathfrak{G}'(\theta_1)|^q+|\mathfrak{G}'(\theta_2)|^q}{6}\bigg)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Thus, the proof is completed. □

5 Special cases and an example

In this section, we give some special cases of newly established inequalities on the basis of the parameters used in the inequalities. We also present an example to show the validity of the given inequality.

From Lemma 1, we have the following special cases:

(i) By setting $\lambda=\frac{1}{8}$, $\mu=\frac{1}{2}$, and $\nu=\frac{7}{8}$, we have the following equality

$$\begin{aligned}
&\frac{1}{8}\bigg[\mathfrak{G}(\theta_1)+3\mathfrak{G}\bigg(\frac{2\theta_1+\theta_2}{3}\bigg)+3\mathfrak{G}\bigg(\frac{\theta_1+2\theta_2}{3}\bigg)+\mathfrak{G}(\theta_2)\bigg]\\
&\quad-\frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^\alpha}\big[J^\alpha_{\theta_1+}\mathfrak{G}(\theta_2)+J^\alpha_{\theta_2-}\mathfrak{G}(\theta_1)\big]\\
&\quad=\frac{(\theta_2-\theta_1)}{2}\int_0^1\Delta(\rho)\big[\mathfrak{G}'\big(\rho\theta_2+(1-\rho)\theta_1\big)-\mathfrak{G}'\big(\rho\theta_1+(1-\rho)\theta_2\big)\big]\,d\rho,
\end{aligned}$$

where

$$\Delta(\rho)=\begin{cases}\rho^{\alpha}-\frac{1}{8}, & \rho\in[0,\frac{1}{3}),\\ \rho^{\alpha}-\frac{1}{2}, & \rho\in[\frac{1}{3},\frac{2}{3}),\\ \rho^{\alpha}-\frac{7}{8}, & \rho\in[\frac{2}{3},1].\end{cases}$$

This is established by Hezenci et al. in [30] and this identity helps us to obtain Newton inequalities for Riemann–Liouville fractional integrals.

(ii) By setting $\mu=\lambda=\nu=\frac{1}{2}$, we have the following new equality

$$\begin{aligned}&\frac{\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^{\alpha}}\left[J^{\alpha}_{\theta_1+}\mathfrak{G}(\theta_2)+J^{\alpha}_{\theta_2-}\mathfrak{G}(\theta_1)\right]\\&\quad=\frac{(\theta_2-\theta_1)}{2}\int_0^1\left(\rho^{\alpha}-\frac{1}{2}\right)\left[\mathfrak{G}'\left(\rho\theta_2+(1-\rho)\theta_1\right)-\mathfrak{G}'\left(\rho\theta_1+(1-\rho)\theta_2\right)\right]d\rho.\end{aligned}$$

This new identity can help us to obtain the bounds of the trapezoidal formula for Riemann–Liouville fractional integrals.

(iii) By setting $\alpha=1$, we have the following equality

$$\begin{aligned}&\frac{1}{2}\left[(1+\lambda-\nu)\left[\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)\right]+(\nu-\lambda)\left[\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)\right]\right]\\&\qquad-\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\mathfrak{G}(\varkappa)\,d\varkappa\\&\quad=\frac{(\theta_2-\theta_1)}{2}\int_0^1\Delta(\rho)\left[\mathfrak{G}'\left(\rho\theta_2+(1-\rho)\theta_1\right)-\mathfrak{G}'\left(\rho\theta_1+(1-\rho)\theta_2\right)\right]d\rho,\end{aligned}$$

where

$$\Delta(\rho)=\begin{cases}\rho-\lambda, & \rho\in[0,\frac{1}{3}),\\ \rho-\mu, & \rho\in[\frac{1}{3},\frac{2}{3}),\\ \rho-\nu, & \rho\in[\frac{2}{3},1].\end{cases}$$

This equality was established by You et al. in [31, Corollary 2].

From Theorem 3, we have the following special cases:

(i) By setting $\lambda=\frac{1}{8}$, $\mu=\frac{1}{2}$, and $\nu=\frac{7}{8}$, we have the following fractional Newton inequality

$$\begin{aligned}&\left|\frac{1}{8}\left[\mathfrak{G}(\theta_1)+3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right)+3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right)+\mathfrak{G}(\theta_2)\right]\right.\\&\qquad\left.-\frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^{\alpha}}\left[J^{\alpha}_{\theta_1+}\mathfrak{G}(\theta_2)+J^{\alpha}_{\theta_2-}\mathfrak{G}(\theta_1)\right]\right|\\&\quad\leq\frac{\theta_2-\theta_1}{2}\left[B_1(\alpha)+B_2(\alpha)+B_2(\alpha)\right]\left(\left|\mathfrak{G}'(\theta_1)\right|+\left|\mathfrak{G}'(\theta_2)\right|\right),\end{aligned}$$

where

$$B_1(\alpha)=\int_0^{\frac{1}{3}}\left|\rho^\alpha-\frac{1}{8}\right|d\rho=\begin{cases}\frac{2\alpha}{1+\alpha}(\frac{1}{8})^{1+\frac{1}{\alpha}}+\frac{1}{3^{\alpha+1}(\alpha+1)}-\frac{1}{24}, & 0<\alpha\le\frac{\ln(\frac{1}{8})}{\ln(\frac{1}{3})},\\ \frac{1}{24}-\frac{1}{3^{\alpha+1}(\alpha+1)}, & \alpha>\frac{\ln(\frac{1}{8})}{\ln(\frac{1}{3})},\end{cases}$$

$$B_2(\alpha)=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\rho^\alpha-\frac{1}{2}\right|d\rho=\begin{cases}\frac{2^{1+\alpha}-1}{3^{\alpha+1}(\alpha+1)}-\frac{1}{6}, & 0<\alpha\le\frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})},\\ \frac{\alpha}{1+\alpha}(\frac{1}{2})^{\frac{1}{\alpha}}+\frac{1+2^{1+\alpha}}{3^{\alpha+1}(\alpha+1)}-\frac{1}{2}, & \frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})}<\alpha\le\frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})},\\ \frac{1}{6}-\frac{2^{1+\alpha}-1}{3^{\alpha+1}(\alpha+1)}, & \alpha>\frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})}\end{cases}$$

and

$$B_3(\alpha)=\int_{\frac{2}{3}}^{1}\left|\rho^\alpha-\frac{7}{8}\right|d\rho=\begin{cases}\frac{3^{1+\alpha}-2^{1+\alpha}}{3^{\alpha+1}(\alpha+1)}-\frac{7}{24}, & 0<\alpha\le\frac{\ln(\frac{7}{8})}{\ln(\frac{2}{3})},\\ \frac{2\alpha}{1+\alpha}(\frac{7}{8})^{1+\frac{1}{\alpha}}+\frac{2^{1+\alpha}+3^{1+\alpha}}{3^{\alpha+1}(\alpha+1)}-\frac{35}{24}, & \alpha>\frac{\ln(\frac{7}{8})}{\ln(\frac{2}{3})}.\end{cases}$$

This was established by Hezenci et al. in [30].

(ii) By setting $\mu=\lambda=\nu=\frac{1}{2}$, we have the following fractional trapezoidal-type new inequality

$$\begin{aligned}&\left|\frac{\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^\alpha}\left[J^\alpha_{\theta_1+}\mathfrak{G}(\theta_2)+J^\alpha_{\theta_2-}\mathfrak{G}(\theta_1)\right]\right| \\ &\quad\le\frac{\theta_2-\theta_1}{2}\left[C_1(\alpha)+C_2(\alpha)+C_3(\alpha)\right]\left(\left|\mathfrak{G}'(\theta_1)\right|+\left|\mathfrak{G}'(\theta_2)\right|\right) \\ &\quad=\frac{\theta_2-\theta_1}{2}\left[\frac{\alpha}{1+\alpha}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}+\frac{1}{\alpha+1}-\frac{1}{2}\right]\left(\left|\mathfrak{G}'(\theta_1)\right|+\left|\mathfrak{G}'(\theta_2)\right|\right),\end{aligned}\tag{5.1}$$

where

$$C_1(\alpha)=\int_0^{\frac{1}{3}}\left|\rho^\alpha-\frac{1}{2}\right|d\rho=\begin{cases}\frac{\alpha}{1+\alpha}(\frac{1}{2})^{\frac{1}{\alpha}}+\frac{1}{3^{\alpha+1}(\alpha+1)}-\frac{1}{6}, & 0<\alpha\le\frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})},\\ \frac{1}{6}-\frac{1}{3^{\alpha+1}(\alpha+1)}, & \alpha>\frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})},\end{cases}$$

$$C_2(\alpha)=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\rho^\alpha-\frac{1}{2}\right|d\rho=\begin{cases}\frac{2^{1+\alpha}-1}{3^{\alpha+1}(\alpha+1)}-\frac{1}{6}, & 0<\alpha\le\frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})},\\ \frac{\alpha}{1+\alpha}(\frac{1}{2})^{\frac{1}{\alpha}}+\frac{1+2^{1+\alpha}}{3^{\alpha+1}(\alpha+1)}-\frac{1}{2}, & \frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})}<\alpha\le\frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})},\\ \frac{1}{6}-\frac{2^{1+\alpha}-1}{3^{\alpha+1}(\alpha+1)}, & \alpha>\frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})}\end{cases}$$

and

$$C_3(\alpha)=\int_{\frac{2}{3}}^{1}\left|\rho^\alpha-\frac{1}{2}\right|d\rho=\begin{cases}\frac{3^{1+\alpha}-2^{1+\alpha}}{3^{\alpha+1}(\alpha+1)}-\frac{1}{6}, & 0<\alpha\le\frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})},\\ \frac{\alpha}{1+\alpha}(\frac{1}{2})^{\frac{1}{\alpha}}+\frac{2^{1+\alpha}+3^{1+\alpha}}{3^{\alpha+1}(\alpha+1)}-\frac{5}{6}, & \alpha>\frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})}.\end{cases}$$

(iii) By setting $\alpha=1$, we recapture the inequality established in [31, Corollary 5].

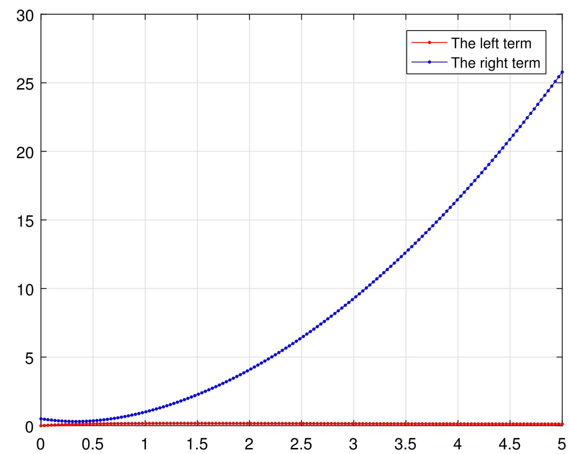

Figure 1 An example of the inequality (5.2) depending on α, computed and plotted with MATLAB

Example 1 Let us consider a function $\mathfrak{G} : [\theta_1, \theta_2] = [0, 1] \to \mathbb{R}$ given by $\mathfrak{G}(\rho) = \rho^2$ in the inequality (5.1). Then, the left-hand side of (5.1) reduces to

$$\begin{aligned} &\left| \frac{\mathfrak{G}(\theta_1) + \mathfrak{G}(\theta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^\alpha} \left[J^\alpha_{\theta_1+} \mathfrak{G}(\theta_2) + J^\alpha_{\theta_2-} \mathfrak{G}(\theta_1) \right] \right| \\ &\quad = \left| \frac{1}{2} - \frac{\alpha}{2} \left[\int_0^1 (1-\rho)^{\alpha-1} \rho^2 \, d\rho + \int_0^1 \rho^{\alpha-1} \rho^2 \, d\rho \right] \right| = \left| \frac{1}{2} - \frac{\alpha^2+\alpha+2}{2(\alpha+1)(\alpha+2)} \right|. \end{aligned}$$

The right hand-side of (5.1) becomes

$$\begin{aligned} &\frac{\theta_2-\theta_1}{2} \left[C_1(\alpha) + C_2(\alpha) + C_3(\alpha) \right] \left(\left| \mathfrak{G}'(\theta_1) \right| + \left| \mathfrak{G}'(\theta_2) \right| \right) \\ &\quad = \frac{\alpha}{1+\alpha} \left(\frac{1}{2} \right)^{\frac{1}{\alpha}} + \frac{1}{\alpha+1} - \frac{1}{2}. \end{aligned}$$

Then, by the inequality, we have

$$\left| \frac{1}{2} - \frac{\alpha^2+\alpha+2}{2(\alpha+1)(\alpha+2)} \right| \le \frac{\alpha}{1+\alpha} \left(\frac{1}{2} \right)^{\frac{1}{\alpha}} + \frac{1}{\alpha+1} - \frac{1}{2}. \tag{5.2}$$

One can see the validity of the inequality (5.2) in Fig. 1.

As can be seen in Fig. 1, the left-hand side of (5.1) in Example 1 is always below the right-hand side of this equation, for all values of $\alpha \in (0, 5]$.

From Theorem 4, we have the following special cases:

(i) By setting $\lambda = \frac{1}{8}$, $\mu = \frac{1}{2}$, and $\nu = \frac{7}{8}$, we have the following fractional Newton inequality

$$\begin{aligned} &\left| \frac{1}{8} \left[\mathfrak{G}(\theta_1) + 3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3} \right) + 3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3} \right) + \mathfrak{G}(\theta_2) \right] \right. \\ &\qquad \left. - \frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^\alpha} \left[J^\alpha_{\theta_1+} \mathfrak{G}(\theta_2) + J^\alpha_{\theta_2-} \mathfrak{G}(\theta_1) \right] \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\theta_2-\theta_1}{2}\Big[B_1^{1-\frac{1}{q}}(\alpha)\Big\{\big(B_4(\alpha)\big|\mathfrak{G}'(\theta_2)\big|^q+\big(B_1(\alpha)-B_4(\alpha)\big)\big|\mathfrak{G}'(\theta_1)\big|^q\big)^{\frac{1}{q}} \\
&\quad+\big(B_4(\alpha)\big|\mathfrak{G}'(\theta_1)\big|^q+\big(B_1(\alpha)-B_4(\alpha)\big)\big|\mathfrak{G}'(\theta_2)\big|^q\big)^{\frac{1}{q}}\Big\} \\
&\quad+B_2^{1-\frac{1}{q}}(\alpha)\Big\{\big(B_5(\alpha)\big|\mathfrak{G}'(\theta_2)\big|^q+\big(B_2(\alpha)-B_5(\alpha)\big)\big|\mathfrak{G}'(\theta_1)\big|^q\big)^{\frac{1}{q}} \\
&\quad+\big(B_5(\alpha)\big|\mathfrak{G}'(\theta_1)\big|^q+\big(B_2(\alpha)-B_5(\alpha)\big)\big|\mathfrak{G}'(\theta_2)\big|^q\big)^{\frac{1}{q}}\Big\} \\
&\quad+B_3^{1-\frac{1}{q}}(\alpha)\Big\{\big(B_6(\alpha)\big|\mathfrak{G}'(\theta_2)\big|^q+\big(B_3(\alpha)-B_6(\alpha)\big)\big|\mathfrak{G}'(\theta_1)\big|^q\big)^{\frac{1}{q}} \\
&\quad+\big(B_6(\alpha)\big|\mathfrak{G}'(\theta_1)\big|^q+\big(B_3(\alpha)-B_6(\alpha)\big)\big|\mathfrak{G}'(\theta_2)\big|^q\big)^{\frac{1}{q}}\Big\}\Big],
\end{aligned}
$$

where

$$
B_4(\alpha)=\int_0^{\frac{1}{3}}\rho\left|\rho^\alpha-\frac{1}{8}\right|d\rho=\begin{cases}\frac{\alpha}{2+\alpha}\left(\frac{1}{8}\right)^{1+\frac{2}{\alpha}}+\frac{1}{3^{\alpha+2}(\alpha+2)}-\frac{1}{144}, & 0<\alpha\leq\frac{\ln(\frac{1}{8})}{\ln(\frac{1}{3})},\\ \frac{1}{144}-\frac{1}{3^{\alpha+2}(\alpha+2)}, & \alpha>\frac{\ln(\frac{1}{8})}{\ln(\frac{1}{3})},\end{cases}
$$

$$
B_5(\alpha)=\int_{\frac{1}{3}}^{\frac{2}{3}}\rho\left|\rho^\alpha-\frac{1}{2}\right|d\rho=\begin{cases}\frac{2^{2+\alpha}-1}{3^{\alpha+2}(\alpha+2)}-\frac{1}{12}, & 0<\alpha\leq\frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})},\\ \frac{\alpha}{2+\alpha}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}+\frac{1+2^{2+\alpha}}{3^{\alpha+2}(\alpha+2)}-\frac{5}{36}, & \frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})}<\alpha\leq\frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})},\\ \frac{1}{12}-\frac{2^{2+\alpha}-1}{3^{\alpha+2}(\alpha+2)}, & \alpha>\frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})}\end{cases}
$$

and

$$
B_6(\alpha)=\int_{\frac{2}{3}}^{1}\rho\left|\rho^\alpha-\frac{7}{8}\right|d\rho=\begin{cases}\frac{3^{2+\alpha}-2^{2+\alpha}}{3^{\alpha+2}(\alpha+2)}-\frac{35}{144}, & 0<\alpha\leq\frac{\ln(\frac{7}{8})}{\ln(\frac{2}{3})},\\ \frac{\alpha}{2+\alpha}\left(\frac{7}{8}\right)^{1+\frac{2}{\alpha}}+\frac{2^{2+\alpha}+3^{2+\alpha}}{3^{\alpha+2}(\alpha+2)}-\frac{91}{144}, & \alpha>\frac{\ln(\frac{7}{8})}{\ln(\frac{2}{3})}.\end{cases}
$$

This was established by Hezenci et al. in [30].

(ii) By setting $\mu=\lambda=\nu=\frac{1}{2}$, we have the following fractional trapezoidal-type new inequality

$$
\begin{aligned}
&\left|\frac{\mathfrak{G}(\theta_1)+\mathfrak{G}(\theta_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^\alpha}\Big[J^\alpha_{\theta_1+}\mathfrak{G}(\theta_2)+J^\alpha_{\theta_2-}\mathfrak{G}(\theta_1)\Big]\right| \\
&\leq \frac{\theta_2-\theta_1}{2}\Big[C_1^{1-\frac{1}{q}}(\alpha)\Big\{\big(C_4(\alpha)\big|\mathfrak{G}'(\theta_2)\big|^q+\big(C_1(\alpha)-C_4(\alpha)\big)\big|\mathfrak{G}'(\theta_1)\big|^q\big)^{\frac{1}{q}} \\
&\quad+\big(C_4(\alpha)\big|\mathfrak{G}'(\theta_1)\big|^q+\big(C_1(\alpha)-C_4(\alpha)\big)\big|\mathfrak{G}'(\theta_2)\big|^q\big)^{\frac{1}{q}}\Big\} \\
&\quad+C_2^{1-\frac{1}{q}}(\alpha)\Big\{\big(C_5(\alpha)\big|\mathfrak{G}'(\theta_2)\big|^q+\big(C_2(\alpha)-C_5(\alpha)\big)\big|\mathfrak{G}'(\theta_1)\big|^q\big)^{\frac{1}{q}} \\
&\quad+\big(C_5(\alpha)\big|\mathfrak{G}'(\theta_1)\big|^q+\big(C_2(\alpha)-C_5(\alpha)\big)\big|\mathfrak{G}'(\theta_2)\big|^q\big)^{\frac{1}{q}}\Big\} \\
&\quad+C_3^{1-\frac{1}{q}}(\alpha)\Big\{\big(C_6(\alpha)\big|\mathfrak{G}'(\theta_2)\big|^q+\big(C_3(\alpha)-C_6(\alpha)\big)\big|\mathfrak{G}'(\theta_1)\big|^q\big)^{\frac{1}{q}} \\
&\quad+\big(C_6(\alpha)\big|\mathfrak{G}'(\theta_1)\big|^q+\big(C_3(\alpha)-C_6(\alpha)\big)\big|\mathfrak{G}'(\theta_2)\big|^q\big)^{\frac{1}{q}}\Big\}\Big],
\end{aligned}
$$

where

$$C_4(\alpha) = \int_0^{\frac{1}{3}} \rho\left|\rho^\alpha - \frac{1}{2}\right| d\rho = \begin{cases} \frac{\alpha}{2+\alpha}(\frac{1}{2})^{1+\frac{2}{\alpha}} + \frac{1}{3^{\alpha+2}(\alpha+2)} - \frac{1}{36}, & 0 < \alpha \leq \frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})}, \\ \frac{1}{36} - \frac{1}{3^{\alpha+2}(\alpha+2)}, & \alpha > \frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})}, \end{cases}$$

$$C_5(\alpha) = \int_{\frac{1}{3}}^{\frac{2}{3}} \rho\left|\rho^\alpha - \frac{1}{2}\right| d\rho = \begin{cases} \frac{2^{2+\alpha}-1}{3^{\alpha+2}(\alpha+2)} - \frac{1}{12}, & 0 < \alpha \leq \frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})}, \\ \frac{\alpha}{2+\alpha}(\frac{1}{2})^{1+\frac{2}{\alpha}} + \frac{1+2^{2+\alpha}}{3^{\alpha+2}(\alpha+2)} - \frac{5}{36}, & \frac{\ln(\frac{1}{2})}{\ln(\frac{1}{3})} < \alpha \leq \frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})}, \\ \frac{1}{12} - \frac{2^{2+\alpha}-1}{3^{\alpha+2}(\alpha+2)}, & \alpha > \frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})} \end{cases}$$

and

$$C_6(\alpha) = \int_{\frac{2}{3}}^{1} \rho\left|\rho^\alpha - \nu\right| d\rho = \begin{cases} \frac{3^{2+\alpha}-2^{2+\alpha}}{3^{\alpha+2}(\alpha+2)} - \frac{5}{36}, & 0 < \alpha \leq \frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})}, \\ \frac{\alpha}{2+\alpha}(\frac{1}{2})^{1+\frac{2}{\alpha}} + \frac{2^{2+\alpha}+3^{2+\alpha}}{3^{\alpha+2}(\alpha+2)} - \frac{13}{36}, & \alpha > \frac{\ln(\frac{1}{2})}{\ln(\frac{2}{3})}. \end{cases}$$

(iii) By setting $\alpha = 1$, we recapture the inequality established in [31, Corollary 6].

From Theorem 5, we have the following special cases:

(i) By setting $\lambda = \frac{1}{8}$, $\mu = \frac{1}{2}$, and $\nu = \frac{7}{8}$, we have the following fractional Newton inequality

$$\begin{aligned} &\left|\frac{1}{8}\left[\mathfrak{G}(\theta_1) + 3\mathfrak{G}\left(\frac{2\theta_1+\theta_2}{3}\right) + 3\mathfrak{G}\left(\frac{\theta_1+2\theta_2}{3}\right) + \mathfrak{G}(\theta_2)\right]\right. \\ &\quad\left. - \frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^\alpha}\left[J_{\theta_1+}^\alpha \mathfrak{G}(\theta_2) + J_{\theta_2-}^\alpha \mathfrak{G}(\theta_1)\right]\right| \\ &\leq \frac{\theta_2-\theta_1}{2}\left[\left(A_7\left(\frac{1}{8},\alpha,p\right) + A_8\left(\frac{7}{8},\alpha,p\right)\right)\right. \\ &\quad \times \left\{\left(\frac{5|\mathfrak{G}'(\theta_1)|^q + |\mathfrak{G}'(\theta_2)|^q}{18}\right)^{\frac{1}{q}} + \left(\frac{|\mathfrak{G}'(\theta_1)|^q + 5|\mathfrak{G}'(\theta_2)|^q}{18}\right)^{\frac{1}{q}}\right\} \\ &\quad\left. + 2A_9\left(\frac{1}{2},\alpha,p\right)\left(\frac{|\mathfrak{G}'(\theta_1)|^q + |\mathfrak{G}'(\theta_2)|^q}{6}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

This was established by Hezenci et al. in [30].

(ii) By setting $\mu = \lambda = \nu = \frac{1}{2}$, we have the following fractional trapezoidal-type new inequality

$$\begin{aligned} &\left|\frac{\mathfrak{G}(\theta_1) + \mathfrak{G}(\theta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\theta_2-\theta_1)^\alpha}\left[J_{\theta_1+}^\alpha \mathfrak{G}(\theta_2) + J_{\theta_2-}^\alpha \mathfrak{G}(\theta_1)\right]\right| \\ &\leq \frac{\theta_2-\theta_1}{2}\left[\left(A_7\left(\frac{1}{2},\alpha,p\right) + A_8\left(\frac{1}{2},\alpha,p\right)\right)\right. \\ &\quad \times \left\{\left(\frac{5|\mathfrak{G}'(\theta_1)|^q + |\mathfrak{G}'(\theta_2)|^q}{18}\right)^{\frac{1}{q}} + \left(\frac{|\mathfrak{G}'(\theta_1)|^q + 5|\mathfrak{G}'(\theta_2)|^q}{18}\right)^{\frac{1}{q}}\right\} \\ &\quad\left. + 2A_9\left(\frac{1}{2},\alpha,p\right)\left(\frac{|\mathfrak{G}'(\theta_1)|^q + |\mathfrak{G}'(\theta_2)|^q}{6}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

6 Conclusion

Using the RLFIs, we illustrated several novel Simpson's second-type inequalities for differentiable convex functions. Additionally, it is demonstrated that the newly established inequalities are a continuation of those that already existed. We used specific choices for the parameters in the new inequalities because we attained some known inequalities for these choices. It is important to note that equivalent inequalities can also be obtained using Hadamard, Conformable, and Katugampola fractional operators as well as fractional operators with an exponential kernel. Future workers will be able to obtain comparable inequalities for multiple convexity types and coordinated convexity on fractals, which is a fascinating and novel challenge.

Declarations

Competing interests

The authors declare no competing interests.

Author contributions

All authors wrote and reviewed the manuscript. M.A. and H.B. prepared Figure 1.

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References

1. Hermite, C.: Sur deux limites d'une integrale de finie. Mathesis **82** (1883)
2. Hadamard, J.: Etude sur les fonctions entiees et en particulier d'une fonction consideree par Riemann. J. Math. Pures Appl. **58**, 171–215 (1893)
3. Dragomir, S.S., Agarwal, P.R.: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoid formula. Appl. Math. Lett. **11**, 91–95 (1998)
4. Kirmaci, U.S.: Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. Appl. Math. Comput. **147**, 137–146 (2004)
5. Qi, F., Xi, B.Y.: Some Hermite–Hadamard type inequalities for differentiable convex functions and applications. Hacet. J. Math. Stat. **42**, 243–257 (2013)
6. Sarikaya, M.Z., Set, E., Yaldiz, H., Başak, N.: Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**, 2403–2407 (2013)
7. Set, E.: New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. Comput. Math. Appl. **63**, 1147–1154 (2012)
8. İşcan, İ., Wu, S.: Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals. Appl. Math. Comput. **238**, 237–244 (2014)
9. Sarikaya, M.Z., Yildrim, H.: On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. Miskolc Math. Notes **17**, 1049–1059 (2016)
10. Sarikaya, M.Z., Set, E., Özdemir, M.E.: On new inequalities of Simpson's type for s-convex functions. Comput. Math. Appl. **60**, 2191–2199 (2010)
11. Peng, C., Zhou, C., Du, T.S.: Riemann-Liouville fractional Simpson's inequalities through generalized (m, h_1, h_2)-preinvexity. Ital. J. Pure Appl. Math. **38**, 345–367 (2017)
12. Chen, J., Huang, X.: Some new inequalities of Simpson's type for s-convex functions via fractional integrals. Filomat **31**, 4989–4997 (2017)
13. Sarikaya, M.Z., Ertugral, F.: On the generalized Hermite-Hadamard inequalities. An. Univ. Craiova, Ser. Mat. Inform. **47**, 193–213 (2020)
14. Zhao, D., Ali, M.A., Kashuri, A., Budak, H.: Generalized fractional integral inequalities of Hermite–Hadamard type for harmonically convex functions. Adv. Differ. Equ. **2020**, 137, 1–14 (2020)
15. Budak, H., Hezenci, F., Kara, H.: On parameterized inequalities of Ostrowski and Simpson type for convex functions via generalized fractional integral. Math. Methods Appl. Sci. **44**(17) 12522–12536 (2021). https://doi.org/10.1002/mma.7558
16. Sitthiwirattham, T., Nonlaopon, K., Ali, M.A., Budak, H.: Riemann-Liouville fractional Newton's type inequalities for differentiable convex functions. Fractal Fract. **6**, 175 (2022)
17. Awan, M.U., Talib, S., Chu, Y.M., Noor, M.A., Noor, K.I.: Some new refinements of Hermite–Hadamard-type inequalities involving-Riemann–Liouville fractional integrals and applications. Math. Probl. Eng. **2020**, 3051920 (2020)

18. Du, T.S., Luo, C.Y., Yu, B.: Certian quantum estimates on the parameterized integral inequalities and their applications. J. Math. Inequal. **15**, 201–228 (2021)
19. Du, T.S., Zhuo, T.C.: On the fractional double inclusion relations having exponential kernels via interval-valued co-ordinated convex mappings. Chaos Solitons Fractals **156**, 111846 (2022)
20. Du, T.S., Luo, C.Y., Cao, Z.J.: On the Bullen-type inequalities via generalized fractional integrals and their applications. Fractals **29**, 2150188 (2021)
21. Kashuri, A., Liko, R.: Generalized trapezoidal type integral inequalities and their applications. J. Anal. **28**, 1023–1043 (2020)
22. Khan, M.A., Iqbal, A., Suleman, M., Chu, Y.M.: Hermite–Hadamard type inequalities for fractional integrals via Green's function. J. Inequal. Appl. **2018**, 161, 1–15 (2018)
23. Khan, M.A., Ali, T., Dragomir, S.S., Sarikaya, M.Z.: Hermite–Hadamard type inequalities for conformable fractional integrals. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **112**, 1033–1048 (2018)
24. Set, E., Choi, J., Gözpinar, A.: Hermite-Hadamard type inequalities for the generalized *k*-fractional integral operators. J. Inequal. Appl. **2017**, 206, 1–17 (2017)
25. Tunc, M.: On new inequalities for *h*-convex functions via Riemann-Liouville fractional integration. Filomat **27**, 559–565 (2013)
26. Vivas-Cortez, M., Ali, M.A., Kashuri, A., Budak, H.: Generalizations of fractional Hermite-Hadamard-Mercer like inequalities for convex functions. AIMS Math. **6**, 9397–9421 (2021)
27. Zhao, D., Ali, M.A., Kashuri, A., Budak, H., Sarikaya, M.Z.: Hermite–Hadamard-type inequalities for the interval-valued approximately *h*-convex functions via generalized fractional integrals. J. Inequal. Appl. **2020**, 222, 1–38 (2020)
28. Gorenflo, R., Mainardi, F.: Fractional Calculus: Integral and Differential Equations of Fractional Order. Springer, Wien (1997)
29. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
30. Hezenci, F., Budak, H., Kara, H.: On new version of Newton's inequalities for Riemann-Liouville fractional integrals. Rocky Mt. J. Math. (2022, in press)
31. You, X.X., Ali, M.A., Budak, H., Vivas-Cortez, M., Qaisar, S.: Some parameterized quantum Simpson's and quantum Newton's integral inequalities via quantum differentiable convex mappings. Math. Probl. Eng. **2021**, 5526726 (2021)