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Dynamics of plate equations with time delay driven by additive noise in \mathbb{R}^n

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Abstract

This paper is concerned with the asymptotic behavior of solutions for plate equations with delay blurred by additive noise in \mathbb{R}^n . First, we obtain the uniform compactness of pullback random attractors of the problem, then derive the upper semicontinuity of the attractors.

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1 Introduction

Delays in differential systems are used for mathematical modeling in many applications to describe the dynamics influenced by events from the past. It is known that differential equations with delay appears in physics, biology, and other disciplines, and the time delay is considered in the model of the systems [7, 12]. In particular, these equations are applied to mathematical modeling in many applications to describe the dynamics influenced by events from the past, see [15, 16, 18]. Most noteworthy is that the attractors of deterministic differential equations with a time delay have been studied in [11], while the stochastic case has been studied in [6, 29].

This paper is concerned with the following plate equations with time delay blurred by additive noise in \mathbb{R}^n :

$$\begin{cases} \partial_{tt}u + \alpha\partial_tu + \partial_{xxxx}(u) + \partial_{xxxx}u + \lambda u + F(u(x,t),x) \\ \quad = f(u(t-\rho,x),x) + g(x,t) + \epsilon h(x)\frac{dW}{dt}, \quad t > \tau, \\ u_\tau(x,s) := u(x,\tau+s) = \phi(x,s), \quad \partial_tu_\tau(x,s) = \partial_t\phi(x,s), \end{cases} \quad (1.1)$$

where $\tau \in \mathbb{R}$, $x \in \mathbb{R}^n$, $s \in [-\rho, 0]$, $\epsilon \in (0, 1]$, α, λ are positive constants, the time delay $\rho > 0$ is a constant, the conditions F, f are satisfied (see Sect. 3), $g(x, \cdot) \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, $h \in H^2(\mathbb{R}^n)$ and $\phi \in C([\tau - \rho, \tau], H^2(\mathbb{R}^n))$, W is a two-side real-value Wiener process on a complete probability space that will be specified later.

For the deterministic case, many authors obtained the existence of global attractors in [2, 8–10, 30–33, 40]. For the stochastic case, Ma and Shen investigated the existence of

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random attractors for plate equations in bounded domains in [14, 19, 20]. Moreover, in the entire space, Yao obtained the existence of random attractors for plate equations in [34–39]. Wang investigated the global existence as well as long-term dynamics for a wide class of lattice plate equations on the entire integer set with nonlinear damping driven by infinite-dimensional nonlinear noise in [27].

For the case $\rho \equiv 0$ in (1.1), we derived the existence of results in [35]. However, as far as we know, there is little literature dealing with stochastic time-delay plate equations. Also, in [28], Wang and Ma studied the existence of pullback attractors for the nonautonomous suspension-bridge equation with time delay.

Motivated by the literature above, we study the dynamics of the delay plate equations. The main features of the work are summarized as follows: (i) We prove that (1.1) generates random dynamical systems; (ii) We show the existence of random attractors for (1.1); (iii) We obtain the convergence of random attractors for (1.1) as $\rho \rightarrow 0$ or $\epsilon \rightarrow 0$. A major difficulty in the proof process is to prove the existence random attractors for (1.1), the reason for this is that Sobolev embeddings are no longer compact. To overcome this, we use the uniform estimates and the splitting technique ([26]).

This paper is organized as follows. In the next section, we recall some basic concepts on the theory of random dynamical systems. We then prove an abstract result for the upper semicontinuity of random attractors for stochastic delay equations. In Sect. 3, we establish the continuous random dynamical system for (1.1). Some necessary estimates are given in Sect. 4. We then prove the existence of pullback attractors for (1.1) in Sect. 5. In Sects. 6 and 7, we further prove the upper semicontinuity of attractors when $\epsilon \rightarrow 0$ and $\rho \rightarrow 0$.

2 Mathematical setting and notation

Now, we recall some notations and propositions on the theory of random dynamical systems, the reader is referred to [1, 3, 4, 13, 17, 21–24].

Denote $(\Omega, \mathcal{F}, \mathbb{P})$ as the probability space and for $t \in \mathbb{R}$, $\omega \in \Omega$,

$$\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t).$$

Let $(X, \|\cdot\|_X)$ be a separable Hilbert space, and let $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ be an ergodic metric dynamical system

Proposition 2.1 ([24]) *Let \mathcal{D} be an inclusion closed collection of some families of nonempty subsets of X , and Φ be a continuous cocycle on X over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$. Then, Φ has a unique \mathcal{D} -pullback random attractor \mathcal{A} in \mathcal{D} if Φ is \mathcal{D} -pullback asymptotically compact in X and Φ has a closed measurable \mathcal{D} -pullback absorbing set K in \mathcal{D} .*

We denote by C_X the space $C([-\rho, 0], X)$ with the sup-norm

$$\|u\|_{C_X} = \sup_{s \in [-\rho, 0]} \|u\|_X, \quad u \in C([-\rho, 0], X).$$

Denote by $(Y, \|\cdot\|)$ a Banach space that satisfies that the injection $X \subset Y$ is continuous, we also denote by $C_{X,Y}$ the Banach space $C_X \cap C^1([-\rho, 0], Y)$ with the norm $\|\cdot\|_{C_{X,Y}}$

$$\|y\|_{C_{X,Y}}^2 = \|\nu\|_{C_Y}^2 + \|u\|_{C_X}^2, \quad y = (u, \nu)^\top, \quad u \in C_X, \nu \in C_Y. \quad (2.1)$$

Denote $-\Delta$ the Laplace operator, $A = \Delta^2$ and the Hilbert spaces $V_r = D(A^{\frac{r}{4}})$ endowed with inner product and norm

$$(u, v)_r = (A^{\frac{r}{4}}u, A^{\frac{r}{4}}v), \quad \|\cdot\|_r = \|A^{\frac{r}{4}}\cdot\|.$$

In particular, $V_0 = L^2(\mathbb{R}^n)$, $V_2 = H^2(\mathbb{R}^n)$.

3 Random dynamical system

In this section, we discuss the assumptions on F, f , and g and define a continuous cocycle in $C_{V_2, V_0}(\mathbb{R}^n)$ for (1.1). Assume $F(x, \cdot) \in C^2(\mathbb{R})$, f, g satisfy the following conditions:

Let $\tilde{F}(r, x) = \int_0^r F(s, x) ds$ for $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$|F(s, x)| \leq c_1 |s|^p + \eta_1(x), \quad \eta_1 \in L^2(\mathbb{R}^n), \quad (3.1)$$

$$F(s, x)s - c_2 \tilde{F}(s, x) \geq \eta_2(x), \quad \eta_2 \in L^1(\mathbb{R}^n), \quad (3.2)$$

$$\tilde{F}(s, x) \geq c_3 |s|^{p+1} - \eta_3(x), \quad \eta_3 \in L^1(\mathbb{R}^n), \quad (3.3)$$

$$\left| \frac{\partial F}{\partial s}(s, x) \right| \leq \varpi, \quad (3.4)$$

$$f(0, x) = 0, \quad \text{and} \quad |f(s_1, x) - f(s_2, x)| \leq l_f |s_1 - s_2|, \quad (3.5)$$

where $\varpi, l_f > 0$, $1 \leq p \leq \frac{n+4}{n-4}$ and $c_i > 0$. It follows from (3.1) and (3.2) that

$$\tilde{F}(s, x) \leq c(|s|^2 + |s|^{p+1} + \eta_1^2 + \eta_2). \quad (3.6)$$

Assume g satisfies

$$\int_{-\infty}^0 e^{\sigma s} \|g(\cdot, s + \tau)\|^2 ds < \infty, \quad \forall \tau \in \mathbb{R}, \quad (3.7)$$

which implies that

$$\lim_{r \rightarrow \infty} \int_{-\infty}^{\tau} \int_{|x| \geq r} e^{\sigma s} |g(\cdot, s)|^2 dx ds = 0, \quad \forall \tau \in \mathbb{R}, \quad (3.8)$$

where σ is a positive constant.

For $Y = (u, v)^{\top} \in C_{V_2, V_0}(\mathbb{R}^n)$, set

$$\begin{aligned} \|Y\|_{C_{V_2, V_0}(\mathbb{R}^n)} &= (\|u\|_{C_{V_2}}^2 + \|v\|_{C_{V_0}}^2)^{\frac{1}{2}} \\ &= ((\lambda + \delta^2 - \delta\alpha) \|u\|^2 + \|v\|^2 + (1 - \delta) \|\Delta u\|^2)^{\frac{1}{2}}. \end{aligned} \quad (3.9)$$

In addition, we see that $\|\cdot\|_{C_{V_2, V_0}(\mathbb{R}^n)}$ is equivalent to $\|\cdot\|_{C_{V_2, V_0}(\mathbb{R}^n)}$ in (2.1).

Let $\xi = \partial_t u + \delta u$, for $x \in \mathbb{R}^n$, $s \in [-\rho, 0]$, hence problem (1.1) is equivalent to

$$\begin{cases} \frac{du}{dt} + \delta u = \xi, \\ \frac{d\xi}{dt} + (\alpha + A - \delta)\xi + \lambda u + F(x, u(t, x)) \\ \quad = [\delta(\alpha + A - \delta) - A]u + f(u(t - \rho, x), x) + g(x, t) + \epsilon h(x) \frac{dW}{dt}, \\ u_{\tau}(x, s) := u(x, \tau + s) = \phi(x, s), \quad \xi_{\tau}(x, s) = \partial_t \phi(x, s) + \delta \phi(x, s). \end{cases} \quad (3.10)$$

Let $z(\theta_t \omega) = hy(\theta_t \omega)$, where y satisfies

$$y(\omega) = - \int_{-\infty}^0 e^s(\omega)(s) ds.$$

From [5], we know that for every $\omega \in \Omega$, $y(\theta_t \omega)$ is continuous.

Let $z(\theta_t \omega) = hy(\theta_t \omega)$. We have the following lemma on $z(\theta_t \omega)$:

Lemma 3.1 ([32]) *For $\forall \varepsilon > 0$, there exists a random variable $\chi : \Omega \rightarrow \mathbb{R}^+$, such that for $\forall t \in \mathbb{R}$, $\omega \in \Omega$,*

$$\begin{aligned} \|z(\theta_t \omega)\| &\leq e^{\varepsilon|t|} \chi(\omega) \|h\|, \\ \|\nabla z(\theta_t \omega)\| &\leq e^{\varepsilon|t|} \chi(\omega) \|\nabla h\|, \\ \|\Delta z(\theta_t \omega)\| &\leq e^{\varepsilon|t|} \chi(\omega) \|\Delta h\|, \end{aligned}$$

where

$$e^{-\varepsilon|t|} \chi(\omega) \leq \chi(\theta_t \omega) \leq e^{\varepsilon|t|} \chi(\omega).$$

Denote $v(t) = \xi(t) - \varepsilon z(\theta_t \omega)$, then (3.10) is equivalent to

$$\left\{ \begin{array}{l} \frac{du}{dt} + \delta u = v + \varepsilon z(\theta_t \omega), \\ \frac{dv}{dt} - (\delta - \alpha - A)v - [\delta(-\delta + \alpha + A) - \lambda - A]u - \varepsilon[1 - (\alpha + A - \delta)]z(\theta_t \omega) \\ \quad + F(u(t, x), x) = f(u(t - \rho, x), x) + g(x, t), \\ u_\tau(x, s) := u(x, \tau + s) = \phi(x, s), \\ v_\tau(x, s) = \partial_t \phi(x, s) + \delta \phi(x, s) - \varepsilon z(\theta_t \omega) := \psi(x, s). \end{array} \right. \quad (3.11)$$

For given $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $\phi \in C_{V_2}(\mathbb{R}^n)$, $\psi \in C_{V_0}(\mathbb{R}^n)$, a solution of (3.11) will be written as $(u(\cdot, \tau, \omega, \phi), v(\cdot, \tau, \omega, \psi))$. As usual, the segments of $u(\cdot, \tau, \omega, \phi)$ and $v(\cdot, \tau, \omega, \psi)$ on $[t - \rho, t]$ are written as $u^t(\cdot, \tau, \omega, \phi)$ and $v^t(\cdot, \tau, \omega, \psi)$, respectively; that is,

$$u_t(s, \tau, \omega, \phi) = u(t + s, \tau, \omega, \phi), \quad \text{for all } s \in [-\rho, 0];$$

$$v_t(s, \tau, \omega, \psi) = v(t + s, \tau, \omega, \psi), \quad \text{for all } s \in [-\rho, 0].$$

Under conditions (3.1)–(3.5), for $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $\phi \in C_{V_2}(\mathbb{R}^n)$, $\psi \in C_{V_0}(\mathbb{R}^n)$, problem (3.11) has a unique continuous solution $(u(\cdot, \tau, \omega, \phi), v(\cdot, \tau, \omega, \psi)) : [\tau - \rho, \infty] \rightarrow C_{V_2, V_0}(\mathbb{R}^n)$, and the segment $u_t(\cdot, \tau, \omega, \phi)$ of u is $(\mathcal{F}, \mathcal{B}(C_{V_2}(\mathbb{R}^n)))$ -measurable in $\omega \in \Omega$ and continuous with respect to ϕ in $C_{V_2}(\mathbb{R}^n)$; the segment $v_t(\cdot, \tau, \omega, \psi)$ of v is $(\mathcal{F}, \mathcal{B}(C_{V_0}(\mathbb{R}^n)))$ -measurable in $\omega \in \Omega$ and continuous with respect to ψ in $C_{V_0}(\mathbb{R}^n)$.

Define $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times C_{V_2, V_0}(\mathbb{R}^n) \rightarrow C_{V_2, V_0}(\mathbb{R}^n)$ by

$$\Phi(t, \tau, \omega, (\phi, \psi))(\cdot) = (u_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \phi), v_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \psi)), \quad (3.12)$$

where $(t, \tau, \omega, (\phi, \psi)) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times C_{V_2, V_0}(\mathbb{R}^n)$, $u_{t+\tau}(s, \tau, \theta_{-\tau}\omega, \phi) = u(t + \tau + s, \tau, \theta_{-\tau}\omega, \phi)$ for $s \in [-\rho, 0]$; $v_{t+\tau}(s, \tau, \theta_{-\tau}\omega, \psi) = v(t + \tau + s, \tau, \theta_{-\tau}\omega, \psi)$ for $s \in [-\rho, 0]$. Then, Φ is a continuous cocycle on $C_{V_2, V_0}(\mathbb{R}^n)$ over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$.

Let $D = \{D(\tau, \omega) \subseteq C_{V_2, V_0}(\mathbb{R}^n) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of $C_{V_2, V_0}(\mathbb{R}^n)$ satisfying

$$\lim_{t \rightarrow +\infty} e^{-\gamma t} \|D(\tau - t, \theta_{-t}\omega)\|_{C_{V_2, V_0}(\mathbb{R}^n)} = 0, \quad \forall \gamma > 0, \quad (3.13)$$

where $\|D(\tau - t, \theta_{-t}\omega)\|_{C_{V_2, V_0}(\mathbb{R}^n)} = \sup_{(u, v) \in D(\tau - t, \theta_{-t}\omega)} \|(u, v)\|_{C_{V_2, V_0}(\mathbb{R}^n)}$. Let \mathcal{D} be the set of all families $D = \{D(\tau, \omega) \subseteq C_{V_2, V_0}(\mathbb{R}^n) : \tau \in \mathbb{R}, \omega \in \Omega\}$ that satisfies (3.13).

For later purposes, we assume $\delta \in (0, 1)$ satisfies

$$1 - \delta > 0, \quad \alpha - \delta > 0, \quad \lambda + \delta^2 - \delta\alpha > 0. \quad (3.14)$$

In addition,

$$\lambda > \frac{16l_f^2 + \delta(\alpha - \delta)^2}{\delta(\alpha - \delta)}. \quad (3.15)$$

Under (3.15), assume σ satisfies

$$\sigma = \min \left\{ \delta, \alpha - \delta, \frac{c_2 \delta}{2}, \frac{1}{\rho} \ln \frac{\delta(\lambda + \delta^2 - \delta\alpha)(\alpha - \delta)}{16l_f^2} \right\}. \quad (3.16)$$

4 Uniform estimates

We will obtain some necessary estimates of solutions for (3.11) in this section.

Lemma 4.1 Assume that (3.1)–(3.5), (3.7), (3.14), and (3.16) hold. Then, for $\forall \varsigma, \tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \varsigma) > 0$ such that for $\forall t \geq T$,

$$\begin{aligned} & \|Y(t+s, \tau - t, \theta_{-\tau}\omega, Y_0)\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 + \int_{\tau-t}^{t+s} e^{\sigma(r-t-s)} \|\nu(r, \tau - t, \theta_{-\tau}\omega, \psi)\|^2 dr \\ & \times \int_{\tau-t}^{t+s} e^{\sigma(r-t-s)} \|\Delta u(r, \tau - t, \theta_{-\tau}\omega, \phi)\|^2 dr \\ & + \int_{\tau-t}^{t+s} e^{\sigma(r-t-s)} \|\Delta v(r, \tau - t, \theta_{-\tau}\omega, \psi)\|^2 dr \\ & \leq M + M \int_{-\infty}^{t-\tau} e^{\sigma(r+\tau-t)} \|g(x, r+\tau)\|^2 dr \\ & + M\epsilon^2 \int_{-\infty}^{t-\tau} e^{\sigma(r+\tau-t)} (1 + \|\Delta z(\theta_r\omega)\|^2 + \|z(\theta_r\omega)\|^2 \\ & + \|\nabla z(\theta_r\omega)\|^2 + \|z(\theta_r\omega)\|_{H^2}^{p+1}) dr, \end{aligned} \quad (4.1)$$

where $Y_0 = (\phi, \psi)^\top \in D(\tau - t, \theta_{-t}\omega)$ and M is a positive constant independent of τ, ω, D , and ϵ , but dependent on λ, σ, α , and δ .

Proof Taking the inner product with (3.11)₂ by v in $L^2(\mathbb{R}^n)$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|^2 &= -(1-\delta)(Au, v) - (Av, v) - (\alpha-\delta)(v, v) - (\lambda + \delta^2 - \delta\alpha)(u, v) \\ &\quad - \epsilon(Az(\theta_t\omega), v) + \epsilon(1-\alpha+\delta)(z(\theta_t\omega), v) + (g(x, t), v) \\ &\quad - (F(u(t, x), x), v) + (f(u(t-\rho, x), x), v). \end{aligned} \quad (4.2)$$

By simple calculation, we can obtain the following estimates for the right-hand side of (4.2):

$$(u, v) \geq \frac{1}{2} \frac{d}{dt} \|u\|^2 + \frac{3\delta}{4} \|u\|^2 - \frac{\epsilon^2}{3\delta} \|z(\theta_t\omega)\|^2, \quad (4.3)$$

$$-(Au, v) \leq -\frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 - \frac{3\delta}{4} \|\Delta u\|^2 + \frac{\epsilon^2}{3\delta} \|\Delta z(\theta_t\omega)\|^2, \quad (4.4)$$

$$\epsilon(1-\alpha+\delta)(z(\theta_t\omega), v) \leq c \|z(\theta_t\omega)\|^2 + \frac{\alpha-\delta}{8} \|v\|^2, \quad (4.5)$$

$$-\epsilon(Az(\theta_t\omega), v) = -\epsilon(\Delta z(\theta_t\omega), \Delta v) \leq \frac{\epsilon^2}{2} \|\Delta z(\theta_t\omega)\|^2 + \frac{1}{2} \|\Delta v\|^2, \quad (4.6)$$

$$(g(x, t), v) \leq c \|g(x, t)\|^2 + \frac{\alpha-\delta}{16} \|v\|^2, \quad (4.7)$$

$$\begin{aligned} (F(u(t, x), x), v) &= \left(F(u(t, x), x), \frac{du}{dt} + \delta u - \epsilon z(\theta_t\omega) \right) \\ &= \frac{d}{dt} \int_{\mathbb{R}^n} \widetilde{F}(x, u) dx + \delta(F(u(t, x), x), u) \\ &\quad - \epsilon(F(u(t, x), x), z(\theta_t\omega)). \end{aligned} \quad (4.8)$$

By (3.2) we obtain

$$(F(u(t, x), x), u) \geq c_2 \int_{\mathbb{R}^n} \widetilde{F}(u, x) dx + \int_{\mathbb{R}^n} \eta_2(x) dx. \quad (4.9)$$

From (3.1) and (3.3), we obtain

$$\begin{aligned} \epsilon(F(u(t, x), x), z(\theta_t\omega)) &\leq \epsilon \|\eta_1(x)\| \|z(\theta_t\omega)\| + c_1 \epsilon \left(\int_{\mathbb{R}^n} |u|^{p+1} dx \right)^{\frac{p}{p+1}} \|z(\theta_t\omega)\|_{p+1} \\ &\leq \epsilon \|\eta_1(x)\| \|z(\theta_t\omega)\| + c_1 \epsilon \left(\int_{\mathbb{R}^n} (\widetilde{F}(x, u) + \eta_3(x)) dx \right)^{\frac{p}{p+1}} \|z(\theta_t\omega)\|_{p+1} \\ &\leq \frac{1}{2} \|\eta_1(x)\|^2 + \frac{\epsilon^2}{2} \|z(\theta_t\omega)\|^2 + \frac{\delta c_2}{2} \int_{\mathbb{R}^n} \widetilde{F}(x, u) dx \\ &\quad + \frac{\delta c_2}{2} \int_{\mathbb{R}^n} \eta_3(x) dx + c \epsilon^2 \|z(\theta_t\omega)\|_{H^2}^{p+1}. \end{aligned} \quad (4.10)$$

From (3.5), we have

$$(f(u(t-\rho, x), x), v) \leq l_f \|u(t-\rho)\| \|v\| \leq \frac{\alpha-\delta}{16} \|v\|^2 + \frac{4l_f^2}{\alpha-\delta} \|u(t-\rho)\|^2. \quad (4.11)$$

Equations (4.3)–(4.11), together with (4.2), imply

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left((\delta^2 + \lambda - \delta\alpha) \|u\|^2 + \|v\|^2 + (1 - \delta) \|\Delta u\|^2 + 2 \int_{\mathbb{R}^n} \tilde{F}(u, x) dx \right) \\
& \leq -\frac{3}{4}(\alpha - \delta) \|v\|^2 - \frac{3}{4}\delta(\delta^2 + \lambda - \delta\alpha) \|u\|^2 - \frac{3}{4}\delta(1 - \delta) \|\Delta u\|^2 \\
& \quad - \frac{\delta c_2}{2} \int_{\mathbb{R}^n} \tilde{F}(u, x) dx - \frac{1}{2} \|\Delta v\|^2 + \frac{4l_f^2}{\alpha - \delta} \|u(t - \rho)\|^2 + c\epsilon^2 (1 + \|\Delta z(\theta_t \omega)\|^2 \\
& \quad + \|\nabla z(\theta_t \omega)\|^2 + \|z(\theta_t \omega)\|_{H^2}^{p+1} + \|z(\theta_t \omega)\|^2) + c \|g(x, t)\|^2,
\end{aligned} \tag{4.12}$$

and together with (3.16) we obtain

$$\begin{aligned}
& \frac{d}{dt} \left(\|Y\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 + 2 \int_{\mathbb{R}^n} \tilde{F}(u, x) dx \right) + \sigma \left(\|Y\|_{C_{V_2, V_0}}^2 + 2 \int_{\mathbb{R}^n} \tilde{F}(u, x) dx \right) \\
& \quad + \frac{1}{2}\delta(\delta^2 + \lambda - \delta\alpha) \|u\|^2 + \|\Delta v\|^2 + \frac{1}{2}(\alpha - \delta) \|v\|^2 + \frac{1}{2}\delta(1 - \delta) \|\Delta u\|^2 \\
& \leq c\epsilon^2 (1 + \|\nabla z(\theta_t \omega)\|^2 + \|\Delta z(\theta_t \omega)\|^2 + \|z(\theta_t \omega)\|^2 + \|z(\theta_t \omega)\|_{H^2}^{p+1}) \\
& \quad + \frac{8l_f^2}{\alpha - \delta} \|u(t - \rho)\|^2 + c \|g(x, t)\|^2.
\end{aligned} \tag{4.13}$$

Substituting τ by $\tau - t$, and integrating (4.13) between $[\tau - t, \iota + s]$, we obtain

$$\begin{aligned}
& e^{\sigma(\iota+s)} \left(\|Y(\iota + s, \tau - t, \theta_{-\tau} \omega, \varphi_0)\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 + 2 \int_{\mathbb{R}^n} \tilde{F}(u(\iota + s, \tau - t, \theta_{-\tau} \omega, \phi), x) dx \right) \\
& \quad + \frac{1}{2}(\alpha - \delta) \int_{\tau-t}^{\iota+s} e^{\sigma r} \|v(r, \tau - t, \theta_{-\tau} \omega, \psi)\|^2 dr \\
& \quad + \frac{1}{2}\delta(\delta^2 + \lambda - \delta\alpha) \int_{\tau-t}^{\iota+s} e^{\sigma r} \|u(r, \tau - t, \theta_{-\tau} \omega, \phi)\|^2 dr \\
& \quad + \frac{1}{2}\delta(1 - \delta) \int_{\tau-t}^{\iota+s} e^{\sigma r} \|\Delta u(r, \tau - t, \theta_{-\tau} \omega, \phi)\|^2 dr \\
& \quad + \int_{\tau-t}^{\iota+s} e^{\sigma r} \|\Delta v(r, \tau - t, \theta_{-\tau} \omega, \psi)\|^2 dr \\
& \leq e^{\sigma(\tau-t)} \left(\|Y_0\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 + 2 \int_{\mathbb{R}^n} \tilde{F}(x, \phi) dx \right) \\
& \quad + c\epsilon^2 \int_{\tau-t}^{\iota+s} e^{\sigma r} (1 + \|\Delta z(\theta_{r-\tau} \omega)\|^2 + \|z(\theta_{r-\tau} \omega)\|^2 \\
& \quad + \|\nabla z(\theta_{r-\tau} \omega)\|^2 + \|z(\theta_{r-\tau} \omega)\|_{H^2}^{p+1}) dr \\
& \quad + \frac{8l_f^2}{\alpha - \delta} \int_{\tau-t}^{\iota+s} e^{\sigma r} \|u(r - \rho, \tau - t, \theta_{-\tau} \omega, \phi)\|^2 dr \\
& \quad + c \int_{\tau-t}^{\iota+s} e^{\sigma r} \|g(x, r)\|^2 dr.
\end{aligned} \tag{4.14}$$

Note that

$$\begin{aligned}
& \int_{\tau-t}^{\tau+s} e^{\sigma r} \|u(r - \rho, \tau - t, \theta_{-\tau}\omega, \phi)\|^2 dr \\
&= \int_{\tau-t-\rho}^{\tau-t} e^{\sigma(r+\rho)} \|u(r, \tau - t, \theta_{-\tau}\omega, \phi)\|^2 dr \\
&\quad + \int_{\tau-t}^{\tau+s-\rho} e^{\sigma(r+\rho)} \|u(r, \tau - t, \theta_{-\tau}\omega, \phi)\|^2 dr \\
&\leq \frac{1}{\sigma} e^{\sigma(\tau-t+\rho)} \|\phi\|_{C_{V_2}(\mathbb{R}^n)}^2 + e^{\sigma\rho} \int_{\tau-t}^{\tau+s} e^{\sigma r} \|u(r, \tau - t, \theta_{-\tau}\omega, \phi)\|^2 dr. \tag{4.15}
\end{aligned}$$

By (4.14), (4.15), and (3.16) we obtain

$$\begin{aligned}
& \|Y(\iota + s, \tau - t, \theta_{-\tau}\omega, \varphi_0)\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 + 2 \int_{\mathbb{R}^n} \tilde{F}(u(\iota + s, \tau - t, \theta_{-\tau}\omega, \phi), x) dx \\
&+ \frac{1}{2}(\alpha - \delta) \int_{\tau-t}^{\tau+s} e^{\sigma(r-\iota-s)} \|\nu(r, \tau - t, \theta_{-\tau}\omega, \psi)\|^2 dr \\
&+ \frac{1}{2}\delta(1 - \delta) \int_{\tau-t}^{\tau+s} e^{\sigma(r-\iota-s)} \|\Delta u(r, \tau - t, \theta_{-\tau}\omega, \phi)\|^2 dr \\
&+ \int_{\tau-t}^{\tau+s} e^{\sigma(r-\iota-s)} \|\Delta \nu(r, \tau - t, \theta_{-\tau}\omega, \psi)\|^2 dr \\
&\leq ce^{\sigma(\tau-t-\iota-s+\rho)} \left(\|Y_0\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 + 2 \int_{\mathbb{R}^n} \tilde{F}(x, \phi) dx \right) \\
&+ c \int_{-\iota}^{\tau-\iota} e^{\sigma(r+\tau-\iota)} \|g(x, r + \tau)\|^2 dr \\
&+ c\epsilon^2 \int_{-\iota}^{\tau-\iota} e^{\sigma(r+\tau-\iota)} (1 + \|\Delta z(\theta_r\omega)\|^2 + \|\nabla z(\theta_r\omega)\|^2 \\
&+ \|z(\theta_r\omega)\|_{H^2}^{p+1} + \|z(\theta_r\omega)\|^2) dr. \tag{4.16}
\end{aligned}$$

Combination of Lemma 3.1 and (3.7) implies that

$$\begin{aligned}
& c\epsilon^2 \int_{-\iota}^{\tau-\iota} e^{\sigma(r+\tau-\iota)} (1 + \|\Delta z(\theta_r\omega)\|^2 + \|z(\theta_r\omega)\|^2 + \|\nabla z(\theta_r\omega)\|^2 + \|z(\theta_r\omega)\|_{H^2}^{p+1}) dr \\
&+ \frac{8}{\alpha - \delta} \int_{-\iota}^{\tau-\iota} e^{\sigma(r+\tau-\iota)} \|g(x, r + \tau)\|^2 dr \\
&\leq c\epsilon^2 \int_{-\infty}^{\tau-\iota} e^{\sigma(r+\tau-\iota)} (1 + \|\Delta z(\theta_r\omega)\|^2 + \|z(\theta_r\omega)\|^2 + \|\nabla z(\theta_r\omega)\|^2 + \|z(\theta_r\omega)\|_{H^2}^{p+1}) dr \\
&+ c \int_{-\infty}^{\tau-\iota} e^{\sigma(r+\tau-\iota)} \|g(x, r + \tau)\|^2 dr \\
&< +\infty. \tag{4.17}
\end{aligned}$$

Equation (3.6) yields

$$\int_{\mathbb{R}^n} \tilde{F}(\phi, x) dx \leq c(1 + \|\phi\|_{C_{V_2}(\mathbb{R}^n)}^{p+1}). \tag{4.18}$$

We can deduce that

$$\lim_{t \rightarrow +\infty} ce^{\sigma(\tau-t-\iota-s+\rho)} \left(\|Y_0\|_{C_{V_2,V_0}(\mathbb{R}^n)}^2 + 2 \int_{\mathbb{R}^n} \tilde{F}(x, \phi) dx \right) = 0,$$

which together with (3.3), (4.16), and (4.17) yields (4.1). \square

Let ρ be a smooth function on \mathbb{R}^+ such that $0 \leq \rho(s) \leq 1$ for all $s \in \mathbb{R}^+$, and

$$\rho(s) = 0 \quad \text{for } |s| \leq \frac{1}{2}; \quad \text{and} \quad \rho(s) = 1 \quad \text{for } |s| \geq 1.$$

For every $k \in \mathbb{N}$, let

$$\rho_k \triangleq \rho_k(x) = \rho(x/k), \quad x \in \mathbb{R}^n.$$

We also assume that for all $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, $|\nabla \rho_k| \leq \frac{1}{k} c_4$, $|\Delta \rho_k| \leq \frac{1}{k} c_5$, $|\Delta \nabla \rho_k| \leq \frac{1}{k} c_6$, $|\Delta^2 \rho_k| \leq \frac{1}{k} c_7$, where c_4, c_5, c_6 , and c_7 are positive constants independent of k .

Given $k \geq 1$, denote $\mathbb{H}_k = \{x \in \mathbb{R}^n : |x| < k\}$ and $\mathbb{R}^n \setminus \mathbb{H}_k$ the complement of \mathbb{H}_k .

Lemma 4.2 Suppose (3.1)–(3.5), (3.7), (3.8), (3.14), and (3.16) hold. Then, for $\forall \tau \in \mathbb{R}$, $s \in [-\rho, 0]$, $\omega \in \Omega$, there exist $\tilde{R} = \tilde{R}(\tau, \omega, \varepsilon) \geq 1$ and $T = T(\tau, \omega, D, \varepsilon) > 0$, such that for $\forall k \geq \tilde{R}$, $t \geq T$,

$$\|Y(\tau + s, \tau - t, \theta_{-\tau}\omega, Y_0)\|_{H^2(\mathbb{R}^n \setminus \mathbb{H}_k) \times L^2(\mathbb{R}^n \setminus \mathbb{H}_k)}^2 \leq \varepsilon. \quad (4.19)$$

Proof Multiplying (3.11)₂ with $\rho_k(x)v$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho_k |v|^2 dx + (\alpha - \delta) \int_{\mathbb{R}^n} \rho_k |v|^2 dx &= -(\lambda + \delta^2 - \delta\alpha) \int_{\mathbb{R}^n} \rho_k \cdot u \cdot v dx \\ &\quad - (1 - \delta) \int_{\mathbb{R}^n} \rho_k \cdot A u \cdot v dx - \int_{\mathbb{R}^n} (\rho_k A v) v dx + \epsilon(1 - \alpha + \delta) \int_{\mathbb{R}^n} (\rho_k z(\theta_t \omega)) v dx \\ &\quad - \epsilon \int_{\mathbb{R}^n} \rho_k \cdot A z(\theta_t \omega) \cdot v dx + \int_{\mathbb{R}^n} (\rho_k g(x, t)) v dx - \int_{\mathbb{R}^n} (\rho_k F(u, x)) v dx \\ &\quad + \int_{\mathbb{R}^n} (\rho_k f(u(t - \rho, x), x)) v dx. \end{aligned} \quad (4.20)$$

For the terms on the right-hand side of (4.20), using Young's inequality and the interpolation inequality

$$\|\nabla v\| \leq \varsigma \|v\| + C_\varsigma \|\Delta v\|, \quad \forall \varsigma > 0,$$

we have

$$\begin{aligned} \int_{\mathbb{R}^n} \rho_k \cdot u \cdot v dx &= \int_{\mathbb{R}^n} \rho_k u \left(\frac{du}{dt} + \delta u - \epsilon z(\theta_t \omega) \right) dx \\ &= \int_{\mathbb{R}^n} \rho_k \left(\frac{1}{2} \frac{d}{dt} u^2 + \delta u^2 - \epsilon z(\theta_t \omega) u \right) dx \\ &\geq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho_k |u|^2 dx + \frac{3\delta}{4} \int_{\mathbb{R}^n} \rho_k |u|^2 dx \\ &\quad - \frac{\epsilon^2}{2\delta} \int_{\mathbb{R}^n} \rho_k |z(\theta_t \omega)|^2 dx, \end{aligned} \quad (4.21)$$

$$\begin{aligned}
& - \int_{\mathbb{R}^n} \rho_k \cdot A u \cdot v \, dx \\
&= - \int_{\mathbb{R}^n} (\Delta^2 u \cdot \rho_k) \cdot \left(\frac{du}{dt} + \delta u - \epsilon z(\theta_t \omega) \right) \, dx \\
&= - \int_{\mathbb{R}^n} \Delta u \cdot \Delta \left(\rho_k \left(\frac{du}{dt} + \delta u - \epsilon z(\theta_t \omega) \right) \right) \, dx \\
&= - \int_{\mathbb{R}^n} \Delta u \cdot \left(\Delta \rho_k \cdot v + 2\nabla \rho_k \cdot \nabla v + \rho_k \cdot \Delta \left(\frac{du}{dt} + \delta u - \epsilon z(\theta_t \omega) \right) \right) \, dx \\
&\leq \frac{c_5}{k} (\|\Delta u\|^2 + \|v\|^2) + \frac{c_4}{k} (\|\Delta u\|^2 + 2\zeta^2 \|v\|^2 + 2C_\zeta^2 \|\Delta v\|^2) \\
&\quad - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho_k |\Delta u|^2 \, dx - \frac{\delta}{2} \int_{\mathbb{R}^n} \rho_k |\Delta u|^2 \, dx + \frac{\epsilon^2}{2\delta} \int_{\mathbb{R}^n} \rho_k |\Delta z(\theta_t \omega)|^2 \, dx, \tag{4.22}
\end{aligned}$$

$$\begin{aligned}
& - \int_{\mathbb{R}^n} \rho_k \cdot A v \cdot v \, dx = - \int_{\mathbb{R}^n} \Delta^2 v \cdot \rho_k \cdot v \, dx = - \int_{\mathbb{R}^n} \Delta v \cdot \Delta(\rho_k \cdot v) \, dx \\
&= - \int_{\mathbb{R}^n} \Delta v \cdot (\Delta \rho_k \cdot v + 2\nabla \rho_k \cdot \nabla v + \rho_k \cdot \Delta v) \, dx \\
&\leq \frac{c_5}{2k} (\|v\|^2 + \|\Delta v\|^2) + \frac{c_4}{k} (\|\Delta v\|^2 + 2C_\zeta^2 \|\Delta v\|^2 + 2\zeta^2 \|v\|^2) \\
&\quad - \int_{\mathbb{R}^n} \rho_k |\Delta v|^2 \, dx, \tag{4.23}
\end{aligned}$$

$$\begin{aligned}
& - \epsilon \int_{\mathbb{R}^n} \rho_k \cdot A z(\theta_t \omega) \cdot v \, dx \\
&= - \epsilon \int_{\mathbb{R}^n} \Delta^2 z(\theta_t \omega) \cdot \rho_k \cdot v \, dx \\
&= - \epsilon \int_{\mathbb{R}^n} \Delta z(\theta_t \omega) \cdot \Delta(\rho_k \cdot v) \, dx \\
&= - \epsilon \int_{\mathbb{R}^n} \Delta z(\theta_t \omega) \cdot (\Delta \rho_k \cdot v + 2\nabla \rho_k \cdot \nabla v + \rho_k \cdot \Delta v) \, dx \\
&\leq \frac{c_5 \epsilon}{k} \int_{\mathbb{R}^n} |\Delta z(\theta_t \omega) \cdot v| \, dx + \frac{2c_4 \epsilon}{k} \int_{\mathbb{R}^n} |\Delta z(\theta_t \omega) \cdot \nabla v| \, dx \\
&\quad + \epsilon \int_{\mathbb{R}^n} \rho_k |\Delta z(\theta_t \omega)| \cdot |\Delta v| \, dx \\
&\leq \frac{c_5 \epsilon}{2k} (\|\Delta z(\theta_t \omega)\|^2 + \|v\|^2) + \frac{2c_4 \epsilon}{k} \|\Delta z(\theta_t \omega)\| (\zeta \|v\| + C_\zeta \|\Delta v\|) \\
&\quad + \epsilon \int_{\mathbb{R}^n} \rho_k |\Delta z(\theta_t \omega)| \cdot |\Delta v| \, dx \\
&\leq \frac{c_5 \epsilon}{2k} (\|v\|^2 + \|\Delta z(\theta_t \omega)\|^2) + \frac{c_4 \epsilon}{k} (\|\Delta z(\theta_t \omega)\|^2 + 2C_\zeta^2 \|\Delta v\|^2 + 2\zeta^2 \|v\|^2) \\
&\quad + \int_{\mathbb{R}^n} \rho_k |\Delta v|^2 \, dx + \frac{\epsilon^2}{4} \int_{\mathbb{R}^n} \rho_k |\Delta z(\theta_t \omega)|^2 \, dx, \tag{4.24}
\end{aligned}$$

$$\epsilon(1 - \alpha + \delta) \int_{\mathbb{R}^n} \rho_k z(\theta_t \omega) v \, dx \leq c \epsilon^2 \int_{\mathbb{R}^n} \rho_k |z(\theta_t \omega)|^2 \, dx + \frac{3(\alpha - \delta)}{16} \int_{\mathbb{R}^n} \rho_k |v|^2 \, dx, \tag{4.25}$$

$$\int_{\mathbb{R}^n} \rho_k g(x, t) v \, dx \leq c \int_{\mathbb{R}^n} \rho_k |g(x, t)|^2 \, dx + \frac{\alpha - \delta}{4} \int_{\mathbb{R}^n} \rho_k |v|^2 \, dx. \tag{4.26}$$

$$\begin{aligned}
\int_{\mathbb{R}^n} \rho_k F(u, x) v dx &= \int_{\mathbb{R}^n} \rho_k F(u, x) \left(\frac{du}{dt} + \delta u - \epsilon z(\theta_t \omega) \right) dx \\
&= \frac{d}{dt} \int_{\mathbb{R}^n} \rho_k \tilde{F}(u, x) dx + \delta \int_{\mathbb{R}^n} \rho_k F(u, x) u dx \\
&\quad - \epsilon \int_{\mathbb{R}^n} \rho_k F(u, x) z(\theta_t \omega) dx.
\end{aligned} \tag{4.27}$$

By (3.1)–(3.3), we have

$$\delta \int_{\mathbb{R}^n} \rho_k F(u, x) u dx \geq c_2 \delta \int_{\mathbb{R}^n} \rho_k \tilde{F}(u, x) dx + \delta \int_{\mathbb{R}^n} \rho_k \eta_2(x) dx, \tag{4.28}$$

$$\begin{aligned}
\epsilon \int_{\mathbb{R}^n} \rho_k F(u, x) z(\theta_t \omega) dx &\leq \epsilon \int_{\mathbb{R}^n} \rho_k (c_1 |u|^p + \eta_1(x)) |z(\theta_t \omega)| dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}^n} \rho_k |\eta_1(x)|^2 dx + \frac{\epsilon^2}{2} \int_{\mathbb{R}^n} \rho_k |z(\theta_t \omega)|^2 dx \\
&\quad + c\epsilon^2 \int_{\mathbb{R}^n} \rho_k |z(\theta_t \omega)|^{p+1} dx \\
&\quad + \frac{c_2 \delta}{2} \int_{\mathbb{R}^n} \rho_k (\tilde{F}(u, x) + \eta_3(x)) dx.
\end{aligned} \tag{4.29}$$

From (3.5) we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^n} \rho_k \cdot f(u(t - \rho, x), x) \cdot v dx &\leq l_f \int_{\mathbb{R}^n} \rho_k |u(t - \rho)| \cdot |v| dx \\
&\leq \frac{4l_f^2}{\alpha - \delta} \int_{\mathbb{R}^n} \rho_k |u(t - \rho)|^2 dx \\
&\quad + \frac{\alpha - \delta}{16} \int_{\mathbb{R}^n} \rho_k |v|^2 dx.
\end{aligned} \tag{4.30}$$

Then, it follows from (4.20)–(4.30) and (3.16) that

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^n} \rho_k ((\delta^2 + \lambda - \delta\alpha) |u|^2 + |v|^2 + (1 - \delta) |\Delta u|^2 + 2\tilde{F}(u, x)) dx \\
&\quad + \sigma \int_{\mathbb{R}^n} \rho_k ((\delta^2 + \lambda - \delta\alpha) |u|^2 + |v|^2 + (1 - \delta) |\Delta u|^2 + 2\tilde{F}(u, x)) dx \\
&\leq \frac{c}{k} (\|v\|^2 + \|\Delta v\|^2 + \|\Delta u\|^2 + \|\Delta z(\theta_t \omega)\|^2) \\
&\quad + \frac{8l_f^2}{\alpha - \delta} \int_{\mathbb{R}^n} \rho_k |u(t - \rho)|^2 dx - \frac{\delta(\lambda + \delta^2 - \delta\alpha)}{2} \int_{\mathbb{R}^n} \rho_k |u|^2 dx + c \int_{\mathbb{R}^n} \rho_k |g(x, t)|^2 dx \\
&\quad + c\epsilon^2 \int_{\mathbb{R}^n} \rho_k (1 + |\Delta z(\theta_t \omega)|^2 + |z(\theta_t \omega)|^{p+1}) dx.
\end{aligned} \tag{4.31}$$

Multiplying (4.31) by $e^{\sigma t}$ and integrating between $\tau - t$ and $\tau + s$, then substituting ω by $\theta_{-\tau} \omega$ and rearranging, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho_k (|v(\tau + s, \tau - t, \theta_{-\tau} \omega, \psi)|^2 + (\delta^2 + \lambda - \delta\alpha) |u(\tau + s, \tau - t, \theta_{-\tau} \omega, \phi)|^2 \\
&\quad + (1 - \delta) |\Delta u(\tau + s, \tau - t, \theta_{-\tau} \omega, \phi)|^2 + 2\tilde{F}(u(\tau + s, \tau - t, \theta_{-\tau} \omega, \phi), x) dx
\end{aligned}$$

$$\begin{aligned}
&\leq e^{-\sigma(t+s)} \int_{\mathbb{R}^n} \rho_k(|\psi|^2 + (\lambda + \delta^2 - \delta\alpha)|\phi|^2 + (1 - \delta)|\Delta\phi|^2 + 2\tilde{F}(x, \phi)) dx \\
&+ \frac{c}{k} \int_{\tau-t}^{\tau+s} e^{\sigma(r-\tau-s)} (\|\Delta v(r, \tau-t, \theta_{-\tau}\omega, \psi)\|^2 + \|v(r, \tau-t, \theta_{-\tau}\omega, \psi)\|^2 \\
&+ \|\Delta u(r, \tau-t, \theta_{-\tau}\omega, \phi)\|^2) dr + \frac{c}{k} \int_{-t}^0 e^{\sigma(r-s)} \|\Delta z(\theta_r\omega)\| dr \\
&+ c \int_{-\infty}^{\tau} e^{\sigma(r-s)} \int_{|x| \geq \frac{k}{2}} |g(x, r+\tau)|^2 dx dr + c\epsilon^2 \int_{-\infty}^0 e^{\sigma(r-s)} \int_{|x| \geq \frac{k}{2}} (1 + |\Delta z(\theta_r\omega)|^2 \\
&+ |\nabla z(\theta_r\omega)|^2 + |z(\theta_r\omega)|^{p+1}) dx dr. \tag{4.32}
\end{aligned}$$

Since $(\phi, \psi)^\top \in D(\tau - t, \theta_{-t}\omega) \in \mathcal{D}$ together with (3.6) we know that there exists $\tilde{T}_1 = \tilde{T}_1(\tau, \epsilon, \omega, D) > 0$, such that for $\forall t > \tilde{T}_1$,

$$e^{-\sigma(t+s)} \int_{\mathbb{R}^n} \rho_k(|\psi|^2 + (\lambda + \delta^2 - \delta\alpha)|\phi|^2 + (1 - \delta)|\Delta\phi|^2 + 2\tilde{F}(x, \phi)) dx \leq \epsilon. \tag{4.33}$$

From Lemma 4.1, there are $\tilde{T}_2 = \tilde{T}_2(\tau, \epsilon, \omega, D) > 0$ and $\tilde{R}_1 = \tilde{R}_1(\epsilon, \omega, D) > 1$, such that for $\forall t > \tilde{T}_2, k > \tilde{R}_1$,

$$\begin{aligned}
&\frac{c}{k} \int_{\tau-t}^{\tau+s} e^{\sigma(r-\tau-s)} (\|v(r, \tau-t, \theta_{-\tau}\omega, \psi)\|^2 \\
&+ \|\Delta v(r, \tau-t, \theta_{-\tau}\omega, \psi)\|^2 + \|\Delta u(r, \tau-t, \theta_{-\tau}\omega, \phi)\|^2) dr \\
&\leq \epsilon. \tag{4.34}
\end{aligned}$$

By Lemma 3.1, there are $\tilde{T}_3 = \tilde{T}_3(\epsilon, \omega) > 0, \tilde{R}_2 = \tilde{R}_2(\epsilon, \omega) > 1$, such that for $\forall t > \tilde{T}_3, k > \tilde{R}_2$,

$$\begin{aligned}
&c\epsilon^2 \int_{-\infty}^0 e^{\sigma(r-s)} \int_{|x| \geq \frac{k}{2}} (1 + |\Delta z(\theta_r\omega)|^2 + |\nabla z(\theta_r\omega)|^2 + |z(\theta_r\omega)|^2 + |z(\theta_r\omega)|^{p+1}) dx dr \\
&+ \frac{c}{k} \int_{-t}^0 e^{\sigma(r-s)} \|\Delta z(\theta_r\omega)\| dr \leq \epsilon. \tag{4.35}
\end{aligned}$$

By (3.8), there exists $\tilde{R}_3 = \tilde{R}_3(\tau, \epsilon) > 1$, such that for $\forall k > \tilde{R}_3$,

$$c \int_{-\infty}^{\tau} e^{\sigma(r-s)} \int_{|x| \geq \frac{k}{2}} |g(x, r+\tau)|^2 dx dr \leq \epsilon. \tag{4.36}$$

Letting $\tilde{R} = \max\{\tilde{R}_1, \tilde{R}_2, \tilde{R}_3\}, \tilde{T} = \max\{\tilde{T}_1, \tilde{T}_2, \tilde{T}_3\}$, together with (4.32)–(4.36), for $\forall t > \tilde{T}, k > \tilde{R}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho_k(x) (|v(\tau+s, \tau-t, \theta_{-\tau}\omega, \psi)|^2 + (\delta^2 + \lambda - \delta\alpha)|u(\tau+s, \tau-t, \theta_{-\tau}\omega, \phi)|^2 \\
&+ (1 - \delta)|\Delta u(\tau+s, \tau-t, \theta_{-\tau}\omega, \phi)|^2 + 2\tilde{F}(x, u(\tau+s, \tau-t, \theta_{-\tau}\omega, \phi))) dx \\
&\leq 4\epsilon, \tag{4.37}
\end{aligned}$$

which together with (3.3) implies (4.19). \square

For $\forall x \in \mathbb{R}^n$ and $k \geq 1$, denote

$$\begin{cases} \widehat{u}(t, \tau, \omega, \widehat{\phi}) = \widehat{\rho}_k u(t, \tau, \omega, \phi), \\ \widehat{v}(t, \tau, \omega, \widehat{\psi}) = \widehat{\rho}_k v(t, \tau, \omega, \psi), \end{cases} \quad (4.38)$$

where $\widehat{\rho}_k = 1 - \rho_k$. Then, for $k \geq 1, x \in \mathbb{R}^n \setminus \mathbb{H}_k$, we have $\widehat{u}(t, \tau, \omega, \widehat{\phi}) = \widehat{v}(t, \tau, \omega, \widehat{\psi}) = 0$. In addition, there is some constant $c > 0$ independent of $k \geq 1$, such that $\|\widehat{u}\|_{H^2(\mathbb{R}^n)} \leq c\|u\|_{H^2(\mathbb{R}^n)}$, $\|\widehat{v}\|_{L^2(\mathbb{R}^n)} \leq c\|v\|_{L^2(\mathbb{R}^n)}$. Accordingly, together with (3.11) and (4.38), we obtain

$$\begin{cases} \frac{d\widehat{u}}{dt} + \delta\widehat{u} = \widehat{v} + \epsilon\widehat{\rho}_k z(\theta_t \omega), \\ \frac{d\widehat{v}}{dt} + (\alpha - \delta)\widehat{v} + (\delta^2 + \lambda - \delta\alpha)\widehat{u} + (1 - \delta)A\widehat{u} + A\widehat{v} \\ = \epsilon(1 - \alpha + \delta)\widehat{\rho}_k z(\theta_t \omega) \\ - \epsilon\widehat{\rho}_k Az(\theta_t \omega) + \widehat{\rho}_k g(x, t) - \widehat{\rho}_k F(u, x) + \widehat{\rho}_k f(u(t - \rho, x), x) \\ + 4(1 - \delta)\Delta\nabla\widehat{\rho}_k \nabla u + 6(1 - \delta)\Delta\widehat{\rho}_k \Delta u \\ + 4(1 - \delta)\nabla\widehat{\rho}_k \Delta\nabla u + (1 - \delta)uA\widehat{\rho}_k \\ + 4\Delta\nabla\widehat{\rho}_k \nabla v + 6\Delta\widehat{\rho}_k \Delta v + 4\nabla\widehat{\rho}_k \Delta\nabla v + vA\widehat{\rho}_k, \\ \widetilde{u}_\tau(s, x) = \widehat{\rho}_k(x)\phi(s, x), \quad \widetilde{v}_\tau(s, x) = \widehat{\rho}_k(x)\psi(s, x), \quad x \in \mathbb{R}^n, s \in [-\rho, 0], \\ \widetilde{u}_\tau(s, x) = 0, \quad \widetilde{v}_\tau(s, x) = 0, \quad x \in \mathbb{R}^n \setminus \mathbb{H}_k, s \in [-\rho, 0]. \end{cases} \quad (4.39)$$

Considering the eigenvalue problem

$$\lambda\widehat{u} = A\widehat{u} \quad \text{in } \mathbb{H}_k, \quad \text{with } \frac{\partial\widehat{u}}{\partial n} = \widehat{u} = 0 \quad \text{on } \partial\mathbb{H}_k, \quad (4.40)$$

it is easy to see that eigenfunctions $\{e_i\}_{i \in \mathbb{N}}$ and eigenvalues $\{\lambda_i\}_{i \in \mathbb{N}}$ of (4.40) satisfy:

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_i \leq \dots, \quad \lambda_i \rightarrow +\infty \quad (i \rightarrow +\infty).$$

For given n , assume $X_n = \text{span}\{e_1, \dots, e_n\}$, $P_n : L^2(\mathbb{H}_k) \rightarrow X_n$.

Lemma 4.3 Suppose (3.1)–(3.5), (3.7), (3.14), and (3.16) hold. Then, for $\forall \omega \in \Omega, \tau \in \mathbb{R}, s \in [-\rho, 0], D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $\widehat{R} = \widehat{R}(\tau, \omega, \varepsilon) \geq 1$, $\widehat{T} = \widehat{T}(\tau, \omega, D, \varepsilon) > 0$, $N = N(\tau, \omega, \varepsilon) > 0$, such that for $\forall t \geq \widehat{T}, n \geq N, k \geq \widehat{R}$,

$$\|(I - P_n)\widehat{Y}(\tau + s, \tau - t, \theta_{-\tau}\omega, \widehat{Y}_0)\|_{H^2(\mathbb{H}_k) \times L^2(\mathbb{H}_k)}^2 \leq \varepsilon. \quad (4.41)$$

Proof Denote $\widehat{u}_{n,2} = (I - P_n)\widehat{u}$, $\widehat{u}_{n,1} = P_n\widehat{u}$, $\widehat{v}_{n,2} = (I - P_n)\widehat{v}$, $\widehat{v}_{n,1} = P_n\widehat{v}$. Multiplying (4.39)₁ with $I - P_n$, we obtain

$$\widehat{v}_{n,2} = \frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \epsilon(I - P_n)\widehat{\rho}_k(x)z(\theta_t \omega). \quad (4.42)$$

Multiplying (4.39)₂ with $I - P_n$ then taking the inner product with $\widehat{v}_{n,2}$ in $L^2(\mathbb{H}_k)$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\widehat{v}_{n,2}\|^2 + (\alpha - \delta) \|\widehat{v}_{n,2}\|^2 \\ &= -(\delta^2 + \lambda - \delta\alpha)(\widehat{u}_{n,2}, \widehat{v}_{n,2}) - (1 - \delta)(A\widehat{u}_{n,2}, \widehat{v}_{n,2}) - (A\widehat{v}_{n,2}, \widehat{v}_{n,2}) \end{aligned}$$

$$\begin{aligned}
& + \epsilon(\delta - \alpha + 1)(\widehat{\rho}_k z(\theta_t \omega), \widehat{v}_{n,2}) - \epsilon(\widehat{\rho}_k A z(\theta_t \omega), \widehat{v}_{n,2}) + (\widehat{\rho}_k g(x, t), \widehat{v}_{n,2}) \\
& - (\widehat{\rho}_k F(u, x), \widehat{v}_{n,2}) + (\widehat{\rho}_k f(u(t - \rho, x), x), \widehat{v}_{n,2}) + (4(1 - \delta)\Delta \nabla \widehat{\rho}_k \nabla u \\
& + 6(1 - \delta)\Delta \widehat{\rho}_k \Delta u + 4(1 - \delta)\nabla \widehat{\rho}_k \Delta \nabla u + (1 - \delta)u A \widehat{\rho}_k, \widehat{v}_{n,2}) \\
& + (4\Delta \nabla \widehat{\rho}_k \nabla v + 6\Delta \widehat{\rho}_k \Delta v + 4\nabla \widehat{\rho}_k \Delta \nabla v + v A \widehat{\rho}_k, \widehat{v}_{n,2}). \tag{4.43}
\end{aligned}$$

For the right-hand side of (4.43), by simple calculation, we obtain the following estimates

$$(\widehat{u}_{n,2}, \widehat{v}_{n,2}) \geq \frac{1}{2} \frac{d}{dt} \|\widehat{u}_{n,2}\|^2 + \frac{3\delta}{4} \|\widehat{u}_{n,2}\|^2 - c\epsilon^2 \|(I - P_n)\widehat{\rho}_k z(\theta_t \omega)\|^2, \tag{4.44}$$

$$-(A\widehat{u}_{n,2}, \widehat{v}_{n,2}) \leq -\frac{1}{2} \frac{d}{dt} \|\Delta \widehat{u}_{n,2}\|^2 - \frac{\delta}{2} \|\Delta \widehat{u}_{n,2}\|^2 + c\epsilon^2 \|(I - P_n)\Delta(\widehat{\rho}_k z(\theta_t \omega))\|^2. \tag{4.45}$$

From (3.1), choosing $\theta = \frac{n(p-1)}{4(p+1)}$, we obtain

$$\begin{aligned}
(\widehat{\rho}_k(x)F(x, u), \widehat{v}_{n,2}) & \leq c_1 \int_{\mathbb{R}^n} \widehat{\rho}_k(x)|u|^k |\widehat{v}_{n,2}| dx + \int_{\mathbb{R}^n} \widehat{\rho}_k(x)|\eta_1(x)| |\widehat{v}_{n,2}| dx \\
& \leq c_1 \|u\|_{p+1}^p \|\widehat{v}_{n,2}\|_{p+1} + \|\eta_1\| \|\widehat{v}_{n,2}\| \\
& \leq c_1 \|u\|_{p+1}^p \|\Delta \widehat{v}_{n,2}\|^\theta \|\widehat{v}_{n,2}\|^{1-\theta} + \lambda_{n+1}^{-\frac{1}{2}} \|\eta_1\| \|\Delta \widehat{v}_{n,2}\| \\
& \leq c_1 \lambda_{n+1}^{\frac{\theta-1}{2}} \|u\|_{H^2}^p \|\Delta \widehat{v}_{n,2}\| + \lambda_{n+1}^{-\frac{1}{2}} \|\eta_1\| \|\Delta \widehat{v}_{n,2}\| \\
& \leq \lambda_{n+1}^{-\frac{1}{2}} \|\Delta \widehat{v}_{n,2}\| (c_1 \lambda_{n+1}^{\frac{\theta}{2}} \|u\|_{H^2}^p + \|\eta_1\|) \\
& \leq \frac{1}{6} \|\Delta \widehat{v}_{n,2}\|^2 + \frac{3}{2} \lambda_{n+1}^{-1} (c_1 \lambda_{n+1}^{\frac{\theta}{2}} \|u\|_{H^2}^p + \|\eta_1\|)^2. \tag{4.46}
\end{aligned}$$

It follows from the Cauchy inequality and Young's inequality that

$$-\epsilon(\widehat{\rho}_k A z(\theta_t \omega), \widehat{v}_{n,2}) \leq \frac{1}{6} \|\Delta \widehat{v}_{n,2}\|^2 + \frac{3\epsilon^2}{2} \|(I - P_n)\widehat{\rho}_k \Delta z(\theta_t \omega)\|^2, \tag{4.47}$$

$$\epsilon(1 - \alpha + \delta)(\widehat{\rho}_k z(\theta_t \omega), \widehat{v}_{n,2}) \leq \frac{\alpha - \delta}{8} \|\widehat{v}_{n,2}\|^2 + c\epsilon^2 \|(I - P_n)\widehat{\rho}_k z(\theta_t \omega)\|^2, \tag{4.48}$$

$$(\widehat{\rho}_k g(x, t), \widehat{v}_{n,2}) \leq \frac{\alpha - \delta}{8} \|\widehat{v}_{n,2}\|^2 + c\|(I - P_n)\widehat{\rho}_k g(x, t)\|^2, \tag{4.49}$$

$$\begin{aligned}
& (1 - \delta)(4\Delta \nabla \widehat{\rho}_k \cdot \nabla u + 6\Delta \widehat{\rho}_k \cdot \Delta u + 4\nabla \widehat{\rho}_k \cdot \Delta \nabla u + u A \widehat{\rho}_k, \widehat{v}_{n,2}) \\
& \leq \frac{4c_6(1 - \delta)}{k} \lambda_{n+1}^{-\frac{1}{4}} \|\Delta u\| \cdot \|\widehat{v}_{n,2}\| + \frac{6c_5(1 - \delta)}{k} \|\Delta u\| \cdot \|\widehat{v}_{n,2}\| \\
& + \frac{4c_4(1 - \delta)}{k} \lambda_{n+1}^{-\frac{1}{4}} \|\Delta u\| \cdot \|\Delta \widehat{v}_{n,2}\| + \frac{c_7(1 - \delta)}{k} \|u\| \cdot \|\widehat{v}_{n,2}\| \\
& \leq c \lambda_{n+1}^{-\frac{1}{2}} \|\Delta u\|^2 + \frac{1}{3} \|\Delta \widehat{v}_{n,2}\|^2 + \frac{\alpha - \delta}{8} \|\widehat{v}_{n,2}\|^2 + \frac{c}{k} (\|u\|^2 + \|\Delta u\|^2), \tag{4.50}
\end{aligned}$$

$$(4\Delta \nabla \widehat{\rho}_k \cdot \nabla v + 6\Delta \widehat{\rho}_k \cdot \Delta v + 4\nabla \widehat{\rho}_k \cdot \Delta \nabla v + v A \widehat{\rho}_k, \widehat{v}_{n,2})$$

$$\leq \frac{4c_6}{k} \lambda_{n+1}^{-\frac{1}{4}} \|\Delta v\| \cdot \|\widehat{v}_{n,2}\| + \frac{6c_5}{k} \|\Delta v\| \cdot \|\widehat{v}_{n,2}\|$$

$$\begin{aligned}
& + \frac{4c_4}{k} \lambda_{n+1}^{-\frac{1}{4}} \|\Delta\nu\| \cdot \|\Delta\widehat{\nu}_{n,2}\| + \frac{c_7}{k} \|\nu\| \cdot \|\widehat{\nu}_{n,2}\| \\
& \leq c \lambda_{n+1}^{-\frac{1}{2}} \|\Delta\nu\|^2 + \frac{1}{3} \|\Delta\widehat{\nu}_{n,2}\|^2 + \frac{\alpha - \delta}{16} \|\widehat{\nu}_{n,2}\|^2 + \frac{c}{k} (\|\nu\|^2 + \|\Delta\nu\|^2), \tag{4.51}
\end{aligned}$$

$$\begin{aligned}
(\widehat{\rho}_k f(u(t-\rho, x), x), \widehat{\nu}_{n,2}) & \leq l_f \| (I - P_n) \widehat{u}(t-\rho, x) \| \cdot \|\widehat{\nu}_{n,2}\| \\
& \leq \frac{\alpha - \delta}{16} \|\widehat{\nu}_{n,2}\|^2 + \frac{4l_f^2}{\alpha - \delta} \|\widehat{u}_{n,2}(t-\rho, x)\|^2. \tag{4.52}
\end{aligned}$$

Therefore, by (3.16), (4.43)–(4.52), and the fact $\eta_1 \in L^2(\mathbb{R}^n)$, $\lambda_n \rightarrow \infty$, there are $\widehat{N}_1 = \widehat{N}_1(\varepsilon) > 0$, $\widehat{R}_1 = \widehat{R}_1(\varepsilon) > 0$ such that for $\forall n > \widehat{N}_1$, $k > \widehat{R}_1$,

$$\begin{aligned}
& \frac{d}{dt} (\|\widehat{\nu}_{n,2}\|^2 + (\lambda + \delta^2 - \delta\alpha) \|\widehat{u}_{n,2}\|^2 + (1 - \delta) \|\Delta\widehat{u}_{n,2}\|^2) \\
& \leq -\sigma (\|\widehat{\nu}_{n,2}\|^2 + (\lambda + \delta^2 - \delta\alpha) \|\widehat{u}_{n,2}\|^2 + (1 - \delta) \|\Delta\widehat{u}_{n,2}\|^2) \\
& \quad + \frac{8l_f^2}{\alpha - \delta} \|\widehat{u}_{n,2}(t-\rho, x)\|^2 - \frac{\delta}{2} (\lambda + \delta^2 - \delta\alpha) \|\widehat{u}_{n,2}\|^2 \\
& \quad + c\varepsilon^2 (\|(I - P_n) \widehat{\rho}_k z(\theta_t \omega)\|^2 + \|(I - P_n) \Delta(\widehat{\rho}_k z(\theta_t \omega))\|^2 + \|(I - P_n) \widehat{\rho}_k \Delta z(\theta_t \omega)\|^2) \\
& \quad + c \|(I - P_n) \widehat{\rho}_k g(x, t)\|^2 + \frac{c}{k} \lambda_{n+1}^{-\frac{1}{2}} (\|\Delta\nu\|^2 + \|\Delta u\|^2) \\
& \quad + \frac{c}{k} (\|u\|^2 + \|\Delta u\|^2 + \|\nu\|^2 + \|\Delta\nu\|^2) \\
& \quad + 3\lambda_{n+1}^{-1} (c_1 \lambda_{n+1}^{\frac{6}{2}} \|u\|_{H^2}^p + \|\eta_1\|)^2 \\
& \leq -\sigma (\|\widehat{\nu}_{n,2}\|^2 + (\lambda + \delta^2 - \delta\alpha) \|\widehat{u}_{n,2}\|^2 + (1 - \delta) \|\Delta\widehat{u}_{n,2}\|^2) \\
& \quad + \frac{8l_f^2}{\alpha - \delta} \|\widehat{u}_{n,2}(t-\rho, x)\|^2 - \frac{\delta}{2} (\lambda + \delta^2 - \delta\alpha) \|\widehat{u}_{n,2}\|^2 \\
& \quad + c \|(I - P_n) \widehat{\rho}_k g(x, t)\|^2 + \frac{c\varepsilon}{k} (\|\Delta\nu\|^2 + \|\Delta u\|^2) \\
& \quad + \frac{c}{k} (\|u\|^2 + \|\Delta u\|^2 + \|\nu\|^2 + \|\Delta\nu\|^2) \\
& \quad + \varepsilon (1 + \|u\|_{H^2}^{2p} + |y(\theta_t \omega)|^2). \tag{4.53}
\end{aligned}$$

Using a similar calculation with (4.14) and (4.15), and combining with (3.16) we have that

$$\begin{aligned}
& \|\widehat{\nu}_{n,2}(\tau + s, \tau - t, \theta_{-\tau} \omega, \widehat{\psi})\|^2 + (\lambda + \delta^2 - \delta\alpha) \|\widehat{u}_{n,2}(\tau + s, \tau - t, \theta_{-\tau} \omega, \widehat{\phi})\|^2 \\
& \quad + (1 - \delta) \|\Delta\widehat{u}_{n,2}(\tau + s, \tau - t, \theta_{-\tau} \omega, \widehat{\phi})\|^2 \\
& \leq e^{-\sigma(t+s)} (\|(I - P_n) \widehat{\rho}_k(x) \widehat{\psi}\|^2 \\
& \quad + (\delta^2 + \lambda - \delta\alpha) \|(I - P_n) \widehat{\rho}_k(x) \widehat{\phi}\|^2 + (1 - \delta) \|(I - P_n) \widehat{\rho}_k(x) \Delta\widehat{\phi}\|^2) \\
& \quad + c \int_{\tau-t}^{\tau+s} e^{\sigma(r-\tau-s)} \|(I - P_n) \widehat{\rho}_k(x) g(x, t)\|^2 dr \\
& \quad + \frac{c\varepsilon}{k} \int_{\tau-t}^{\tau+s} e^{\sigma(r-\tau-s)} (\|\Delta\nu(r, \tau - t, \theta_{-\tau} \omega, \psi)\|^2 + \\
& \quad + \|\Delta u(r, \tau - t, \theta_{-\tau} \omega, \phi)\|^2) dr
\end{aligned}$$

$$\begin{aligned}
& + \frac{c}{k} \int_{\tau-t}^{\tau+s} e^{\sigma(r-\tau-s)} (\|u(r, \tau-t, \theta_{-\tau}\omega, \phi)\|^2 + \|\Delta u(r, \tau-t, \theta_{-\tau}\omega, \phi)\|^2 \\
& + \|v(r, \tau-t, \theta_{-\tau}\omega, \psi)\|^2 + \|\Delta v(r, \tau-t, \theta_{-\tau}\omega, \psi)\|^2) dr \\
& + \varepsilon \int_{\tau-t}^{\tau+s} e^{\sigma(r-\tau-s)} (1 + |y(\theta_{r-\tau}\omega)|^2 + \|u(r, \tau-t, \theta_{-\tau}\omega, \phi)\|_{H^2}^{2p}) dr. \tag{4.54}
\end{aligned}$$

Since $(\hat{\phi}, \hat{\psi})^\top \in D(\tau-t, \theta_{-\tau}\omega) \in \mathcal{D}$, then there are $\widehat{T}_1 = \widehat{T}_1(\tau, \varepsilon, D, \omega) > 0$, $\widehat{R}_1 = \widehat{R}_1(\tau, \varepsilon, \omega) > 1$, such that for $t > \widehat{T}_1$, $k > \widehat{R}_1$

$$\begin{aligned}
& e^{-\sigma(t+s)} (\|(I - P_n)\widehat{\rho}_k(x)\widehat{\psi}\|^2 \\
& + (\delta^2 + \lambda - \delta\alpha)(\|(I - P_n)\widehat{\rho}_k(x)\widehat{\phi}\|^2 + (1 - \delta)(\|(I - P_n)\widehat{\rho}_k(x)\Delta\widehat{\phi}\|^2)) \leq \varepsilon. \tag{4.55}
\end{aligned}$$

From (3.7), there exists $\widehat{N} = \widehat{N}(\tau, \varepsilon, \omega) > 0$, such that for $\forall n > \widehat{N}$

$$c \int_{\tau-t}^{s+\tau} e^{\sigma(r-\tau-s)} \|(I - P_n)\widehat{\rho}_k g(x, t)\|^2 dr \leq \varepsilon. \tag{4.56}$$

From Lemma 4.1, there are $\widehat{T}_2 = \widehat{T}_2(\tau, \varepsilon, D, \omega) > 0$, $\widehat{R}_2(\tau, \varepsilon, \omega) > 1$, such that for $\forall t > \widehat{T}_2$, $k > \widehat{R}_2$,

$$\begin{aligned}
& \frac{c\varepsilon}{k} \int_{\tau-t}^{s+\tau} e^{\sigma(r-\tau-s)} (\|\Delta u(r, \tau-t, \theta_{-\tau}\omega, \phi)\|^2 + \|\Delta v(r, \tau-t, \theta_{-\tau}\omega, \psi)\|^2) dr \\
& + \frac{c}{k} \int_{\tau-t}^{\tau+s} e^{\sigma(r-\tau-s)} (\|u(r, \tau-t, \theta_{-\tau}\omega, \phi)\|^2 + \|v(r, \tau-t, \theta_{-\tau}\omega, \psi)\|^2 \\
& + \|\Delta u(r, \tau-t, \theta_{-\tau}\omega, \phi)\|^2 \\
& + \|\Delta v(r, \tau-t, \theta_{-\tau}\omega, \psi)\|^2) dr \leq \varepsilon. \tag{4.57}
\end{aligned}$$

By Lemma 4.1, there is $\widehat{T}_3 = \widehat{T}_3(\tau, \varepsilon, D, \omega) > 0$, for $\forall t > \widehat{T}_3$

$$\int_{\tau-t}^{s+\tau} e^{\sigma(r-\tau-s)} (1 + |y(\theta_{r-\tau}\omega)|^2 + \|u(r, \tau-t, \theta_{-\tau}\omega, \phi)\|_{H^2}^{2p}) dr < \infty, \tag{4.58}$$

which together with (4.54)–(4.57) gives the desired result (4.41). \square

5 Existence of random attractors

In this section, we establish the existence and uniqueness of random attractors for problem (3.11). We can easily obtain the existence of random absorbing sets of Φ from Lemma 4.1.

Lemma 5.1 Suppose (3.1)–(3.5), (3.7), (3.14), and (3.16) hold. Then, for $\forall \epsilon \in (0, 1]$, $\omega \in \Omega$, $\tau \in \mathbb{R}$, the cocycle Φ has a random absorbing set $K_\epsilon = \{K_\epsilon(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ as

$$K_\epsilon(\tau, \omega) = \left\{ Y \in C_{V_2, V_0(\mathbb{R}^n)} : \|Y\|_{C_{V_2, V_0(\mathbb{R}^n)}}^2 \leq R_\epsilon(\tau, \omega) \right\},$$

with $Y = (u, v)^\top$ and

$$\begin{aligned} R_\epsilon(\tau, \omega) = & M + M \int_{-\infty}^0 e^{\sigma r} \|g(x, r + \tau)\|^2 dr \\ & + M\epsilon^2 \int_{-\infty}^0 e^{\sigma r} (1 + \|\Delta z(\theta_r \omega)\|^2 + \|z(\theta_r \omega)\|^2 \\ & + \|\nabla z(\theta_r \omega)\|^2 + \|z(\theta_r \omega)\|_{H^2}^{p+1}) dr. \end{aligned} \quad (5.1)$$

Next, we establish the asymptotic compactness of the cocycle Φ in $C_{V_2, V_0}(\mathbb{R}^n)$.

Lemma 5.2 Suppose (3.1)–(3.5), (3.7), (3.8), (3.14), and (3.16) hold. Then, the cocycle Φ is \mathcal{D} -pullback asymptotically compact in $C_{V_2, V_0}(\mathbb{R}^n)$.

Proof For $\forall \tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, we will prove that the sequence $\{Y_\tau(\cdot, \tau - t_n, \theta_{-\tau} \omega, Y_{0,n})\}$ has a convergent subsequence in $C_{V_2, V_0}(\mathbb{R}^n)$ whenever $t_n \rightarrow \infty$ and $Y_{0,n} \in D(\tau - t_n, \theta_{-\tau} \omega)$.

From Lemma 4.1, $\{Y_\tau(\cdot, \tau - t_n, \theta_{-\tau} \omega, Y_{0,n})\}$ is bounded in $C_{V_2, V_0}(\mathbb{R}^n)$; which means that for $\forall \tau \in \mathbb{R}$, $\omega \in \Omega$, there is $\widehat{N}_1 = \widehat{N}_1(\tau, \omega, D) > 0$ for $\forall n > \widehat{N}_1$,

$$\|Y_\tau(\cdot, \tau - t_n, \theta_{-\tau} \omega, Y_{0,n})\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 \leq R_\epsilon(\tau, \omega). \quad (5.2)$$

In addition, from Lemma 4.2 we know that there are $k_1 = k_1(\tau, \varepsilon, \omega) > 0$, $\widehat{N}_2 = \widehat{N}_2(\tau, D, \varepsilon, \omega) > 0$, for $\forall n \geq \widehat{N}_2$ and fixed $s \in [-\rho, 0]$,

$$\|Y(\tau + s, \tau - t_n, \theta_{-\tau} \omega, Y_{0,n})\|_{H^2(\mathbb{R}^n \setminus \mathbb{H}_{k_1}) \times L^2(\mathbb{R}^n \setminus \mathbb{H}_{k_1})}^2 \leq \varepsilon. \quad (5.3)$$

From Lemma 4.3, there exist $N = N(\tau, \varepsilon, \omega) > 0$, $k_2 = k_2(\tau, \varepsilon, \omega) \geq k_1$, $\widehat{N}_3 = \widehat{N}_3(\tau, D, \varepsilon, \omega) > 0$, for $\forall n \geq \widehat{N}_3$ and fixed $s \in [-\rho, 0]$,

$$\|(I - P_N)\widehat{Y}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \widehat{Y}_{0,n})\|_{H^2(\mathbb{H}_{2k_2}) \times L^2(\mathbb{H}_{2k_2})}^2 \leq \varepsilon. \quad (5.4)$$

By (4.38) and (5.2), we see that $\{P_N \widehat{Y}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \widehat{Y}_{0,n})\}$ is bounded in $P_N H^2(\mathbb{H}_{2k_2}) \times L^2(\mathbb{H}_{2k_2})$, together with (5.4) we find $\{\widehat{Y}(\tau + s, \tau - t_n, \theta_{-\tau} \omega, \widehat{Y}_{0,n})\}$ is precompact in $H^2(\mathbb{H}_{2k_2}) \times L^2(\mathbb{H}_{2k_2})$.

Note that $\widehat{\rho_{k_2}} = 1$ for $|x| \leq \frac{k_2}{2}$. By (4.38), we obtain $\{Y(\tau + s, \tau - t_n, \theta_{-\tau} \omega, Y_{0,n})\}$ is precompact in $H^2(\mathbb{H}_{k_2}) \times L^2(\mathbb{H}_{k_2})$, which together with (5.3) shows the precompactness of this sequence in $C_{V_2, V_0}(\mathbb{R}^n)$ for fixed $s \in [-\rho, 0]$. \square

As an immediate consequence of Proposition 2.1, Lemma 5.1, and Lemma 5.2, we have

Theorem 5.1 Assume that (3.1)–(3.5), (3.7), (3.8), (3.14), and (3.16) hold. Then, for every $\epsilon \in (0, 1]$, the continuous cocycle Φ associated with (3.11) has a unique \mathcal{D} -pullback attractor $\mathcal{A}_\epsilon = \{\mathcal{A}_\epsilon(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $C_{V_2, V_0}(\mathbb{R}^n)$.

6 Upper semicontinuity of attractors as $\epsilon \rightarrow 0$

In this section, we establish the upper semicontinuity of random attractors of the plate Eq. (3.11) with delay driven by additive noise when $\epsilon \rightarrow 0$. We write the solution and the corresponding cocycle of (3.11) as u^ϵ, v^ϵ and Φ_ϵ , respectively.

In Sect. 5, we obtained that Φ_ϵ has a \mathcal{D} -pullback attractor $\mathcal{A}_\epsilon \in \mathcal{D}$ in $C_{V_2, V_0}(\mathbb{R}^n)$ and random absorbing set $K_\epsilon = \{K_\epsilon(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ with $K_\epsilon(\tau, \omega) \subseteq K(\tau, \omega)$ for all $\epsilon \in (0, 1]$, where for every $\tau \in \mathbb{R}, \omega \in \Omega$,

$$K(\tau, \omega) = \{Y \in C_{V_2, V_0}(\mathbb{R}^n) : \|Y\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 \leq R(\tau, \omega)\}$$

and

$$\begin{aligned} R(\tau, \omega) &= M + M \int_{-\infty}^0 e^{\sigma r} \|g(x, r + \tau)\|^2 dr \\ &\quad + M \int_{-\infty}^0 e^{\sigma r} (1 + \|\Delta z(\theta_r \omega)\|^2 + \|z(\theta_r \omega)\|^2 + \|\nabla z(\theta_r \omega)\|^2 + \|z(\theta_r \omega)\|_{H^2}^{p+1}) dr, \end{aligned}$$

where $Y = (u, v)^\top$.

From Lemma 5.1 we know for $\forall \tau \in \mathbb{R}, \omega \in \Omega$,

$$\bigcup_{0 < \epsilon \leq 1} \mathcal{A}_\epsilon(\tau, \omega) \subseteq \bigcup_{0 < \epsilon \leq 1} K_\epsilon(\omega) \subseteq K(\tau, \omega). \quad (6.1)$$

As $\epsilon = 0$, the problem (3.11) reduces to a deterministic one:

$$\begin{cases} \frac{du}{dt} + \delta u = v, \\ \frac{dv}{dt} = (\delta - A - \alpha)v + [\delta(A - \delta + \alpha) - A - \lambda]u - F(u(t, x), x) \\ \quad + f(u(t - \rho, x), x) + g(x, t), \quad x \in \mathbb{R}^n, \quad t > \tau, \\ u_\tau(x, s) = \phi(x, s), \quad v_\tau(x, s) = \partial_t \phi(x, s) + \delta \phi(x, s) := \hat{\psi}(x, s). \end{cases} \quad (6.2)$$

Accordingly, by Theorem 5.1 the cocycle Φ_0 generated by (6.2) has a unique random attractor $\mathcal{A}_0 = \{\mathcal{A}_0(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$ in $C_{V_2, V_0}(\mathbb{R}^n)$ and a random absorbing set $K_0 = \{K_0(\tau) : \tau \in \mathbb{R}\}$, where

$$\mathcal{D}_0 = \left\{ D = \left\{ D(\tau) \subseteq C_{V_2, V_0}(\mathbb{R}^n) : \tau \in \mathbb{R}, \lim_{t \rightarrow +\infty} e^{-\gamma t} \|D(\tau - t)\|_{C_{V_2, V_0}(\mathbb{R}^n)} = 0, \forall \gamma > 0 \right\} \right\}$$

and

$$K_0(\tau) = \{Y \in C_{V_2, V_0}(\mathbb{R}^n) : \|Y\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 \leq R_0(\tau)\}, \quad (6.3)$$

where

$$R_0(\tau) = M + M \int_{-\infty}^0 e^{\sigma r} \|g(x, r + \tau)\|^2 dr. \quad (6.4)$$

Note that $R_0(\tau)$ corresponds to the number $R_\epsilon(\tau, \omega)$ given by (5.1) with $\epsilon = 0$. From Lemma 5.1 and (6.3) and (6.4) we have that for $\forall \tau \in \mathbb{R}, \omega \in \Omega$,

$$\limsup_{\epsilon \rightarrow 0} R_\epsilon(\tau, \omega) = R_0(\tau), \quad \text{and} \quad \limsup_{\epsilon \rightarrow 0} \|K_\epsilon(\tau, \omega)\| = \|K_0(\tau)\|. \quad (6.5)$$

Now, we will establish the convergence of the solutions of (3.11) as $\epsilon \rightarrow 0$ to obtain the upper semicontinuity of the \mathcal{D} -pullback attractor \mathcal{A}_ϵ .

Lemma 6.1 Let $Y^\epsilon = (u^\epsilon, v^\epsilon)$ and $Y = (u, v)$ be the solutions of (3.11) and (6.2) with initial values $Y_0^\epsilon = (\phi^\epsilon, \psi^\epsilon)$ and $Y_0 = (\phi, \psi)$, respectively. Assume that (3.1)–(3.5) and (3.14) hold. If $\lim_{\epsilon \rightarrow 0} (\phi^\epsilon, \psi^\epsilon) = (\phi, \psi) \in C_{V_2, V_0}(\mathbb{R}^n)$, for $\forall \tau \in \mathbb{R}, \omega \in \Omega, T > 0, t \in [\tau, \tau + T]$,

$$\|Y_t^\epsilon(\cdot, \tau, \omega, Y_0^\epsilon) - Y_t(\cdot, \tau, Y_0)\|_{C_{V_2, V_0}(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (6.6)$$

Proof Let $(u^\epsilon(t, \tau, \omega, \phi^\epsilon), v^\epsilon(t, \tau, \omega, \psi^\epsilon))$ be the solution of (3.11) and $\tilde{u} = u^\epsilon - u, \tilde{v} = v^\epsilon - v$. Then, by (3.11) and (6.2) we have

$$\begin{cases} \frac{d\tilde{u}}{dt} + \delta\tilde{u} = \tilde{v} + \epsilon z(\theta_t \omega), \\ \frac{d\tilde{v}}{dt} = (\delta - A - \alpha)\tilde{v} + [\delta(-\delta + A + \alpha) - \lambda - A]\tilde{u} - (F(u^\epsilon, x) - F(u, x)) \\ \quad + (f(u^\epsilon(t - \rho, x), x) - f(u(t - \rho, x), x)) + \epsilon[1 - (\alpha + A - \delta)]z(\theta_t \omega), \\ \tilde{u}_\tau(s, x) = \phi^\epsilon(s, x) - \phi(s, x), \quad \tilde{v}_\tau(s, x) = \psi^\epsilon(s, x) - \psi(s, x). \end{cases} \quad (6.7)$$

Taking the inner product of (6.7)₂ with \tilde{v} , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\tilde{v}\|^2 + (\lambda + \delta^2 - \delta\alpha)\|\tilde{u}\|^2 + (1 - \delta)\|\Delta\tilde{u}\|^2) \\ & \leq -\frac{3}{4}\delta(\lambda + \delta^2 - \delta\alpha)\|\tilde{u}\|^2 - \frac{3}{4}\delta(1 - \delta)\|\Delta\tilde{u}\|^2 - \frac{3}{4}(\alpha - \delta)\|\tilde{v}\|^2 \\ & \quad - (F(u^\epsilon, x) - F(u, x), \tilde{v}) + (f(u^\epsilon(t - \rho, x), x) - f(u(t - \rho, x), x), \tilde{v}) \\ & \quad + c\epsilon^2(1 + \|z(\theta_t \omega)\|^2 + \|\Delta z(\theta_t \omega)\|^2). \end{aligned} \quad (6.8)$$

By (3.4), we obtain

$$|(F(u^\epsilon, x) - F(u, x), \tilde{v})| \leq c\|\tilde{u}\|^2 + c\|\tilde{v}\|^2. \quad (6.9)$$

From (3.5), we have

$$\begin{aligned} & |(f(u^\epsilon(t - \rho, x), x) - f(u(t - \rho, x), x), \tilde{v})| \leq l_f \|\tilde{u}(t - \rho, x)\| \cdot \|\tilde{v}\| \\ & \leq c\|\tilde{u}(t - \rho, x)\|^2 + c\|\tilde{v}\|^2, \end{aligned}$$

which together with (6.7)–(6.9) gives

$$\begin{aligned} & \frac{d}{dt} (\|\tilde{v}\|^2 + (\lambda + \delta^2 - \delta\alpha)\|\tilde{u}\|^2 + (1 - \delta)\|\Delta\tilde{u}\|^2) \\ & \leq c(\|\tilde{v}\|^2 + (\lambda + \delta^2 - \delta\alpha)\|\tilde{u}\|^2 + (1 - \delta)\|\Delta\tilde{u}\|^2) + c\|\tilde{u}(t - \rho, x)\|^2 \\ & \quad + c\epsilon^2(1 + \|z(\theta_t \omega)\|^2 + \|\Delta z(\theta_t \omega)\|^2). \end{aligned} \quad (6.10)$$

Integrating (6.10) between τ and t we obtain

$$\begin{aligned} & \|\tilde{v}(t)\|^2 + (\lambda + \delta^2 - \delta\alpha)\|\tilde{u}(t)\|^2 + (1 - \delta)\|\Delta\tilde{u}(t)\|^2 \\ & \leq \|\tilde{v}(\tau)\|^2 + (\lambda + \delta^2 - \delta\alpha)\|\tilde{u}(\tau)\|^2 + (1 - \delta)\|\Delta\tilde{u}(\tau)\|^2 \end{aligned}$$

$$\begin{aligned}
& + c \int_{\tau}^t (\|\tilde{v}(r)\|^2 + (\lambda + \delta^2 - \delta\alpha) \|\tilde{u}(r)\|^2 + (1 - \delta) \|\Delta \tilde{u}(r)\|^2) dr \\
& + c \int_{\tau}^t \|\tilde{u}(r - \rho, x)\|^2 dr + c\epsilon^2 \int_{\tau}^t (1 + \|\Delta z(\theta_r \omega)\|^2 + \|z(\theta_r \omega)\|^2) dr. \tag{6.11}
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{\tau}^t \|\tilde{u}(r - \rho, x)\|^2 dr &= \int_{\tau-\rho}^{t-\rho} \|\tilde{u}(r, x)\|^2 dr = \int_{\tau-\rho}^{\tau} \|\tilde{u}(r, x)\|^2 dr + \int_{\tau}^{t-\rho} \|\tilde{u}(r, x)\|^2 dr \\
&\leq \rho \|\phi^\epsilon - \phi\|_{C_{V_2}(\mathbb{R}^n)} + \int_{\tau}^t \|\tilde{u}(r, x)\|^2 dr. \tag{6.12}
\end{aligned}$$

Thus, for $\forall t \in [\tau, \tau + T]$, from (6.11), (6.12), and (3.9) we have that

$$\begin{aligned}
\|\tilde{Y}(t)\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 &\leq c \|Y_0^\epsilon - Y_0\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 + c \int_{\tau}^t \|\tilde{Y}(r)\|^2 dr \\
&\quad + c\epsilon^2 \int_{\tau}^t (1 + \|\Delta z(\theta_r \omega)\|^2 + \|z(\theta_r \omega)\|^2) dr.
\end{aligned}$$

Accordingly, for $\forall t \in [\tau, \tau + T]$, we obtain

$$\begin{aligned}
&\|\tilde{Y}(t, \tau, \omega, \tilde{Y}_0)\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 \\
&\leq c \|Y_0^\epsilon - Y_0\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 + c\epsilon^2 \int_{\tau}^t (1 + \|\Delta z(\theta_r \omega)\|^2 + \|z(\theta_r \omega)\|^2) dr. \tag{6.13}
\end{aligned}$$

Hence, if $\lim_{\epsilon \rightarrow 0} (\phi^\epsilon, \psi^\epsilon) = (\phi, \hat{\psi}) \in C_{V_2, V_0}(\mathbb{R}^n)$, then

$$\|Y_0^\epsilon - Y_0\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0 \tag{6.14}$$

and hence by (6.13), for $\forall t \in [\tau, \tau + T]$,

$$\sup_{-\rho \leq s \leq 0} \|\tilde{Y}(t + s, \tau, \omega, \tilde{Y}_0)\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0, \tag{6.15}$$

which implies (6.6). \square

Now, we establish the uniform compactness of \mathcal{A}_ϵ in $C_{V_2, V_0}(\mathbb{R}^n)$.

Lemma 6.2 Suppose (3.1)–(3.5), (3.7), (3.8), (3.14), and (3.16) hold. Then, for $\forall \tau \in \mathbb{R}, \omega \in \Omega, \bigcup_{0 < \epsilon \leq 1} \mathcal{A}_\epsilon(\tau, \omega)$ is precompact in $C_{V_2, V_0}(\mathbb{R}^n)$.

Proof Given $\epsilon \in (0, 1]$. First, from (6.4), Lemma 4.2, and the invariance of $\mathcal{A}_\epsilon(\tau, \omega)$, we know that for $\forall \epsilon > 0, \tau \in \mathbb{R}, \omega \in \Omega$, there is $r_0 = r_0(\omega, \epsilon) \geq 1$ such that

$$\int_{|x| \geq k_0} (\|u(x)\|^2 + \|\Delta u(x)\|^2 + \|v(x)\|^2) dx \leq \epsilon, \quad \text{for all } (u, v) \in \bigcup_{0 < \epsilon \leq 1} \mathcal{A}_\epsilon(\tau, \omega). \tag{6.16}$$

Secondly, from (6.1), Lemma 5.2, Lemma 4.3, and the invariance of $\mathcal{A}_\epsilon(\tau, \omega)$, we know that there exists $k_1 = k_1(\omega, \epsilon) \geq k_0$ such that for $\forall k \geq k_1$, the set $\bigcup_{0 < \epsilon \leq 1} \mathcal{A}_\epsilon(\tau, \omega)$ is precompact

in $C_{V_2, V_0}(\mathbb{H}_k)$, which together with (6.16) implies that $\bigcup_{0 < \epsilon \leq 1} \mathcal{A}_\epsilon(\tau, \omega)$ is precompact in $C_{V_2, V_0}(\mathbb{R}^n)$. \square

Now, we are ready to prove the upper semicontinuity of the \mathcal{A}_ϵ as $\epsilon \rightarrow 0$. In fact, it is an immediate consequence of Theorem 3.2 in [25] based on (6.5) and Lemmas 6.1 and 6.2.

Theorem 6.1 *Assume that (3.1)–(3.5), (3.7), (3.8), (3.14), and (3.16) hold. Then, for $\forall \tau \in \mathbb{R}$, $\omega \in \Omega$,*

$$\lim_{\epsilon \rightarrow 0} d_{C_{V_2, V_0}(\mathbb{R}^n)}(\mathcal{A}_\epsilon(\tau, \omega), \mathcal{A}(\tau)) = 0.$$

7 Upper semicontinuity of attractors as $\rho \rightarrow 0$

In this section, we establish the upper semicontinuity of random attractors of the plate Eq. (3.11) when the delay ρ approaches zero for a fixed $\epsilon \in (0, 1]$. We write the solution and the corresponding cocycle of (3.11) as u^ρ , v^ρ and Φ^ρ , respectively.

For given $\tau \in \mathbb{R}$, $\omega \in \Omega$, denote

$$K^\rho(\tau, \omega) = \{Y \in C_{V_2, V_0}(\mathbb{R}^n) : \|Y\|_{C_{V_2, V_0}(\mathbb{R}^n)}^2 \leq R^\rho(\tau, \omega)\}, \quad (7.1)$$

where $R^\rho(\tau, \omega)$ is equal to the right-hand side of (5.1). From (7.1) and Lemma 5.1 we know that, for all $\rho \in (0, 1]$, $K^\rho = \{K^\rho(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a \mathcal{D} -pullback absorbing set of Φ^ρ in $C_{V_2, V_0}(\mathbb{R}^n)$. In addition, for $\forall \tau \in \mathbb{R}$, $\omega \in \Omega$, Φ^ρ has a \mathcal{D} -pullback attractor $\mathcal{A}^\rho \in \mathcal{D}$ in $C_{V_2, V_0}(\mathbb{R}^n)$,

$$\mathcal{A}^\rho(\tau, \omega) \subseteq K^\rho(\tau, \omega). \quad (7.2)$$

As $\rho = 0$, the stochastic delay system (3.11) becomes a stochastic system without delay given by

$$\begin{cases} \frac{du}{dt} + \delta u = v + \epsilon z(\theta_t \omega), \\ \frac{dv}{dt} = (\delta - A - \alpha)v + [\delta(-\delta + A + \alpha) - A - \lambda]u + \epsilon[1 - (\alpha - \delta + A)]z(\theta_t \omega) \\ \quad - F(u(x, t), x) + f(u(x, t), x) + g(x, t), \\ u_\tau(x, s) = \phi(x, s), \quad v_\tau(x, s) = \partial_t \phi(x, s) + \delta \phi(x, s) - \epsilon z(\theta_t \omega) := \psi(x, s). \end{cases} \quad (7.3)$$

Accordingly, by Theorem 5.1 the cocycle Φ^0 generated by (7.3) is readily verified to admit a unique \mathcal{D}^0 -pullback attractor $\mathcal{A}^0 = \{\mathcal{A}^0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}^0$ in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and a \mathcal{D}^0 -pullback absorbing set $K^0 = \{K^0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, where

$$\mathcal{D}^0 = \{D = \left\{ D(\tau, \omega) \subseteq H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n), \lim_{t \rightarrow +\infty} e^{-\gamma t} \|D(\tau - t, \theta_{-t} \omega)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} = 0 \right\}\},$$

and

$$K^0(\tau, \omega) = \{Y | \|Y\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}^2 \leq R^0(\tau, \omega)\}, \quad (7.4)$$

with $R^0(\tau, \omega)$ given by the right-hand side of (5.1).

From (7.1) and (7.4), we see that for $\forall \tau \in \mathbb{R}$, $\omega \in \Omega$,

$$\limsup_{\rho \rightarrow 0} \|K^\rho(\tau, \omega)\|_{C_{V_2, V_0}(\mathbb{R}^n)} = \|K^0(\tau, \omega)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}. \quad (7.5)$$

Lemma 7.1 Suppose $Y^\rho = (u^\rho, v^\rho)^\top$ and $Y = (u, v)^\top$ are the solutions of (3.11) and (7.3) with initial values $Y_0^\rho = (\phi^\rho, \psi^\rho)^\top$ and $Y_0 = (\phi, \psi)^\top$, respectively. Assume that (3.1)–(3.5) and (3.14) hold. If $\lim_{\rho \rightarrow 0} \sup_{-\rho \leq s \leq 0} \|Y_0^\rho(s) - Y_0\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} = 0$, then for $\forall \tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$, $t \in [\tau, \tau + T]$,

$$\sup_{-\rho \leq s \leq 0} \|Y^\rho(t + s, \tau, \omega, Y_0^\rho) - Y(t, \tau, \omega, Y_0)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } \rho \rightarrow 0. \quad (7.6)$$

Proof For $\forall s \in [-\rho, 0]$, $t \geq \tau$, let $\widehat{v} = v^\rho(t + s) - v(t)$ and $\widehat{u} = u^\rho(t + s) - u(t)$, where $v^\rho = \partial_t u^\rho + \delta u^\rho - \epsilon z(\theta_t \omega)$ and $v = \partial_t u + \delta u - \epsilon z(\theta_t \omega)$ with $\psi^\rho(s) = \partial_t \phi^\rho(s) + \delta \phi^\rho(s) - \epsilon z(\theta_t \omega)$ and $\psi(s) = \partial_t \phi(s) + \delta \phi(s) - \epsilon z(\theta_t \omega)$.

By (3.11) and (7.3) we see that for $t > \tau - s$ and $s \in [-\rho, 0]$,

$$\begin{aligned} \frac{d\widehat{v}}{dt} &= (\delta - A - \alpha)\widehat{v} + [\delta(-\delta + \alpha + A) - A - \lambda]\widehat{u} \\ &\quad + \epsilon[1 - (\alpha - \delta + A)][z(\theta_{t+s}\omega) - z(\theta_t\omega)] \\ &\quad - (F(u^\rho(t + s, x), x) - F(u(t, x), x)) + (f(u^\rho(t + s - \rho, x), x) - f(u(t, x), x)) \\ &\quad + g(x, t + s) - g(x, t). \end{aligned} \quad (7.7)$$

Taking the inner product of (7.7) with \widehat{v} , then

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|\widehat{v}\|^2 + (\lambda + \delta^2 - \delta\alpha)\|\widehat{u}\|^2 + (1 - \delta)\|\Delta \widehat{u}\|^2) \\ &\leq -\frac{3}{4}(\alpha - \delta)\|\widehat{v}\|^2 - \frac{3}{4}\delta(\lambda + \delta^2 - \delta\alpha)\|\widehat{u}\|^2 - \frac{3}{4}\delta(1 - \delta)\|\Delta \widehat{u}\|^2 \\ &\quad - (F(x), u^\rho(t + s, x), F(u(t, x), x), \widehat{v}) + (f(u^\rho(t + s - \rho, x), x) - f(u(t, x), x), \tilde{v}) \\ &\quad + \frac{1}{2}\|g(x, t + s) - g(x, t)\|^2 + c\epsilon^2(1 + \|\Delta(z(\theta_{t+s}\omega) - z(\theta_t\omega))\|^2 \\ &\quad + \|z(\theta_{t+s}\omega) - z(\theta_t\omega)\|^2). \end{aligned} \quad (7.8)$$

From (3.4), we obtain

$$|(F(u^\rho(t + s, x), x) - F(u(t, x), x), \widehat{v})| \leq c\|\widehat{u}\|^2 + c\|\widehat{v}\|^2. \quad (7.9)$$

By (3.5), we obtain

$$\begin{aligned} &|(f(u^\rho(t + s - \rho, x), x) - f(u(t, x), x), \widehat{v})| \\ &\leq l_f \|u^\rho(t + s - \rho, x) - u(t, x)\| \cdot \|\widehat{v}\| \\ &\leq c \|u^\rho(t + s - \rho, x) - u(t, x)\|^2 + c\|\widehat{v}\|^2, \end{aligned}$$

which along with (7.7)–(7.9) implies

$$\begin{aligned} & \frac{d}{dt} (\|\widehat{v}\|^2 + (1-\delta)\|\Delta\widehat{u}\|^2 + (\lambda + \delta^2 - \delta\alpha)\|\widehat{u}\|^2) \\ & \leq c(\|\widehat{v}\|^2 + (1-\delta)\|\Delta\widehat{u}\|^2 + (\lambda + \delta^2 - \delta\alpha)\|\widehat{u}\|^2) + c\|u^\rho(t+s-\rho, x) - u(t, x)\|^2 \\ & \quad + \|g(x, t+s) - g(x, t)\|^2 + c\epsilon^2(1 + \|z(\theta_{t+s}\omega) - z(\theta_t\omega)\|^2 \\ & \quad + \|\Delta(z(\theta_{t+s}\omega) - z(\theta_t\omega))\|^2). \end{aligned} \quad (7.10)$$

Integrating (7.10) over $(\tau-s, t)$ with $t \in [\tau, \tau+T]$, we obtain that

$$\begin{aligned} & \|\widehat{v}(t)\|^2 + (1-\delta)\|\Delta\widehat{u}(t)\|^2 + (\lambda + \delta^2 - \delta\alpha)\|\widehat{u}(t)\|^2 \\ & \leq \|\widehat{v}(\tau-s)\|^2 + (1-\delta)\|\Delta\widehat{u}(\tau-s)\|^2 + (\lambda + \delta^2 - \delta\alpha)\|\widehat{u}(\tau-s)\|^2 \\ & \quad + c \int_{\tau-s}^t \|u^\rho(r-\rho+s, x) - u(r, x)\|^2 dr + \int_{\tau-s}^t \|g(x, r+s) - g(x, r)\|^2 dr \\ & \quad + c\epsilon^2 \int_{\tau-s}^t (1 + \|\Delta(z(\theta_{r+s}\omega) - z(\theta_r\omega))\|^2 + \|z(\theta_{r+s}\omega) - z(\theta_r\omega)\|^2) dr. \end{aligned} \quad (7.11)$$

Note that

$$\begin{aligned} & \int_{\tau-s}^t \|u^\rho(r-\rho+s, x) - u(r, x)\|^2 dr \\ & = \int_{\tau-s}^{\tau-s+\rho} \|u^\rho(r-\rho+s, x) - u(r, x)\|^2 dr + \int_{\tau-s+\rho}^t \|u^\rho(r-\rho+s, x) - u(r, x)\|^2 dr \\ & \leq 2\rho \sup_{-\rho \leq s \leq 0} \|\phi^\rho(s) - \phi\|^2 + 2 \int_{\tau-s}^{\tau+\rho-s} \|u(r, x) - \phi\|^2 dr \\ & \quad + 2 \int_{\tau-s}^t \|\widehat{u}(r)\|^2 dr + 2 \int_{\tau-s}^t \|u(r+\rho, x) - u(r, x)\|^2 dr. \end{aligned} \quad (7.12)$$

Thus, for $\forall t \in [\tau, \tau+T]$ with $t > \tau-s$, from (7.11), (7.12), and (3.9) we have that

$$\begin{aligned} & \|\widehat{Y}(t)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}^2 \\ & \leq \|\widehat{Y}(\tau-s)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}^2 + c \int_{\tau-s}^t \|\widehat{Y}(r)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}^2 dr \\ & \quad + \int_{\tau}^{\tau+T} \|g(x, r+s) - g(x, r)\|^2 dr + c\rho \sup_{-\rho \leq s \leq 0} \|\phi^\rho(s) - \phi\|^2 \\ & \quad + c \int_{\tau}^{\tau+2\rho} \|u(r, x) - \phi\|^2 dr + c \int_{\tau}^{\tau+T} \|u(r+\rho, x) - u(r, x)\|^2 dr \\ & \quad + c\epsilon^2 \int_{\tau-s}^t (1 + \|\Delta(z(\theta_{r+s}\omega) - z(\theta_r\omega))\|^2 + \|z(\theta_{r+s}\omega) - z(\theta_r\omega)\|^2) dr. \end{aligned} \quad (7.13)$$

From Lemma 3.1, given $\eta > 0$, there is $\rho_1 \in (0, 1]$ for $\forall \rho \leq \rho_1, s \in [-\rho, 0], r \in [\tau, \tau+T]$,

$$\|z(\theta_{r+s}\omega) - z(\theta_r\omega)\| \leq \eta. \quad (7.14)$$

By $\lim_{\rho \rightarrow 0} \int_{\tau}^{\tau+2\rho} \|u(r, x) - \phi\|^2 dr = 0$, we see for $\forall \rho \leq \rho_2$, there is $\rho_2 \leq \rho_1$ such that

$$\int_{\tau}^{\tau+2\rho} \|u(r, x) - \phi\|^2 dr \leq \eta. \quad (7.15)$$

Note that u is uniformly continuous from $[\tau, \tau + 1 + T]$ to $H^2(\mathbb{R}^n)$, and we obtain that there is $\rho_3 \leq \rho_2$ such that for $\forall \rho \leq \rho_3$, $r \in [\tau, \tau + T]$,

$$\|u(r + \rho, x) - u(r, x)\| \leq \eta. \quad (7.16)$$

Since $g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$ one obtains that

$$\lim_{s \rightarrow 0} \int_{\tau}^{\tau+T} \|g(x, r+s) - g(x, r)\|^2 dr = 0, \quad (7.17)$$

which implies that there is $\rho_4 \leq \rho_3$ such that for $\forall \rho \leq \rho_4$, $s \in [-\rho, 0]$,

$$\int_{\tau}^{\tau+T} \|g(x, r+s) - g(x, r)\|^2 dr \leq \eta. \quad (7.18)$$

It follows from (7.13)–(7.18) that

$$\begin{aligned} \|\widehat{Y}(t)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}^2 &\leq c \int_{\tau-s}^t \|\widehat{Y}(r)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}^2 dr + \|\widehat{Y}(\tau-s)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}^2 \\ &\quad + c\rho \sup_{-\rho \leq s \leq 0} \|\phi^\rho(s) - \phi\|^2 + c\eta. \end{aligned} \quad (7.19)$$

Accordingly, we have

$$\|\widehat{Y}(t)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}^2 \leq c \|\widehat{Y}(\tau-s)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}^2 + c\rho \sup_{-\rho \leq s \leq 0} \|\phi^\rho(s) - \phi\|^2 + c\eta. \quad (7.20)$$

In addition,

$$\begin{aligned} \|\widehat{Y}(\tau-s)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}^2 &= \|Y^\rho(\tau) - Y(\tau-s)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}^2 \\ &= \|\nu^\rho(\tau) - \nu(\tau-s)\|^2 + (\delta^2 + \lambda - \delta\alpha) \|u^\rho(\tau) - u(\tau-s)\|^2 \\ &\quad + (1-\delta) \|\Delta(u^\rho(\tau) - u(\tau-s))\|^2 \\ &\leq 2\|\nu(\tau-s) - \psi\|^2 + 2\|\nu^\rho(\tau) - \psi\|^2 \\ &\quad + 2(\delta^2 + \lambda - \delta\alpha) (\|u^\rho(\tau) - \phi\|^2 + \|u(\tau-s) - \phi\|^2) \\ &\quad + 2(1-\delta) (\|\Delta(u^\rho(\tau) - \phi)\|^2 + \|\Delta(u(\tau-s) - \phi)\|^2), \end{aligned}$$

which together with (7.16) shows that there is $\rho_5 \leq \rho_4$ such that for $\forall \rho \leq \rho_5$, $s \in [-\rho, 0]$,

$$\|\widehat{Y}(\tau-s)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}^2 \leq c \sup_{-\rho \leq s \leq 0} \|Y_0^\rho(s) - Y_0\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} + c\eta. \quad (7.21)$$

It follows from (7.20) and (7.21) that for $\forall \rho \leq \rho_5$, $t \in [\tau, \tau + T]$ with $t > \tau - s$ and $s \in [-\rho, 0]$,

$$\begin{aligned} & \|Y^\rho(t+s, \tau, \omega, Y_0^\rho) - Y(t, \tau, \omega, Y_0)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \\ & \leq c \sup_{-\rho \leq s \leq 0} \|Y_0^\rho(s) - Y_0\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} + c\eta. \end{aligned} \quad (7.22)$$

For $t \in [\tau, \tau - s]$, we set $r = t - \tau$. Thus, we have $\tau - \rho \leq t + s \leq \tau$, $t = r + \tau$. One can easily obtain that there is $\rho_6 \leq \rho_5$ such that for $\forall \rho \leq \rho_6$, $t \in [\tau, \tau - s]$,

$$\begin{aligned} & \|Y^\rho(t+s, \tau, \omega, Y_0^\rho) - Y(t, \tau, \omega, Y_0)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \\ & \leq c \sup_{-\rho \leq s \leq 0} \|Y_0^\rho(s) - Y_0\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} + c\eta. \end{aligned} \quad (7.23)$$

Hence, by (7.22) and (7.23) we obtain that

$$\begin{aligned} & \|Y^\rho(t+s, \tau, \omega, Y_0^\rho) - Y(t, \tau, \omega, Y_0)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \\ & \leq c \sup_{-\rho \leq s \leq 0} \|Y_0^\rho(s) - Y_0\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} + c\eta, \end{aligned}$$

which gives (7.6). \square

Lemma 7.2 Suppose (3.1)–(3.5), (3.7), (3.8), (3.14), and (3.16) hold. If $\rho_n \rightarrow 0$, $Y_n \in \mathcal{A}^{\rho_n}(\tau, \omega)$, then there exists a subsequence $\{Y_{n_m}\}$ of $\{Y_n\}$ and $Y \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that

$$\lim_{m \rightarrow \infty} \sup_{-\rho_{n_m} \leq s \leq 0} \|Y_{n_m}(s) - Y\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} = 0, \quad (7.24)$$

where $Y = (u, v)^\top$.

Proof Denote $\{t_n\}_{n=1}^\infty$ as a sequence and $t_n \rightarrow \infty$, $n \rightarrow \infty$. By the invariance of \mathcal{A}^{ρ_n} , there is $\widehat{Y}_n \in \mathcal{A}^{\rho_n}(\tau - t_n, \theta_{-t_n}\omega)$ such that

$$Y_n = \Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n). \quad (7.25)$$

By (7.2), we have $\widehat{Y}_n \in K^{\rho_n}(\tau - t_n, \theta_{-t_n}\omega)$. By the uniform estimates obtained in Sect. 5, one can verify:

- (i) $\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(0)$ is precompact in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.
- (ii) For given $\forall \eta > 0$, there is $N_1 \geq 1$ such that for $\forall n \geq N_1$, $s \in [-\rho_n, 0]$,

$$\|\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(s) - \Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(0)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \leq \eta.$$

From (i) we see that there is $Y \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that

$$\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(0) \rightarrow Y \quad \text{in } H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n).$$

Thus, there is $N_2 \geq N_1$ such that for $\forall n \geq N_2$,

$$\|\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(0) - Y\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \leq \eta, \quad (7.26)$$

together with (ii), for $\forall n \geq N_2$ and $s \in [-\rho_n, 0]$, we have

$$\begin{aligned} & \|\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(s) - Y\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \\ & \leq \|\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(s) - \Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(0)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \\ & \quad + \|\Phi^{\rho_n}(t_n, \tau - t_n, \theta_{-t_n}\omega, \widehat{Y}_n)(0) - Y\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \leq 2\eta, \end{aligned} \quad (7.27)$$

which along with (7.25) implies (7.24). \square

Finally, we will prove the upper semicontinuity of \mathcal{A}^ρ as $\rho \rightarrow 0$.

Theorem 7.1 Assume that (3.1)–(3.5), (3.7), (3.8), (3.14), and (3.16) hold. Then, for $\forall \tau \in \mathbb{R}$, $\omega \in \Omega$,

$$\lim_{\rho \rightarrow 0} d_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}(\mathcal{A}^\rho(\tau, \omega), \mathcal{A}^0(\tau)) = 0, \quad (7.28)$$

where $d_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}$ is defined for $\forall E \subseteq C_{V_2, V_0}(\mathbb{R}^n)$, $S \subseteq H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ as

$$d_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}(E, S) = \sup_{\varphi \in E} \inf_{x \in S} \sup_{-\rho \leq s \leq 0} \|\varphi(s) - x\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}.$$

Proof Let $Y_0^{\rho_n} \in C_{V_2, V_0}(\mathbb{R}^n)$ and $Y_0 \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ with $\sup_{-\rho_n \leq s \leq 0} \|Y_0^{\rho_n}(s) - Y_0\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \rightarrow 0$ as $n \rightarrow \infty$, $\rho_n \rightarrow 0$. We have from Lemma 7.1 that for $\forall \tau \in \mathbb{R}$, $\omega \in \Omega$, $t \geq \tau$,

$$\sup_{-\rho_n \leq s \leq 0} \|\Phi^{\rho_n}(t, \tau, \omega, Y_0^{\rho_n})(s) - \Phi^0(t, \tau, \omega, Y_0)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (7.29)$$

By (7.4), (7.5), (7.29), and Lemma 7.2 we obtain (7.28) from Theorem 2.1 in [29] immediately. \square

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