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On Fejér's inequality: generalizations and applications

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Abstract

This paper deals with generalizations and refinements of Fejér's inequality with some new applications. We introduce a real mapping $\mathcal{M}_f^{\omega}(t)$ and obtain some its functional properties. We present several inequalities in connection with fractional integrals and their refinements by monotone functions. We also give some inequalities related to Euler's gamma and beta functions. Furthermore, we obtain some upper and lower bounds for generalized logarithmic mean and the expected value of random variables by using the arithmetic mean.

MSC: 26A33; 26A51; 26D10; 26D15

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1 Introduction and preliminaries

Lipót Fejér (1880–1959) in 1906 [18], while studying trigonometric polynomials, discovered the following integral inequalities, which later became known as Fejér's inequalities (in some references, they are called the left and right inequalities):

$$\mathcal{F}\left(\frac{a+b}{2}\right)\int_a^b \mathcal{G}(x)\,dx \leq \int_a^b \mathcal{F}(x)\mathcal{G}(x)\,dx \leq \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}\int_a^b \mathcal{G}(x)\,dx, \tag{1}$$

where $\mathcal{F}$ is a convex function ([37]) in the interval (a,b), and $\mathcal{G}$ is a positive function in the same interval such that

$$\mathcal{G}(a+t)=\mathcal{G}(b-t), \quad 0\leq t\leq \frac{a+b}{2},$$

i.e., $y=\mathcal{G}(x)$ is a symmetric curve with respect to the straight line containing the point $(\frac{a+b}{2},0)$ and is normal to the x-axis.

For more results about the Fejér's inequalities, see [7, 17, 24, 29, 38, 40, 41, 45, 50, 55] and references therein. In fact, Fejér's inequality (1) is the weighted version of celebrated Hermite–Hadamard's inequality for convex function $f:[a,b]\to\mathbb{R}$:

$$\mathcal{F}\left(\frac{a+b}{2}\right)\leq \frac{1}{b-a}\int_a^b \mathcal{F}(x)\,dx \leq \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}. \tag{2}$$

Note that inequality (2) was published by Charles Hermite (1822–1901) in *Mathesis* **(3)** ([23], 1883, p. 82) but nowhere mentioned in the mathematical literature at that time. Ten years later, Jacques Hadamard in 1893 [21] proved (2) and apparently was not aware of Hermite's result. In honor of these two mathematicians, inequality (2) is known as Hermite–Hadamard's inequality. For more historical details, results, and applications about (2), see [9, 10, 14, 26, 30, 32, 34, 39, 43, 59] and references therein.

Definition 1.1 We say that a nonnegative function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex or $f \in SX(h, I)$ if for a nonnegative function $h : (0,1) \subseteq J \subseteq \mathbb{R} \to \mathbb{R}$ ($h \not\equiv 0$) and for all $x, y \in I$ and $\alpha \in (0,1)$, we have

$$\mathcal{F}\bigl(\alpha x + (1-\alpha)y\bigr) \le h(\alpha)\mathcal{F}(x) + h(1-\alpha)\mathcal{F}(y).$$

A function f is said to be h-concave or $f \in SV(h, I)$ if the above inequality is reversed.

The class of h-convex functions includes the classes of Godunova–Levin functions [19] known as $Q(I)$ (see [15, 31]), s-convex functions in the second sense [5] known as K_s^2, and p-convex functions [15] known as $P(I)$ (see [13, 33]). Note that if $h(\alpha) = \alpha$, then all nonnegative convex functions belong to $SX(h, I)$, and all nonnegative concave functions belong to $SV(h, I)$. Also, if $h(\alpha) = \frac{1}{\alpha}$, $h(\alpha) = 1$, and $h(\alpha) = \alpha^s$, where $s \in (0,1)$, then $Q(I) = SX(h, I)$, $P(I) \subseteq SX(h, I)$, and $K_s^2 \subseteq SX(h, I)$, respectively. Note in this paper, the function h is assumed to be integrable on $[0,1]$.

The mapping $\mathcal{M}_f^{\omega}(t)$ For two real numbers $a < b$, consider integrable functions $f : [a,b] \to \mathbb{R}$ and $\omega : [a,b] \to \mathbb{R}^+ \cup \{0\}$. Define the mapping $\mathcal{M}_f^{\omega}(t) : [0,1] \to \mathbb{R}$ as

$$\mathcal{M}_f^{\omega}(t) = \int_a^{m_t(\mathcal{L},\mathcal{R})} f(x)\omega(x)\,dx + \int_{M_t(\mathcal{L},\mathcal{R})}^b f(x)\omega(x)\,dx,$$

where

$$m_t(\mathcal{L},\mathcal{R}) = \min\bigl\{\mathcal{L}(t), \mathcal{R}(t)\bigr\}, \qquad M_t(\mathcal{L},\mathcal{R}) = \max\bigl\{\mathcal{L}(t), \mathcal{R}(t)\bigr\},$$

where $\mathcal{L}(t) : [0,1] \to [a,b]$ and $\mathcal{R}(t) : [0,1] \to [a,b]$ are defined as

$$\mathcal{L}(t) = tb + (1-t)a, \qquad \mathcal{R}(t) = ta + (1-t)b$$

for $t \in [0,1]$. Note that

$$\mathcal{M}_f^{1}(t) = \int_a^{m_t(\mathcal{L},\mathcal{R})} f(x)\,dx + \int_{M_t(\mathcal{L},\mathcal{R})}^b f(x)\,dx,$$

where by 1 we mean $\omega \equiv 1$.

We will frequently use the following lemma.

Lemma 1.1 *Consider two functions $f : [a,b] \to \mathbb{R}$ and $\omega : [a,b] \to \mathbb{R}^+ \cup \{0\}$. Also, for any $s \in [0,1]$, define the bifunction $A_s : [a,b] \times [a,b] \to [a,b]$ as $A_s(x,y) = sx + (1-s)y$ for $x, y \in [a,b]$. Then for all $t \in [0,1]$,*

(i) $m_t(\mathcal{L},\mathcal{R}) + M_t(\mathcal{L},\mathcal{R}) = \mathcal{L}(t) + \mathcal{R}(t) = a + b$;
(ii) $M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R}) = |\mathcal{L}(t) - \mathcal{R}(t)| = |1 - 2t|(b - a)$;
(iii) $f(m_t(\mathcal{L},\mathcal{R})) + f(M_t(\mathcal{L},\mathcal{R})) = [f \circ \mathcal{L}](t) + [f \circ \mathcal{R}](t)$;
(iv) $A_s(m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})) + A_s(M_t(\mathcal{L},\mathcal{R}), m_t(\mathcal{L},\mathcal{R})) = a + b$.

If ω is symmetric on $[a,b]$ with respect to $\frac{a+b}{2}$, then:

(v) *It is symmetric on the interval $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$ with respect to $\frac{a+b}{2}$*;
(vi) *We have the following identities*:

$$\begin{aligned}[\omega \circ A_s]\big(m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})\big) &= [\omega \circ A_s]\big(M_t(\mathcal{L},\mathcal{R}), m_t(\mathcal{L},\mathcal{R})\big) \\ &= [\omega \circ A_s]\big(\mathcal{L}(t), \mathcal{R}(t)\big) = [\omega \circ A_s]\big(\mathcal{R}(t), \mathcal{L}(t)\big);\end{aligned}$$

(vii) *We have the following integral equalities*:

$$\int_a^{m_t(\mathcal{L},\mathcal{R})} \omega(x)\,dx = \int_{M_t(\mathcal{L},\mathcal{R})}^b \omega(x)\,dx, \qquad \int_{m_t(\mathcal{L},\mathcal{R})}^{\frac{a+b}{2}} \omega(x)\,dx = \int_{\frac{a+b}{2}}^{M_t(\mathcal{L},\mathcal{R})} \omega(x)\,dx.$$

Proof The proof of (i), (iii), and (iv) is obvious, and for (ii), we can use the identities

$$\min\{x,y\} = \frac{x + y - |y - x|}{2},$$

and

$$\max\{x,y\} = \frac{x + y + |y - x|}{2}.$$

For (v), suppose $a \le m_t(\mathcal{L},\mathcal{R}) \le \frac{a+b}{2} \le M_t(\mathcal{L},\mathcal{R}) \le b$, and so for $x \in [m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$, from (i) we have

$$\omega\big(m_t(\mathcal{L},\mathcal{R}) + M_t(\mathcal{L},\mathcal{R}) - x\big) = \omega(a + b - x) = \omega(x),$$

since ω is symmetric with respect to $\frac{a+b}{2}$. For (vi), we just need to consider the following equalities:

$$\begin{aligned}[\omega \circ A_s]\big(\mathcal{R}(t), \mathcal{L}(t)\big) &= \omega\big(s\big(tb + (1-t)a\big) + (1-s)\big(ta + (1-t)b\big)\big) \\ &= \omega\big(t\big(sb + (1-s)a\big) + (1-t)\big(sa + (1-s)b\big)\big) \\ &= \omega\big((1-t)\big(sb + (1-s)a\big) + t\big(sa + (1-s)b\big)\big) \\ &= \omega\big(s\big(ta + (1-t)b\big) + (1-s)\big(tb + (1-t)a\big)\big) \\ &= [\omega \circ A_s]\big(\mathcal{L}(t), \mathcal{R}(t)\big).\end{aligned}$$

Finally, for (vii), it suffices to use the change of variable $u = a + b - x$ and (i). □

By Lemma 1.1 we obtain the following basic properties for the mapping $\mathcal{M}_f^{\omega}(t)$.

Proposition 1 *Consider two functions $f : [a,b] \to \mathbb{R}$ and $\omega : [a,b] \to \mathbb{R}^+ \cup \{0\}$. Then*:

(i) *For all* $t \in [0,1]$,

$$\mathcal{M}_f^{\omega}(t) = \mathcal{M}_f^{\omega}(1-t),$$

which shows that $\mathcal{M}_f^{\omega}(t)$ *is symmetric on* $[a,b]$ *with respect to* $\frac{1}{2}$.

(ii) *For symmetric* ω *on* $[a,b]$ *with respect to* $\frac{a+b}{2}$ *and* $p,q \geq 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *we have*

$$\left|\mathcal{M}_f^{\omega}(t)\right| \leq \|f\|_p\|\omega\|_q.$$

Also, if $m_t(\mathcal{L},\mathcal{R}) = \mathcal{L}(t)$, *then*

$$\left|\mathcal{M}_f^{\omega}(t)\right| \leq \left(\frac{1}{2}\right)^{\frac{1}{q}}\left[t(b-a)\right]^{\frac{1}{p}}\|\omega\|_q\|f\|_{\infty},$$

and if $m_t(\mathcal{L},\mathcal{R}) = \mathcal{R}(t)$, *then*

$$\left|\mathcal{M}_f^{\omega}(t)\right| \leq \left(\frac{1}{2}\right)^{\frac{1}{q}}\left[(1-t)(b-a)\right]^{\frac{1}{p}}\|\omega\|_q\|f\|_{\infty}.$$

(iii) *Suppose that the function* $(f\omega)(x) = f(x)\omega(x)$ *is convex on* $[a,b]$. *If* $m_t(\mathcal{L},\mathcal{R}) = \mathcal{R}(t)$ *for some* $t \in [0,1)$, *then the function* $\frac{\mathcal{M}_f^{\omega}(t)}{1-t}$ *is convex. Also, if* $m_t(\mathcal{L},\mathcal{R}) = \mathcal{L}(t)$ *for some* $t \in (0,1]$, *then the function* $\frac{\mathcal{M}_f^{\omega}(t)}{t}$ *is convex.*

(iv) *Suppose that* f *and* ω *are two continuous functions on* $[a,b]$. *If* f *is nonnegative (nonpositive) on* $[a,b]$, *then the function* $\mathcal{M}_f^{\omega}(t)$ *is increasing (decreasing) on* $[0,\frac{1}{2})$ *and is decreasing (increasing) on* $(\frac{1}{2},1]$. *Also,* $\mathcal{M}_f^{\omega}(t)$ *has a relative extreme point at* $t=\frac{1}{2}$. *If* $\omega \not\equiv 0$, *then corresponding to any* $x \in [a,b]\setminus\{\frac{a+b}{2}\}$ *satisfying*

$$f(x)+f(a+b-x) = 0,$$

there exists a critical point for $\mathcal{M}_f^{\omega}(t)$.

Proof (i) This is a consequence of the facts $\mathcal{L}(1-t) = \mathcal{R}(t)$ and $\mathcal{R}(1-t) = \mathcal{L}(t)$.

(ii) Since $m_t(\mathcal{L},\mathcal{R}) \leq \frac{a+b}{2} \leq M_t(\mathcal{L},\mathcal{R})$, using Hölder's inequality [37] and statements (iii) and (vii) in Lemma 1.1, we obtain the following inequalities:

$$\begin{aligned}
&\left|\mathcal{M}_f^{\omega}(t)\right| \\
&\quad\leq \left(\int_a^{m_t(\mathcal{L},\mathcal{R})}\left|f(x)\right|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^{m_t(\mathcal{L},\mathcal{R})}\left|\omega(x)\right|^q\,dx\right)^{\frac{1}{q}} \\
&\qquad + \left(\int_{M_t(\mathcal{L},\mathcal{R})}^b\left|f(x)\right|^p\,dx\right)^{\frac{1}{p}}\left(\int_{M_t(\mathcal{L},\mathcal{R})}^b\left|\omega(x)\right|^q\,dx\right)^{\frac{1}{q}} \\
&\quad= \left(\int_a^{m_t(\mathcal{L},\mathcal{R})}\left|\omega(x)\right|^q\,dx\right)^{\frac{1}{q}}\left[\left(\int_a^{m_t(\mathcal{L},\mathcal{R})}\left|f(x)\right|^p\,dx\right)^{\frac{1}{p}} + \left(\int_{M_t(\mathcal{L},\mathcal{R})}^b\left|f(x)\right|^p\,dx\right)^{\frac{1}{p}}\right] \\
&\quad\leq \frac{1}{2}\left(\int_a^{m_t(\mathcal{L},\mathcal{R})}\left|\omega(x)\right|^q\,dx\right)^{\frac{1}{q}}\left[\left(\int_a^{m_t(\mathcal{L},\mathcal{R})}\left|f(x)\right|^p\,dx\right)^{\frac{1}{p}} + \left(\int_{M_t(\mathcal{L},\mathcal{R})}^b\left|f(x)\right|^p\,dx\right)^{\frac{1}{p}}\right].
\end{aligned}$$

This proves (ii).

(iii) We just prove the first part. Considering the changes of variable $x = ta + (1-t)u$ and $x = tb + (1-t)u$ in two integrals of $\mathcal{H}(t) = \frac{\mathcal{M}_f^{\omega}(t)}{1-t}$, we obtain that

$$\mathcal{H}(t) = \int_a^b (f\omega)\big(ta + (1-t)x\big)\,dx + \int_a^b (f\omega)\big(tb + (1-t)x\big)\,dx.$$

Now for $t_1, t_2 \in [0,1)$ and nonnegative α, β with $\alpha + \beta = 1$, we have

$$\begin{aligned}
\mathcal{H}(\alpha t_1 + \beta t_2) &= \int_a^b (f\omega)\big((\alpha t_1 + \beta t_2)a + \big(1-(\alpha t_1 + \beta t_2)\big)x\big)\,dx \\
&\quad + \int_a^b (f\omega)\big((\alpha t_1 + \beta t_2)b + \big(1-(\alpha t_1 + \beta t_2)\big)x\big)\,dx \\
&= \int_a^b (f\omega)\big(\alpha\big(t_1 a + (1-t_1)x\big)\big)\,dx + \int_a^b (fw)\big(\beta\big(t_2 a + (1-t_2)x\big)\big)\,dx \\
&\quad + \int_a^b (f\omega)\big(\alpha\big(t_1 b + (1-t_1)x\big)\big)\,dx + \int_a^b (fw)\big(\beta\big(t_2 b + (1-t_2)x\big)\big)\,dx \\
&\le \alpha\left[\int_a^b (f\omega)\big(t_1 a + (1-t_1)x\big)\,dx + \int_a^b (f\omega)\big(t_1 b + (1-t_1)x\big)\,dx\right] \\
&\quad + \beta\left[\int_a^b (f\omega)\big(t_2 a + (1-t_2)x\big)\,dx + \int_a^b (f\omega)\big(t_2 b + (1-t_2)x\big)\,dx\right] \\
&= \alpha\mathcal{H}(t_1) + \beta\mathcal{H}(t_2).
\end{aligned}$$

(iv) It suffices to apply the following result, which is obtained by using Leibniz integral rule [35] along with the fact that ω is symmetric on $[a,b]$ with respect to $\frac{a+b}{2}$:

$$\frac{1}{b-a}\frac{d\mathcal{M}_f^{\omega}}{dt}(t) = \begin{cases} [\omega\circ\mathcal{L}](t)\{[f\circ\mathcal{L}](t) + [f\circ\mathcal{R}](t)\}, & t\in[0,\frac{1}{2}), \\ -[\omega\circ\mathcal{R}](t)\{[f\circ\mathcal{L}](t) + [f\circ\mathcal{R}](t)\}, & t\in(\frac{1}{2},1). \end{cases} \qquad \square$$

2 Generalization and refinement of Fejér's inequality

The following result presents a new and generalized type of the celebrated Fejér's inequality in connection with h-convex functions.

Theorem 2.1 *Consider two integrable functions* $f : [a,b] \to \mathbb{R}$ *and* $w : [a,b] \to \mathbb{R}^+ \cup \{0\}$ *such that* f *is* h*-convex and* ω *is symmetric with respect to* $\frac{a+b}{2}$. *For all* $t \in [0,1]$, *we have the following inequality*:

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right)\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \omega(x)\,dx \qquad (3)\\
&\le \int_a^b f(x)\omega(x)\,dx - \mathcal{M}_f^{\omega}(t) \\
&\le \frac{|\mathcal{R}(t)-\mathcal{L}(t)|[f\circ\mathcal{L}](t) + [f\circ\mathcal{R}](t)}{(\mathcal{L}(t)-\mathcal{R}(t))}\int_{\mathcal{R}(t)}^{\mathcal{L}(t)} h\left(\frac{x-\mathcal{R}(t)}{\mathcal{L}(t)-\mathcal{R}(t)}\right)\omega(x)\,dx \\
&= \frac{|\mathcal{R}(t)-\mathcal{L}(t)|([f\circ\mathcal{L}](t) + [f\circ\mathcal{R}](t)])}{(\mathcal{R}(t)-\mathcal{L}(t))}\int_{\mathcal{L}(t)}^{\mathcal{R}(t)} h\left(\frac{x-\mathcal{L}(t)}{\mathcal{R}(t)-\mathcal{L}(t)}\right)\omega(x)\,dx.
\end{aligned}$$

Proof According Lemma 1.1, we know that for any $t \in [0,1]$, the function ω is symmetric on the interval $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$ with respect to

$$\frac{m_t(\mathcal{L},\mathcal{R}) + M_t(\mathcal{L},\mathcal{R})}{2} = \frac{a+b}{2}.$$

Since f is an h-convex function on $[a,b]$, this property is induced to $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$ for any $t \in [0,1]$. So if we consider Theorems 3 and 5 in [4] in the case that f is h-convex on $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$ and ω is symmetric on interval $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$ with respect to $\frac{a+b}{2}$, then we have

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})} f\left(\frac{m_t(\mathcal{L},\mathcal{R}) + M_t(\mathcal{L},\mathcal{R})}{2}\right) \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \omega(x)\,dx \\
&\quad\leq \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(x)\omega(x)\,dx \\
&\quad\leq \big[M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big]\big[f\big(m_t(\mathcal{L},\mathcal{R})\big) + f\big(M_t(\mathcal{L},\mathcal{R})\big)\big] \\
&\qquad\times \int_0^1 h(s)[\omega \circ A_s]\big(m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})\big)\,ds.
\end{aligned}$$

Now statements (ii), (iii), and (v) in Lemma 1.1 imply that

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \omega(x)\,dx \\
&\quad\leq \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(x)\omega(x)\,dx \\
&\quad\leq \big|\mathcal{R}(t) - \mathcal{L}(t)\big|\big([f\circ\mathcal{L}](t) + [f\circ\mathcal{R}](t)\big) \int_0^1 h(s)[\omega\circ A_s]\big(\mathcal{L}(t),\mathcal{R}(t)\big)\,ds \\
&\quad= \big|\mathcal{R}(t) - \mathcal{L}(t)\big|\big([f\circ\mathcal{L}](t) + [f\circ\mathcal{R}](t)\big) \int_0^1 h(s)[\omega\circ A_s]\big(\mathcal{R}(t),\mathcal{L}(t)\big)\,ds.
\end{aligned} \tag{4}$$

By the definition of mapping $\mathcal{M}_f^{\omega}$ it is not hard to see that the following identity holds:

$$\int_a^b f(x)\omega(x)\,dx - \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(x)\omega(x)\,dx = \mathcal{M}_f^{\omega}(t) \tag{5}$$

for all $t \in [0,1]$. On the other hand, apply the changes of variable $u = A_s(\mathcal{L}(t),\mathcal{R}(t))$ or $u = A_s(\mathcal{R}(t),\mathcal{L}(t))$ in two last integrals in (4) and consider that

$$\begin{aligned}
&\frac{1}{\mathcal{L}(t) - \mathcal{R}(t)} \int_{\mathcal{R}(t)}^{\mathcal{L}(t)} h\left(\frac{x - \mathcal{R}(t)}{\mathcal{L}(t) - \mathcal{R}(t)}\right)\omega(x)\,dx \\
&\quad= \frac{1}{\mathcal{R}(t) - \mathcal{L}(t)} \int_{\mathcal{L}(t)}^{\mathcal{R}(t)} h\left(\frac{x - \mathcal{L}(t)}{\mathcal{R}(t) - \mathcal{L}(t)}\right)\omega(x)\,dx.
\end{aligned} \tag{6}$$

Finally, by the above explanations and using (5) and (6) in (4), we get the desired result. □

Corollaries and remarks

Multiplying the inequalities (a consequence of the h-convexity of f and some statements in Lemma 1.1)

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{A_s(m_t(\mathcal{L},\mathcal{R}),M_t(\mathcal{L},\mathcal{R})) + A_s(M_t(\mathcal{L},\mathcal{R}),m_t(\mathcal{L},\mathcal{R}))}{2}\right) \\ &\le h\left(\frac{1}{2}\right)\left[f\circ A_s\big(m_t(\mathcal{L},\mathcal{R}),M_t(\mathcal{L},\mathcal{R})\big) + f\circ A_s\big(M_t(\mathcal{L},\mathcal{R}),m_t(\mathcal{L},\mathcal{R})\big)\right] \\ &\le h\left(\frac{1}{2}\right)H(s)\left[f\big(m_t(\mathcal{L},\mathcal{R})\big) + f\big(M_t(\mathcal{L},\mathcal{R})\big)\right] \\ &= h\left(\frac{1}{2}\right)H(s)\big([f\circ\mathcal{L}](t) + [f\circ\mathcal{R}](t)\big), \end{aligned}$$

by $[\omega\circ A_s](m_t(\mathcal{L},\mathcal{R}),M_t(\mathcal{L},\mathcal{R}))$ $(s,t\in[0,1])$ and then integrating with respect to variable $s\in[0,1]$, we obtain the following presentation of generalized Fejér's inequality:

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)\,dx \\ &\le \int_a^b f(x)\omega(x)\,dx - \mathcal{M}_f^{\omega}(t) \\ &\le \frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{2}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}H\left(\frac{M_t(\mathcal{L},\mathcal{R})-x}{M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})}\right)\omega(x)\,dx \\ &= \frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{2}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}H\left(\frac{x-m_t(\mathcal{L},\mathcal{R})}{M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})}\right)\omega(x)\,dx, \end{aligned} \tag{7}$$

where $H(s)=h(s)+h(1-s)$, $s\in[0,1]$. Since

$$\begin{aligned} &\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}H\left(\frac{M_t(\mathcal{L},\mathcal{R})-x}{M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})}\right)\omega(x)\,dx \\ &= 2\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}h\left(\frac{M_t(\mathcal{L},\mathcal{R})-x}{M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})}\right)\omega(x)\,dx, \end{aligned}$$

we conclude that (3) and (7) are equivalent, and the difference is in presentation and consequences.

Inequalities (3) and (7) generalize many Fejér-type inequalities obtained for h-convex functions in the literature. In these inequalities, taking $h(s)=s^{\alpha}$ $(\alpha\in(0,1])$, $h(s)=\frac{1}{s}$ $(s\in(0,1))$, $h(s)=1$, and $h(s)=s$, we obtain generalized Fejér-type inequalities for s-convex functions in the second sense, Godunova–Levin functions, P-functions, and convex functions, respectively. However, setting $h(s)=s$ in (3) and (7), we get the following inequalities, respectively:

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)\,dx \\ &\le \int_a^b f(x)\omega(x)\,dx - \mathcal{M}_f^{\omega}(t) \end{aligned} \tag{8}$$

$$\leq \frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{|\mathcal{R}(t)-\mathcal{L}(t)|}\int_{\mathcal{L}(t)}^{\mathcal{R}(t)}\big(x-\mathcal{L}(t)\big)\omega(x)\,dx$$
$$=\frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{|\mathcal{L}(t)-\mathcal{R}(t)|}\int_{\mathcal{R}(t)}^{\mathcal{L}(t)}\big(x-\mathcal{R}(t)\big)\omega(x)\,dx,$$

and

$$f\left(\frac{a+b}{2}\right)\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)\,dx\leq\int_a^b f(x)\omega(x)\,dx-\mathcal{M}_f^{\omega}(t)$$
$$\leq\frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{2}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)\,dx. \tag{9}$$

Inequality (8) is a new generalized Fejér-type inequality, and inequality (9) is a straight generalization of the classic Fejér's inequality related to the convex functions.

Inequalities (3) and (7) coincide for the case $\omega\equiv 1$. However, this gives a new generalized form of the Hermite–Hadamard inequality related to the class of h-convex functions and its subclasses as well (see also [27, 33, 46, 47]):

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\leq\frac{1}{(b-a)|1-2t|}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}f(x)\,dx \tag{10}$$
$$\leq\big([f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)\big)\int_0^1 h(s)\,ds$$
$$=\left(\frac{[f\circ\mathcal{L}](t)+[f\circ\mathcal{R}](t)}{2}\right)\int_0^1 H(s)\,ds.$$

If we set $t=0,1$ in (3) ($\mathcal{M}_f^{\omega}(0)=\mathcal{M}_f^{\omega}(1)=0$), then we recapture the following Fejér-type inequality related to the h-convex functions obtained in [4]:

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\int_a^b\omega(x)\,dx\leq\int_a^b f(x)\omega(x)\,dx \tag{11}$$
$$\leq(b-a)\big[f(a)+f(b)\big]\int_0^1 h(s)\omega\big(sa+(1-s)b\big)\,ds,$$

and also with this assumption in (7), we obtain a new h-convex version of Fejér's inequality:

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\int_a^b\omega(x)\,dx\leq\int_a^b f(x)\omega(x)\,dx \tag{12}$$
$$\leq\frac{f(a)+f(b)}{2}\int_a^b H\left(\frac{b-x}{b-a}\right)\omega(x)\,dx$$
$$=\frac{f(a)+f(b)}{2}\int_a^b H\left(\frac{x-a}{b-a}\right)\omega(x)\,dx.$$

Taking $\omega\equiv 1$ in (11) and (12), we recapture the following result obtained in [46]:

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\big[f(a)+f(b)\big]\int_0^1 h(s)\,ds,$$

which is the Hermite–Hadamard inequality related to h-convex functions.

Now for $n \in \mathbb{N}$, $n \geq 3$, consider the following relations:

$$a = \frac{a+(n-1)a}{n} \leq \frac{b+(n-1)a}{n} = \frac{a+(n-1)b}{n} + \frac{n-2}{n}(a-b) \leq \frac{a+(n-1)b}{n} \leq b,$$

and

$$\omega\left(\frac{a+(n-1)a}{n} + \frac{b+(n-1)a}{n} - x\right) = \omega(a+b-x) = \omega(x),$$

showing that ω is symmetric on $[\frac{b+(n-1)a}{n} + \frac{a+(n-1)b}{n}]$ with respect to $\frac{a+b}{2}$. So if we set $t = \frac{1}{n}$ in (3), then we have

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \int_{\frac{b+(n-1)a}{n}}^{\frac{a+(n-1)b}{n}} \omega(x)\,dx \\ &\leq \int_{\frac{b+(n-1)a}{n}}^{\frac{a+(n-1)b}{n}} f(x)\omega(x)\,dx \\ &\leq \frac{n-2}{n}(b-a)\left[f\left(\frac{b+(n-1)a}{n}\right) + f\left(\frac{a+(n-1)b}{n}\right)\right] \\ &\quad \times \int_0^1 h(s)[\omega \circ A_s]\left(\frac{b+(n-1)a}{n}, \frac{a+(n-1)b}{n}\right) ds, \end{aligned} \tag{13}$$

and also in (7), we have

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \int_{\frac{b+(n-1)a}{n}}^{\frac{a+(n-1)b}{n}} \omega(x)\,dx \\ &\leq \int_{\frac{b+(n-1)a}{n}}^{\frac{a+(n-1)b}{n}} f(x)\omega(x)\,dx \\ &\leq \frac{f(\frac{b+(n-1)a}{n}) + f(\frac{a+(n-1)b}{n})}{2} \int_{\frac{b+(n-1)a}{n}}^{\frac{a+(n-1)b}{n}} H\left(\left(\frac{n}{n-2}\right)\left(\frac{b-x}{b-a} - \frac{1}{n}\right)\right)\omega(x)\,dx. \end{aligned} \tag{14}$$

Inequalities (13) and (14) cover many Hermite–Hadamard-type and Fejér-type inequalities of this kind for all $n \geq 3$. For example, consider $n = 3, 4$ and $h(s) = s$ in (13) and (14) to obtain the following Fejér-type inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_{\frac{b+2a}{3}}^{\frac{a+2b}{3}} \omega(x)\,dx &\leq \int_{\frac{b+2a}{3}}^{\frac{a+2b}{3}} f(x)\omega(x)\,dx \\ &\leq \left[f\left(\frac{b+2a}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] \int_{\frac{b+2a}{3}}^{\frac{a+2b}{3}} \omega(x)\,dx, \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_{\frac{b+3a}{4}}^{\frac{a+3b}{4}} \omega(x)\,dx &\leq \int_{\frac{b+3a}{4}}^{\frac{a+3b}{4}} f(x)\omega(x)\,dx \\ &\leq \frac{f(\frac{b+3a}{4}) + f(\frac{a+3b}{4})}{2} \int_{\frac{b+3a}{4}}^{\frac{a+3b}{4}} \omega(x)\,dx. \end{aligned}$$

Inequalities of this kind of can be found in [12, 53, 54] and references therein. For different types of $h(s)$, ω, and $n > 4$, we may obtain many inequalities of Hermite–Hadamard and Fejér types.

In a more particular case, if we consider $\omega \equiv 1$ and $h(s) = s$ simultaneously in (3) and (7), then we have the following result:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)|1-2t|} \int_{m_t(\mathcal{L},\mathcal{R})}^{m_t(\mathcal{L},\mathcal{R})} f(x)\,dx \leq \frac{1}{2}\big([f \circ \mathcal{L}](t) + [f \circ \mathcal{R}](t)\big) \tag{15}$$

for $t \in [0,1] \setminus \frac{1}{2}$, which is a new presentation for Hermite–Hadamard's inequality dependent on variable t. In particular, for $t = 0, 1$, we recapture the classical Hermite–Hadamard inequality. Also, for any $n \in \mathbb{N} \setminus \{1,2\}$, we obtain that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{n}{(b-a)(n-2)} \int_{\frac{b+(n-1)a}{n}}^{\frac{a+(n-1)b}{n}} f(x)\,dx \\ &\leq \frac{1}{2}\left[f\left(\frac{b+(n-1)a}{n}\right) + f\left(\frac{a+(n-1)b}{n}\right)\right], \end{aligned}$$

which gives a Hermite–Hadamard-type inequality depending on n.

We continue that for fixed $t \in [0,1]$ and $x \in [a,b]$, the function

$$\omega(x) = \big(M_t(\mathcal{L},\mathcal{R}) - x\big)\big(x - m_t(\mathcal{L},\mathcal{R})\big)$$

is symmetric on $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$ with respect to $\frac{a+b}{2}$. Also by Lemma 1.1, we obtain that (the details are omitted)

$$\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \big(M_t(\mathcal{L},\mathcal{R}) - x\big)\big(x - m_t(\mathcal{L},\mathcal{R})\big)\,dx = \frac{|1-2t|^3(b-a)^3}{6}$$

and

$$\int_0^1 h(s)[\omega \circ A_s]\big(\mathcal{L}(t),\mathcal{R}(t)\big)\,ds = |1-2t|^2(b-a)^2 \int_0^1 h(s)s(1-s)\,ds.$$

So if f is an h-convex function on $[a,b]$, then according to Theorem 2.1, we have the following Hermite–Hadamard-type inequality for h-convex functions:

$$\begin{aligned} \frac{1}{12h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{|1-2t|^3(b-a)^3} \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(x)\big(M_t(\mathcal{L},\mathcal{R}) - x\big)\big(x - m_t(\mathcal{L},\mathcal{R})\big)\,dx \\ &\leq \big([f \circ \mathcal{L}](t) + [f \circ \mathcal{R}](t)\big) \int_0^1 h(s)s(1-s)\,ds. \end{aligned}$$

In the particular case of $h(s) = s$ and $t = 0, 1$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{6}{(b-a)^3} \int_a^b f(x)(b-x)(x-a)\,dx \leq \frac{f(a)+f(b)}{2},$$

which was obtained in [11] for convex function f defined on $[a,b]$.

Finally, for a convex function $f : [a,b] \to \mathbb{R}$, by the definition of $\mathcal{M}_f^1(t)$, applying some suitable changes of variable in integrals and using Lemma (1.1), we obtain the following inequality:

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right) - \left|t-\frac{1}{2}\right| \{f(a)+f(b)\} \\
& \leq \int_a^b f(x)\,dx - \left|t-\frac{1}{2}\right| \{[f\circ\mathcal{L}](t) + [f\circ\mathcal{R}](t)\} \\
& \leq \frac{1}{b-a}\mathcal{M}_f^1(t) \leq t(1-t)\big[f(a)+f(b)\big] + 2\max\{t^2,(1-t)^2\} \int_a^b f(x)\,dx,
\end{aligned}
$$

which gives another refinement for the classical Hermite–Hadamard inequality. We encourage the interested reader to work on other particular cases of inequalities obtained in this section and compare the results with previous ones in the literature.

3 Fractional integrals

In this section, we introduce a new class of fractional integrals. Investigating its properties is the subject of our next works. Here we just consider some particular cases, which are known types of fractional integrals in the literature (see [20, 25, 28, 42, 44]) and find some Hermite–Hadamard-type inequalities for them by using generalized Fejér inequality obtained in the previous section.

For $t \in [0,1] \setminus \{\frac{1}{2}\}$, consider a bifunction $G : [m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})] \times [m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})] \to \mathbb{R}^+ \cup \{0\}$ and define the following class of fractional integrals:

$$
\mathcal{F}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x) = \int_{m_t(\mathcal{L},\mathcal{R})}^x G(x,u)f(u)\,du, \quad x > m_t(\mathcal{L},\mathcal{R}),
$$

and

$$
\mathcal{F}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x) = \int_x^{M_t(\mathcal{L},\mathcal{R})} G(x,u)f(u)\,du, \quad x < M_t(\mathcal{L},\mathcal{R}),
$$

if the above integrals exist.

Now we discuss three special cases of $\mathcal{F}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x)$ and $\mathcal{F}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x)$ and obtain some results in connection with Theorem 2.1.

(1) For $\alpha > 0$, considering

$$
G(x,u) = \frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}|x-u|\right), \quad x,u \in \big[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})\big],
$$

in $\mathcal{F}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x)$ and $\mathcal{F}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x)$, we obtain the following class of fractional integrals:

$$
\mathcal{I}^\alpha_{m_t(\mathcal{L},\mathcal{R})^+}[f](x) = \frac{1}{\alpha}\int_{m_t(\mathcal{L},\mathcal{R})}^x \exp\left(-\frac{1-\alpha}{\alpha}(x-u)\right)f(u)\,du, \quad x > m_t(\mathcal{L},\mathcal{R}),
$$

and

$$
\mathcal{I}^\alpha_{M_t(\mathcal{L},\mathcal{R})^-}[f](x) = \frac{1}{\alpha}\int_x^{M_t(\mathcal{L},\mathcal{R})} \exp\left(-\frac{1-\alpha}{\alpha}(u-x)\right)f(u)\,du, \quad x < M_t(\mathcal{L},\mathcal{R}),
$$

which generalize $\mathcal{I}^{\alpha}_{a^+}[f](x)$ and $\mathcal{I}^{\alpha}_{b^-}[f](x)$, presented and discussed in [2] (consider $t = 0, 1$ and $\alpha \in (0,1)$ in the above integrals).

To obtain Hermite–Hadamard's inequality related to the classes $\mathcal{I}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x)$ and $\mathcal{I}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x)$, in Theorem 2.1, consider

$$\omega(x) = \frac{1}{\alpha}\left[\exp\left(-\frac{1-\alpha}{\alpha}\big(M_t(\mathcal{L},\mathcal{R}) - x\big)\right) + \exp\left(-\frac{1-\alpha}{\alpha}\big(x - m_t(\mathcal{L},\mathcal{R})\big)\right)\right]$$

for $x \in [m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$ and also

$$\rho(t) = -\frac{(1-\alpha)|1-2t|(b-a)}{\alpha}, \quad t \in [0,1) \setminus \left\{\frac{1}{2}\right\}.$$

Obviously, ω is a nonnegative function, and using Lemma 1.1, it is not hard to see that $M_t(\mathcal{L},\mathcal{R}) - x = a + b - x - m_t(\mathcal{L},\mathcal{R})$ and $x - m_t(\mathcal{L},\mathcal{R}) = M_t(\mathcal{L},\mathcal{R}) - (a + b - x)$. So

$$\begin{aligned}
\omega(x) &= \frac{1}{\alpha}\left[\exp\left(-\frac{1-\alpha}{\alpha}\big(M_t(\mathcal{L},\mathcal{R}) - x\big)\right) + \exp\left(-\frac{1-\alpha}{\alpha}\big(x - m_t(\mathcal{L},\mathcal{R})\big)\right)\right] \\
&= \frac{1}{\alpha}\left[\exp\left(-\frac{1-\alpha}{\alpha}\big(a + b - x - m_t(\mathcal{L},\mathcal{R})\big)\right)\right. \\
&\quad \left. + \exp\left(-\frac{1-\alpha}{\alpha}\big(M_t(\mathcal{L},\mathcal{R}) - (a + b - x)\big)\right)\right] \\
&= \omega(a + b - x) = w\big(m_t(\mathcal{L},\mathcal{R}) + M_t(\mathcal{L},\mathcal{R}) - x\big),
\end{aligned}$$

which implies that ω is symmetric on $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$ with respect to $\frac{a+b}{2}$. It follows that

$$\begin{aligned}
\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \omega(x)\,dx &= \frac{2}{1-\alpha}\left[1 - \exp\left(-\frac{1-\alpha}{\alpha}\big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)\right)\right] \\
&= \frac{2}{1-\alpha}\left[1 - \exp\left(-\frac{1-\alpha}{\alpha}\big(|1-2t|(b-a)\big)\right)\right] \\
&= \frac{2}{1-\alpha}\big[1 - [\exp \circ \rho](t)\big].
\end{aligned} \tag{16}$$

Also,

$$\int_a^b f(x)\omega(x)\,dx - \mathcal{M}_f^{\omega}(t) = \mathcal{I}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\big(M_t(\mathcal{L},\mathcal{R})\big) + \mathcal{I}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\big(m_t(\mathcal{L},\mathcal{R})\big), \tag{17}$$

and again by Lemma 1.1

$$\begin{aligned}
&\int_0^1 h(s)[\omega \circ A_s]\big(m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})\big)\,ds \\
&\quad = \frac{1}{\alpha}\int_0^1 H(s)\exp\left(-\frac{(1-\alpha)s}{\alpha}(b-a)|1-2t|\right)ds \\
&\quad = \frac{1}{\alpha}\int_0^1 H(s)\big[\exp \circ (s\rho)\big](t)\,ds.
\end{aligned} \tag{18}$$

Finally, from (16)–(18) we have

$$\begin{aligned}
&\frac{1-[\exp\circ\rho](t)}{h(\frac{1}{2})(1-\alpha)} f\left(\frac{a+b}{2}\right) \\
&\quad\leq \mathcal{I}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\big(M_t(\mathcal{L},\mathcal{R})\big) + \mathcal{I}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\big(m_t(\mathcal{L},\mathcal{R})\big) \\
&\quad\leq \frac{|1-2t|(b-a)[f\circ\mathcal{L}(t)+f\circ\mathcal{R}(t)]}{\alpha} \int_0^1 H(s)\big[\exp\circ(s\rho)\big](t)\,ds
\end{aligned} \tag{19}$$

for $t \in [0,1] \setminus \{\frac{1}{2}\}$, which is the h-convex type of Hermite–Hadamard's inequality in connection with this class of fractional integrals. In particular, if we set $h(s) = s$ in (19), then from the fact that

$$\int_0^1 \big[\exp\circ(s\rho)\big](t)\,ds = \frac{\alpha}{(1-\alpha)|1-2t|(b-a)}\big(1-[\exp\circ\rho](t)\big)$$

we obtain

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right) \\
&\quad\leq \frac{(1-\alpha)[\mathcal{I}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f](M_t(\mathcal{L},\mathcal{R})) + \mathcal{I}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f](m_t(\mathcal{L},\mathcal{R}))]}{2(1-[\exp\circ\rho](t))} \\
&\quad\leq \frac{f\circ\mathcal{L}(t)+f\circ\mathcal{R}(t)}{2}
\end{aligned} \tag{20}$$

for $t \in [0,1] \setminus \{\frac{1}{2}\}$, which gives a convex type of Hermite–Hadamard's inequality in connection with $\mathcal{I}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f](M_t(\mathcal{L},\mathcal{R}))$ and $\mathcal{I}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f](m_t(\mathcal{L},\mathcal{R}))$. In a more particular case for $t = 0, 1$ in (20), we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{(1-\alpha)[\mathcal{I}^{\alpha}_{a^+}f(b) + \mathcal{I}^{\alpha}_{b^-}f(a)]}{2(1-\exp(-\frac{(1-\alpha)(b-a)}{\alpha}))} \leq \frac{f(a)+f(b)}{2},$$

which was obtained in [2] (Theorem 1). Finally, letting $\alpha \to 1^-$ in (20), we recapture (15), since

$$\lim_{\alpha\to 1^-} \frac{1-\alpha}{2(1-\exp\circ(\rho)](t))} = \frac{1}{2(b-a)|1-2t|},$$

for all $t \in [0,1] \setminus \{\frac{1}{2}\}$.

(2) In $\mathcal{F}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x)$ and $\mathcal{F}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x)$ for $\alpha > 0$, consider

$$G(x,u) = \frac{1}{\Gamma(\alpha)}|x-u|^{\alpha-1}, \quad x,u \in \big[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})\big].$$

So we obtain the following generalized Riemann–Liouville fractional integrals:

$$\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_{m_t(\mathcal{L},\mathcal{R})}^{x} (x-u)^{\alpha-1}f(u)\,du, \quad x > m_t(\mathcal{L},\mathcal{R}),$$

and

$$\mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^{M_t(\mathcal{L},\mathcal{R})}(u-x)^{\alpha-1}f(u)\,dt, \quad M_t(\mathcal{L},\mathcal{R}) < x.$$

The fractional integrals $\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}f(x)$ and $\mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}f(x)$ in the particular cases $t = 0, 1$ reduce to $J^{\alpha}_{a^+}f(x)$ and $J^{\alpha}_{b^-}f(x)$, respectively, which are known as the Riemann–Liouville fractional integrals in the literature. Now in Theorem 2.1 consider

$$\omega(x) = \frac{(M_t(\mathcal{L},\mathcal{R})-x)^{\alpha-1} + (x-m_t(\mathcal{L},\mathcal{R}))^{\alpha-1}}{\Gamma(\alpha)}, \quad x \in \big[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})\big].$$

By using Lemma 1.1 it is not hard to see that w is symmetric on $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$ with respect to $\frac{a+b}{2}$ and also is nonnegative. Also, the following results hold:

$$\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)\,dx = \frac{2(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R}))^{\alpha}}{\Gamma(\alpha+1)} = \frac{2(b-a)^{\alpha}|1-2t|^{\alpha}}{\Gamma(\alpha+1)},$$
$$\int_a^b f(x)\omega(x)\,dx - \mathcal{M}_f^{\omega}(t) = \mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\big(M_t(\mathcal{L},\mathcal{R})\big) + \mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\big(m_t(\mathcal{L},\mathcal{R})\big),$$

and

$$\int_0^1 h(s)[\omega\circ A_s]\big(m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})\big)\,ds = \big(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})\big)^{\alpha-1}\int_0^1 H(s)s^{\alpha-1}\,ds.$$

The above results altogether imply that

$$\begin{aligned}
&\frac{1}{h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \\
&\quad\le \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}|1-2t|^{\alpha}}\big[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\big(M_t(\mathcal{L},\mathcal{R})\big) + \mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\big(m_t(\mathcal{L},\mathcal{R})\big)\big] \\
&\quad\le \alpha\big[f\circ\mathcal{L}(t) + f\circ\mathcal{R}(t)\big]\int_0^1 H(s)s^{\alpha-1}\,ds
\end{aligned} \tag{21}$$

for $t \in [0,1]\setminus\{\frac{1}{2}\}$. In the case $h(s) = s$, from (21) we get the following inequality, which is a generalization of inequality (2.1) obtained in [48]:

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}|1-2t|^{\alpha}}\big[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\big(M_t(\mathcal{L},\mathcal{R})\big) + \mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\big(m_t(\mathcal{L},\mathcal{R})\big)\big] \\
&\le \frac{f\circ\mathcal{L}(t) + f\circ\mathcal{R}(t)}{2}.
\end{aligned}$$

Also, for $\alpha = 1$, we obtain a generalization of inequality (2.1) presented in [46]:

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \le \frac{1}{(b-a)|1-2t|}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}f(x)\,dx \le \big[f\circ\mathcal{L}(t) + f\circ\mathcal{R}(t)\big]\int_0^1 h(s)\,ds.$$

Furthermore, for $t = 0, 1$ in (21), we have the following Hermite–Hadamard-type inequality via Riemann–Liouville fractional integrals, which is comparable with inequality (6) in

[56]:

$$\frac{1}{h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathcal{J}^{\alpha}_{a^+}[f](a)+\mathcal{J}^{\alpha}_{b^-}[f](b)\right]\le\alpha\left[f(a)+f(b)\right]\int_0^1 H(s)s^{\alpha-1}\,ds.$$

Also, for

$$\omega(u)=G\left(\frac{a+b}{2},u\right)=\frac{|u-\frac{a+b}{2}|^{\alpha-1}}{\Gamma(\alpha)},\quad u\in[a,b],$$

in $\mathcal{F}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x)$ and $\mathcal{F}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x)$, by Theorem 2.1 we have another inequality related to the classical Riemann–Liouville fractional integrals with respect to the midpoint of the interval $[a,b]$:

$$\begin{aligned}\frac{1}{2^{\alpha}h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)&\le\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathcal{J}^{\alpha}_{a^+}[f]\left(\frac{a+b}{2}\right)+\mathcal{J}^{\alpha}_{a^+}[f]\left(\frac{a+b}{2}\right)\right]\\&\le\alpha\left[f(a)+f(b)\right]\int_0^1 h(s)\left|\frac{1}{2}-s\right|^{\alpha-1}ds.\end{aligned}$$

Now for the particular case $h(s)=s$, we get

$$f\left(\frac{a+b}{2}\right)\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\mathcal{J}^{\alpha}_{a^+}[f]\left(\frac{a+b}{2}\right)+\mathcal{J}^{\alpha}_{a^+}[f]\left(\frac{a+b}{2}\right)\right]\le\frac{f(a)+f(b)}{2},$$

which is equivalent to inequality (2.1) (obtained for positive functions) in [49].

(3) If we consider

$$G(x,u)=\frac{1}{\Gamma(\alpha)}|x-u|^{\alpha-1}\ln^{\beta}\left(\frac{\gamma}{|x-u|}\right),\quad x,u\in\left[m_t(\mathcal{L},\mathcal{R}),M_t(\mathcal{L},\mathcal{R})\right],$$

in $\mathcal{F}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x)$ and $\mathcal{F}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x)$ for $\alpha>0$, $\gamma>M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})$, and $\beta\ge 0$, then we obtain the following class of fractional integrals, which in the particular case $t=0,1$ have been presented and discussed in [44] as operators with power-logarithmic kernels:

$$\mathcal{K}^{\alpha,\beta}_{m_t(\mathcal{L},\mathcal{R})^+}[f](x)=\frac{1}{\Gamma(\alpha)}\int_{m_t(\mathcal{L},\mathcal{R})}^{x}(x-u)^{\alpha-1}\ln^{\beta}\left(\frac{\gamma}{x-u}\right)f(u)\,du,\quad x>m_t(\mathcal{L},\mathcal{R}),$$

and

$$\mathcal{K}^{\alpha,\beta}_{M_t(\mathcal{L},\mathcal{R})^-}[f](x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{M_t(\mathcal{L},\mathcal{R})}(u-x)^{\alpha-1}\ln^{\beta}\left(\frac{\gamma}{u-x}\right)f(u)\,du,\quad x<M_t(\mathcal{L},\mathcal{R}).$$

Here we obtain a Hermite–Hadamard-type inequality for fractional integrals $\mathcal{K}^{\alpha,1}_{a^+}[f](x)$ and $\mathcal{K}^{\alpha,1}_{a^+}[f](x)$ as a particular case of the above fractional integrals. For any $x\in[m_t(\mathcal{L},\mathcal{R}),M_t(\mathcal{L},\mathcal{R})]$, define

$$\omega(x)=\frac{(M_t(\mathcal{L},\mathcal{R})-x)^{\alpha-1}\ln(\frac{\gamma}{M_t(\mathcal{L},\mathcal{R})-x})+(x-m_t(\mathcal{L},\mathcal{R}))^{\alpha-1}\ln(\frac{\gamma}{x-m_t(\mathcal{L},\mathcal{R})})}{\Gamma(\alpha)}.$$

The function w is nonnegative on $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$ and symmetric with respect to $\frac{a+b}{2}$. Also, we have the following results:

$$\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \omega(x)\,dx = \frac{2(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R}))^\alpha}{\Gamma(\alpha+1)}\left[\ln\left(\frac{\gamma}{M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})}\right) + \frac{1}{\alpha}\right],$$
$$\int_a^b f(x)\omega(x)\,dx - \mathcal{M}_f^\omega(t) = \mathcal{K}^{\alpha,1}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\big(M_t(\mathcal{L},\mathcal{R})\big) + \mathcal{K}^{\alpha,1}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\big(m_t(\mathcal{L},\mathcal{R})\big),$$

and

$$\int_0^1 h(s)[\omega \circ A_s]\big(\mathcal{R}(t), \mathcal{L}(t)\big)\,ds$$
$$= \frac{(M_t(\mathcal{L},\mathcal{R}) - M_t(\mathcal{L},\mathcal{R}))^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 H(s)s^{\alpha-1}\ln\left(\frac{\gamma}{s(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R}))}\right)ds.$$

So from Theorem 2.1 we have

$$\frac{1}{h(\frac{1}{2})} f\left(\frac{a+b}{2}\right)\left[\ln\left(\frac{\gamma}{M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})}\right) + \frac{1}{\alpha}\right]$$
$$\leq \frac{\Gamma(\alpha+1)}{(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R}))^\alpha}\left[\mathcal{K}^{\alpha,1}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\big(M_t(\mathcal{L},\mathcal{R})\big) + \mathcal{K}^{\alpha,1}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\big(m_t(\mathcal{L},\mathcal{R})\big)\right]$$
$$\leq \alpha\big[f\circ\mathcal{L}(t) + f\circ\mathcal{R}(t)\big]\int_0^1 H(s)s^{\alpha-1}\ln\left(\frac{\gamma}{s(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R}))}\right)ds$$

for $t \in [0,1] \setminus \{\frac{1}{2}\}$. Now if we set $h(s) = s$ in the above inequality, then we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha|1-2t|^\alpha\mathcal{R}}\left[\mathcal{K}^{\alpha,1}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\big(M_t(\mathcal{L},\mathcal{R})\big) + \mathcal{K}^{\alpha,1}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\big(m_t(\mathcal{L},\mathcal{R})\big)\right]$$
$$\leq \frac{[f\circ\mathcal{L}(t) + f\circ\mathcal{R}(t)]}{2},$$

where $\mathcal{R} = \ln(\frac{\gamma}{(b-a)|1-2t|}) + \frac{1}{\alpha}$. In the particular case of $t = 0, 1$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\mathcal{R}}\left[\mathcal{K}^{\alpha,1}_{a^+}[f](b) + \mathcal{K}^{\alpha,1}_{b^-}[f](a)\right] \leq \frac{f(a)+f(b)}{2}.$$

Finally, we recommend the interested readers to study [20, 25, 28, 44] and references therein to get more inequalities and results.

4 Refinements for Hermite–Hadamard's inequality by monotone functions

In this section, we obtain some refinements of Hermite–Hadamard's inequality by using fractional integrals discussed in the previous section, provided that the considered functions are nonnegative and monotone. We focus on the Riemann–Liouville fractional integrals, but results can be extended to many classes of fractional integrals. We need the following result, which is a consequence of Theorem 1 in [1] (see also [6, 22]).

Theorem 4.1 *If f_1 and f_2 are nonnegative increasing functions on* $[0,1]$, *then*

$$\int_0^1 f_1(x)\,dx \int_0^1 f_2(x)\,dx \leq \int_0^1 f_1(x)f_2(x)\,dx.$$

Here we give some refinements for Hermite–Hadamard's inequality by using fractional integrals for h-convex functions.

Theorem 4.2 *Suppose that* $f:[a,b]\to\mathbb{R}$ *is an integrable* h*-convex function and* $t\in[0,1]\setminus\{\frac{1}{2}\}$. *Then*:

(i) *For* $\alpha\geq 1$, *we have the following inequality for nonnegative and increasing* f:

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \\
&\quad\leq \frac{1}{|1-2t|(b-a)}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(u)\,du \\
&\quad\leq \frac{\Gamma(\alpha+1)}{2|1-2t|^{\alpha}(b-a)^{\alpha}}\left[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\big(M_t(\mathcal{L},\mathcal{R})\big)+\mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\big(m_t(\mathcal{L},\mathcal{R})\big)\right] \\
&\quad\leq \alpha\left[\frac{f\circ\mathcal{L}(t)+f\circ\mathcal{R}(t)}{2}\right]\int_0^1 H(s)s^{\alpha-1}\,ds.
\end{aligned}\tag{22}$$

(ii) *For any* $\alpha>0$, *we have*

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \\
&\quad\leq \frac{\Gamma(\alpha+1)}{2|1-2t|^{\alpha}(b-a)^{\alpha}}\left[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\big(M_t(\mathcal{L},\mathcal{R})\big)+\mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\big(m_t(\mathcal{L},\mathcal{R})\big)\right] \\
&\quad\leq \frac{\alpha}{|1-2t|(b-a)}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(u)\,du.
\end{aligned}\tag{23}$$

Proof It is clear that $a\leq m_t(\mathcal{L},\mathcal{R})\leq M_t(\mathcal{L},\mathcal{R})\leq b$ for all $t\in[0,1]$. So

$$f\big(sm_t(\mathcal{L},\mathcal{R})+(1-s)M_t(\mathcal{L},\mathcal{R})\big)\leq h(s)f\big(m_t(\mathcal{L},\mathcal{R})\big)+h(1-s)f\big(M_t(\mathcal{L},\mathcal{R})\big),$$

and

$$f\big(sM_t(\mathcal{L},\mathcal{R})+(1-s)m_t(\mathcal{L},\mathcal{R})\big)\leq h(s)f\big(M_t(\mathcal{L},\mathcal{R})\big)+h(1-s)f\big(m_t(\mathcal{L},\mathcal{R})\big).$$

From these inequalities we have

$$\begin{aligned}
&f\big(sm_t(\mathcal{L},\mathcal{R})+(1-s)M_t(\mathcal{L},\mathcal{R})\big)+f\big(sM_t(\mathcal{L},\mathcal{R})+(1-s)m_t(\mathcal{L},\mathcal{R})\big) \\
&\quad\leq H(s)\big[f\big(m_t(\mathcal{L},\mathcal{R})\big)+f\big(M_t(\mathcal{L},\mathcal{R})\big)\big].
\end{aligned}\tag{24}$$

Multiplying both sides of (24) by $s^{\alpha-1}$, integrating the resulting inequality with respect to variable s over $[0,1]$, and using Theorem 4.1, imply the following result:

$$\begin{aligned}
&\int_0^1 s^{\alpha-1}\,ds\cdot\int_0^1 f\big(sm_t(\mathcal{L},\mathcal{R})+(1-s)M_t(\mathcal{L},\mathcal{R})\big)\,ds \\
&\quad+\int_0^1 s^{\alpha-1}\,ds\cdot\int_0^1 f\big(sM_t(\mathcal{L},\mathcal{R})+(1-s)m_t(\mathcal{L},\mathcal{R})\big)\,ds
\end{aligned}$$

$$\begin{aligned}
&\le \int_0^1 s^{\alpha-1} f\bigl(sm_t(\mathcal{L},\mathcal{R})+(1-s)M_t(\mathcal{L},\mathcal{R})\bigr)\,ds \\
&\quad+\int_0^1 s^{\alpha-1} f\bigl(sM_t(\mathcal{L},\mathcal{R})+(1-s)m_t(\mathcal{L},\mathcal{R})\bigr)\,ds \\
&\le \bigl[f\bigl(m_t(\mathcal{L},\mathcal{R})\bigr)+f\bigl(M_t(\mathcal{L},\mathcal{R})\bigr)\bigr]\int_0^1 s^{\alpha-1}H(s)\,ds.
\end{aligned}$$

Using the changes of variable $u = sm_t(\mathcal{L},\mathcal{R}) + (1-s)M_t(\mathcal{L},\mathcal{R})$ and $u = sM_t(\mathcal{L},\mathcal{R}) + (1-s)m_t(\mathcal{L},\mathcal{R})$, respectively, in the above integrals, we obtain that

$$\begin{aligned}
&\frac{2}{M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(u)\,du \\
&\quad\le \frac{\alpha}{(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R}))^{\alpha}} \\
&\qquad\times\biggl[\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\bigl(M_t(\mathcal{L},\mathcal{R})-u\bigr)^{\alpha-1}f(u)\,du+\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\bigl(u-m_t(\mathcal{L},\mathcal{R})\bigr)^{\alpha-1}f(u)\,du\biggr] \\
&\quad\le \alpha\bigl[f\bigl(m_t(\mathcal{L},\mathcal{R})\bigr)+f\bigl(M_t(\mathcal{L},\mathcal{R})\bigr)\bigr]\int_0^1 H(s)s^{\alpha-1}\,ds.
\end{aligned}$$

Now since f is h-convex on $[m_t(\mathcal{L},\mathcal{R}),M_t(\mathcal{L},\mathcal{R})]$, by the left part of (10), Lemma 1.1, and the definition of $\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f](M_t(\mathcal{L},\mathcal{R}))$ and $\mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f](m_t(\mathcal{L},\mathcal{R}))$ we have

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})}f\biggl(\frac{a+b}{2}\biggr) \\
&\quad\le \frac{1}{M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(u)\,du \\
&\quad\le \frac{\Gamma(\alpha+1)}{2(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R}))^{\alpha}}\bigl[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\bigl(M_t(\mathcal{L},\mathcal{R})\bigr)+\mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\bigl(m_t(\mathcal{L},\mathcal{R})\bigr)\bigr] \\
&\quad\le \alpha\biggl[\frac{f\circ\mathcal{L}(t)+f\circ\mathcal{R}(t)}{2}\biggr]\int_0^1 H(s)s^{\alpha-1}\,ds,
\end{aligned}$$

which is equivalent to inequality (22). It follows that

$$\begin{aligned}
f\biggl(\frac{a+b}{2}\biggr)&=f\biggl(\frac{m_t(\mathcal{L},\mathcal{R})+M_t(\mathcal{L},\mathcal{R})}{2}\biggr) \\
&=f\biggl(\frac{A_s(m_t(\mathcal{L},\mathcal{R}),M_t(\mathcal{L},\mathcal{R}))+A_s(M_t(\mathcal{L},\mathcal{R}),m_t(\mathcal{L},\mathcal{R}))}{2}\biggr) \\
&\le h\biggl(\frac{1}{2}\biggr)\bigl[f\bigl(A_s\bigl(m_t(\mathcal{L},\mathcal{R}),M_t(\mathcal{L},\mathcal{R})\bigr)\bigr)+f\bigl(A_s\bigl(M_t(\mathcal{L},\mathcal{R}),m_t(\mathcal{L},\mathcal{R})\bigr)\bigr)\bigr].
\end{aligned}$$

Multiplying the above inequalities by $s^{\alpha-1}$ and then integrating with respect to variable s over $[0,1]$ lead to

$$\begin{aligned}
\int_0^1 f\biggl(\frac{a+b}{2}\biggr)s^{\alpha-1}\,ds\le h\biggl(\frac{1}{2}\biggr)\biggl[&\int_0^1 f\circ A_s\bigl(M_t(\mathcal{L},\mathcal{R}),m_t(\mathcal{L},\mathcal{R})\bigr)s^{\alpha-1}\,ds \\
&+\int_0^1 f\circ A_s\bigl(m_t(\mathcal{L},\mathcal{R}),M_t(\mathcal{L},\mathcal{R})\bigr)s^{\alpha-1}\,ds\biggr]
\end{aligned}$$

$$\leq h\left(\frac{1}{2}\right)\left[\int_0^1 f\circ A_s\bigl(M_t(\mathcal{L},\mathcal{R}),m_t(\mathcal{L},\mathcal{R})\bigr)\,ds + \int_0^1 f\circ A_s\bigl(m_t(\mathcal{L},\mathcal{R}),M_t(\mathcal{L},\mathcal{R})\bigr)\,ds\right].$$

This implies that

$$\begin{aligned}
&\frac{1}{\alpha h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\\
&\quad\leq \frac{\Gamma(\alpha)}{(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R}))^{\alpha}}\bigl[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\bigl(M_t(\mathcal{L},\mathcal{R})\bigr)+\mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\bigl(m_t(\mathcal{L},\mathcal{R})\bigr)\bigr]\\
&\quad\leq \frac{2}{M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(u)\,du,
\end{aligned}$$

which is equivalent to inequality (23). □

From Theorem 4.2 we have the following refinement for h-convex version of Hermite–Hadamard's inequality.

Corollary 4.1 *Suppose that $f:[a,b]\to\mathbb{R}$ is a nonnegative increasing h-convex function and integrable on $[a,b]$. Consider $t\in[0,1]\setminus\{\frac{1}{2}\}$ and $\alpha\geq 1$. Then*

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \qquad (25)\\
&\quad\leq \frac{1}{|1-2t|(b-a)}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(u)\,du\\
&\quad\leq \frac{\Gamma(\alpha+1)}{2|1-2t|^{\alpha}(b-a)^{\alpha}}\bigl[\mathcal{J}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\bigl(M_t(\mathcal{L},\mathcal{R})\bigr)+\mathcal{J}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\bigl(m_t(\mathcal{L},\mathcal{R})\bigr)\bigr]\\
&\quad\leq \frac{\alpha}{|1-2t|(b-a)}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(u)\,du\leq \alpha\bigl[f\circ\mathcal{L}(t)+f\circ\mathcal{R}(t)\bigr]\int_0^1 h(s)\,ds.
\end{aligned}$$

Remark 1 (i) If h_1 and h_2 are two nonnegative decreasing functions defined on $[0,1]$ with upper bounds B_1 and B_2, respectively, then the functions B_1-h_1 and B_2-h_2 are nonnegative increasing functions defined on $[0,1]$. So we have that

$$\int_0^1 \bigl(B_1-h_1(x)\bigr)\,dx\int_0^1\bigl(B_2-h_2(x)\bigr)\,dx\leq\int_0^1\bigl(B-h_1(x)\bigr)\bigl(B-h_2(x)\bigr)\,dx,$$

which with some calculations gives

$$\int_0^1 h_1(x)\,dx\int_0^1 h_2(x)\,dx\leq\int_0^1 h_1(x)h_2(x)\,dx.$$

This implies that inequality (25) can be obtained if $f:[a,b]\to\mathbb{R}$ is a nonnegative upper bounded decreasing h-convex function, $0<\alpha\leq 1$, and $t\in[0,1]\setminus\{\frac{1}{2}\}$.

(ii) Multiplying both sides of (24) by the nonnegative increasing function

$$g(s)=\exp\left(-\frac{1-\alpha}{\alpha}s\bigl(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})\bigr)\right)$$

for fixed $t \in [0,1] \setminus \{\frac{1}{2}\}$, integrating the resulting inequality with respect to variable s over $[0,1]$, and using Theorem 4.1, we obtain the following refinement of (19):

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \\
&\quad \leq \frac{1}{|1-2t|(b-a)} \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(u)\,du \\
&\quad \leq \frac{1-\alpha}{2(1-[\exp\circ\rho](t))} \bigl[\mathcal{I}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\bigl(M_t(\mathcal{L},\mathcal{R})\bigr) + \mathcal{I}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\bigl(m_t(\mathcal{L},\mathcal{R})\bigr)\bigr] \\
&\quad \leq \frac{\rho(t)}{[\exp\circ\rho](t)-1} \left[\frac{f\circ\mathcal{L}(t)+f\circ\mathcal{R}(t)}{2}\right] \int_0^1 H(s)\bigl[\exp\circ(s\rho)\bigr](t)\,ds.
\end{aligned}$$

In the case $h(s)=s$, we have the following refinement of Hermite–Hadamard's inequality by using another kind of fractional integrals:

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{1}{|1-2t|(b-a)} \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(u)\,du \\
&\leq \frac{1-\alpha}{2(1-[\exp\circ\rho](t))} \bigl[\mathcal{I}^{\alpha}_{m_t(\mathcal{L},\mathcal{R})^+}[f]\bigl(M_t(\mathcal{L},\mathcal{R})\bigr) + \mathcal{I}^{\alpha}_{M_t(\mathcal{L},\mathcal{R})^-}[f]\bigl(m_t(\mathcal{L},\mathcal{R})\bigr)\bigr] \\
&\leq \frac{f\circ\mathcal{L}(t)+f\circ\mathcal{R}(t)}{2}.
\end{aligned}$$

If we let $\alpha \to 1^-$, then we recapture (15).

5 Gamma and beta functions

In this section, we present some inequalities and results related to gamma and beta functions. Especially, considering appropriate functions in Theorem 2.1 along with some calculations, we give a simple proof of the well-known Stirling formula as well.

The Euler integral of the second kind, i.e., gamma function [8], is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad \operatorname{Re}(x) > 0.$$

Consider the function $f(x) = \ln\Gamma(x)$, $x \in (0,+\infty)$, which is convex ($\Gamma(x)$ is log-convex). To see this (see also [3]), we have

$$(\ln\Gamma)''(x) = \frac{\Gamma''(x)\Gamma(x) - (\Gamma'(x))^2}{(\Gamma(x))^2} > 0,$$

which follows by using the Cauchy–Schwarz inequality [52] for

$$\langle f,g\rangle = \int_0^\infty f(t)g(t)t^{x-1}e^{-t}\,dt \quad \bigl(f(t)=\ln(t), g\equiv 1\bigr)$$

and the fact that

$$\Gamma^{(n)}(x) = \int_0^\infty t^{x-1}e^{-t}\bigl[\ln(t)\bigr]^n\,dt \quad (n\text{th derivative}).$$

Now in Theorem 2.1, consider $h(s) = s$, $t = 0, 1$, $b = a + 1$ for $a \in (0, +\infty)$, and a symmetric function $\omega : [a, a + 1] \to (0, +\infty)$ with respect to $a + \frac{1}{2}$. Then we obtain the following inequality:

$$\Gamma\left(a + \frac{1}{2}\right) \leq \exp\left(\frac{1}{\mathcal{K}} \int_a^{a+1} \omega(x) \ln \Gamma(x)\, dx\right) \leq \sqrt{\Gamma(a)\Gamma(a+1)}, \tag{26}$$

where $\mathcal{K} = \int_a^{a+1} \omega(x)\, dx$. In the particular case $\omega \equiv 1$, by the Raabe formula [36]

$$\int_a^{a+1} \ln \Gamma(x)\, dx = \ln \sqrt{2\pi} + a \ln(a) - a$$

and inequality (26) we have

$$\Gamma\left(a + \frac{1}{2}\right) \leq \sqrt{2\pi}\left(\frac{a}{e}\right)^a \leq \sqrt{\Gamma(a)\Gamma(a+1)} \tag{27}$$

for all $a \in (0, +\infty)$. By applying Wendel's inequality ([58])

$$\left(\frac{a}{a+s}\right)^{1-s} \leq \frac{\Gamma(a+s)}{a^s \Gamma(a)} \leq 1$$

in (27) for $s = \frac{1}{2}$, we get

$$\sqrt{\frac{a}{a + \frac{1}{2}}} \leq \frac{\Gamma(a + \frac{1}{2})}{a^{\frac{1}{2}} \Gamma(a)} \leq \frac{\sqrt{2\pi a}(\frac{a}{e})^r}{\Gamma(a+1)} \leq 1. \tag{28}$$

So we can extract two results from inequality (28) by using the squeeze theorem [51]. The first one is

$$\lim_{a \to \infty} \frac{\Gamma(a + \frac{1}{2})}{a^{\frac{1}{2}} \Gamma(a)} = 1,$$

and the second is a generalization of Stirling's formula [16]

$$\Gamma(a+1) \approx \sqrt{2\pi a}\left(\frac{a}{e}\right)^a \quad \text{as } a \to \infty.$$

For the case $a \in \mathbb{N}$, we recapture the classic Stirling formula:

$$a! \approx \sqrt{2\pi a}\left(\frac{a}{e}\right)^a \quad \text{as } a \to \infty.$$

The Euler integral of the first kind is known as the beta function [3]:

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad \operatorname{Re}(x) > 0, \operatorname{Re}(y) > 0.$$

To obtain some results in connection with beta function by Fejér's inequality, consider

$$\begin{cases} f(x) = (x - m_t(\mathcal{L},\mathcal{R}))^r, & 0 < m_t(\mathcal{L},\mathcal{R}) \le x \le M_t(\mathcal{L},\mathcal{R}), r \in [1,\infty), \\ \omega(x) = \frac{(M_t(\mathcal{L},\mathcal{R})-x)^{P-1}(x-m_t(\mathcal{L},\mathcal{R}))^{P-1}}{(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R}))^p}, & 0 < m_t(\mathcal{L},\mathcal{R}) \le x \le M_t(\mathcal{L},\mathcal{R}), \\ h(s) = s^k, & 0 \le k \le 1, s > 0, \end{cases}$$

where $0 < a < b$, $p > 0$, and $t \in [0,1] \setminus \{\frac{1}{2}\}$. From Example 7 in [57] we deduce that f is an h-convex function on $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$. Also, it is not hard to see that ω is symmetric on $[m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})]$ with respect to $\frac{a+b}{2}$. The following results hold:

$$\begin{aligned} \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \omega(x)\,dx &= \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p-1} \int_0^1 x^{p-1}(1-x)^{p-1}\,dx \\ &= \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p-1} \beta(p,p), \\ \int_a^b f(x)\omega(x)\,dx - \mathcal{M}_f^{\omega}(t) &= \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} f(x)\omega(x)\,dx \\ &= \frac{(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R}))^{2p+r-1}}{(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R}))^p} \int_0^1 x^{p-1}(1-x)^{p+r-1}\,dx \\ &= \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p+r-1} \beta(p,p+r). \end{aligned}$$

Also,

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= \left(\frac{a+b}{2} - m_t(\mathcal{L},\mathcal{R})\right)^r = \left(\frac{M_t(\mathcal{L},\mathcal{R}) + m_t(\mathcal{L},\mathcal{R})}{2} - m_t(\mathcal{L},\mathcal{R})\right)^r \\ &= \frac{(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R}))^r}{2^r}. \end{aligned}$$

Note that by Lemma 1.1 we have

$$[f \circ \mathcal{L}](t) + [f \circ \mathcal{R}](t) = \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^r.$$

It follows by some calculations that

$$\begin{aligned} &[\omega \circ A_s]\big(\mathcal{L}(t),\mathcal{R}(t)\big)\,ds \\ &\quad = \omega\big(s\mathcal{L}(t) + (1-s)\mathcal{R}(t)\big) \\ &\quad = \frac{(M_t(\mathcal{L},\mathcal{R}) - (s\mathcal{L}(t) + (1-s)\mathcal{R}(t)))^{p-1}(s\mathcal{L}(t) + (1-s)\mathcal{R}(t) - m_t(\mathcal{L},\mathcal{R}))^{p-1}}{(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R}))^p} \\ &\quad = \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p-2} s^{p-1}(1-s)^{p-1}, \end{aligned}$$

and so

$$\begin{aligned} \int_0^1 h(s)[\omega \circ A_s]\big(\mathcal{L}(t),\mathcal{R}(t)\big)\,ds &= \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p-2} \int_0^1 s^{k+p-1}(1-s)^{p-1}\,ds \\ &= \big(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\big)^{p-2} \beta(k+p,p). \end{aligned}$$

Finally, by the above results and Theorem 2.1 we obtain that

$$\begin{aligned}&\frac{1}{2(\frac{1}{2})^k}\cdot\frac{(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R}))^r}{2^r}\big(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})\big)^{p-1}\beta(p,p)\\ &\quad\le\big(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})\big)^{p+r-1}\beta(p,p+r)\\ &\quad\le\big(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})\big)^{r+1}\big(M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})\big)^{p-2}\beta(k+p,p),\end{aligned}$$

which implies the following inequalities related to the beta function:

$$2^{k-r-1}\beta(p,p)\le t\beta(p,p+r)+(1-t)2^{k-r-1}\beta(p,p)\le\beta(p,p+r)\le\beta(k+p,p) \tag{29}$$

for $t\in[0,1]\setminus\{\frac{1}{2}\}$, $0\le k\le 1$, and $r\in[1,\infty)$.

Remark 2 For the case that $f(x)=(M_t(\mathcal{L},\mathcal{R})-x)^r$, with the same argument as above, we recapture (29) because $\beta(p,p+r)=\beta(p+r,p)$.

In particular, if we set $k=1$ and $t=0,1$, then we get

$$\frac{1}{2^r}\beta(p,p)\le\beta(p+r,p)\le\beta(1+p,p)=\frac{1}{2}\beta(p,p) \tag{30}$$

for $p>0$ and $r\in[1,\infty)$. From (30) and the characterization $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ we obtain that

$$\frac{1}{2^r}\le\frac{\Gamma(2p)\Gamma(p+r)}{\Gamma(p)\Gamma(2p+r)}\le\frac{1}{2}$$

for $p>0$ and $r\in[1,\infty)$. In a more particular case, for any $p>0$, we have the following result:

$$\frac{1}{2}\Gamma(p)\Gamma(2p+1)=\Gamma(2p)\Gamma(p+1).$$

6 Special means and random variables

In this section, we discuss a generalization of the well-known inequality including the arithmetic and logarithmic means by using our results in Sect. 2. Also, we give some upper and lower bounds for the expected value of random variables.

6.1 Bounds for generalized logarithmic mean

For $a,b>0$ and $a\neq b$, there are two well-known means:

$$\begin{aligned}&A(a,b)=\frac{a+b}{2},\quad\text{arithmetic mean},\\ &L_r(a,b)=\left[\frac{b^{r+1}-a^{r+1}}{(b-a)(r+1)}\right]^{\frac{1}{r}},\quad r\in\mathbb{R}\setminus\{-1,0\},\text{generalized logarithmic mean}.\end{aligned}$$

Consider $x\in[a,b]$ and define

$$\begin{cases}f(x)=x^n, & n\in(-\infty,-1)\cup(-1,0]\cup[1,\infty),\\ h(t)=t^p, & p\le 1,\\ \omega(x)=(x-\frac{a+b}{2})^{2k}, & k\in\mathbb{N}\cup\{0\}.\end{cases}$$

According to Example 7 in [57], f is h-convex, and also, obviously, ω is symmetric with respect to $\frac{a+b}{2}$. According to Theorem 2.1, we get to the following inequalities:

$$\begin{aligned}
&2^{p-1}\left(\frac{a+b}{2}\right)^n \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \left(x-\frac{a+b}{2}\right)^{2k} dx \\
&\leq \int_a^b x^n \left(x-\frac{a+b}{2}\right)^{2k} dx - \mathcal{M}_f^{\omega}(t) \\
&\leq \left|\mathcal{L}(t)-\mathcal{R}(t)\right|\left[\left(\mathcal{L}(t)\right)^n + \left(\mathcal{R}(t)\right)^n\right] \int_0^1 s^p \left(A_s\left(\mathcal{L}(t),\mathcal{R}(t)\right) - \frac{a+b}{2}\right)^{2k} ds.
\end{aligned} \tag{31}$$

We consider two cases for For $k \in \mathbb{N} \cup \{0\}$.

(i) In the case $k = 0$ in (31), we obtain that

$$2^{p-1}\left(\frac{a+b}{2}\right)^n \leq \frac{M_t(\mathcal{L},\mathcal{R})^{n+1} - m_t(\mathcal{L},\mathcal{R})^{n+1}}{(n+1)|1-2t|(b-a)} \leq \frac{(\mathcal{L}(t))^n + (\mathcal{R}(t))^n}{p+1},$$

which implies that

$$2^{p-1}A^n(a,b) \leq L_n^n\left(m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})\right) \leq \frac{2A((\mathcal{L}(t))^n, (\mathcal{R}(t))^n)}{p+1}. \tag{32}$$

Inequality (32) includes some new bounds for the generalized logarithmic mean by the arithmetic mean. In particular, for $p = 1$, we have

$$A^n(a,b) \leq L_n^n\left(m_t(\mathcal{L},\mathcal{R}), M_t(\mathcal{L},\mathcal{R})\right) \leq A\left(\left(\mathcal{L}(t)\right)^n, \left(\mathcal{R}(t)\right)^n\right). \tag{33}$$

Furthermor,e for $t = 0, 1$, we have the following result, which is known in the literature (see also [14]):

$$A^n(a,b) \leq L_n^n(a,b) \leq A\left(a^n, b^n\right). \tag{34}$$

(ii) For the case $k > 0$, by some calculations (the details are omitted) it follows that

$$\begin{aligned}
&\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} \left(x-\frac{a+b}{2}\right)^{2k} dx = \frac{2^{-2k}|\mathcal{L}(t)-\mathcal{R}(t)|^{2k+1}}{2k+1}, \\
&\int_a^b x^n \left(x-\frac{a+b}{2}\right)^{2k} dx - \mathcal{M}_f^{\omega}(t) \\
&= \int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})} x^n \left(x-\frac{a+b}{2}\right)^{2k} dx \\
&= \sum_{i=0}^{2k-1} \frac{(-1)^i \prod_{j=0}^{i-1}(2k-j)[M_t(\mathcal{L},\mathcal{R})^{n+1+i} - m_t(\mathcal{L},\mathcal{R})^{n+1+i}]}{2^{2k-i}\prod_{l=1}^{i+1}(n+l)} \left(M_t(\mathcal{L},\mathcal{R}) - m_t(\mathcal{L},\mathcal{R})\right)^{2k-i} \\
&\quad + \frac{2k!}{\prod_{l=1}^{2k+1}(n+l)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 s^p[\omega\circ A_s]\big(\mathcal{L}(t),\mathcal{R}(t)\big)\,ds\\
&= |\mathcal{L}-\mathcal{R}|^{2k}\int_0^1 s^p\left(s-\frac{1}{2}\right)^{2k}\\
&= \big|\mathcal{L}(t)-\mathcal{R}(t)\big|^{2k}\left(\sum_{i=0}^{2k-1}\frac{(-1)^i(\frac{1}{2})^{2k-i}\prod_{j=0}^{i-1}(2k-j)}{\prod_{l=1}^{i+1}(p+l)}+\frac{2k!}{\prod_{l=1}^{2k+1}(p+l)}\right),
\end{aligned}$$

where in the case $i=0$, we use the equality $\prod_{j=0}^{-1}(2k-j)=1$. So we obtain the following discrete-type inequality, which gives a refinement and generalization of inequality (34):

$$\begin{aligned}
&\frac{2^{p-2k+1}}{2k+1}\left(\frac{a+b}{2}\right)^n\\
&\leq \sum_{i=0}^{2k-1}\frac{(-1)^i\prod_{j=0}^{i-1}(2k-j)[M_t(\mathcal{L},\mathcal{R})^{n+1+i}-m_t(\mathcal{L},\mathcal{R})^{n+1+i}]}{2^{2k-i}|\mathcal{L}(t)-\mathcal{R}(t)|^{i+1}\prod_{l=1}^{i+1}(n+l)}\\
&\quad+\frac{2k!}{|\mathcal{L}(t)-\mathcal{R}(t)|^{2k+1}\prod_{l=1}^{2k+1}(n+l)}\\
&\leq \big[\big(\mathcal{L}(t)\big)^n+\big(\mathcal{R}(t)\big)^n\big]\left(\sum_{i=0}^{2k-1}\frac{(-1)^i(\frac{1}{2})^{2k-i}\prod_{j=0}^{i-1}(2k-j)}{\prod_{l=1}^{i+1}(p+l)}+\frac{2k!}{\prod_{l=1}^{2k+1}(p+l)}\right).
\end{aligned}$$

6.2 Bounds for the expected value of random variables

For $0<a<b$, let $\omega:[a,b]\to[0,+\infty)$ be a continuous probability density function related to a continuous random variable X symmetric with respect to $\frac{a+b}{2}$. Also, for $r\in\mathbb{R}$ and $t\in[0,1]$, define the generalized expected value of random variable X as

$$E_r^t(X)=\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}x^r\omega(x)\,dx,$$

where the integral is assumed to be finite.

If we consider $f(x)=x^r$ for $r\geq 1$ and $x\in[a,b]$, then from remarks and corollaries after Theorem 2.1 we have

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})}\left(\frac{a+b}{2}\right)^r\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)\,dx\\
&\leq E_r^t(X)\\
&\leq \big[\big(m_t(\mathcal{L},\mathcal{R})\big)^r+\big(M_t(\mathcal{L},\mathcal{R})\big)^r\big]\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}h\left(\frac{x-m_t(\mathcal{L},\mathcal{R})}{M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})}\right)\omega(x)\,dx\\
&= \big[\big(m_t(\mathcal{L},\mathcal{R})\big)^r+\big(M_t(\mathcal{L},\mathcal{R})\big)^r\big]\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}h\left(\frac{M_t(\mathcal{L},\mathcal{R})-x}{M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})}\right)\omega(x)\,dx,
\end{aligned}$$

which generally gives bounds for the generalized expected value of random variable X. Now in the particular case $h(s) = s$, we get

$$\begin{aligned} &A^r(a,b)\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)\,dx \\ &\quad\le E_r^t(X)\le \frac{(m_t(\mathcal{L},\mathcal{R}))^r+(M_t(\mathcal{L},\mathcal{R}))^r}{M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\bigl(x-m_t(\mathcal{L},\mathcal{R})\bigr)\omega(x)\,dx \\ &\quad= \frac{(m_t(\mathcal{L},\mathcal{R}))^r+(M_t(\mathcal{L},\mathcal{R}))^r}{M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\bigl(M_t(\mathcal{L},\mathcal{R})-x\bigr)\omega(x)\,dx \\ &\quad= \frac{(m_t(\mathcal{L},\mathcal{R}))^r+(M_t(\mathcal{L},\mathcal{R}))^r}{M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})}\biggl[E_1^t(X)-\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}m_t(\mathcal{L},\mathcal{R})\omega(x)\,dx\biggr] \\ &\quad= \frac{(m_t(\mathcal{L},\mathcal{R}))^r+(M_t(\mathcal{L},\mathcal{R}))^r}{M_t(\mathcal{L},\mathcal{R})-m_t(\mathcal{L},\mathcal{R})}\biggl[\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}M_t(\mathcal{L},\mathcal{R})\omega(x)\,dx-E_1^t(X)\biggr]. \end{aligned} \tag{35}$$

Since

$$E_1^t(X)=\frac{a+b}{2}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)\,dx=\frac{m_t(\mathcal{L},\mathcal{R})+M_t(\mathcal{L},\mathcal{R})}{2}\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)\,dx,$$

from (35) we obtain the following interesting result:

$$\begin{aligned} &A^r(a,b)\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)\,dx \\ &\quad\le E_r^t(X)\le A\bigl(\bigl(m_t(\mathcal{L},\mathcal{R})\bigr)^r,\bigl(M_t(\mathcal{L},\mathcal{R})\bigr)^r\bigr)\int_{m_t(\mathcal{L},\mathcal{R})}^{M_t(\mathcal{L},\mathcal{R})}\omega(x)\,dx. \end{aligned}$$

By taking $\omega\equiv 1$ we recapture inequality (33).

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