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A new class of computationally efficient algorithms for solving fixed-point problems and variational inequalities in real Hilbert spaces

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Abstract

A family of inertial extragradient-type algorithms is proposed for solving convex pseudomonotone variational inequality with fixed-point problems, where the involved mapping for the fixed point is a ρ-demicontractive mapping. Under standard hypotheses, the generated iterative sequence achieves strong convergence to the common solution of the variational inequality and fixed-point problem. Some special cases and sufficient conditions that guarantee the validity of the hypotheses of the convergence statements are also discussed. Numerical applications in detail illustrate the theoretical results and comparison with existing methods.

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1 Introduction

The objective for studying a common solution problem is its potential application to mathematical models with fixed-point constraints. This is especially true in real-world applications like signal processing, network resource allocation, and image recovery. This is extremely important for signal analysis, composite reduction, optimization techniques, and image-recovery problems; see for example, [1, 13, 20, 21, 27]. Let us look at both problems highlighted by this study. Let $\mathcal{D}$ to be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{E}$ with the inner product $\langle\cdot,\cdot\rangle$, and induced norm $\|\cdot\|$. This study contributes significantly by investigating the convergence analysis of iterative algorithms for handling variational inequality problems and fixed-point problems in real Hilbert spaces. Let $\mathcal{N}:\mathcal{D}\to\mathcal{E}$ be an operator. Then, the variational inequality problem [29] is defined in the following manner:

$$\text{Find}\quad \varpi^*\in\mathcal{D}\quad \text{such that } \langle\mathcal{N}(\varpi^*), r-\varpi^*\rangle\ge 0,\quad \forall r\in\mathcal{D}. \tag{VIP}$$

Consider VI($\mathcal{D}$,$\mathcal{N}$) to describe the solution set of the problem (VIP). Variational inequalities are used in a number of areas, including partial differential equations, optimization, engineering, applied mathematics, and economics (see [12, 14–17, 24, 30]). The variational inequality problem is important in applied sciences. Many researchers have investigated not only the existence and stability of solutions, but also iterative methods for solving such problems. Projection methods, in particular, are crucial for determining the numerical solution to variational inequalities. Several authors have proposed projection methodologies to solve the problem [3, 4, 10, 11, 18, 25, 26, 33, 40–42]) and others in [5–9, 34–38]. The projection technique, which is computed on the feasible set $\mathcal{D}$, is used by most of the algorithms to solve the problem. The extragradient method was developed by Korpelevich [18] and Antipin [2]. The method has the following form:

$$\begin{cases} s_1 \in \mathcal{D} \quad \text{and} \quad 0 < \hbar < \frac{1}{L}, \\ r_k = P_{\mathcal{D}}[s_k - \hbar\mathcal{N}(s_k)], \\ s_{k+1} = P_{\mathcal{D}}[s_k - \hbar\mathcal{N}(r_k)]. \end{cases} \tag{1.1}$$

This method needs to compute two projections on the feasible set $\mathcal{D}$ for each iteration. In fact, if the feasible set $\mathcal{D}$ has a sophisticated structure, the computing efficacy of the chosen method may decline.

The first is the subgradient extragradient method developed by Censor et al. [10]. This method takes the following form:

$$\begin{cases} s_1 \in \mathcal{D} \quad \text{and} \quad 0 < \hbar < \frac{1}{L}, \\ r_k = P_{\mathcal{D}}[s_k - \hbar\mathcal{N}(s_k)], \\ s_{k+1} = P_{\mathcal{E}_k}[s_k - \hbar\mathcal{N}(r_k)], \end{cases} \tag{1.2}$$

where

$$\mathcal{E}_k = \{z \in \mathcal{E} : \langle s_k - \hbar\mathcal{N}(s_k) - r_k, z - r_k \rangle \le 0\}.$$

Following that, we will look at the strong convergence analysis of Tseng's extragradient method [33], which uses only one projection per iteration:

$$\begin{cases} s_1 \in \mathcal{D} \quad \text{and} \quad 0 < \hbar < \frac{1}{L}, \\ r_k = P_{\mathcal{D}}[s_k - \hbar\mathcal{N}(s_k)], \\ s_{k+1} = r_k - \hbar[\mathcal{N}(r_k) - \mathcal{N}(s_k)]. \end{cases} \tag{1.3}$$

In terms of computing, the technique (1.3) is especially efficient since it only requires one solution to a minimization problem each iteration. As a result, the method (1.3) is not more computationally expensive, but it performs better in most situations. Suppose that $\mathcal{M} : \mathcal{E} \to \mathcal{E}$ is a mapping. The fixed-point problem for a mapping $\mathcal{M}$ is defined by:

$$\mathcal{M}(\varpi^*) = \varpi^*. \tag{FP}$$

The solution set of the fixed-point problem (FP) is represented by the set Fix($\mathcal{M}$). Most of the methods for solving problem (FP) are derived from the basic Mann iteration, specifically from $s_1 \in \mathcal{E}$, which generates sequence $\{s_{k+1}\}$ for every $k \geq 1$ by

$$s_{k+1} = \sigma_k s_k + (1 - \sigma_k)\mathcal{M}s_k. \tag{1.4}$$

To accomplish weak convergence, the variable sequence $\{\sigma_k\}$ must adhere to certain criteria. The Halpern iteration is another structured iterative method that is more effective at achieving strong convergence in infinite-dimensional Hilbert spaces. The iterative sequence is as follows:

$$s_{k+1} = \sigma_k s_1 + (1 - \sigma_k)\mathcal{M}s_k, \tag{1.5}$$

where $s_1 \in \mathcal{E}$ and the sequence $\sigma_k \subset (0; 1)$ is nonsummable and gradually declining, i.e.,

$$\sigma_k \to 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \sigma_k = +\infty.$$

Furthermore, the viscosity algorithm [23], in which the cost mapping $\mathcal{M}$ is iteratively combined with a contraction mapping, is a generic variant of the Halpern iteration. In addition to the Halpern iteration, there is a general form of it, namely the viscosity algorithm [23], in which the cost mapping $\mathcal{M}$ is merged with a contraction mapping in the iterates. Finally, the hybrid steepest-descent approach published in [39] is another methodology that provides significant convergence.

Tan et al. [31, 32] developed an innovative numerical algorithm, the extragradient viscosity algorithm, for solving variational inequalities involving a fixed-point problem constraint of a ρ-demicontractive mapping using the extragradient algorithm [10, 18] and the Mann-type technique [22]. The authors showed the strong convergence of all methods under the condition that the operator is monotone and meets the Lipschitz condition. These techniques have the advantage of being numerically estimated using optimization tools, as shown in [31, 32].

The fundamental issue with these methods is that they rely on viscosity and Mann-type techniques to achieve strong convergence. As is known, achieving strong convergence is important for iterative sequences, especially in infinite-dimensional domains. There are a few methods with strong convergence that use inertial schemes. The Mann and viscosity procedures may be difficult to estimate from an algorithmic perspective, affecting the algorithm's convergence speed and usefulness. These algorithms increase the number of numerical and computational steps, making the system more complicated.

As a result, the following straightforward question arises:

> *Is it possible to design self-adaptive strongly convergent inertial extragradient algorithms that do not rely on Mann- and viscosity-type methods for solving variational inequalities and fixed-point problems?*

We respond to the above question by constructing two strong convergence extragradient-type algorithms for solving monotone variational inequalities and the ρ-demicontractive fixed-point problem in real Hilbert spaces, inspired by the studies described in [31, 32]. Furthermore, we avoid employing any hybrid schemes, such as the Mann-type scheme and the viscosity scheme, to achieve the strong convergence of these methods.

We presented novel algorithms with strong convergence that make use of inertial mechanisms.

The paper is divided into sections. Section 2 gives some basic results. Section 3 introduces four different methods and confirms their convergence analysis. Finally, Sect. 4 provides some numerical data to demonstrate the practical use of the methods presented.

2 Preliminaries

Let $\mathcal{D}$ be a nonempty, closed, and convex subset of $\mathcal{E}$, the real Hilbert space. For any $s, r \in \mathcal{E}$, we have

(i) $\|s+r\|^2 = \|s\|^2 + 2\langle s, r\rangle + \|r\|^2$;

(ii) $\|s+r\|^2 \leq \|s\|^2 + 2\langle r, s+r\rangle$;

(iii) $\|bs + (1-b)r\|^2 = b\|s\|^2 + (1-b)\|r\|^2 - b(1-b)\|s-r\|^2$.

A *metric projection* $P_{\mathcal{D}}(s)$ of $s \in \mathcal{E}$ is defined by

$$P_{\mathcal{D}}(s) = \arg\min\big\{\|s-r\| : r \in \mathcal{D}\big\}.$$

It is well known that $P_{\mathcal{D}}$ is nonexpansive and possesses the following important properties:

(1) $\langle s - P_{\mathcal{D}}(s), r - P_{\mathcal{D}}(s)\rangle \leq 0$, $\forall r \in \mathcal{D}$;

(2) $\|P_{\mathcal{D}}(s) - P_{\mathcal{D}}(r)\|^2 \leq \langle P_{\mathcal{D}}(s) - P_{\mathcal{D}}(r), s - r\rangle$, $\forall r \in \mathcal{D}$.

Definition 2.1 Let $\mathcal{M} : \mathcal{E} \to \mathcal{E}$ be a nonlinear mapping with $\mathrm{Fix}(\mathcal{M}) \neq \emptyset$. Then, $I - \mathcal{M}$ is said to be demiclosed at zero if for any $\{s_k\}$ in $\mathcal{E}$. Then, the following statement is true:

$$s_k \rightharpoonup s \quad \text{and} \quad (I-\mathcal{M})s_k \to 0 \quad \Rightarrow \quad s \in \mathrm{Fix}(\mathcal{M}).$$

Definition 2.2 Let $\mathcal{N} : \mathcal{D} \to \mathcal{D}$ be an operator. It is said to be:

(1) *monotone* if

$$\big\langle \mathcal{N}(s_1) - \mathcal{N}(s_2), s_1 - s_2\big\rangle \geq 0, \quad \forall s_1, s_2 \in \mathcal{D};$$

(2) *Lipschitz-continuous* with constant $L > 0$ such that

$$\big\|\mathcal{N}(s_1) - \mathcal{N}(s_2)\big\| \leq L\|s_1 - s_2\|, \quad \forall s_1, s_2 \in \mathcal{D};$$

(3) *sequentially weakly continuous* if a sequence $\{\mathcal{N}(s_k)\}$ convergent weakly to $\mathcal{N}(s)$ for any sequence $\{s_k\}$ convergent weakly to s.

Definition 2.3 Suppose the $\mathcal{M} : \mathcal{D} \to \mathcal{D}$ is a mapping and $\mathrm{Fix}(\mathcal{M}) \neq \emptyset$. It is said to be:

(1) ρ-demicontractive if for any fixed number $0 \leq \rho < 1$ such that

$$\big\|\mathcal{M}(s_1) - s_2\big\|^2 \leq \|s_1 - s_2\|^2 + \rho\big\|(I-\mathcal{M})(s_1)\big\|^2, \quad \forall s_2 \in \mathrm{Fix}(\mathcal{M}), s_1 \in \mathcal{E};$$

or equivalently

$$\big\langle \mathcal{M}(s_1) - s_1, s_1 - s_2\big\rangle \leq \frac{\rho - 1}{2}\big\|s_1 - \mathcal{M}(s_1)\big\|^2, \quad \forall s_2 \in \mathrm{Fix}(\mathcal{M}), s_1 \in \mathcal{E}.$$

Lemma 2.4 ([19]) *Let $\mathcal{N}:\mathcal{E}\to\mathcal{E}$ be a L-Lipschitz continuous and monotone operator on $\mathcal{D}$. Take $\mathcal{M}=P_{\mathcal{D}}(I-\hbar\mathcal{N})$, with $\hbar>0$. If $\{s_k\}$ is a sequence in $\mathcal{E}$ that satisfies $s_k\rightharpoonup q$ and $s_k-\mathcal{N}(s_k)\to 0$, then $q\in \mathrm{VI}(\mathcal{D},\mathcal{N})=\mathrm{Fix}(\mathcal{M})$.*

Lemma 2.5 ([28]) *Suppose that $\{c_k\}\subset[0,+\infty)$, $\{d_k\}\subset(0,1)$, and $\{e_k\}\subset\mathbb{R}$ sequences satisfies the following criteria:*

$$c_{k+1}\le(1-d_k)c_k+d_ke_k,\quad \forall k\in\mathbb{N},\ and\ \sum_{k=1}^{+\infty}d_k=+\infty.$$

If $\limsup_{j\to+\infty} r_{k_j}\le 0$ for any subsequence $\{c_{k_j}\}$ of $\{c_j\}$ such that

$$\liminf_{j\to+\infty}(c_{k_j+1}-c_{k_j})\ge 0,$$

then $\lim_{k\to\infty}c_k=0$.

3 Main results

In this section, we examine the convergence of four novel inertial extragradient algorithms for solving the fixed-point and variational inequality problems in detail. First, we consider the given algorithms. In order to confirm the strong convergence, it is assumed that the following conditions are satisfied:

($\mathcal{N}$1) The common solution set is denoted by $\mathrm{Fix}(\mathcal{M})\cap\mathrm{VI}(\mathcal{D},\mathcal{N})$ and it is nonempty;

($\mathcal{N}$2) The operator $\mathcal{N}:\mathcal{E}\to\mathcal{E}$ is monotone;

($\mathcal{N}$3) The operator $\mathcal{N}:\mathcal{E}\to\mathcal{E}$ is Lipschitz continuous;

($\mathcal{N}$4) The mapping $\mathcal{M}:\mathcal{E}\to\mathcal{E}$ is ρ-demicontractive for $0\le\rho<1$ and demiclosed at zero;

($\mathcal{N}$5) The operator $\mathcal{N}:\mathcal{E}\to\mathcal{E}$ is sequentially weakly continuous.

Algorithm 1 (Inertial Subgradient Extragradient Method With Constant Step-Size Rule)

STEP 0: Take $s_0,s_1\in\mathcal{D}$, $\ell\in(0,1)$, $0<\hbar<\frac{1}{L}$ and a sequence $\{\varsigma_k\}\subset(0,1-\rho)$ that satisfies the following condition:

$$\lim_{k\to+\infty}\varsigma_k=0\quad\text{and}\quad\sum_{k=1}^{+\infty}\varsigma_k=+\infty.$$

STEP 1: Calculate

$$q_k=s_k+\ell_k(s_k-s_{k-1})-\varsigma_k\big[s_k+\ell_k(s_k-s_{k-1})\big],$$

where ℓ_k is defined as follows:

$$0\le\ell_k\le\hat{\ell}_k\quad\text{and}\quad \hat{\ell}_k=\begin{cases}\min\{\frac{\ell}{2},\frac{\chi_k}{\|s_k-s_{k-1}\|}\} & \text{if } s_k\ne s_{k-1},\\ \frac{\ell}{2} & \text{else.}\end{cases}\tag{3.1}$$

Moreover, take a sequence $\chi_k=\circ(\varsigma_k)$ satisfying the condition $\lim_{k\to+\infty}\frac{\chi_k}{\varsigma_k}=0$.

STEP 2: Calculate

$$r_k = P_{\mathcal{D}}\big(q_k - \hbar\mathcal{N}(q_k)\big).$$

If $q_k = r_k$, then STOP. Otherwise, go to **STEP 3**.

STEP 3: Construct a half-space first

$$\mathcal{E}_k = \big\{z \in \mathcal{E} : \big\langle q_k - \hbar\mathcal{N}(q_k) - r_k, z - r_k\big\rangle \leq 0\big\}$$

and $p_k = P_{\mathcal{E}_k}(q_k - \hbar\mathcal{N}(r_k))$.

STEP 4: For any sequence $\sigma_k \subset (0, 1 - \rho)$. Calculate

$$s_{k+1} = (1 - \sigma_k)p_k + \sigma_k\mathcal{M}(p_k).$$

Set $k := k + 1$ and go to **STEP 1**.

Algorithm 2 (Inertial Subgradient Extragradient Method With Nonmonotone Step-Size Rule)

STEP 0: Take $s_0, s_1 \in \mathcal{D}$, $\ell \in (0, 1)$, $\mu \in (0, 1)$, $\hbar_1 > 0$. Moreover, a sequence $\{\beth_k\}$ such that $\sum_{k=1}^{\infty} \beth_k < +\infty$, and a sequence $\{\varsigma_k\} \subset (0, 1 - \rho)$ that satisfies the following condition:

$$\lim_{k\to+\infty} \varsigma_k = 0 \quad \text{and} \quad \sum_{k=1}^{+\infty} \varsigma_k = +\infty.$$

STEP 1: Calculate

$$q_k = s_k + \ell_k(s_k - s_{k-1}) - \varsigma_k\big[s_k + \ell_k(s_k - s_{k-1})\big],$$

where ℓ_k is defined as follows:

$$0 \leq \ell_k \leq \hat{\ell}_k \quad \text{and} \quad \hat{\ell}_k = \begin{cases} \min\{\frac{\ell}{2}, \frac{\chi_k}{\|s_k - s_{k-1}\|}\} & \text{if } s_k \neq s_{k-1}, \\ \frac{\ell}{2} & \text{otherwise.} \end{cases} \tag{3.2}$$

Moreover, a sequence $\chi_k = \circ(\varsigma_k)$ satisfying the condition $\lim_{k\to+\infty} \frac{\chi_k}{\varsigma_k} = 0$.

STEP 2: Calculate

$$r_k = P_{\mathcal{D}}\big(q_k - \hbar_k\mathcal{N}(q_k)\big).$$

If $q_k = r_k$, then STOP. Otherwise, go to **STEP 3**.

STEP 3: Create a half-space first

$$\mathcal{E}_k = \big\{z \in \mathcal{E} : \big\langle q_k - \hbar_k\mathcal{N}(q_k) - r_k, z - r_k\big\rangle \leq 0\big\}$$

and calculate $p_k = P_{\mathcal{E}_k}(q_k - \hbar_k\mathcal{N}(r_k))$.

STEP 4: For any sequence $\sigma_k \subset (0, 1 - \rho)$. Calculate

$$s_{k+1} = (1 - \sigma_k)p_k + \sigma_k\mathcal{M}(p_k).$$

STEP 5: Calculate

$$\hbar_{k+1} = \begin{cases} \min\{\hbar_k + \beth_k, \frac{\mu\|q_k - r_k\|^2 + \mu\|p_k - r_k\|^2}{2[\langle \mathcal{N}(q_k) - \mathcal{N}(r_k), p_k - r_k\rangle]}\} & \text{if } \langle \mathcal{N}(q_k) - \mathcal{N}(r_k), p_k - r_k\rangle > 0, \\ \hbar_k + \beth_k, & \text{otherwise.} \end{cases} \quad (3.3)$$

Set $k := k + 1$ and go to **STEP 1**.

Lemma 3.1 *A sequence* $\{\hbar_k\}$ *that is generated by the expression* (3.3) *is convergent to* $\hbar$ *and bounded by* $\min\{\frac{\mu}{L}, \hbar_1\} \le \hbar \le \hbar_1 + P$, *where*

$$P = \sum_{k=1}^{+\infty} \beth_k.$$

Proof It is given that $\langle \mathcal{N}(q_k) - \mathcal{N}(r_k), p_k - r_k\rangle > 0$, such that

$$\begin{aligned} \frac{\mu(\|q_k - r_k\|^2 + \|p_k - r_k\|^2)}{2\langle \mathcal{N}(q_k) - \mathcal{N}(r_k), p_k - r_k\rangle} &\ge \frac{2\mu\|q_k - r_k\|\|p_k - r_k\|}{2\|\mathcal{N}(q_k) - \mathcal{N}(r_k)\|\|p_k - r_k\|} \\ &\ge \frac{2\mu\|q_k - r_k\|\|p_k - r_k\|}{2L\|q_k - r_k\|\|p_k - r_k\|} \\ &\ge \frac{\mu}{L}. \end{aligned} \quad (3.4)$$

By definition of $\hbar_{k+1}$, we have

$$\min\left\{\frac{\mu}{L}, \hbar_1\right\} \le \hbar_k \le \hbar_1 + P.$$

Let

$$[\hbar_{k+1} - \hbar_k]^+ = \max\{0, \hbar_{k+1} - \hbar_k\}$$

and

$$[\hbar_{k+1} - \hbar_k]^- = \max\{0, -(\hbar_{k+1} - \hbar_k)\}.$$

By using the definition of $\{\hbar_k\}$, we have

$$\sum_{k=1}^{+\infty} (\hbar_{k+1} - \hbar_k)^+ = \sum_{k=1}^{+\infty} \max\{0, \hbar_{k+1} - \hbar_k\} \le P < +\infty. \quad (3.5)$$

This implies that the series $\sum_{k=1}^{+\infty} (\hbar_{k+1} - \hbar_k)^+$ is convergent. Following that, we must demonstrate the convergence of

$$\sum_{k=1}^{+\infty} (\hbar_{k+1} - \hbar_k)^-.$$

Let us consider that $\sum_{k=1}^{+\infty} (\hbar_{k+1} - \hbar_k)^- = +\infty$. Thus, we obtain

$$\hbar_{k+1} - \hbar_k = (\hbar_{k+1} - \hbar_k)^+ - (\hbar_{k+1} - \hbar_k)^-.$$

Thus, we obtain

$$\hbar_{k+1} - \hbar_1 = \sum_{k=0}^{k} (\hbar_{k+1} - \hbar_k) = \sum_{k=0}^{k} (\hbar_{k+1} - \hbar_k)^+ - \sum_{k=0}^{k} (\hbar_{k+1} - \hbar_k)^-. \tag{3.6}$$

By allowing $k \to +\infty$ in the formulation (3.6), we obtain $\hbar_k \to -\infty$ as $k \to \infty$. This is a logical contradiction. Due to the convergence of the series $\sum_{k=0}^{k} (\hbar_{k+1} - \hbar_k)^+$ and $\sum_{k=0}^{k} (\hbar_{k+1} - \hbar_k)^-$ taking $k \to +\infty$ in (3.6), we obtain $\lim_{k\to\infty} \hbar_k = \hbar$. This completes the proof of the lemma. □

Algorithm 3 (Inertial Tseng's Extragradient Method With Constant Step-Size Rule)

STEP 0: Consider $s_0, s_1 \in \mathcal{D}$, $\ell \in (0,1)$, $\mu \in (0,1)$, $0 < \hbar < \frac{1}{L}$ and a sequence $\{\varsigma_k\} \subset (0, 1 - \rho)$ that satisfies the following condition:

$$\lim_{k\to+\infty} \varsigma_k = 0 \quad \text{and} \quad \sum_{k=1}^{+\infty} \varsigma_k = +\infty.$$

STEP 1: Calculate

$$q_k = s_k + \ell_k(s_k - s_{k-1}) - \varsigma_k\big[s_k + \ell_k(s_k - s_{k-1})\big],$$

where ℓ_k is defined as follows:

$$0 \le \ell_k \le \hat{\ell}_k \quad \text{and} \quad \hat{\ell}_k = \begin{cases} \min\{\frac{\ell}{2}, \frac{\chi_k}{\|s_k - s_{k-1}\|}\} & \text{if } s_k \ne s_{k-1}, \\ \frac{\ell}{2} & \text{else.} \end{cases} \tag{3.7}$$

Moreover, a sequence $\chi_k = \circ(\varsigma_k)$ satisfying the condition $\lim_{k\to+\infty} \frac{\chi_k}{\varsigma_k} = 0$.

STEP 2: Calculate

$$r_k = P_{\mathcal{D}}\big(q_k - \hbar \mathcal{N}(q_k)\big).$$

If $q_k = r_k$, then STOP. Otherwise, go to **STEP 3**.

STEP 3: Calculate

$$p_k = r_k + \hbar\big[\mathcal{N}(q_k) - \mathcal{N}(r_k)\big].$$

STEP 4: For any sequence $\sigma_k \subset (0, 1 - \rho)$. Calculate

$$s_{k+1} = (1 - \sigma_k)p_k + \sigma_k \mathcal{M}(p_k).$$

Set $k := k + 1$ and go back to **STEP 1**.

Algorithm 4 (Inertial Tseng's Extragradient Method With Nonmonotone Step-Size Rule)

STEP 0: Consider $s_0, s_1 \in \mathcal{D}$, $\ell \in (0,1)$, $\mu \in (0,1)$, $\hbar_1 > 0$. Moreover, $\{\daleth_k\}$ such that $\sum_{k=1}^{\infty} \daleth_k < +\infty$ and a sequence $\{\varsigma_k\} \subset (0, 1-\rho)$ that satisfies the following condition:

$$\lim_{k \to +\infty} \varsigma_k = 0 \quad \text{and} \quad \sum_{k=1}^{+\infty} \varsigma_k = +\infty.$$

STEP 1: Calculate

$$q_k = s_k + \ell_k(s_k - s_{k-1}) - \varsigma_k\big[s_k + \ell_k(s_k - s_{k-1})\big],$$

where ℓ_k is defined as follows:

$$0 \le \ell_k \le \hat{\ell}_k \quad \text{and} \quad \hat{\ell}_k = \begin{cases} \min\{\frac{\ell}{2}, \frac{\chi_k}{\|s_k - s_{k-1}\|}\} & \text{if } s_k \neq s_{k-1}, \\ \frac{\ell}{2} & \text{otherwise.} \end{cases} \tag{3.8}$$

Moreover, a sequence $\chi_k = \circ(\varsigma_k)$ satisfying the condition $\lim_{k \to +\infty} \frac{\chi_k}{\varsigma_k} = 0$.

STEP 2: Calculate

$$r_k = P_{\mathcal{D}}\big(q_k - \hbar_k \mathcal{N}(q_k)\big).$$

If $q_k = r_k$, then STOP. Otherwise, go to **STEP 3**.

STEP 3: Calculate

$$p_k = r_k + \hbar_k\big[\mathcal{N}(q_k) - \mathcal{N}(r_k)\big].$$

STEP 4: For any sequence $\sigma_k \subset (0, 1-\rho)$. Calculate

$$s_{k+1} = (1 - \sigma_k)p_k + \sigma_k \mathcal{M}(p_k).$$

STEP 5: Compute

$$\begin{cases} \min\{\hbar_k + \daleth_k, \frac{\mu\|q_k - r_k\|}{\|\mathcal{N}(q_k) - A(r_k)\|}\} & \text{if } \mathcal{N}(q_k) \neq \mathcal{N}(r_k), \\ \hbar_k + \daleth_k, & \text{otherwise.} \end{cases} \tag{3.9}$$

Set $k := k + 1$ and go back to **STEP 1**.

Lemma 3.2 *A sequence* $\{\hbar_k\}$ *that is generated by the expression* (3.9) *is decreasing monotonically and bounded by* $\min\{\frac{\mu}{L}, \hbar_1\} \le \hbar \le \hbar_1 + P$, *where*

$$P = \sum_{k=1}^{+\infty} \daleth_k.$$

Proof It is given that the mapping $\mathcal{N}$ is Lipschitz continuous. Thus, we have

$$\frac{\mu\|q_k - r_k\|}{\|\mathcal{N}(q_k) - \mathcal{N}(r_k)\|} \ge \frac{\mu\|q_k - r_k\|}{L\|q_k - r_k\|} \ge \frac{\mu}{L}. \tag{3.10}$$

The remainder of the proof is similar to that of Lemma 3.1. □

Lemma 3.3 *Let $\mathcal{N}:\mathcal{E}\to\mathcal{E}$ be an operator that satisfies the conditions ($\mathcal{N}$1)–($\mathcal{N}$5). Suppose that $\{s_k\}$ is a sequence generated by Algorithm 2. For any $\varpi^*\in \mathrm{VI}(\mathcal{D},\mathcal{N})$, we have*

$$\|p_k-\varpi^*\|^2\le\|q_k-\varpi^*\|^2-\left(1-\frac{\mu\hbar_k}{\hbar_{k+1}}\right)\|q_k-r_k\|^2-\left(1-\frac{\mu\hbar_k}{\hbar_{k+1}}\right)\|p_k-r_k\|^2.$$

Proof Consider that

$$\begin{aligned}\|p_k-\varpi^*\|^2&=\|P_{\mathcal{E}_k}[q_k-\hbar_k\mathcal{N}(r_k)]-\varpi^*\|^2\\&=\|P_{\mathcal{E}_k}[q_k-\hbar_k\mathcal{N}(r_k)]+[q_k-\hbar_k\mathcal{N}(r_k)]-[q_k-\hbar_k\mathcal{N}(r_k)]-\varpi^*\|^2\\&=\|[q_k-\hbar_k\mathcal{N}(r_k)]-\varpi^*\|^2+\|P_{\mathcal{E}_k}[q_k-\hbar_k\mathcal{N}(r_k)]-[q_k-\hbar_k\mathcal{N}(r_k)]\|^2\\&\quad+2\langle P_{\mathcal{E}_k}[q_k-\hbar_k\mathcal{N}(r_k)]-[q_k-\hbar_k\mathcal{N}(r_k)],[q_k-\hbar_k\mathcal{N}(r_k)]-\varpi^*\rangle.\end{aligned}\tag{3.11}$$

It is given that $\varpi^*\in \mathrm{VI}(\mathcal{D},\mathcal{N})\subset\mathcal{D}\subset\mathcal{E}_k$, we have

$$\begin{aligned}&\|P_{\mathcal{E}_k}[q_k-\hbar_k\mathcal{N}(r_k)]-[q_k-\hbar_k\mathcal{N}(r_k)]\|^2\\&\quad+\langle P_{\mathcal{E}_k}[q_k-\hbar_k\mathcal{N}(r_k)]-[q_k-\hbar_k\mathcal{N}(r_k)],[q_k-\hbar_k\mathcal{N}(r_k)]-\varpi^*\rangle\\&=\langle[q_k-\hbar_k\mathcal{N}(r_k)]-P_{\mathcal{E}_k}[q_k-\hbar_k\mathcal{N}(r_k)],\varpi^*-P_{\mathcal{E}_k}[q_k-\hbar_k\mathcal{N}(r_k)]\rangle\le 0.\end{aligned}\tag{3.12}$$

Moreover, we have

$$\begin{aligned}&\langle P_{\mathcal{E}_k}[q_k-\hbar_k\mathcal{N}(r_k)]-[q_k-\hbar_k\mathcal{N}(r_k)],[q_k-\hbar_k\mathcal{N}(r_k)]-\varpi^*\rangle\\&\quad\le-\|P_{\mathcal{E}_k}[q_k-\hbar_k\mathcal{N}(r_k)]-[q_k-\hbar_k\mathcal{N}(r_k)]\|^2.\end{aligned}\tag{3.13}$$

By combining expressions (3.11) and (3.13), we obtain

$$\begin{aligned}\|p_k-\varpi^*\|^2&\le\|q_k-\hbar_k\mathcal{N}(r_k)-\varpi^*\|^2-\|P_{\mathcal{E}_k}[q_k-\hbar_k\mathcal{N}(r_k)]-[q_k-\hbar_k\mathcal{N}(r_k)]\|^2\\&\le\|q_k-\varpi^*\|^2-\|q_k-p_k\|^2+2\hbar_k\langle\mathcal{N}(r_k),\varpi^*-p_k\rangle.\end{aligned}\tag{3.14}$$

Furthermore, we have

$$\langle\mathcal{N}(\varpi^*),y-\varpi^*\rangle-\langle\mathcal{N}(y),y-\varpi^*\rangle\le 0,\quad\forall y\in\mathcal{D}.$$

Since $\varpi^*\in \mathrm{VI}(\mathcal{D},\mathcal{N})$, we obtain

$$\langle\mathcal{N}(y),y-\varpi^*\rangle\ge 0,\quad\forall y\in\mathcal{D}.$$

By substituting $y=r_k\in\mathcal{D}$, we have

$$\langle\mathcal{N}(r_k),r_k-\varpi^*\rangle\ge 0.$$

Thus, this implies that

$$\langle\mathcal{N}(r_k),\varpi^*-p_k\rangle=\langle\mathcal{N}(r_k),\varpi^*-r_k\rangle+\langle\mathcal{N}(r_k),r_k-p_k\rangle\le\langle\mathcal{N}(r_k),r_k-p_k\rangle.\tag{3.15}$$

We obtain by combining formulas (3.14) and (3.15)

$$\begin{aligned}\|p_k - \varpi^*\|^2 &\le \|q_k - \varpi^*\|^2 - \|q_k - p_k\|^2 + 2\hbar_k\langle \mathcal{N}(r_k), r_k - p_k\rangle \\ &\le \|q_k - \varpi^*\|^2 - \|q_k - r_k + r_k - p_k\|^2 + 2\hbar_k\langle \mathcal{N}(r_k), r_k - p_k\rangle \\ &\le \|q_k - \varpi^*\|^2 - \|q_k - r_k\|^2 - \|r_k - p_k\|^2 \\ &\quad + 2\langle q_k - \hbar_k\mathcal{N}(r_k) - r_k, p_k - r_k\rangle. \end{aligned} \tag{3.16}$$

By using expression $p_k = P_{\mathcal{E}_k}[q_k - \hbar_k\mathcal{N}(r_k)]$, we have

$$\begin{aligned}&2\langle q_k - \hbar_k\mathcal{N}(r_k) - r_k, p_k - r_k\rangle \\ &\quad = 2\langle q_k - \hbar_k\mathcal{N}(q_k) - r_k, p_k - r_k\rangle + 2\hbar_k\langle \mathcal{N}(q_k) - \mathcal{N}(r_k), p_k - r_k\rangle \\ &\quad \le \frac{\hbar_k}{\hbar_{k+1}} 2\hbar_{k+1}\langle \mathcal{N}(q_k) - \mathcal{N}(r_k), p_k - r_k\rangle \\ &\quad \le \frac{\mu\hbar_k}{\hbar_{k+1}}\|q_k - r_k\|^2 + \frac{\mu\hbar_k}{\hbar_{k+1}}\|p_k - r_k\|^2. \end{aligned} \tag{3.17}$$

From expressions (3.16) and (3.17) we can obtain

$$\begin{aligned}&\|p_k - \varpi^*\|^2 \\ &\quad \le \|q_k - \varpi^*\|^2 - \|q_k - r_k\|^2 - \|r_k - p_k\|^2 + \frac{\hbar_k}{\hbar_{k+1}}\left[\mu\|q_k - r_k\|^2 + \mu\|p_k - r_k\|^2\right] \\ &\quad \le \|q_k - \varpi^*\|^2 - \left(1 - \frac{\mu\hbar_k}{\hbar_{k+1}}\right)\|q_k - r_k\|^2 - \left(1 - \frac{\mu\hbar_k}{\hbar_{k+1}}\right)\|p_k - r_k\|^2. \end{aligned} \tag{3.18}$$

□

Lemma 3.4 *Let $\mathcal{N} : \mathcal{E} \to \mathcal{E}$ be an operator that satisfies the conditions ($\mathcal{N}$1)–($\mathcal{N}$5). Suppose that $\{s_k\}$ is a sequence generated by the Algorithm 1. For any $\varpi^* \in \mathrm{VI}(\mathcal{D}, \mathcal{N})$, we have*

$$\|p_k - \varpi^*\|^2 \le \|q_k - \varpi^*\|^2 - (1 - \hbar L)\|q_k - r_k\|^2 - (1 - \hbar L)\|p_k - r_k\|^2.$$

Proof The proof is similar to the proof of Lemma 3.3. □

Lemma 3.5 *Let $\mathcal{N} : \mathcal{E} \to \mathcal{E}$ be an operator that satisfies the conditions ($\mathcal{N}$1)–($\mathcal{N}$5). Suppose that $\{s_k\}$ is a sequence generated by the Algorithm 4. For any $\varpi^* \in \mathrm{VI}(\mathcal{D}, \mathcal{N})$, we have*

$$\|p_k - \varpi^*\|^2 \le \|q_k - \varpi^*\|^2 - \left(1 - \mu^2\frac{\hbar_k^2}{\hbar_{k+1}^2}\right)\|q_k - r_k\|^2.$$

Proof Consider the following:

$$\begin{aligned}&\|p_k - \varpi^*\|^2 \\ &\quad = \|r_k + \hbar_k[\mathcal{N}(s_k) - \mathcal{N}(r_k)] - \varpi^*\|^2 \\ &\quad = \|r_k - \varpi^*\|^2 + \hbar_k^2\|\mathcal{N}(s_k) - \mathcal{N}(r_k)\|^2 + 2\hbar_k\langle r_k - \varpi^*, \mathcal{N}(s_k) - \mathcal{N}(r_k)\rangle \end{aligned}$$

$$\begin{aligned}
&= \left\|r_k + s_k - s_k - \varpi^*\right\|^2 + \hbar_k^2 \left\|\mathcal{N}(s_k) - \mathcal{N}(r_k)\right\|^2 + 2\hbar_k \left\langle r_k - \varpi^*, \mathcal{N}(s_k) - \mathcal{N}(r_k)\right\rangle \\
&= \|r_k - s_k\|^2 + \left\|s_k - \varpi^*\right\|^2 + 2\left\langle r_k - s_k, s_k - \varpi^*\right\rangle \\
&\quad + \hbar_k^2 \left\|\mathcal{N}(s_k) - \mathcal{N}(r_k)\right\|^2 + 2\hbar_k \left\langle r_k - \varpi^*, \mathcal{N}(s_k) - \mathcal{N}(r_k)\right\rangle \\
&= \left\|s_k - \varpi^*\right\|^2 + \|r_k - s_k\|^2 + 2\left\langle r_k - s_k, r_k - \varpi^*\right\rangle + 2\langle r_k - s_k, s_k - r_k\rangle \\
&\quad + \hbar_k^2 \left\|\mathcal{N}(s_k) - \mathcal{N}(r_k)\right\|^2 + 2\hbar_k \left\langle r_k - \varpi^*, \mathcal{N}(s_k) - \mathcal{N}(r_k)\right\rangle.
\end{aligned} \tag{3.19}$$

Furthermore, we can write

$$\left\langle s_k - \hbar_k \mathcal{N}(s_k) - r_k, y - r_k\right\rangle \le 0, \quad \forall y \in \mathcal{D}. \tag{3.20}$$

For given $\varpi^* \in \mathrm{VI}(\mathcal{D}, \mathcal{N})$, we can write

$$\left\langle s_k - r_k, \varpi^* - r_k\right\rangle \le \hbar_k \left\langle \mathcal{N}(s_k), \varpi^* - r_k\right\rangle. \tag{3.21}$$

By combining expressions (3.19) and (3.21), we can obtain

$$\begin{aligned}
&\left\|p_k - \varpi^*\right\|^2 \\
&\quad \le \left\|s_k - \varpi^*\right\|^2 + \|r_k - s_k\|^2 + 2\hbar_k \left\langle \mathcal{N}(s_k), \varpi^* - r_k\right\rangle - 2\langle s_k - r_k, s_k - r_k\rangle \\
&\qquad + \hbar_k^2 \left\|\mathcal{N}(s_k) - \mathcal{N}(r_k)\right\|^2 - 2\hbar_k \left\langle \mathcal{N}(s_k) - \mathcal{N}(r_k), \varpi^* - r_k\right\rangle \\
&\quad = \left\|s_k - \varpi^*\right\|^2 - \|s_k - r_k\|^2 + \hbar_k^2 \left\|\mathcal{N}(s_k) - \mathcal{N}(r_k)\right\|^2 - 2\hbar_k \left\langle \mathcal{N}(r_k), r_k - \varpi^*\right\rangle.
\end{aligned} \tag{3.22}$$

By use of the notion of a mapping $\mathcal{N}$ on $\mathcal{D}$, we can obtain

$$\left\langle \mathcal{N}\left(\varpi^*\right), y - \varpi^*\right\rangle - \left\langle \mathcal{N}(y), y - \varpi^*\right\rangle \le 0, \quad \forall y \in \mathcal{D}.$$

By using $\varpi^* \in \mathrm{VI}(\mathcal{D}, \mathcal{N})$, we obtain

$$\left\langle \mathcal{N}(y), y - \varpi^*\right\rangle \ge 0, \quad \forall y \in \mathcal{D}.$$

By substituting $y = r_k \in \mathcal{D}$, we can write

$$\left\langle \mathcal{N}(r_k), r_k - \varpi^*\right\rangle \ge 0. \tag{3.23}$$

From expressions (3.22) and (3.23) we can obtain

$$\begin{aligned}
\left\|p_k - \varpi^*\right\|^2 &\le \left\|s_k - \varpi^*\right\|^2 - \|s_k - r_k\|^2 + \mu^2 \frac{\hbar_k^2}{\hbar_{k+1}^2} \|s_k - r_k\|^2 \\
&= \left\|s_k - \varpi^*\right\|^2 - \left(1 - \mu^2 \frac{\hbar_k^2}{\hbar_{k+1}^2}\right) \|s_k - r_k\|^2.
\end{aligned} \tag{3.24}$$

□

Lemma 3.6 *Let $\mathcal{N} : \mathcal{E} \to \mathcal{E}$ be a map that fulfils the criteria $(\mathcal{N}1)$–$(\mathcal{N}5)$. Suppose that $\{s_k\}$ is a sequence created due to Algorithm 3. For any $\varpi^* \in \mathrm{VI}(\mathcal{D}, \mathcal{N})$, we have*

$$\|p_k - \varpi^*\|^2 \le \|q_k - \varpi^*\|^2 - (1 - \hbar^2 L^2)\|q_k - r_k\|^2.$$

Proof The proof is similar to the proof of Lemma 3.5. □

Theorem 3.7 *Let $\mathcal{N} : \mathcal{E} \to \mathcal{E}$ be an operator that satisfies the conditions $(\mathcal{N}1)$–$(\mathcal{N}5)$. Then, the sequence $\{s_k\}$ generated by the Algorithm 2 converges strongly to some $\varpi^* \in \mathrm{VI}(\mathcal{D}, \mathcal{N}) \cap \mathrm{Fix}(\mathcal{M})$, where $\varpi^* = P_{\mathrm{VI}(\mathcal{D},\mathcal{N})\cap \mathrm{Fix}(\mathcal{M})}(0)$.*

Proof **Claim 1: The sequence $\{\mathbf{s_k}\}$ is bounded.**
It is given that

$$s_{k+1} = (1 - \sigma_k)p_k + \sigma_k \mathcal{M}(p_k).$$

To use this result of the sequence $\{s_{k+1}\}$, we derive

$$\begin{aligned} \|s_{k+1} - \varpi^*\|^2 &= \|(1 - \sigma_k)p_k + \sigma_k \mathcal{M}(p_k) - \varpi^*\|^2 \\ &= \|p_k - \varpi^*\|^2 + 2\sigma_k\langle p_k - \varpi^*, \mathcal{M}(p_k) - p_k\rangle + \sigma_k^2\|\mathcal{M}(p_k) - p_k\|^2 \\ &\le \|p_k - \varpi^*\|^2 + \sigma_k(\rho - 1)\|\mathcal{M}(p_k) - p_k\|^2 + \sigma_k^2\|\mathcal{M}(p_k) - p_k\|^2 \\ &= \|p_k - \varpi^*\|^2 - \sigma_k(1 - \rho - \sigma_k)\|\mathcal{M}(p_k) - p_k\|^2. \end{aligned} \tag{3.25}$$

By using the value of $\{q_k\}$, we obtain

$$\begin{aligned} \|q_k - \varpi^*\| &= \|s_k + \ell_k(s_k - s_{k-1}) - \varsigma_k s_k - \ell_k \varsigma_k(s_k - s_{k-1}) - \varpi^*\| \\ &= \|(1 - \varsigma_k)(s_k - \varpi^*) + (1 - \varsigma_k)\ell_k(s_k - s_{k-1}) - \varsigma_k \varpi^*\| \qquad (3.26) \\ &\le (1 - \varsigma_k)\|s_k - \varpi^*\| + (1 - \varsigma_k)\ell_k\|s_k - s_{k-1}\| + \varsigma_k\|\varpi^*\| \\ &\le (1 - \varsigma_k)\|s_k - \varpi^*\| + \varsigma_k K_1, \qquad (3.27) \end{aligned}$$

for some fixed number K_1 we have

$$(1 - \varsigma_k)\frac{\ell_k}{\varsigma_k}\|s_k - s_{k-1}\| + \|\varpi^*\| \le K_1.$$

It is given that $\hbar_k \to \hbar$ such that there exists a fixed number $\vartheta \in (0, 1 - \mu)$ such that

$$\lim_{k\to\infty}\left(1 - \frac{\mu \hbar_k}{\hbar_{k+1}}\right) = 1 - \mu > \vartheta > 0.$$

As a result, there exists a finite natural number $N_1 \in \mathbb{N}$ such that

$$\left(1 - \frac{\mu \hbar_k}{\hbar_{k+1}}\right) > \vartheta > 0, \quad \forall k \ge N_1. \tag{3.28}$$

By using Lemma 3.3, we can write

$$\|p_k - \varpi^*\|^2 \le \|q_k - \varpi^*\|^2, \quad \forall k \ge N_1. \tag{3.29}$$

From expressions (3.25), (3.27), and (3.29) we infer that

$$\|s_{k+1} - \varpi^*\| \le (1 - \varsigma_k)\|s_k - \varpi^*\| + \varsigma_k K_1 - \sigma_k(1 - \rho - \sigma_k)\|\mathcal{M}(p_k) - p_k\|^2. \tag{3.30}$$

It is considered that $\{\sigma_k\} \subset (0, 1 - \rho)$ such that

$$\begin{aligned}\|s_{k+1} - \varpi^*\| &\le (1 - \varsigma_k)\|s_k - \varpi^*\| + \varsigma_k K_1 \\ &\le \max\{\|s_k - \varpi^*\|, K_1\} \\ &\vdots \\ &\le \max\{\|s_{N_1} - \varpi^*\|, K_1\}.\end{aligned} \tag{3.31}$$

This implies that the sequence $\{s_k\}$ is a bounded sequence.

Claim 2:

$$\begin{aligned}&\left(1 - \frac{\mu \hbar_k}{\hbar_{k+1}}\right)\|q_k - r_k\|^2 + \left(1 - \frac{\mu \hbar_k}{\hbar_{k+1}}\right)\|p_k - r_k\|^2 \\ &\quad + \sigma_k(1 - \rho - \sigma_k)\|\mathcal{M}(p_k) - p_k\|^2 \\ &\le \|s_k - \varpi^*\|^2 - \|s_{k+1} - \varpi^*\|^2 + \varsigma_k K_2,\end{aligned} \tag{3.32}$$

for some $K_2 > 0$. By using the definition of $\{s_{k+1}\}$, we have

$$\begin{aligned}\|s_{k+1} - \varpi^*\|^2 &= \|(1 - \sigma_k)p_k + \sigma_k \mathcal{M}(p_k) - \varpi^*\|^2 \\ &= \|p_k - \varpi^*\|^2 + 2\sigma_k\langle p_k - \varpi^*, \mathcal{M}(p_k) - p_k\rangle + \sigma_k^2\|\mathcal{M}(p_k) - p_k\|^2 \\ &\le \|p_k - \varpi^*\|^2 + \sigma_k(\rho - 1)\|\mathcal{M}(p_k) - p_k\|^2 + \sigma_k^2\|\mathcal{M}(p_k) - p_k\|^2 \\ &= \|p_k - \varpi^*\|^2 - \sigma_k(1 - \rho - \sigma_k)\|\mathcal{M}(p_k) - p_k\|^2.\end{aligned} \tag{3.33}$$

By using the expression (3.18), we can derive

$$\|p_k - \varpi^*\|^2 \le \|q_k - \varpi^*\|^2 - \left(1 - \frac{\mu \hbar_k}{\hbar_{k+1}}\right)\|q_k - r_k\|^2 - \left(1 - \frac{\mu \hbar_k}{\hbar_{k+1}}\right)\|p_k - r_k\|^2. \tag{3.34}$$

Thus, the expression (3.27) implies that

$$\begin{aligned}\|q_k - \varpi^*\|^2 &\le (1 - \varsigma_k)^2\|s_k - \varpi^*\|^2 + \varsigma_k^2 K_1^2 + 2K_1\varsigma_k(1 - \varsigma_k)\|s_k - \varpi^*\| \\ &\le \|s_k - \varpi^*\|^2 + \varsigma_k[\varsigma_k K_1^2 + 2K_1(1 - \varsigma_k)\|s_k - \varpi^*\|] \\ &\le \|s_k - \varpi^*\|^2 + \varsigma_k K_2,\end{aligned} \tag{3.35}$$

for $K_2 > 0$. Combining expressions (3.33), (3.34), and (3.35), we obtain

$$\begin{aligned}\|s_{k+1} - \varpi^*\|^2 &\le \|s_k - \varpi^*\|^2 + \varsigma_k K_2 - \sigma_k(1-\rho-\sigma_k)\|\mathcal{M}(p_k) - p_k\|^2 \\ &\quad - \left(1 - \frac{\mu\hbar_k}{\hbar_{k+1}}\right)\|q_k - r_k\|^2 - \left(1 - \frac{\mu\hbar_k}{\hbar_{k+1}}\right)\|p_k - r_k\|^2. \end{aligned} \tag{3.36}$$

Claim 3:

By using the definition of $\{q_k\}$, we obtain

$$\begin{aligned}&\|q_k - \varpi^*\|^2 \\ &= \|s_k + \ell_k(s_k - s_{k-1}) - \varsigma_k s_k - \ell_k\varsigma_k(s_k - s_{k-1}) - \varpi^*\|^2 \\ &= \|(1-\varsigma_k)(s_k - \varpi^*) + (1-\varsigma_k)\ell_k(s_k - s_{k-1}) - \varsigma_k\varpi^*\|^2 \\ &\le \|(1-\varsigma_k)(s_k - \varpi^*) + (1-\varsigma_k)\ell_k(s_k - s_{k-1})\|^2 + 2\varsigma_k\langle -\varpi^*, q_k - \varpi^*\rangle \\ &= (1-\varsigma_k)^2\|s_k - \varpi^*\|^2 + (1-\varsigma_k)^2\ell_k^2\|s_k - s_{k-1}\|^2 \\ &\quad + 2\ell_k(1-\varsigma_k)^2\|s_k - \varpi^*\|\|s_k - s_{k-1}\| + 2\varsigma_k\langle -\varpi^*, q_k - s_{k+1}\rangle + 2\varsigma_k\langle -\varpi^*, s_{k+1} - \varpi^*\rangle \\ &\le (1-\varsigma_k)\|s_k - \varpi^*\|^2 + \ell_k^2\|s_k - s_{k-1}\|^2 + 2\ell_k(1-\varsigma_k)\|s_k - \varpi^*\|\|s_k - s_{k-1}\| \\ &\quad + 2\varsigma_k\|\varpi^*\|\|q_k - s_{k+1}\| + 2\varsigma_k\langle -\varpi^*, s_{k+1} - \varpi^*\rangle \\ &= (1-\varsigma_k)\|s_k - \varpi^*\|^2 + \varsigma_k\bigg[\ell_k\|s_k - s_{k-1}\|\frac{\ell_k}{\varsigma_k}\|s_k - s_{k-1}\| \\ &\quad + 2(1-\varsigma_k)\|s_k - \varpi^*\|\frac{\ell_k}{\varsigma_k}\|s_k - s_{k-1}\| + 2\|\varpi^*\|\|q_k - s_{k+1}\| \\ &\quad + 2\langle \varpi^*, \varpi^* - s_{k+1}\rangle\bigg]. \end{aligned} \tag{3.37}$$

Combining expressions (3.29) and (3.37), we obtain

$$\begin{aligned}&\|s_{k+1} - \varpi^*\|^2 \\ &\le (1-\varsigma_k)\|s_k - \varpi^*\|^2 + \varsigma_k\bigg[\ell_k\|s_k - s_{k-1}\|\frac{\ell_k}{\varsigma_k}\|s_k - s_{k-1}\| \\ &\quad + 2(1-\varsigma_k)\|s_k - \varpi^*\|\frac{\ell_k}{\varsigma_k}\|s_k - s_{k-1}\| + 2\|\varpi^*\|\|q_k - s_{k+1}\| \\ &\quad + 2\langle \varpi^*, \varpi^* - s_{k+1}\rangle\bigg]. \end{aligned} \tag{3.38}$$

Claim 4: The sequence $\|\mathbf{s_k} - \varpi^*\|^2$ converges to zero.

Suppose that

$$p_k := \|s_k - \varpi^*\|^2$$

and

$$e_k := \ell_k \|s_k - s_{k-1}\| \frac{\ell_k}{\varsigma_k} \|s_k - s_{k-1}\| + 2(1-\varsigma_k)\|s_k - \varpi^*\| \frac{\ell_k}{\varsigma_k} \|s_k - s_{k-1}\| \\ + 2\|\varpi^*\| \|q_k - s_{k+1}\| + 2\langle \varpi^*, \varpi^* - s_{k+1}\rangle.$$

Then, **Claim 4** can be rewritten as follows:

$$p_{k+1} \le (1-\varsigma_k)p_k + \varsigma_k e_k.$$

Indeed, from Lemma 2.5, it suffices to prove that $\limsup_{j\to\infty} e_{k_j} \le 0$ for any subsequence $\{p_{k_j}\}$ of $\{p_k\}$ satisfying

$$\liminf_{j\to+\infty}(p_{k_j+1} - p_{k_j}) \ge 0.$$

This is comparable to demonstrating that

$$\limsup_{j\to\infty}\langle \varpi^*, \varpi^* - s_{k_j+1}\rangle \le 0$$

and

$$\limsup_{j\to\infty}\|q_{k_j} - s_{k_j+1}\| \le 0,$$

in each subsequence $\{\|s_{k_j} - \varpi^*\|\}$ of $\{\|s_k - \varpi^*\|\}$ reassuring

$$\liminf_{j\to+\infty}\left(\|s_{k_j+1} - \varpi^*\| - \|s_{k_j} - \varpi^*\|\right) \ge 0.$$

Suppose that $\{\|s_{k_j} - \varpi^*\|\}$ is a subsequence of $\{\|s_k - \varpi^*\|\}$ satisfying

$$\liminf_{j\to+\infty}\left(\|s_{k_j+1} - \varpi^*\| - \|s_{k_j} - \varpi^*\|\right) \ge 0.$$

Then,

$$\liminf_{j\to+\infty}\left(\|s_{k_j+1} - \varpi^*\|^2 - \|s_{k_j} - \varpi^*\|^2\right) \\ = \liminf_{j\to+\infty}\left(\|s_{k_j+1} - \varpi^*\| - \|s_{k_j} - \varpi^*\|\right)\left(\|s_{k_j+1} - \varpi^*\| + \|s_{k_j} - \varpi^*\|\right) \ge 0. \tag{3.39}$$

By using Claim 2 that

$$\limsup_{j\to\infty}\left[\left(1 - \frac{\mu\hbar_{k_j}}{\hbar_{k_j+1}}\right)\|q_{k_j} - r_{k_j}\|^2 \right. \\ \left. + \left(1 - \frac{\mu\hbar_{k_j}}{\hbar_{k_j+1}}\right)\|p_{k_j} - r_{k_j}\|^2 + \sigma_{k_j}(1-\rho-\sigma_{k_j})\|\mathcal{M}(p_{k_j}) - p_{k_j}\|^2\right] \\ \le \limsup_{j\to\infty}\left[\|s_{k_j} - \varpi^*\|^2 - \|s_{k_j+1} - \varpi^*\|^2\right] + \limsup_{j\to\infty}\varsigma_{k_j}K_2$$

$$= -\liminf_{j\to\infty}\left[\left\|s_{k_j+1}-\varpi^*\right\|^2-\left\|s_{k_j}-\varpi^*\right\|^2\right]$$
$$\leq 0, \tag{3.40}$$

the above relationship suggests that

$$\lim_{j\to\infty}\|q_{k_j}-r_{k_j}\|=0, \qquad \lim_{j\to\infty}\|p_{k_j}-r_{k_j}\|=0, \qquad \lim_{j\to\infty}\left\|\mathcal{M}(p_{k_j})-p_{k_j}\right\|=0. \tag{3.41}$$

Thus, we obtain

$$\lim_{j\to\infty}\|p_{k_j}-q_{k_j}\|=0. \tag{3.42}$$

Next, we compute the following:

$$\begin{aligned}\|q_{k_j}-s_{k_j}\| &= \left\|s_{k_j}+\ell_{k_j}(s_{k_j}-s_{k_j-1})-\varsigma_{k_j}\left[s_{k_j}+\ell_{k_j}(s_{k_j}-s_{k_j-1})\right]-s_{k_j}\right\| \\ &\leq \ell_{k_j}\|s_{k_j}-s_{k_j-1}\|+\varsigma_{k_j}\|s_{k_j}\|+\ell_{k_j}\varsigma_{k_j}\|s_{k_j}-s_{k_j-1}\| \\ &= \varsigma_{k_j}\frac{\ell_{k_j}}{\varsigma_{k_j}}\|s_{k_j}-s_{k_j-1}\|+\varsigma_{k_j}\|s_{k_j}\|+\varsigma_{k_j}^2\frac{\ell_{k_j}}{\varsigma_{k_j}}\|s_{k_j}-s_{k_j-1}\| \longrightarrow 0.\end{aligned} \tag{3.43}$$

This, together with $\lim_{j\to\infty}\|p_{k_j}-q_{k_j}\|=0$, yields that

$$\lim_{j\to\infty}\|p_{k_j}-s_{k_j}\|=0. \tag{3.44}$$

From the definition of $s_{k_j+1}=(1-\sigma_{k_j})p_{k_j}+\sigma_{k_j}\mathcal{M}(p_{k_j})$, one sees that

$$\lim_{j\to\infty}\|s_{k_j+1}-p_{k_j}\|=\sigma_{k_j}\left\|\mathcal{M}(p_{k_j})-p_{k_j}\right\|\leq(1-\rho)\left\|\mathcal{M}(p_{k_j})-p_{k_j}\right\|. \tag{3.45}$$

Thus, we obtain

$$\lim_{j\to\infty}\|s_{k_j+1}-p_{k_j}\|=0. \tag{3.46}$$

The above expression implies that

$$\lim_{j\to\infty}\|s_{k_j}-s_{k_j+1}\|\leq\lim_{j\to\infty}\|s_{k_j}-p_{k_j}\|+\lim_{j\to\infty}\|p_{k_j}-s_{k_j+1}\|=0 \tag{3.47}$$

and

$$\lim_{j\to\infty}\|q_{k_j}-s_{k_j+1}\|\leq\lim_{j\to\infty}\|q_{k_j}-p_{k_j}\|+\lim_{j\to\infty}\|p_{k_j}-s_{k_j+1}\|=0. \tag{3.48}$$

This implies that the sequence $\{s_{k_j}\}$ is a bounded sequence. We can infer that $\{s_{k_j}\}$ weakly converges to some $\hat{u}\in\mathcal{E}$. By using the value $\varpi^*=P_{\mathrm{VI}(\mathcal{D},\mathcal{N})\cap\mathrm{Fix}(\mathcal{M})}(0)$, we have

$$\left\langle 0-\varpi^*, y-\varpi^*\right\rangle\leq 0, \quad \forall y\in\mathrm{VI}(\mathcal{D},\mathcal{N})\cap\mathrm{Fix}(\mathcal{M}). \tag{3.49}$$

From the expression (3.43) it is provided that $\{q_{k_j}\}$ weakly converges to $\hat{u} \in \mathcal{E}$. By using the expression (3.41), $\lim_{k\to\infty} \hbar_k = \hbar$, and Lemma 2.4, one concludes that $\hat{u} \in \mathrm{VI}(\mathcal{D},\mathcal{N})$. It follows from (3.44) that $\{p_{k_j}\}$ weakly converges to $\hat{u} \in \mathcal{E}$. Due to the use of the demiclosedness of $(I - \mathcal{M})$, we derive that $\hat{u} \in \mathrm{Fix}(\mathcal{M})$. This implies that $\hat{u} \in \mathrm{VI}(\mathcal{D},\mathcal{N}) \cap \mathrm{Fix}(\mathcal{M})$. Thus, we obtain

$$\lim_{j\to\infty} \langle \varpi^*, \varpi^* - s_{k_j} \rangle = \langle \varpi^*, \varpi^* - \hat{u} \rangle \leq 0. \tag{3.50}$$

Next, we can use $\lim_{j\to\infty} \|s_{k_j+1} - s_{k_j}\| = 0$. Thus, we can write

$$\begin{aligned} &\limsup_{j\to\infty} \langle \varpi^*, \varpi^* - s_{k_j+1} \rangle \\ &\quad \leq \limsup_{j\to\infty} \langle \varpi^*, \varpi^* - s_{k_j} \rangle + \limsup_{j\to\infty} \langle \varpi^*, s_{k_j} - s_{k_j+1} \rangle \leq 0. \end{aligned} \tag{3.51}$$

By using Claim 3 and Lemma 2.5, we see that $s_k \to \varpi^*$ as $k \to \infty$. This completes the proof of the theorem. □

Theorem 3.8 *Let $\mathcal{N} : \mathcal{E} \to \mathcal{E}$ be an operator that satisfies the conditions* $(\mathcal{N}1)$–$(\mathcal{N}5)$. *Then, the sequence $\{s_k\}$ generated by the Algorithm 4 converges strongly to $\varpi^* \in \mathrm{VI}(\mathcal{D},\mathcal{N}) \cap \mathrm{Fix}(\mathcal{M})$, where $\varpi^* = P_{\mathrm{VI}(\mathcal{D},\mathcal{N})\cap \mathrm{Fix}(\mathcal{M})}(0)$.*

Proof **Claim 1: $\{\mathbf{s_k}\}$ is a bounded sequence.**

Let us consider that

$$s_{k+1} = (1 - \sigma_k)p_k + \sigma_k \mathcal{M}(p_k).$$

By using the definition of a sequence $\{s_{k+1}\}$, we have

$$\begin{aligned} \|s_{k+1} - \varpi^*\|^2 &= \|(1 - \sigma_k)p_k + \sigma_k \mathcal{M}(p_k) - \varpi^*\|^2 \\ &= \|p_k - \varpi^*\|^2 + 2\sigma_k \langle p_k - \varpi^*, \mathcal{M}(p_k) - p_k \rangle + \sigma_k^2 \|\mathcal{M}(p_k) - p_k\|^2 \\ &\leq \|p_k - \varpi^*\|^2 + \sigma_k(\rho - 1)\|\mathcal{M}(p_k) - p_k\|^2 + \sigma_k^2 \|\mathcal{M}(p_k) - p_k\|^2 \\ &= \|p_k - \varpi^*\|^2 - \sigma_k(1 - \rho - \sigma_k)\|\mathcal{M}(p_k) - p_k\|^2. \end{aligned} \tag{3.52}$$

By using the value of $\{q_k\}$, we obtain

$$\begin{aligned} \|q_k - \varpi^*\| &= \|s_k + \ell_k(s_k - s_{k-1}) - \varsigma_k s_k - \ell_k \varsigma_k (s_k - s_{k-1}) - \varpi^*\| \\ &= \|(1 - \varsigma_k)(s_k - \varpi^*) + (1 - \varsigma_k)\ell_k(s_k - s_{k-1}) - \varsigma_k \varpi^*\| \qquad (3.53) \\ &\leq (1 - \varsigma_k)\|s_k - \varpi^*\| + (1 - \varsigma_k)\ell_k\|s_k - s_{k-1}\| + \varsigma_k\|\varpi^*\| \\ &\leq (1 - \varsigma_k)\|s_k - \varpi^*\| + \varsigma_k M_1, \qquad (3.54) \end{aligned}$$

for some fixed number M_1 we have

$$(1 - \varsigma_k)\frac{\ell_k}{\varsigma_k}\|s_k - s_{k-1}\| + \|\varpi^*\| \leq M_1.$$

By using $\hbar_k \to \hbar$ such that $\chi \in (0, 1-\mu^2)$, we have

$$\lim_{k\to\infty}\left(1-\mu^2\frac{\hbar_k^2}{\hbar_{k+1}^2}\right)=1-\mu^2>\chi>0.$$

Thus, there exists some fixed $k_0 \in \mathbb{N}$ such that

$$\left(1-\mu^2\frac{\hbar_k^2}{\hbar_{k+1}^2}\right)>\chi>0,\quad \forall k\geq k_0. \tag{3.55}$$

By using Lemma 3.5, we can rewrite

$$\left\|p_k-\varpi^*\right\|^2\leq\left\|q_k-\varpi^*\right\|^2,\quad \forall k\geq k_0. \tag{3.56}$$

From expressions (3.52), (3.54), and (3.56) we infer that

$$\left\|s_{k+1}-\varpi^*\right\|\leq(1-\varsigma_k)\left\|s_k-\varpi^*\right\|+\varsigma_k M_1-\sigma_k(1-\rho-\sigma_k)\left\|\mathcal{M}(p_k)-p_k\right\|^2. \tag{3.57}$$

Thus, for $\{\sigma_k\}\subset(0,1-\rho)$, we obtain

$$\begin{aligned}\left\|s_{k+1}-\varpi^*\right\|&\leq(1-\varsigma_k)\left\|s_k-\varpi^*\right\|+\varsigma_k M_1\\&\leq\max\left\{\left\|s_k-\varpi^*\right\|,M_1\right\}\\&\ \ \vdots\\&\leq\max\left\{\left\|s_{k_0}-\varpi^*\right\|,M_1\right\}.\end{aligned} \tag{3.58}$$

Consequently, we may infer that the sequence $\{s_k\}$ is a bounded sequence.

Claim 2:

$$\begin{aligned}&\left(1-\mu^2\frac{\hbar_k^2}{\hbar_{k+1}^2}\right)\|q_k-r_k\|^2+\sigma_k(1-\rho-\sigma_k)\left\|\mathcal{M}(p_k)-p_k\right\|^2\\&\quad\leq\left\|s_k-\varpi^*\right\|^2-\left\|s_{k+1}-\varpi^*\right\|^2+\varsigma_k M_2,\end{aligned} \tag{3.59}$$

for some fixed $M_2>0$. Indeed, by using the definition of $\{s_{k+1}\}$, we have

$$\begin{aligned}\left\|s_{k+1}-\varpi^*\right\|^2&=\left\|(1-\sigma_k)p_k+\sigma_k\mathcal{M}(p_k)-\varpi^*\right\|^2\\&=\left\|p_k-\varpi^*\right\|^2+2\sigma_k\langle p_k-\varpi^*,\mathcal{M}(p_k)-p_k\rangle+\sigma_k^2\left\|\mathcal{M}(p_k)-p_k\right\|^2\\&\leq\left\|p_k-\varpi^*\right\|^2+\sigma_k(\rho-1)\left\|\mathcal{M}(p_k)-p_k\right\|^2+\sigma_k^2\left\|\mathcal{M}(p_k)-p_k\right\|^2\\&=\left\|p_k-\varpi^*\right\|^2-\sigma_k(1-\rho-\sigma_k)\left\|\mathcal{M}(p_k)-p_k\right\|^2.\end{aligned} \tag{3.60}$$

By using Lemma 3.5, we obtain

$$\left\|p_k-\varpi^*\right\|^2\leq\left\|q_k-\varpi^*\right\|^2-\left(1-\mu^2\frac{\hbar_k^2}{\hbar_{k+1}^2}\right)\|q_k-r_k\|^2. \tag{3.61}$$

By using expression (3.54), we can obtain

$$\begin{aligned}\|q_k - \varpi^*\|^2 &\leq (1-\varsigma_k)^2\|s_k - \varpi^*\|^2 + \varsigma_k^2 M_1^2 + 2M_1\varsigma_k(1-\varsigma_k)\|s_k - \varpi^*\| \\ &\leq \|s_k - \varpi^*\|^2 + \varsigma_k\big[\varsigma_k M_1^2 + 2M_1(1-\varsigma_k)\|s_k - \varpi^*\|\big] \\ &\leq \|s_k - \varpi^*\|^2 + \varsigma_k M_2, \end{aligned} \tag{3.62}$$

where for some fixed constant $M_2 > 0$. From expressions (3.60), (3.61), and (3.62) we obtain

$$\begin{aligned}\|s_{k+1} - \varpi^*\|^2 \leq{}& \|s_k - \varpi^*\|^2 + \varsigma_k M_2 - \sigma_k(1-\rho-\sigma_k)\|\mathcal{M}(p_k) - p_k\|^2 \\ &- \left(1 - \mu^2\frac{\hbar_k^2}{\hbar_{k+1}^2}\right)\|q_k - r_k\|^2. \end{aligned} \tag{3.63}$$

Claim 3:
Using the value of $\{q_k\}$, we can write as follows:

$$\begin{aligned}&\|q_k - \varpi^*\|^2 \\ &= \|s_k + \ell_k(s_k - s_{k-1}) - \varsigma_k s_k - \ell_k\varsigma_k(s_k - s_{k-1}) - \varpi^*\|^2 \\ &= \|(1-\varsigma_k)(s_k - \varpi^*) + (1-\varsigma_k)\ell_k(s_k - s_{k-1}) - \varsigma_k\varpi^*\|^2 \\ &\leq \|(1-\varsigma_k)(s_k - \varpi^*) + (1-\varsigma_k)\ell_k(s_k - s_{k-1})\|^2 + 2\varsigma_k\langle -\varpi^*, q_k - \varpi^*\rangle \\ &= (1-\varsigma_k)^2\|s_k - \varpi^*\|^2 + (1-\varsigma_k)^2\ell_k^2\|s_k - s_{k-1}\|^2 \\ &\quad + 2\ell_k(1-\varsigma_k)^2\|s_k - \varpi^*\|\|s_k - s_{k-1}\| + 2\varsigma_k\langle -\varpi^*, q_k - s_{k+1}\rangle \\ &\quad + 2\varsigma_k\langle -\varpi^*, s_{k+1} - \varpi^*\rangle \\ &\leq (1-\varsigma_k)\|s_k - \varpi^*\|^2 + \ell_k^2\|s_k - s_{k-1}\|^2 + 2\ell_k(1-\varsigma_k)\|s_k - \varpi^*\|\|s_k - s_{k-1}\| \\ &\quad + 2\varsigma_k\|\varpi^*\|\|q_k - s_{k+1}\| + 2\varsigma_k\langle -\varpi^*, s_{k+1} - \varpi^*\rangle \\ &= (1-\varsigma_k)\|s_k - \varpi^*\|^2 + \varsigma_k\bigg[\ell_k\|s_k - s_{k-1}\|\frac{\ell_k}{\varsigma_k}\|s_k - s_{k-1}\| \\ &\quad + 2(1-\varsigma_k)\|s_k - \varpi^*\|\frac{\ell_k}{\varsigma_k}\|s_k - s_{k-1}\| + 2\|\varpi^*\|\|q_k - s_{k+1}\| \\ &\quad + 2\langle \varpi^*, \varpi^* - s_{k+1}\rangle\bigg]. \end{aligned} \tag{3.64}$$

Combining expressions (3.56) and (3.64), we obtain

$$\begin{aligned}&\|s_{k+1} - \varpi^*\|^2 \\ &\leq (1-\varsigma_k)\|s_k - \varpi^*\|^2 + \varsigma_k\bigg[\ell_k\|s_k - s_{k-1}\|\frac{\ell_k}{\varsigma_k}\|s_k - s_{k-1}\| \\ &\quad + 2(1-\varsigma_k)\|s_k - \varpi^*\|\frac{\ell_k}{\varsigma_k}\|s_k - s_{k-1}\| + 2\|\varpi^*\|\|q_k - s_{k+1}\| \\ &\quad + 2\langle \varpi^*, \varpi^* - s_{k+1}\rangle\bigg]. \end{aligned} \tag{3.65}$$

Claim 4: $\|\mathbf{s_k} - \varpi^*\|^2$ **is a sequence that is convergent to zero.**
Set

$$p_k := \left\|s_k - \varpi^*\right\|^2$$

and

$$\begin{aligned} e_k &:= \ell_k \|s_k - s_{k-1}\| \frac{\ell_k}{\varsigma_k} \|s_k - s_{k-1}\| + 2(1-\varsigma_k)\left\|s_k - \varpi^*\right\| \frac{\ell_k}{\varsigma_k} \|s_k - s_{k-1}\| \\ &\quad + 2\left\|\varpi^*\right\| \|q_k - s_{k+1}\| + 2\langle \varpi^*, \varpi^* - s_{k+1}\rangle. \end{aligned}$$

Then, **Claim 4** can be rewritten as follows:

$$c_{k+1} \leq (1-\varsigma_k)p_k + \varsigma_k e_k.$$

By Lemma 2.5, it suffices to show that $\limsup_{j\to\infty} e_{k_j} \leq 0$ for $\{c_{k_j}\}$ of $\{p_k\}$ such as

$$\liminf_{j\to+\infty}(c_{k_j+1} - c_{k_j}) \geq 0.$$

This seems to be equivalent to stating that

$$\limsup_{j\to\infty}\langle \varpi^*, \varpi^* - s_{k_j+1}\rangle \leq 0$$

and

$$\limsup_{j\to\infty}\|q_{k_j} - s_{k_j+1}\| \leq 0,$$

one from each subsequence $\{\|s_{k_j} - \varpi^*\|\}$ of $\{\|s_k - \varpi^*\|\}$ following that

$$\liminf_{j\to+\infty}\left(\left\|s_{k_j+1} - \varpi^*\right\| - \left\|s_{k_j} - \varpi^*\right\|\right) \geq 0.$$

Suppose that $\{\|s_{k_j} - \varpi^*\|\}$ is a subsequence of $\{\|s_k - \varpi^*\|\}$ satisfying

$$\liminf_{j\to+\infty}\left(\left\|s_{k_j+1} - \varpi^*\right\| - \left\|s_{k_j} - \varpi^*\right\|\right) \geq 0,$$

then, we have

$$\begin{aligned} &\liminf_{j\to+\infty}\left(\left\|s_{k_j+1} - \varpi^*\right\|^2 - \left\|s_{k_j} - \varpi^*\right\|^2\right) \\ &\quad = \liminf_{j\to+\infty}\left(\left\|s_{k_j+1} - \varpi^*\right\| - \left\|s_{k_j} - \varpi^*\right\|\right)\left(\left\|s_{k_j+1} - \varpi^*\right\| + \left\|s_{k_j} - \varpi^*\right\|\right) \geq 0. \end{aligned} \tag{3.66}$$

As a result of Claim 2 that

$$\begin{aligned} &\limsup_{j\to\infty}\left[\left(1 - \frac{\mu^2 \hbar_{k_j}^2}{\hbar_{k_j+1}^2}\right)\|q_{k_j} - r_{k_j}\|^2 + \sigma_{k_j}(1-\rho-\sigma_{k_j})\left\|\mathcal{M}(p_{k_j}) - p_{k_j}\right\|^2\right] \\ &\quad \leq \limsup_{j\to\infty}\left[\left\|s_{k_j} - \varpi^*\right\|^2 - \left\|s_{k_j+1} - \varpi^*\right\|^2\right] + \limsup_{j\to\infty}\varsigma_{k_j} K_2 \end{aligned}$$

$$= -\liminf_{j\to\infty}\left[\left\|s_{k_j+1} - \varpi^*\right\|^2 - \left\|s_{k_j} - \varpi^*\right\|^2\right]$$
$$\leq 0, \tag{3.67}$$

the above relationship implies that

$$\lim_{j\to\infty}\|q_{k_j} - r_{k_j}\| = 0, \qquad \lim_{j\to\infty}\left\|\mathcal{M}(p_{k_j}) - p_{k_j}\right\| = 0. \tag{3.68}$$

It follows that

$$\|p_{k_j} - r_{k_j}\| = \left\|r_{k_j} + \hbar_{k_j}\left[\mathcal{N}(q_{k_j}) - \mathcal{N}(r_{k_j})\right] - r_{k_j}\right\| \leq \hbar_{k_j}L\|q_{k_j} - r_{k_j}\|. \tag{3.69}$$

Thus, we have

$$\lim_{j\to\infty}\|p_{k_j} - r_{k_j}\| = 0. \tag{3.70}$$

The remaining proof is similar to Claim 4 of Theorem 3.7. As a result, we omit it here and this completes the proof of the theorem. □

4 Numerical illustrations

In contrast to some past work in the literature, this part discusses the algorithmic implications of the supplied techniques, as well as a study of how differences in control settings affect the numerical efficacy of the recommended algorithms. All calculations are performed in MATLAB R2018b on an HP i5 Core (TM) i5-6200 laptop with 8.00 GB (7.78 GB usable) RAM.

Example 4.1 Consider that a mapping $\mathcal{N} : \mathbb{R}^m \to \mathbb{R}^m$ is described using

$$\mathcal{N}(u) = Mu + q,$$

where $q = 0$. Moreover, we have

$$M = NN^T + B + D.$$

The matrices $N = \text{rand}(m)$ and $K = \text{rand}(m)$ are chosen randomly, whereas the other two are generated in the following manner:

$$B = 0.5K - 0.5K^T \quad \text{and} \quad D = \text{diag}\big(\text{rand}(m,1)\big).$$

The feasible set $\mathcal{D}$ is interpreted as follows:

$$\mathcal{D} = \left\{u \in \mathbb{R}^m : -10 \leq s_i \leq 10\right\}.$$

It is evident that the mapping $\mathcal{N}$ is monotone and Lipschitz is continuous with the value $L = \|M\|$. Moreover, the function $\mathcal{M} : \mathcal{E} \to \mathcal{E}$ is considered as follows:

$$\mathcal{M}(u) = \frac{1}{2}u.$$

The starting points for these tests are $s_0 = s_1 = (2, 2, \dots, 2)$. The dimension of the Hilbert space is treated differently while studying the behavior of higher-dimension Hilbert spaces. The stopping condition for such experiments is $D_k = \|q_k - r_k\| \leq 10^{-10}$. The initial points to run these experiments are taken as $s_0 = s_1 = (2, 2, \dots, 2)$. The dimension of the Hilbert space is taken differently to study the behavior for higher-dimension Hilbert spaces. The stopping criterion for such experiments is taken as $D_k = \|q_k - r_k\| \leq 10^{-10}$. Figures 1–6 and Tables 1 and 2 illustrate empirical observations for Example 2. The following control criteria are in effect:

(1) Algorithm 2 (**alg-1**): $\hbar_1 = 0.55$, $\ell = 0.45$, $\mu = 0.44$, $\chi_k = \frac{100}{(1+k)^2}$, $\varsigma_k = \frac{1}{(2k+4)}$, $\sigma_k = \frac{k}{(2k+1)}$.
(2) Algorithm 4 (**alg-2**): $\hbar_1 = 0.55$, $\ell = 0.45$, $\mu = 0.44$, $\chi_k = \frac{100}{(1+k)^2}$, $\varsigma_k = \frac{1}{(2k+4)}$, $\sigma_k = \frac{k}{(2k+1)}$.
(3) Algorithm 1 in [31] (**mtalg-1**): $\gamma_1 = 0.55$, $\delta = 0.45$, $\phi = 0.44$, $\ell_k = \frac{1}{(2k+4)}$, $\hbar_k = \frac{1}{2}(1 - \ell_k)$, $\chi_k = \frac{100}{(1+k)^2}$.
(4) Algorithm 2 in [31] (**mtalg-2**): $\gamma_1 = 0.55$, $\delta = 0.45$, $\phi = 0.44$, $\ell_k = \frac{1}{(2k+4)}$, $\hbar_k = \frac{1}{2}(1 - \ell_k)$, $\chi_k = \frac{100}{(1+k)^2}$.
(5) Algorithm 1 in [32] (**vtalg-1**): $\tau_1 = 0.55$, $\ell = 0.45$, $\mu = 0.44$, $\chi_k = \frac{100}{(1+k)^2}$, $\varsigma_k = \frac{1}{(2k+4)}$, $\sigma_k = \frac{k}{(2k+1)}$, $f(u) = \frac{u}{2}$.

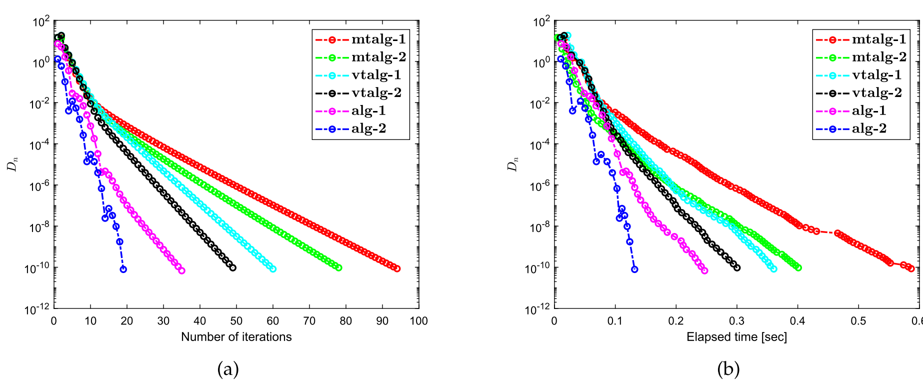


Figure 1 Numerical comparison of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $m = 5$

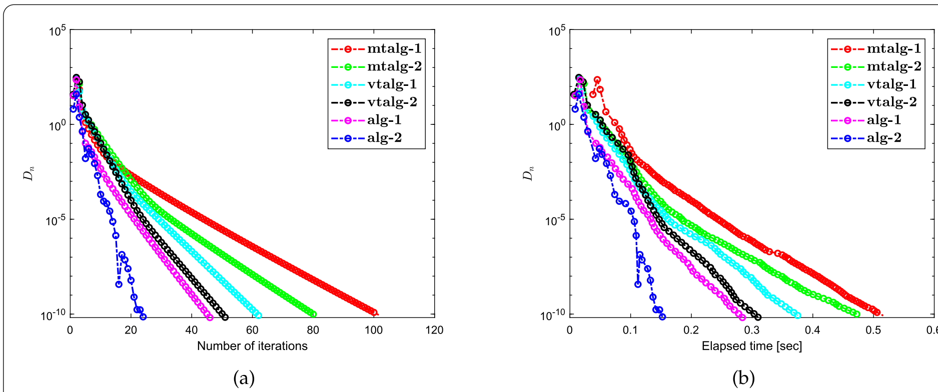
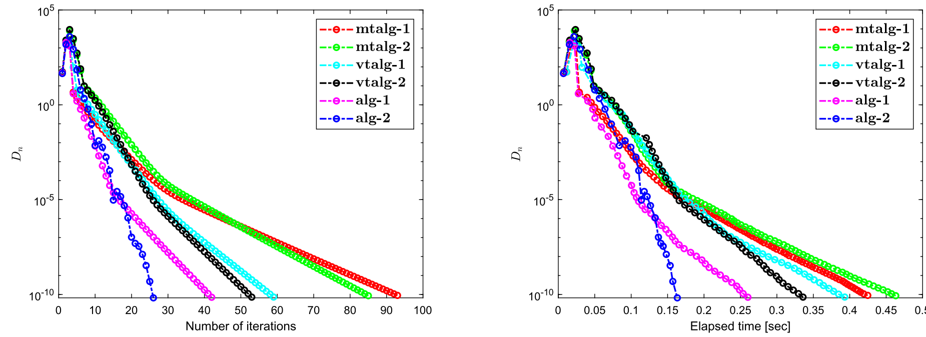
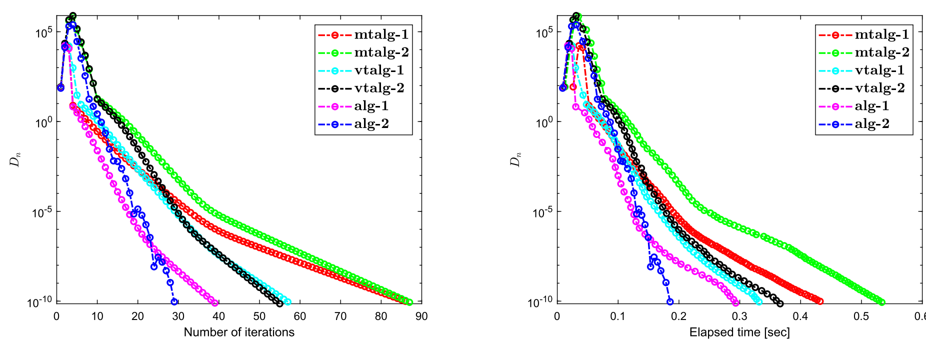


Figure 2 Numerical comparison of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $m = 10$



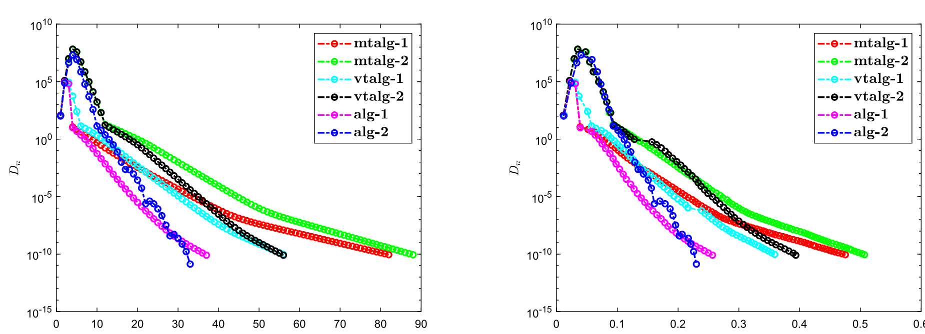
(a) (b)

Figure 3 Numerical comparison of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $m = 20$



(a) (b)

Figure 4 Numerical comparison of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $m = 50$



(a) (b)

Figure 5 Numerical comparison of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $m = 100$

(6) Algorithm 2 in [32] (**vtalg-2**): $\tau_1 = 0.55$, $\ell = 0.45$, $\mu = 0.44$, $\chi_k = \frac{100}{(1+k)^2}$, $\varsigma_k = \frac{1}{(2k+4)}$, $\sigma_k = \frac{k}{(2k+1)}, f(u) = \frac{u}{2}$.

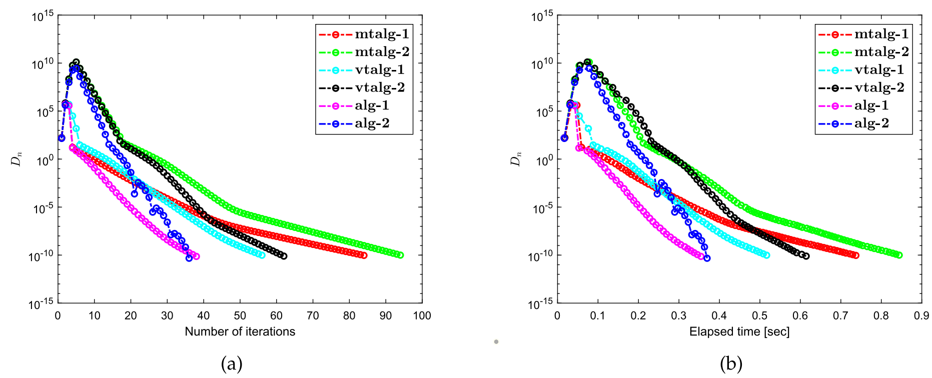


Figure 6 Numerical comparison of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $m = 200$

Table 1 Numerical values for Figs. 1–6

m	The cumulative number of iterations					
	alg-1	**alg-2**	**mtalg-1**	**mtalg-2**	**vtalg-1**	**vtalg-2**
5	35	19	94	78	60	49
10	46	24	102	80	62	51
20	42	25	93	85	59	53
50	39	29	86	87	57	55
100	37	33	84	88	56	56
200	38	36	84	94	56	62

Table 2 Numerical values for Figs. 1–6

m	Time required to complete the task					
	alg-1	**alg-2**	**mtalg-1**	**mtalg-2**	**vtalg-1**	**vtalg-2**
5	0.246841	0.1317703	0.5865135	0.4009539	0.360533465	0.3001653
10	0.284076	0.1523123	0.5159276	0.4722816	0.375091336	0.3097725
20	0.2602246	0.1633652	0.4246998	0.4630932	0.393142367	0.3358743
50	0.293302	0.1854808	0.4320612	0.5335381	0.331728156	0.3663686
100	0.2566573	0.2301228	0.4752024	0.5067862	0.358997537	0.3936471
200	0.3544296	0.3695034	0.7371152	0.8441844	0.516623963	0.6142675

Example 4.2 Consider a nonlinear mapping $\mathcal{N} : \mathcal{R}^2 \to \mathcal{R}^2$ described using

$$\mathcal{N}(u, y) = (u + y + \sin u; -u + y + \sin y).$$

Furthermore, the workable set $\mathcal{D}$ is just a set written as follows:

$$\mathcal{D} = [-1, 1] \times [1, 1].$$

It is simple to demonstrate that $\mathcal{N}$ is monotone and Lipschitz continuous given the constant $L = 3$. Suppose another mapping $\mathcal{M} : \mathcal{R}^2 \to \mathcal{R}^2$ is described as follows:

$$\mathcal{M}(z) = \|E\|^{-1} Ez,$$

Table 3 Numerical values for Figs. 7–14

$s_0 = s_1$	Total number of iterations					
	alg-1	**alg-2**	**mtalg-1**	**mtalg-2**	**vtalg-1**	**vtalg-2**
$(1,1)^T$	49	35	68	75	82	85
$(2,2)^T$	48	36	61	65	77	78
$(1,-1)^T$	44	33	72	83	86	92
$(-2,3)^T$	51	37	65	70	81	79

Table 4 Numerical values for Figs. 7–14

$s_0 = s_1$	Required CPU time					
	alg-1	**alg-2**	**mtalg-1**	**mtalg-2**	**vtalg-1**	**vtalg-2**
$(1,1)^T$	0.2284193	0.1631707	0.2969821	0.3224385	0.3469049	0.3625844
$(2,2)^T$	0.2297859	0.1757931	0.3720656	0.3078242	0.3847476	0.4105755
$(1,-1)^T$	0.1986126	0.1512495	0.3220028	0.3729462	0.3787876	0.4068135
$(-2,3)^T$	0.2380988	0.1703252	0.2690971	0.3069672	0.3448697	0.3428332

where E is defined by:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

The mapping $\mathcal{M}$ is clearly 0-demicontractive, with $\rho = 0$. Depending on the stopping condition, the starting point for this experiment is calculated differently as $D_k = \|q_k - r_k\| \leq 10^{-10}$. Figures 7–14 and Tables 3 and 4 demonstrate quantitative data for Example 4.2. The following control criteria are in effect:

(1) Algorithm 2 (briefly, **alg-1**): $\hbar_1 = 0.45$, $\ell = 0.35$, $\mu = 0.33$, $\chi_k = \frac{10}{(1+k)^2}$, $\varsigma_k = \frac{1}{(3k+6)}$, $\sigma_k = \frac{k}{(3k+1)}$.

(2) Algorithm 4 (briefly, **alg-2**): $\hbar_1 = 0.45$, $\ell = 0.35$, $\mu = 0.33$, $\chi_k = \frac{10}{(1+k)^2}$, $\varsigma_k = \frac{1}{(3k+4)}$, $\sigma_k = \frac{k}{(3k+1)}$.

(3) Algorithm 1 in [31] (briefly, **mtalg-1**): $\gamma_1 = 0.45$, $\delta = 0.35$, $\phi = 0.33$, $\ell_k = \frac{1}{(3k+6)}$, $\hbar_k = \frac{1}{2.5}(1-\ell_k)$, $\chi_k = \frac{10}{(1+k)^2}$.

(4) Algorithm 2 in [31] (briefly, **mtalg-2**): $\gamma_1 = 0.45$, $\delta = 0.35$, $\phi = 0.33$, $\ell_k = \frac{1}{(3k+6)}$, $\hbar_k = \frac{1}{2.5}(1-\ell_k)$, $\chi_k = \frac{10}{(1+k)^2}$.

(5) Algorithm 1 in [32] (briefly, **vtalg-1**): $\tau_1 = 0.45$, $\ell = 0.35$, $\mu = 0.33$, $\chi_k = \frac{10}{(1+k)^2}$, $\varsigma_k = \frac{1}{(3k+6)}$, $\sigma_k = \frac{k}{(3k+1)}$, $f(u) = \frac{u}{2}$.

(6) Algorithm 2 in [32] (briefly, **vtalg-2**): $\tau_1 = 0.45$, $\ell = 0.35$, $\mu = 0.33$, $\chi_k = \frac{10}{(1+k)^2}$, $\varsigma_k = \frac{1}{(3k+6)}$, $\sigma_k = \frac{k}{(3k+1)}$, $f(u) = \frac{u}{2}$.

Example 4.3 Take the following set:

$$\mathcal{D} := \{u \in L^2([0,1]) : \|u\| \leq 1\}.$$

Let $\mathcal{N} : \mathcal{D} \to \mathcal{E}$ be an operator described through

$$\mathcal{N}(u)(t) = \int_0^1 \big(u(t) - H(t,s)f(u(s))\big)\, ds + g(t),$$

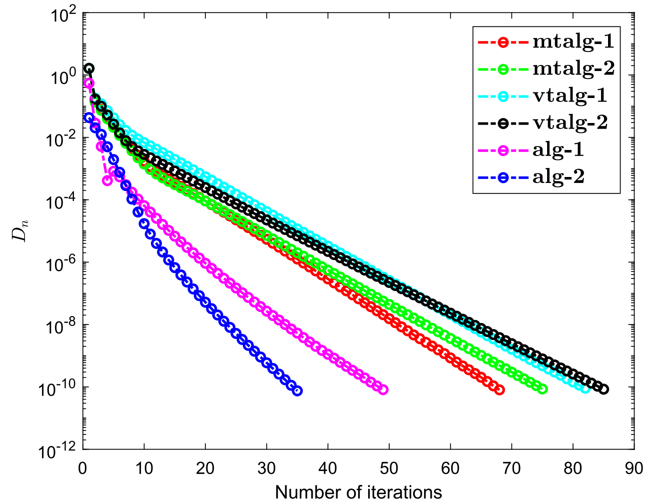


Figure 7 Numerical comparsion of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $s_0 = s_1 = (1,1)^T$

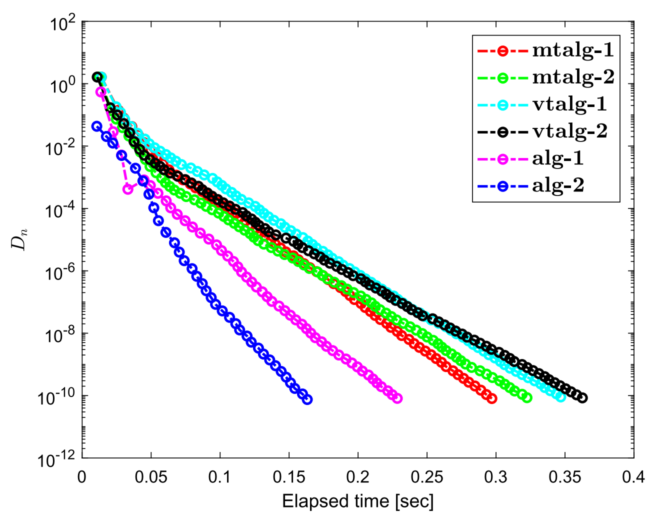


Figure 8 Numerical comparsion of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $s_0 = s_1 = (1,1)^T$

where

$$H(t,s) = \frac{2tse^{(t+s)}}{e\sqrt{e^2-1}}, \qquad f(u) = \cos u, \qquad g(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

In this case, $\mathcal{E} = L^2([0,1])$ denotes a Hilbert space via an inner product

$$\langle u, y\rangle = \int_0^1 u(t)y(t)\,dt, \quad \forall u, y \in \mathcal{E},$$

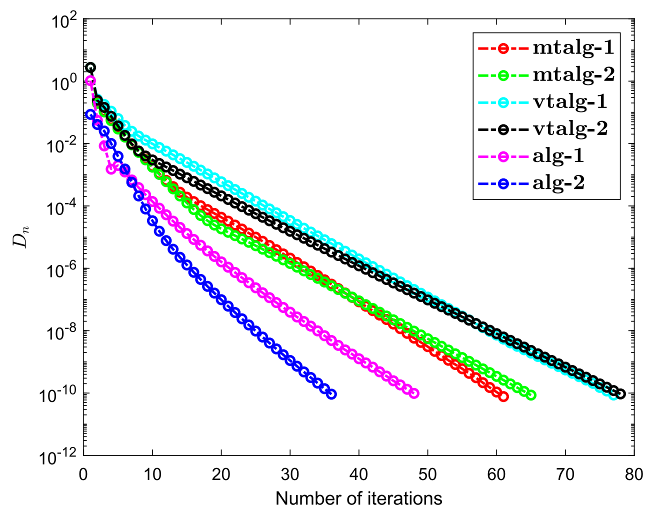


Figure 9 Numerical comparsion of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $s_0 = s_1 = (2,2)^T$

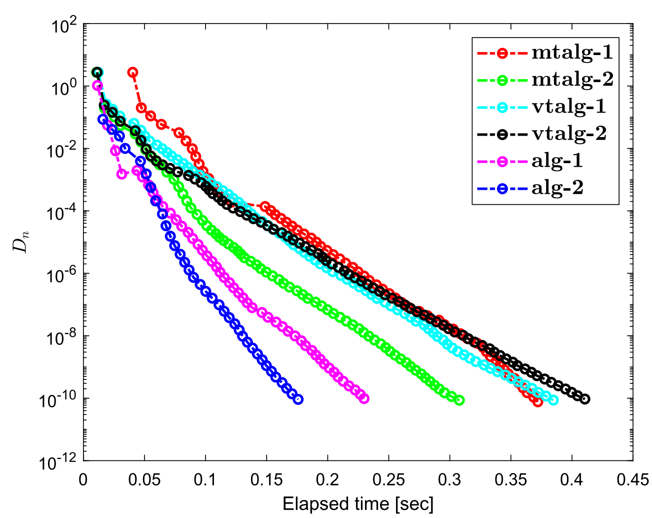


Figure 10 Numerical comparsion of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $s_0 = s_1 = (2,2)^T$

where its induced norm is:

$$\|u\| = \sqrt{\int_0^1 |u(t)|^2 \, dt}.$$

A function $\mathcal{M} : L^2([0,1]) \to L^2([0,1])$ is of the form

$$\mathcal{M}(u)(t) = \int_0^1 tu(s)\, ds, \quad t \in [0,1].$$

A simple calculation suggests that $\mathcal{M}$ is 0-demicontractive. The solution is $\varpi^*(t) = 0$. The stopping condition in this experiment is $D_k = \|q_k - r_k\| \le 10^{-6}$. Figures 15–18 and Tables 5

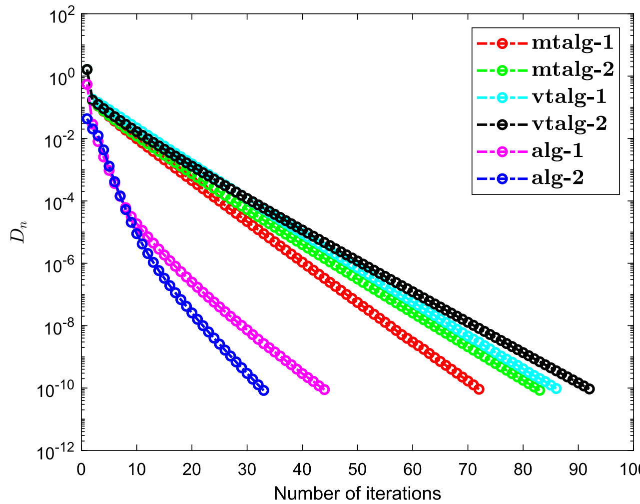


Figure 11 Numerical comparsion of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $s_0 = s_1 = (1, -1)^T$

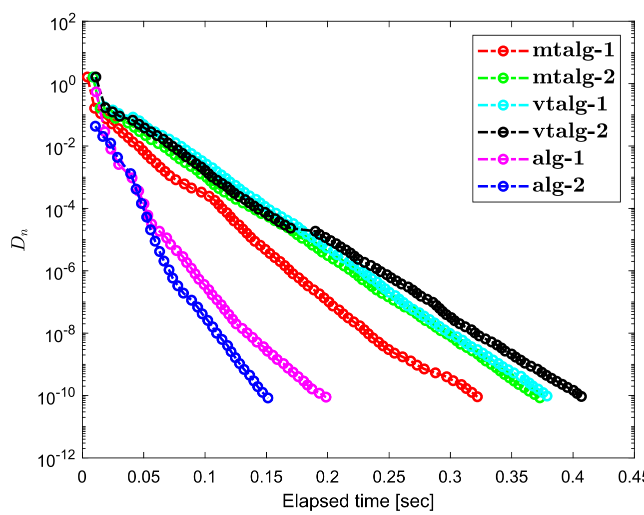


Figure 12 Numerical comparsion of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $s_0 = s_1 = (1, -1)^T$

and 6 illustrate numerical observations for Example 4.3. The following control criteria are in effect:

(1) Algorithm 2 (briefly, **alg-1**): $\hbar_1 = 0.33$, $\ell = 0.66$, $\mu = 0.55$, $\chi_k = \frac{1}{(1+k)^2}$, $\varsigma_k = \frac{1}{(4k+8)}$, $\sigma_k = \frac{k}{(5k+1)}$.

(2) Algorithm 4 (briefly, **alg-2**): $\hbar_1 = 0.33$, $\ell = 0.66$, $\mu = 0.55$, $\chi_k = \frac{1}{(1+k)^2}$, $\varsigma_k = \frac{1}{(4k+8)}$, $\sigma_k = \frac{k}{(5k+1)}$.

(3) Algorithm 1 in [31] (briefly, **mtalg-1**): $\gamma_1 = 0.33$, $\delta = 0.66$, $\phi = 0.55$, $\ell_k = \frac{1}{(4k+8)}$, $\hbar_k = \frac{1}{2}(1 - \ell_k)$, $\chi_k = \frac{1}{(1+k)^2}$.

(4) Algorithm 2 in [31] (briefly, **mtalg-2**): $\gamma_1 = 0.33$, $\delta = 0.66$, $\phi = 0.55$, $\ell_k = \frac{1}{(4k+8)}$, $\hbar_k = \frac{1}{2}(1 - \ell_k)$, $\chi_k = \frac{1}{(1+k)^2}$.

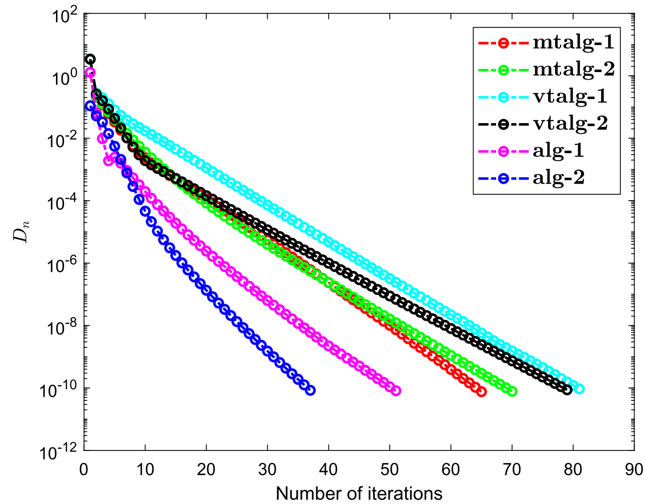


Figure 13 Numerical comparsion of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $s_0 = s_1 = (-2,-3)^T$

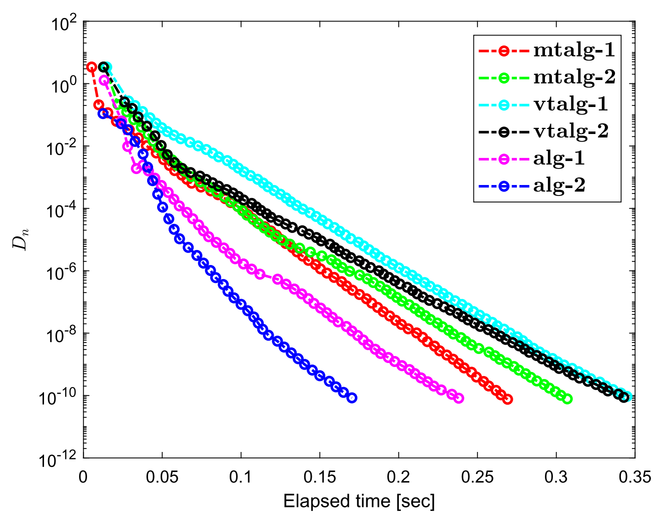


Figure 14 Numerical comparsion of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $s_0 = s_1 = (-2,-3)^T$

(5) Algorithm 1 in [32] (briefly, **vtalg-1**): $\tau_1 = 0.33$, $\ell = 0.66$, $\mu = 0.55$, $\chi_k = \frac{1}{(1+k)^2}$, $\varsigma_k = \frac{1}{(4k+8)}$, $\sigma_k = \frac{k}{(4k+1)}$, $f(u) = \frac{u}{3}$.
(6) Algorithm 2 in [32] (briefly, **vtalg-2**): $\tau_1 = 0.33$, $\ell = 0.66$, $\mu = 0.55$, $\chi_k = \frac{1}{(1+k)^2}$, $\varsigma_k = \frac{1}{(4k+8)}$, $\sigma_k = \frac{k}{(4k+1)}$, $f(u) = \frac{u}{3}$.

Example 4.4 Consider that the feasible set $\mathcal{D}$ is provided by

$$\mathcal{D} := \{u \in L^2([0,1]) : \|u\| \leq 1\}.$$

Table 5 Numerical values for the Figs. 15–18

$s_0 = s_1$	Total number of iterations					
	alg-1	**alg-2**	**mtalg-1**	**mtalg-2**	**vtalg-1**	**vtalg-2**
1	44	33	76	70	66	57
t	42	31	89	84	58	48
sin(t)	45	34	75	64	58	51
cos(t)	47	35	74	94	58	51

Table 6 Numerical values for the Figs. 15–18

$s_0 = s_1$	Required CPU time					
	alg-1	**alg-2**	**mtalg-1**	**mtalg-2**	**vtalg-1**	**vtalg-2**
1	0.1310831	0.1149104	0.2380825	0.2171721	0.1915358	0.178602
t	0.0583617	0.0538350	0.1154974	0.1059993	0.0784157	0.0548289
sin(t)	0.1372786	0.1029274	0.2692971	0.185825	0.1745996	0.1476468
cos(t)	0.1364229	0.1253482	0.2207376	0.2697567	0.172504	0.1452834

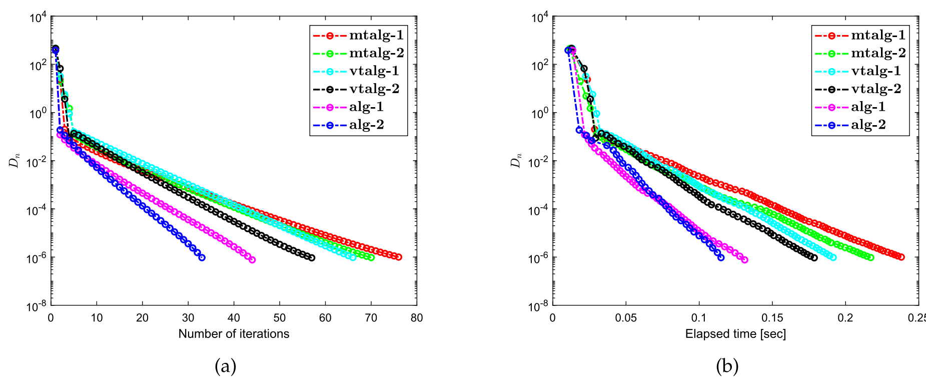


Figure 15 Numerical comparison of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $s_0 = 1$

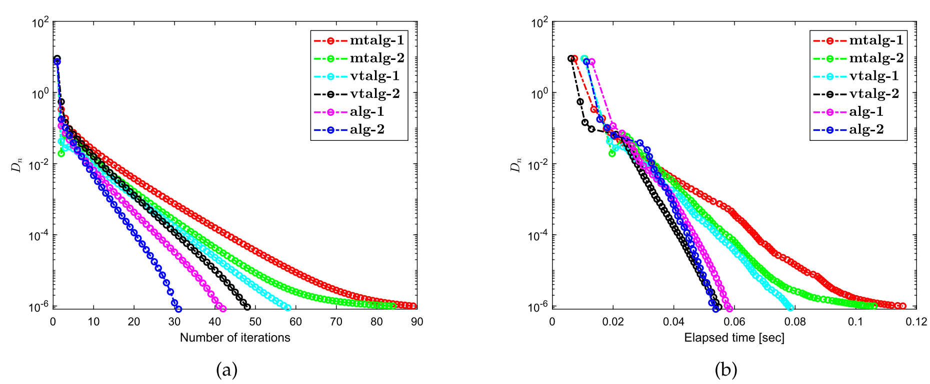


Figure 16 Numerical comparison of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $s_0 = s_1 = t$

Let us design an operator $\mathcal{N} : \mathcal{D} \to \mathcal{E}$ as

$$\mathcal{N}(u)(t) = \max\{u(t), 0\} = \frac{u(t) + |u(t)|}{2}.$$

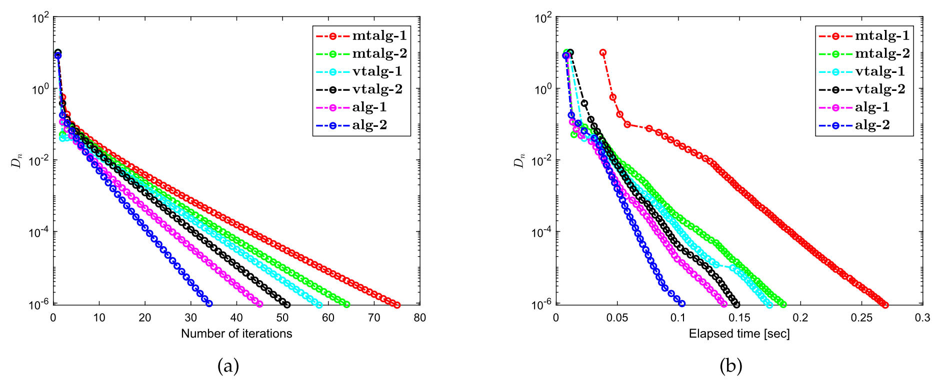


Figure 17 Numerical comparison of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $s_0 = s_1 = \sin(t)$

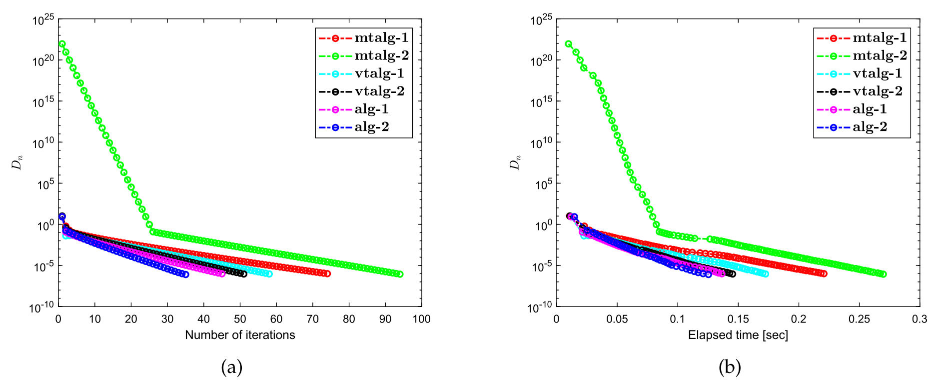


Figure 18 Numerical comparison of Algorithm 2 and Algorithm 4 with Algorithm 1 in [31], Algorithm 2 in [31], Algorithm 1 in [32], and Algorithm 2 in [32] when $s_0 = s_1 = \cos(t)$

Let $\mathcal{E} = L^2([0,1])$ represent a real Hilbert space. Its induced norm and inner product are described by

$$\langle u, y\rangle = \int_0^1 u(t)y(t)\,dt, \quad \forall u, y \in \mathcal{E}$$

and

$$\|u\| = \sqrt{\int_0^1 |u(t)|^2\,dt}.$$

It is trivial to verify that $\mathcal{N}$ is monotone and 1-Lipschitz continuous, and that the projection on $\mathcal{D}$ is naturally straightforward, that is,

$$P_C(u) = \begin{cases} \frac{u}{\|u\|}, & \text{if } \|u\| > 1, \\ u, & \|u\| \le 1. \end{cases}$$

Table 7 Numerical values for Example 4.4

$s_0 = s_1$	Total number of iterations					
	alg-1	**alg-2**	**mtalg-1**	**mtalg-2**	**vtalg-1**	**vtalg-2**
t^2	53	43	88	79	78	66
$\cos(t)$	61	47	93	85	81	73
$\exp(t)$	66	42	91	81	75	78
3^t	65	51	97	87	71	79

Table 8 Numerical values for Example 4.4

$s_0 = s_1$	Required CPU time					
	alg-1	**alg-2**	**mtalg-1**	**mtalg-2**	**vtalg-1**	**vtalg-2**
t^2	0.1745382	0.1275749	1.5867949	1.1684235	1.5915358	1.50273734
$\cos(t)$	0.1976944	0.2557759	1.8563924	1.5462935	1.7394833	1.74387483
$\exp(t)$	1.0575325	0.1547495	1.6949494	1.0997845	1.1872049	1.84874373
3^t	1.0025344	0.1937548	1.2207376	1.7112947	1.1990292	1.95858459

A mapping $\mathcal{M} : L^2([0,1]) \to L^2([0,1])$ takes the following form:

$$\mathcal{M}(u)(t) = \int_0^1 tu(s)\,ds, \quad t \in [0,1].$$

A simple analysis demonstrates that $\mathcal{M}$ is 0-demicontractive. The solution is $\varpi^*(t) = 0$. These trials begin differently by setting a halting requirement $D_k = \|q_k - r_k\| \le 10^{-6}$. Tables 7 and 8 include numerical results for Example 4.4. The following conditions are used as control criteria:

(1) Algorithm 2 (briefly, **alg-1**): $\hbar_1 = 0.25$, $\ell = 0.44$, $\mu = 0.75$, $\chi_k = \frac{1}{(1+k)^2}$, $\varsigma_k = \frac{1}{(5k+10)}$, $\sigma_k = \frac{k}{(2k+1)}$.

(2) Algorithm 4 (briefly, **alg-2**): $\hbar_1 = 0.25$, $\ell = 0.44$, $\mu = 0.75$, $\chi_k = \frac{1}{(1+k)^2}$, $\varsigma_k = \frac{1}{(5k+10)}$, $\sigma_k = \frac{k}{(2k+1)}$.

(3) Algorithm 1 in [31] (briefly, **mtalg-1**): $\gamma_1 = 0.25$, $\delta = 0.44$, $\phi = 0.75$, $\ell_k = \frac{1}{(5k+10)}$, $\hbar_k = \frac{1}{2}(1 - \ell_k)$, $\chi_k = \frac{1}{(1+k)^2}$.

(4) Algorithm 2 in [31] (briefly, **mtalg-2**): $\gamma_1 = 0.25$, $\delta = 0.44$, $\phi = 0.75$, $\ell_k = \frac{1}{(5k+10)}$, $\hbar_k = \frac{1}{2}(1 - \ell_k)$, $\chi_k = \frac{1}{(1+k)^2}$.

(5) Algorithm 1 in [32] (briefly, **vtalg-1**): $\tau_1 = 0.25$, $\ell = 0.44$, $\mu = 0.75$, $\chi_k = \frac{1}{(1+k)^2}$, $\varsigma_k = \frac{1}{(5k+10)}$, $\sigma_k = \frac{k}{(2k+1)}$, $f(u) = \frac{u}{4}$.

(6) Algorithm 2 in [32] (briefly, **vtalg-2**): $\tau_1 = 0.25$, $\ell = 0.44$, $\mu = 0.75$, $\chi_k = \frac{1}{(1+k)^2}$, $\varsigma_k = \frac{1}{(5k+10)}$, $\sigma_k = \frac{k}{(2k+1)}$, $f(u) = \frac{u}{4}$.

5 Conclusion

We proposed four inertial extragradient-type methods to solve the monotone variational inequality problem numerically as well as a fixed-point problem. These methods are viewed as a modified version of the two-step extragradient method. Two strong convergence theorems have been established for the proposed methods. These numerical results were established in order to confirm the numerical effectiveness of the suggested algorithms over the existing methods. These computational results show that the non-monotone variable step-size rule continues to improve the iterative sequence's usefulness in this context.

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Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
WK, writing-original draft preparation. HR, writing-original draft preparation and project administration. PK, methodology and project administration. All authors read and approved the final manuscript.

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