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Fixed point results of Jaggi–Suzuki-type hybrid contractions with applications

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Abstract

In this manuscript, a novel general class of contractions, called Jaggi–Suzuki-type hybrid (G-α-ϕ)-contraction, is introduced and some fixed point theorems that cannot be deduced from their akin in metric spaces are proved. The dominance of this family of contractions is that its contractive inequality can be specialized in various manners, depending on multiple parameters. Nontrivial comparative examples are constructed to validate the assumptions of our obtained theorems. Consequently, a number of corollaries that reduce our result to some prominent results in the literature are highlighted and analyzed. Additionally, we examine Ulam-type stability and well-posedness for the new contraction proposed herein. Finally, one of our obtained corollaries is applied to set up unprecedented existence conditions for solution to a class of integral equations. For future aspects of our results, an open problem is noted concerning the discretized population balance model, whose solution may be analyzed using the techniques established herein.

MSC: 47H10; 54H25; 46L07

Keywords: G-metric; Fixed point; Hybrid contraction; Ulam stability; Integral equation

1 Introduction

The famous Banach contraction in metric spaces (MS) has paved the way for a new dawn in metric fixed point theory, which is proven to have many applications in inequalities, approximation theory, optimization, and so on. Researchers in this area have introduced several new concepts in MS and obtained a wealth of fixed point (*FP*) results for linear and nonlinear contractions. Recently, Noorwali and Yeşilkaya [15] introduced a new notion of hybrid contraction, which combined and unified some existing linear and nonlinear contractions in MS.

On the other hand, Mustafa [11] introduced an extension of MS by the name, generalized MS (or more specifically, G-metric space (G-MS)) and proved some *FP* results for Banach-type contraction mappings. This new generalization was brought to limelight by Mustafa and Sims [14]. Subsequently, Mustafa et al. [12] and several other authors (see, e.g., [1, 5, 6, 10, 13, 17, 18]) obtained some remarkable *FP* results satisfying certain contractive conditions on G-MS. However, Jleli and Samet [7] as well as Samet et al. [19] published observations that most of the *FP* results in *G*-MS are direct consequences of existence results in MS. In fact, Jleli and Samet [7] noted that if a *G*-metric can be reduced to a quasi-metric, then the related *FP* results become the known *FP* results in the context of quasi-MS. Motivated by the latter observation, many investigators (see, e.g., [4, 8]) have developed techniques for establishing *FP* results in *G*-MS that cannot be followed from their analogue ones in ordinary or quasi-MS.

Following the existing literature, we realize that hybrid *FP* results in *G*-MS have not been sufficiently investigated. Hence, motivated by the ideas in [4, 8, 15], we introduce a new concept of Jaggi–Suzuki-type hybrid (G-α-ϕ)-contraction in *G*-MS and prove some related *FP* theorems. An example is given to demonstrate the validity of our result and to show that the main ideas obtained herein do not reduce to any existence result in MS. Some corollaries are presented to show that the concept proposed in this paper is an extension and generalization of some well-known *FP* theorems in *G*-MS. Additionally, Ulam-type stability and well-posedness of this type of hybrid contraction are established and analyzed. Furthermore, one of our obtained corollaries is applied to establish novel existence conditions for the solution of a class of integral equations. For further research and extension of our results, an open problem is highlighted concerning discretized population balance model, whose solution may be analyzed using our established techniques.

2 Preliminaries

In this section, we present some fundamental notations and results that will be deployed subsequently.

All through, every set Φ is considered nonempty, $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ represents the set of real numbers, and $\mathbb{R}_+$ the set of nonnegative real numbers.

Definition 1 ([14]) Let Φ be a nonempty set and let $G : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a function satisfying:

(G_1) $G(r,s,t) = 0$ if $r = s = t$;

(G_2) $0 < G(r,r,s)$ for all $r,s \in \Phi$ with $r \neq s$;

(G_3) $G(r,r,s) \leq G(r,s,t)$ for all $r,s,t \in \Phi$ with $t \neq s$;

(G_4) $G(r,s,t) = G(r,t,s) = G(s,r,t) = \cdots$ (symmetry in all three variables);

(G_5) $G(r,s,t) \leq G(r,a,a) + G(a,s,t)$ for all $r,s,t,a \in \Phi$ (rectangle inequality).

Then the function G is called a generalized metric or, more specifically, a G-metric on Φ, and the pair (Φ, G) is called a G-MS.

Example 1 ([12]) Let (Φ, d) be a usual MS. Then (Φ, G_p) and (Φ, G_m) are G-MS, where

$$G_p(r,s,t) = d(r,s) + d(s,t) + d(r,t) \quad \forall r,s,t \in \Phi, \tag{1}$$

$$G_m(r,s,t) = \max\{d(r,s), d(s,t), d(r,t)\} \quad \forall r,s,t \in \Phi. \tag{2}$$

Definition 2 ([12]) Let (Φ, G) be a G-MS and let $\{r_x\}_{x\in\mathbb{N}}$ be a sequence of points of Φ. We say that $\{r_x\}_{x\in\mathbb{N}}$ is G-convergent to r if $\lim_{x,m\to\infty} G(r, r_x, r_m) = 0$; that is, for any $\epsilon > 0$, there exists $x_0 \in \mathbb{N}$ such that $G(r, r_x, r_m) < \epsilon$, $\forall x, m \geq x_0$. We refer to r as the limit of the sequence $\{r_x\}_{x\in\mathbb{N}}$.

Proposition 1 ([12]) *Let* (Φ, G) *be a* G*-MS. Then the following are equivalent*:

(i) $\{r_x\}_{x\in\mathbb{N}}$ *is G-convergent to* r.
(ii) $G(r, r_x, r_m) \longrightarrow 0$ *as* $x, m \to \infty$.
(iii) $G(r_x, r, r) \longrightarrow 0$ *as* $x \to \infty$.
(iv) $G(r_x, r_x, r) \longrightarrow 0$ *as* $x \to \infty$.

Definition 3 ([12]) Let (Φ, G) be a G-MS. A sequence $\{r_x\}_{x\in\mathbb{N}}$ is called G-Cauchy if, given $\epsilon > 0$, there exists $x_0 \in \mathbb{N}$ such that $G(r_x, r_m, r_l) < \epsilon$, $\forall x, m, l \geq x_0$, that is, $G(r_x, r_m, r_l) \longrightarrow 0$ as $x, m, l \to \infty$.

Proposition 2 ([12]) *In a G-MS* (Φ, G), *the following are equivalent*:
(i) *The sequence* $\{r_x\}_{x\in\mathbb{N}}$ *is G-Cauchy.*
(ii) *For every* $\epsilon > 0$, *there exists* $x_0 \in \mathbb{N}$ *such that* $G(r_x, r_m, r_m) < \epsilon$, $\forall x, m \geq x_0$.

Definition 4 ([12]) Let (Φ, G) and (Φ', G') be two G-MS and let $f : (\Phi, G) \longrightarrow (\Phi', G')$ be a function. Then f is said to be G-continuous at a point $a \in \Phi$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $r, s \in \Phi$ and $G(a, r, s) < \delta \Rightarrow G'(f(a), f(r), f(s)) < \epsilon$. A function f is G-continuous on Φ if and only if it is G-continuous at all $a \in \Phi$.

Proposition 3 ([12]) *Let* (Φ, G) *and* (Φ', G') *be two G-MS. Then a function* $f : (\Phi, G) \longrightarrow (\Phi', G')$ *is said to be G-continuous at a point* $r \in \Phi$ *if and only if it is G-sequentially continuous at* r, *that is, whenever* $\{r_x\}_{x\in\mathbb{N}}$ *is G-convergent to* r, $\{fr_x\}$ *is G-convergent to* fr.

Definition 5 ([12]) A G-MS (Φ, G) is called symmetric G-MS if

$$G(r, r, s) = G(s, r, r) \quad \forall r, s \in \Phi.$$

Proposition 4 ([12]) *Let* (Φ, G) *be a G-MS. Then the function* $G(r, s, t)$ *is jointly continuous in all three of its variables.*

Proposition 5 ([12]) *Every G-MS* (Φ, G) *will define an MS* (Φ, d_G) *by*

$$d_G(r, s) = G(r, s, s) + G(s, r, r) \quad \forall r, s \in \Phi. \tag{3}$$

Note that if (Φ, G) *is a symmetric G-MS, then*

$$(\Phi, d_G) = 2G(r, s, s) \quad \forall r, s \in \Phi. \tag{4}$$

However, if (Φ, G) *is not symmetric, then it holds by the G-metric properties that*

$$\frac{3}{2}G(r, s, s) \leq d_G(r, s) \leq 3G(r, s, s) \quad \forall r, s \in \Phi, \tag{5}$$

and that in general, these inequalities are sharp.

Definition 6 ([12]) A G-MS (Φ, G) is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in (Φ, G) is G-convergent in (Φ, G).

Proposition 6 ([12]) *A G-MS* (Φ, G) *is G-complete if and only if* (Φ, d_G) *is a complete MS.*

Mustafa [11] proved the following result in the framework of G-MS.

Theorem 1 ([11]) *Let* (Φ, G) *be a complete G-MS, and let* $\Gamma : \Phi \longrightarrow \Phi$ *be a self-mapping satisfying the following condition*:

$$G(\Gamma r, \Gamma s, \Gamma t) \leq kG(r, s, t) \tag{6}$$

for all $r, s, t \in \Phi$, *where* $0 \leq k < 1$. *Then* Γ *has a unique FP* (*say* u, *i.e.*, $\Gamma u = u$), *and* Γ *is G-continuous at* u.

Definition 7 ([2]) Let Ψ be the set of all functions $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying:
(i) ϕ is monotone increasing, that is, $p_1 \leq p_2 \Rightarrow \phi(p_1) \leq \phi(p_2)$;
(ii) The series $\sum_{x=0}^{\infty} \phi^x(p)$ is convergent for all $p > 0$.
Then ϕ is called a (c)-comparison function.

Remark 1 If $\phi \in \Psi$, then $\phi(p) < p$ for any $p > 0$, $\phi(0) = 0$ and ϕ is continuous at 0.

Popescu [16] gave the following definitions in the setting of MS.

Definition 8 ([16]) Let $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a function. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is called α-orbital admissible if for all $r \in \Phi$,

$$\alpha(r, \Gamma r) \geq 1 \quad \Rightarrow \quad \alpha\left(\Gamma r, \Gamma^2 r\right) \geq 1.$$

Definition 9 ([16]) Let $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a function. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is called triangular α-orbital admissible if for all $r \in \Phi$, Γ is α-orbital admissible and

$$\alpha(r, s) \geq 1 \quad \text{and} \quad \alpha(s, \Gamma s) \geq 1 \quad \Rightarrow \quad \alpha(r, \Gamma s) \geq 1.$$

We modify the above definitions in the framework of G-MS as follows.

Definition 10 Let $\alpha : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a function. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is called $(G\text{-}\alpha)$-orbital admissible if for all $r \in \Phi$,

$$\alpha\left(r, \Gamma r, \Gamma^2 r\right) \geq 1 \quad \Rightarrow \quad \alpha\left(\Gamma r, \Gamma^2 r, \Gamma^3 r\right) \geq 1.$$

Definition 11 Let $\alpha : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a function. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is called triangular $(G\text{-}\alpha)$-orbital admissible if for all $r \in \Phi$, Γ is $(G\text{-}\alpha)$-orbital admissible and

$$\alpha(r, s, \Gamma s) \geq 1 \quad \text{and} \quad \alpha\left(s, \Gamma s, \Gamma^2 s\right) \geq 1 \quad \Rightarrow \quad \alpha\left(r, \Gamma s, \Gamma^2 s\right) \geq 1.$$

Lemma 1 *Let* $\Gamma : \Phi \longrightarrow \Phi$ *be a triangular*$(G\text{-}\alpha)$*-orbital admissible mapping. If there exists* $r_0 \in \Phi$ *such that* $\alpha(r_0, \Gamma r_0, \Gamma^2 r_0) \geq 1$, *then*

$$\alpha(r_x, r_m, r_l) \geq 1 \quad \forall x, m, l \in \mathbb{N}, \tag{7}$$

where $\{r_x\}_{x \in \mathbb{N}}$ *is a sequence defined by* $r_{x+1} = \Gamma r_x$, $x \in \mathbb{N}$.

Proof Since Γ is (G-α)-orbital admissible mapping and $\alpha(r_0, \Gamma r_0, \Gamma^2 r_0) \geq 1$, then we deduce that $\alpha(r_1, r_2, r_3) = \alpha(\Gamma r_0, \Gamma r_1, \Gamma r_2) \geq 1$. Continuing in this manner, we obtain $\alpha(r_x, r_{x+1}, r_{x+2}) \geq 1$ for all $x \geq 1$. Assume that $\alpha(r_x, r_m, r_{m+1}) \geq 1$, where $m > x$. Since Γ is triangular (G-α)-orbital admissible mapping and $\alpha(r_m, r_{m+1}, r_{m+2}) \geq 1$, then, clearly, $\alpha(r_x, r_{m+1}, r_{m+2}) \geq 1$ for all $m, x \in \mathbb{N}$. This validates our assumption that $\alpha(r_x, r_m, r_{m+1}) \geq 1$. Letting $l = m + 1$ completes the proof. □

Definition 12 ([3]) Let $\alpha : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a mapping. The set Φ is called regular with respect to α if and only if for every sequence $\{r_x\}_{x\in\mathbb{N}}$ in Φ such that $\alpha(r_x, r_{x+1}, r_{x+2}) \geq 1$, for all x and $r_x \to r \in \Phi$ as $x \to \infty$, we have $\alpha(r_x, r, r) \geq 1$ for all x.

Noorwali and Yeşilkaya [15] gave the following definition of Jaggi-type hybrid contraction in MS.

Definition 13 ([15]) Let (Φ, d) be an MS. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is called a Jaggi–Suzuki-type hybrid contraction if there exist $\phi \in \Psi$ and $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}_+$ such that

$$\frac{1}{2}d(r, \Gamma r) \leq d(r, s) \quad \Rightarrow \quad \alpha(r, s)d(\Gamma r, \Gamma s) \leq \phi\bigl(M(r, s)\bigr) \tag{8}$$

for all $r, s \in \Phi$, where

$$M(r, s) = \begin{cases} [\lambda_1(\frac{d(r,\Gamma r)\cdot d(s,\Gamma s)}{d(r,s)})^q + \lambda_2 d(r, s)^q]^{\frac{1}{q}} & \text{for } q > 0, r \neq s; \\ d(r, \Gamma r)^{\lambda_1} \cdot d(s, \Gamma s)^{\lambda_2} & \text{for } q = 0, r, s \in \Phi \backslash \operatorname{Fix}(\Gamma), \end{cases} \tag{9}$$

$\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$, $\lambda_1 < \frac{1}{2}$ and $\operatorname{Fix}(\Gamma) = \{r \in \Phi : \Gamma r = r\}$.

3 Main results

We begin this section by defining the notion of Jaggi–Suzuki-type hybrid (G-α-ϕ)-contraction in G-MS.

Definition 14 Let (Φ, G) be a G-MS. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is called a Jaggi–Suzuki-type hybrid (G-α-ϕ)-contraction if there exist $\phi \in \Psi$ and $\alpha : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ such that

$$\frac{1}{2}G\bigl(r, \Gamma r, \Gamma^2 r\bigr) \leq G(r, s, \Gamma s) \quad \Rightarrow \quad \alpha(r, s, \Gamma s)G\bigl(\Gamma r, \Gamma s, \Gamma^2 s\bigr) \leq \phi\bigl(M(r, s, \Gamma s)\bigr) \tag{10}$$

for all $r, s \in \Phi \backslash \operatorname{Fix}(\Gamma)$, where

$$M(r, s, \Gamma s) = \begin{cases} [\lambda_1(\frac{G(r,\Gamma r,\Gamma^2 r)\cdot G(s,\Gamma s,\Gamma^2 s)}{G(r,s,\Gamma s)})^q + \lambda_2 G(r, s, \Gamma s)^q]^{\frac{1}{q}} & \text{for } q > 0; \\ G(r, \Gamma r, \Gamma^2 r)^{\lambda_1} \cdot G(s, \Gamma s, \Gamma^2 s)^{\lambda_2} & \text{for } q = 0, \end{cases} \tag{11}$$

$\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ and $\operatorname{Fix}(\Gamma) = \{r \in \Phi : \Gamma r = r\}$.

The following is our main result.

Theorem 2 *Let* (Φ, G) *be a complete G-MS and let* $\Gamma : \Phi \longrightarrow \Phi$ *be a Jaggi–Suzuki-type hybrid* (G-α-ϕ)*-contraction. Assume further that*

(i) Γ *is triangular* (G-α)*-orbital admissible*;
(ii) *There exists* $r_0 \in \Phi$ *such that* $\alpha(r_0, \Gamma r_0, \Gamma^2 r_0) \geq 1$;
(iii) *Either* Γ *is continuous or*
(iv) Γ^3 *is continuous and* $\alpha(r, \Gamma r, \Gamma^2 r) \geq 1$ *for each* $r \in \mathrm{Fix}(\Gamma^3)$.

Then Γ *has an FP in* Φ.

Proof Let $r_0 \in \Phi$ be an arbitrary point and define a sequence $\{r_x\}_{x\in\mathbb{N}}$ in Φ by $r_x = \Gamma^x r_0$ for all $x \in \mathbb{N}$. Assume that there exists some $m \in \mathbb{N}$ such that $\Gamma r_m = r_{m+1} = r_m$, then, clearly, r_m is an *FP* of Γ. So, we presume that $r_x \neq r_{x-1}$ for any $x \in \mathbb{N}$. Since Γ is a Jaggi–Suzuki-type hybrid (G-α-ϕ)-contraction, then from (10) we have

$$\begin{aligned}\frac{1}{2}G\bigl(r_{x-1}, \Gamma r_{x-1}, \Gamma^2 r_{x-1}\bigr) = \frac{1}{2}G(r_{x-1}, r_x, \Gamma r_x) &\leq G(r_{x-1}, r_x, \Gamma r_x)\\ &= \alpha(r_{x-1}, r_x, \Gamma r_x)G\bigl(\Gamma r_{x-1}, \Gamma r_x, \Gamma^2 r_x\bigr)\\ &\leq \phi\bigl(M(r_{x-1}, r_x, \Gamma r_x)\bigr). \end{aligned} \tag{12}$$

Owing to the fact that Γ is triangular (G-α)-orbital admissible together with (7) and (12), we have

$$\begin{aligned}G(r_x, r_{x+1}, \Gamma r_{x+1}) &= G(r_x, r_{x+1}, r_{x+2})\\ &\leq \alpha(r_{x-1}, r_x, \Gamma r_x)G\bigl(\Gamma r_{x-1}, \Gamma r_x, \Gamma^2 r_x\bigr)\\ &\leq \phi\bigl(M(r_{x-1}, r_x, \Gamma r_x)\bigr). \end{aligned} \tag{13}$$

We now consider the given cases of (10).

Case 1: For $q > 0$, we obtain

$$\begin{aligned}M(r_{x-1}, r_x, \Gamma r_x) &= \left[\lambda_1\left(\frac{G(r_{x-1}, \Gamma r_{x-1}, \Gamma^2 r_{x-1})G(r_x, \Gamma r_x, \Gamma^2 r_x)}{G(r_{x-1}, r_x, \Gamma r_x)}\right)^q + \lambda_2 G(r_{x-1}, r_x, \Gamma r_x)^q\right]^{\frac{1}{q}}\\ &= \left[\lambda_1\left(\frac{G(r_{x-1}, r_x, r_{x+1})G(r_x, r_{x+1}, r_{x+2})}{G(r_{x-1}, r_x, r_{x+1})}\right)^q + \lambda_2 G(r_{x-1}, r_x, r_{x+1})^q\right]^{\frac{1}{q}}\\ &= \bigl[\lambda_1 G(r_x, r_{x+1}, r_{x+2})^q + \lambda_2 G(r_{x-1}, r_x, r_{x+1})^q\bigr]^{\frac{1}{q}}. \end{aligned}$$

Since ϕ is nondecreasing, if we assume that

$$G(r_{x-1}, r_x, r_{x+1}) \leq G(r_x, r_{x+1}, r_{x+2}),$$

then (13) becomes

$$\begin{aligned}G(r_x, r_{x+1}, r_{x+2}) &\leq \phi\bigl(\bigl[\lambda_1 G(r_x, r_{x+1}, r_{x+2})^q + \lambda_2 G(r_{x-1}, r_x, r_{x+1})^q\bigr]^{\frac{1}{q}}\bigr)\\ &\leq \phi\bigl(\bigl[\lambda_1 G(r_x, r_{x+1}, r_{x+2})^q + \lambda_2 G(r_x, r_{x+1}, r_{x+2})^q\bigr]^{\frac{1}{q}}\bigr)\\ &= \phi\bigl((\lambda_1 + \lambda_2)^{\frac{1}{q}} G(r_x, r_{x+1}, r_{x+2})\bigr)\\ &= \phi\bigl(G(r_x, r_{x+1}, r_{x+2})\bigr) \end{aligned}$$

$$< G(r_x, r_{x+1}, r_{x+2}),$$

which is a contradiction. Therefore, for every $x \in \mathbb{N}$, we have

$$G(r_x, r_{x+1}, r_{x+2}) < G(r_{x-1}, r_x, r_{x+1}),$$

so that (13) becomes

$$\begin{aligned} G(r_x, r_{x+1}, r_{x+2}) &\leq \phi\big(\big[\lambda_1 G(r_x, r_{x+1}, r_{x+2})^q + \lambda_2 G(r_{x-1}, r_x, r_{x+1})^q\big]^{\frac{1}{q}}\big) \\ &\leq \phi\big((\lambda_1 + \lambda_2)^{\frac{1}{q}} G(r_{x-1}, r_x, r_{x+1})\big) \\ &\leq \phi\big(G(r_{x-1}, r_x, r_{x+1})\big). \end{aligned}$$

Continuing inductively, we have

$$G(r_x, r_{x+1}, r_{x+2}) \leq \phi^x\big(G(r_0, r_1, r_2)\big). \tag{14}$$

Now, since

$$G(r_x, r_x, r_{x+1}) \leq G(r_x, r_{x+1}, r_{x+2}) \leq \phi^x\big(G(r_0, r_1, r_2)\big)$$

for all $x \in \mathbb{N}$ with $r_{x+1} \neq r_{x+2}$, then for any $x, m \in \mathbb{N}$ with $x < m$ and by the rectangle inequality, we have

$$\begin{aligned} G(r_x, r_x, r_m) &\leq G(r_x, r_x, r_{x+1}) + G(r_{x+1}, r_{x+1}, r_{x+2}) + \cdots + G(r_{m-1}, r_{m-1}, r_m) \\ &\leq \big(\phi^x + \phi^{x+1} + \phi^{x+2} + \cdots + \phi^{m-1}\big)\big(G(r_0, r_1, r_2)\big) \\ &= \sum_{i=x}^{m-1} \phi^i\big(G(r_0, r_1, r_2)\big) \leq \sum_{i=x}^{\infty} \phi^i\big(G(r_0, r_1, r_2)\big). \end{aligned}$$

Since ϕ is a (c)-comparison function, then the series $\sum_{i=0}^{\infty} \phi^i(G(r_0, r_1, r_2))$ is convergent, and so denoting by $S_p = \sum_{i=0}^{\infty} \phi^i(G(r_0, r_1, r_2))$, we have

$$G(r_x, r_x, r_m) \leq S_{m-1} - S_{x-1}.$$

Hence, as $x, m \to \infty$, we see that

$$G(r_x, r_x, r_m) \longrightarrow 0.$$

Thus $\{r_x\}_{x\in\mathbb{N}}$ is a G-Cauchy sequence in (Φ, G), and so by the completeness of (Φ, G), there exists $t \in \Phi$ such that $\{r_x\}_{x\in\mathbb{N}}$ is G-convergent to t, that is,

$$\lim_{x\to\infty} G(r_x, r_x, t) = 0.$$

We will now show that t is an *FP* of Γ. By the assumption that Γ is continuous, we have

$$\lim_{x\to\infty} G(t, t, \Gamma t) = \lim_{x\to\infty} G(r_{x+1}, r_{x+1}, \Gamma t) = \lim_{x\to\infty} G(\Gamma r_x, \Gamma r_x, \Gamma t)$$

$$= \lim_{x\to\infty} G(\Gamma r_x, \Gamma r_x, \Gamma r_x) = 0,$$

so we get $\Gamma t = t$, that is, t is an *FP* of Γ.

Alternatively, by the assumption that (iv) holds, we have $\Gamma^3 t = \lim \Gamma^3 r_x = t$. To see that $\Gamma t = t$, assume contrary that $\Gamma t \neq t$. Then, by (10) and Proposition 1, we obtain

$$\begin{aligned}&\frac{1}{2}G\bigl(t, \Gamma t, \Gamma^2 t\bigr) \leq G\bigl(t, \Gamma t, \Gamma^2 t\bigr)\\ &\quad\Rightarrow\quad \alpha\bigl(t, \Gamma t, \Gamma^2 t\bigr)G\bigl(\Gamma t, \Gamma^2 t, \Gamma^3 t\bigr) \leq \phi\bigl(M\bigl(t, \Gamma t, \Gamma^2 t\bigr)\bigr).\end{aligned}$$

Now,

$$\begin{aligned}G\bigl(t, \Gamma t, \Gamma^2 t\bigr) = G\bigl(\Gamma^3 t, \Gamma t, \Gamma^2 t\bigr) &\leq \alpha\bigl(t, \Gamma t, \Gamma^2 t\bigr)G\bigl(\Gamma t, \Gamma^2 t, \Gamma^3 t\bigr)\\ &\leq \phi\bigl(M\bigl(t, \Gamma t, \Gamma^2 t\bigr)\bigr) < M\bigl(t, \Gamma t, \Gamma^2 t\bigr),\end{aligned}$$

where

$$\begin{aligned}M\bigl(t, \Gamma t, \Gamma^2 t\bigr) &= \biggl[\lambda_1\biggl(\frac{G(t, \Gamma t, \Gamma^2 t)\cdot G(\Gamma t, \Gamma^2 t, \Gamma^3 t)}{G(t, \Gamma t, \Gamma^2 t)}\biggr)^q + \lambda_2 G\bigl(t, \Gamma t, \Gamma^2 t\bigr)^q\biggr]^{\frac{1}{q}}\\ &= \bigl[\lambda_1 G\bigl(t, \Gamma t, \Gamma^2 t\bigr)^q + \lambda_2 G\bigl(t, \Gamma t, \Gamma^2 t\bigr)^q\bigr]^{\frac{1}{q}}\\ &= \bigl[(\lambda_1 + \lambda_2)G\bigl(t, \Gamma t, \Gamma^2 t\bigr)^q\bigr]^{\frac{1}{q}}\\ &= (\lambda_1 + \lambda_2)^{\frac{1}{q}} G\bigl(t, \Gamma t, \Gamma^2 t\bigr)\\ &= G\bigl(t, \Gamma t, \Gamma^2 t\bigr),\end{aligned}$$

which is a contradiction. Hence, $\Gamma t = t$.

Case 2: For $q = 0$, we have

$$\begin{aligned}M(r_{x-1}, r_x, \Gamma r_x) &= G\bigl(r_{x-1}, \Gamma r_{x-1}, \Gamma^2 r_{x-1}\bigr)^{\lambda_1}\cdot G\bigl(r_x, \Gamma r_x, \Gamma^2 r_x\bigr)^{\lambda_2}\\ &= G(r_{x-1}, r_x, r_{x+1})^{\lambda_1}\cdot G(r_x, r_{x+1}, r_{x+2})^{\lambda_2}.\end{aligned}$$

Now, if $G(r_{x-1}, r_x, r_{x+1}) \leq G(r_x, r_{x+1}, r_{x+2})$, then (13) becomes

$$G(r_x, r_{x+1}, r_{x+2}) < G(r_x, r_{x+1}, r_{x+2}),$$

which is a contradiction. Therefore,

$$G(r_x, r_{x+1}, r_{x+2}) < G(r_{x-1}, r_x, r_{x+1}).$$

Hence, by (13), we have

$$\begin{aligned}G(r_x, r_{x+1}, r_{x+2}) &< \phi\bigl(G(r_{x-1}, r_x, r_{x+1})\bigr) < \phi^2\bigl(G(r_{x-1}, r_x, r_{x+1})\bigr)\\ &< \cdots < \phi^x\bigl(G(r_0, r_1, r_2)\bigr).\end{aligned}$$

By similar argument as the case of $q > 0$, we can show that there exist a G-Cauchy sequence $\{r_x\}_{x\in\mathbb{N}}$ in (Φ, G) and a point t in Φ such that $\lim_{x\to\infty} r_x = t$. Similarly, under the assumption that Γ is continuous and by the uniqueness of limit, we have that $\Gamma t = t$, that is, t is an *FP* of Γ. □

In the result that follows, we examine the uniqueness of *FP* of Γ under certain supplementary assumptions.

Theorem 3 *If in Theorem 2, in the event of $q > 0$, we assume further that (Φ, G) is regular with respect to α and $\alpha(r, s, \Gamma s) \geq 1$ for any $r, s \in \mathrm{Fix}(\Gamma)$, then the FP of Γ is unique.*

Proof Let $v, t \in \mathrm{Fix}(\Gamma)$ be such that $v \neq t$. By replacing this in (10) and noting the additional hypotheses, we have

$$\begin{aligned}
&\frac{1}{2}G\big(t,\Gamma t,\Gamma^2 t\big) \leq G(t,v,\Gamma v)\\
&\quad\Rightarrow G(t,v,\Gamma v) \leq \alpha(t,v,\Gamma v)G\big(\Gamma t,\Gamma v,\Gamma^2 v\big)\\
&\qquad\qquad \leq \phi\big(M(t,v,\Gamma v)\big) < M(t,v,\Gamma v)\\
&\qquad\qquad = \left[\lambda_1\left(\frac{G(t,\Gamma t,\Gamma^2 t)G(v,\Gamma v,\Gamma^2 v)}{G(t,v,\Gamma v)}\right)^q + \lambda_2 G(t,v,\Gamma v)^q\right]^{\frac{1}{q}}\\
&\qquad\qquad = \left[\lambda_1\left(\frac{G(t,t,t)G(v,v,v)}{G(t,v,\Gamma v)}\right)^q + \lambda_2 G(t,v,\Gamma v)^q\right]^{\frac{1}{q}}\\
&\qquad\qquad = \lambda_2^{\frac{1}{q}} G(t,v,\Gamma v)\\
&\qquad\qquad < G(t,v,\Gamma v),
\end{aligned}$$

which is a contradiction. Hence, $v = t$, and so the *FP* of Γ is unique. □

Example 2 Let $\Phi = [-1, 1]$ and let $\Gamma : \Phi \longrightarrow \Phi$ be a self-mapping on Φ defined by

$$\Gamma r = \begin{cases} \frac{r}{5} & \text{if } r \in \{-1, 1\};\\ \frac{1}{5} & \text{if } r \in (-1, 1)\end{cases}$$

for all $r \in \Phi$. Define $G : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ by

$$G(r,s,\Gamma s) = |r-s| + |r-\Gamma s| + |s-\Gamma s| \quad \forall r,s \in \Phi.$$

Then (Φ, G) is a complete G-MS. Define $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by $\phi(p) = \frac{p}{2}$ for all $p \geq 0$ and $\alpha : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ by

$$\alpha(r,s,\Gamma s) = \begin{cases} 1 & \text{if } r,s \in \{-1\} \cup [0,1];\\ 0 & \text{otherwise.}\end{cases} \tag{15}$$

Then, obviously, $\phi \in \Psi$, Γ is triangular(G-α)-orbital admissible, Γ is continuous for all $r \in \Phi$, and Γ^3 is continuous for any $r \in \mathrm{Fix}(\Gamma^3)$. Also, there exists $r_0 = \frac{1}{2} \in \Phi$ such that $\alpha(\frac{1}{2}, \Gamma(\frac{1}{2}), \Gamma^2(\frac{1}{2})) = \alpha(\frac{1}{2}, \frac{1}{5}, \frac{1}{5}) \geq 1$. Hence, conditions (i)-(iv) of Theorem 2 are satisfied.

To see that Γ is a Jaggi–Suzuki-type hybrid$(G\text{-}\alpha\text{-}\phi)$-contraction, notice that $\alpha(r,s,\Gamma s) = 0$ for all $r,s \in (-1,0)$ and $G(\Gamma r,\Gamma s,\Gamma^2 s) = 0$ for all $r,s \in (-1,1)$. Hence, inequality (10) holds for all $r,s \in (-1,1)$.

Now, for $r,s \in \{-1,1\}$, if $r = s = 1$, then $G(\Gamma r,\Gamma s,\Gamma^2 s) = 0$ for all $q \geq 0$. If $r = s = -1$, then letting $\lambda_1 = \frac{2}{5}$, $\lambda_2 = \frac{3}{5}$, and $q = 1$, we obtain

$$\begin{aligned}
\frac{1}{2}G\big(r,\Gamma r,\Gamma^2 r\big) &= \frac{1}{2}G\left(-1,\frac{-1}{5},\frac{1}{5}\right) \\
&= \frac{6}{5} < \frac{8}{5} \\
&= G\left(-1,-1,\frac{-1}{5}\right) = G(r,s,\Gamma s) \\
\Rightarrow \quad \alpha(r,s,\Gamma s)G\big(\Gamma r,\Gamma s,\Gamma^2 s\big) &= \alpha\left(-1,-1,\frac{-1}{5}\right)G\left(\frac{-1}{5},\frac{-1}{5},\frac{1}{5}\right) \\
&= \frac{4}{5} < \frac{6}{5} = \frac{1}{2}\left(\frac{12}{5}\right) \\
&= \frac{1}{2}\left(M\left(-1,-1,\frac{-1}{5}\right)\right) \\
&= \phi\big(M(r,s,\Gamma s)\big).
\end{aligned}$$

Also, for $q = 0$, we have

$$\alpha(r,s,\Gamma s)G\big(\Gamma r,\Gamma s,\Gamma^2 s\big) = \frac{4}{5} < \frac{1}{2}\left(\frac{12}{5}\right) = \phi\big(M(r,s,\Gamma s)\big).$$

If $r \neq s$, then for $r = 1$, $s = -1$, we have

$$\frac{1}{2}G\big(r,\Gamma r,\Gamma^2 r\big) = \frac{1}{2}G\left(1,\frac{1}{5},\frac{1}{5}\right) = \frac{4}{5} < 4 = G\left(1,-1,\frac{-1}{5}\right) = G(r,s,\Gamma s),$$

while for $r = -1$, $s = 1$, we obtain

$$\frac{1}{2}G\big(r,\Gamma r,\Gamma^2 r\big) = \frac{1}{2}G\left(-1,\frac{-1}{5},\frac{1}{5}\right) = \frac{6}{5} < 4 = G\left(-1,1,\frac{1}{5}\right) = G(r,s,\Gamma s).$$

These imply that for all $r,s \in \{-1,1\}$ with $r \neq s$, letting $\lambda_1 = \frac{1}{5}$, $\lambda_2 = \frac{4}{5}$ and $q = 3$, we obtain

$$\begin{aligned}
\alpha(r,s,\Gamma s)G\big(\Gamma r,\Gamma s,\Gamma^2 s\big) &= \alpha\left(1,-1,\frac{-1}{5}\right)G\left(\frac{1}{5},\frac{-1}{5},\frac{1}{5}\right) \\
&= \alpha\left(-1,1,\frac{1}{5}\right)G\left(\frac{-1}{5},\frac{1}{5},\frac{1}{5}\right) \\
&= \frac{4}{5} < \frac{8}{5} = \frac{1}{2}\left(\frac{16}{5}\right) \\
&= \frac{1}{2}\left(M\left(1,-1,\frac{-1}{5}\right)\right) = \frac{1}{2}\left(M\left(-1,1,\frac{1}{5}\right)\right) \\
&= \phi\big(M(r,s,\Gamma s)\big).
\end{aligned}$$

Also, for $q = 0$, we take $\lambda_1 = \lambda_2 = \frac{1}{2}$. Then

$$\begin{aligned}\alpha(r,s,\Gamma s)G(\Gamma r,\Gamma s,\Gamma^2 s) &= \alpha\left(1,-1,\frac{-1}{5}\right)G\left(\frac{1}{5},\frac{-1}{5},\frac{1}{5}\right)\\ &= \alpha\left(-1,1,\frac{1}{5}\right)G\left(\frac{-1}{5},\frac{1}{5},\frac{1}{5}\right)\\ &= \frac{4}{5} < \frac{24}{25} = \frac{1}{2}\left(\frac{48}{25}\right)\\ &= \frac{1}{2}\left(M\left(1,-1,\frac{-1}{5}\right)\right) = \frac{1}{2}\left(M\left(-1,1,\frac{1}{5}\right)\right)\\ &= \phi\big(M(r,s,\Gamma s)\big).\end{aligned}$$

Hence, inequality (10) is satisfied for all $r,s \in \Phi$. Therefore, Γ is a Jaggi–Suzuki-type hybrid (G-α-ϕ)-contraction that satisfies all the hypotheses of Theorem 2 and $r = \frac{1}{5}$ is the *FP* of Γ.

We now demonstrate that our result is independent of the result of Noorwali and Yeşilkaya [15]. Let $\alpha : \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be as given by Definition (13), $r_0 \in \Phi$ be such that $\alpha(r_0, \Gamma r_0) \geq 1$, $\phi(p) = \frac{19p}{40}$ for all $p \geq 0$, and $d : \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be defined by

$$d(r,s) = |r-s| \quad \forall r,s \in \Phi.$$

Consider $r,s \in \{-1,1\}$ and take, for Case 2, $r \neq s$, $\lambda_1 = \frac{2}{5}$, and $\lambda_2 = \frac{3}{5}$. Then from inequality (10) we see that

$$\begin{aligned}&\frac{1}{2}G\big(r,\Gamma r,\Gamma^2 r\big) = \frac{1}{2}G\left(1,\frac{1}{5},\frac{1}{5}\right) = \frac{4}{5} < 4 = G\left(1,-1,\frac{-1}{5}\right) = G(r,s,\Gamma s)\\ \Rightarrow\quad &\alpha(r,s,\Gamma s)G\big(\Gamma r,\Gamma s,\Gamma^2 s\big) = \alpha\left(1,-1,\frac{-1}{5}\right)G\left(\frac{1}{5},\frac{-1}{5},\frac{1}{5}\right)\\ &\qquad = \frac{4}{5} < \frac{24}{25} = \frac{19}{40}\left(\frac{192}{95}\right)\\ &\qquad = \frac{19}{40}\left(M\left(1,-1,\frac{-1}{5}\right)\right)\\ &\qquad = \phi\big(M(r,s,\Gamma s)\big)\end{aligned}$$

for $r = 1$, $s = -1$ and

$$\begin{aligned}&\frac{1}{2}G\big(r,\Gamma r,\Gamma^2 r\big) = \frac{1}{2}G\left(-1,\frac{-1}{5},\frac{1}{5}\right) = \frac{6}{5} < 4 = G\left(-1,1,\frac{1}{5}\right) = G(r,s,\Gamma s)\\ \Rightarrow\quad &\alpha(r,s,\Gamma s)G\big(\Gamma r,\Gamma s,\Gamma^2 s\big) = \alpha\left(-1,1,\frac{1}{5}\right)G\left(\frac{-1}{5},\frac{1}{5},\frac{1}{5}\right)\\ &\qquad = \frac{4}{5} < \frac{89}{100} = \frac{19}{40}\left(\frac{178}{95}\right)\\ &\qquad = \frac{19}{40}\left(M\left(-1,1,\frac{1}{5}\right)\right)\\ &\qquad = \phi\big(M(r,s,\Gamma s)\big)\end{aligned}$$

for $r = -1$, $s = 1$. However, inequality (8) due to Noorwali and Yeşilkaya [15] yields

$$\frac{1}{2}d(r,\Gamma r) = \frac{1}{2}d\left(1,\frac{1}{5}\right) = \frac{1}{2}d\left(-1,\frac{-1}{5}\right) = \frac{2}{5} < 2 = d(1,-1) = d(-1,1) = d(r,s),$$

$$\begin{aligned}\text{but } \alpha(r,s)d(\Gamma r,\Gamma s) &= \alpha(1,-1)d\left(\frac{1}{5},\frac{-1}{5}\right) = \alpha(-1,1)d\left(\frac{-1}{5},\frac{1}{5}\right)\\ &= \frac{2}{5} > \frac{19}{50} = \frac{19}{40}\left(\frac{76}{95}\right)\\ &= \frac{19}{40}\big(M(1,-1)\big) = \frac{19}{40}\big(M(-1,1)\big)\\ &= \phi\big(M(r,s)\big)\end{aligned}$$

for all $r,s \in \{-1,1\}$.

Also, Noorwali and Yeşilkaya [15] declared in Case 1 of Definition (13) that r and s are distinct since $M(r,s)$ is undefined if $r = s$. However, our result is valid for all $r,s \in \Phi\backslash \mathrm{Fix}(\Gamma)$.

The above comparison is illustrated graphically for all $r,s \in \{-1,1\}$ using Figs. 1 and 2.

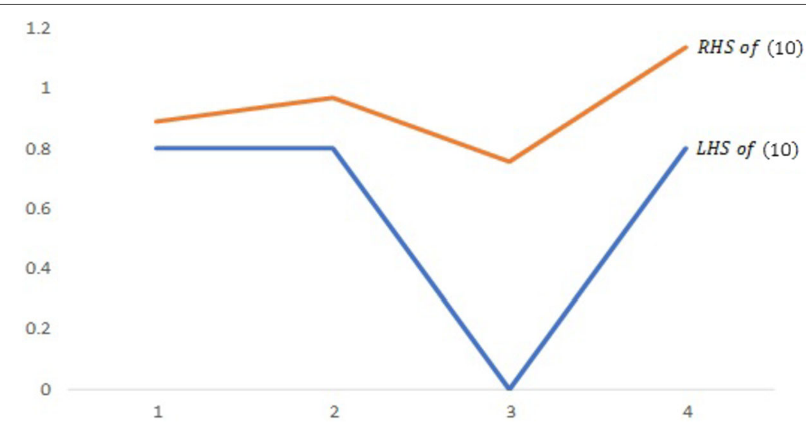

Figure 1 Illustration of contractive inequality (10) for all $r,s \in \{-1,1\}$

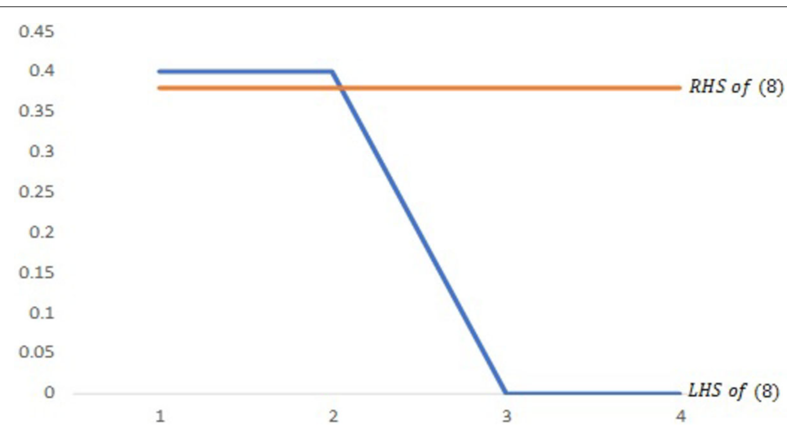

Figure 2 Illustration of contractive inequality (8) for all $r,s \in \{-1,1\}$

Therefore, Jaggi–Suzuki-type hybrid (G-α-ϕ)-contraction is not the Jaggi–Suzuki-type hybrid contraction defined by Noorwali and Yeşilkaya [15], and so Theorem 8 due to Noorwali and Yeşilkaya [15] is not applicable to this example.

Remark 2 Suppose in Definition 14 that $\frac{1}{2}G(r,\Gamma r,\Gamma^2 r) \leq G(r,s,\Gamma s)$ for all $r,s \in \Phi$.

Then we obtain the following consequences of our main result.

Definition 15 ([2]) Let $\Gamma : \Phi \longrightarrow \Phi$ and $\alpha : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be two mappings. Then Γ is said to be α-admissible if for all $r,s,t \in \Phi$,

$$\alpha(r,s,t) \geq 1 \quad \Rightarrow \quad \alpha(\Gamma r,\Gamma s,\Gamma t) \geq 1.$$

Definition 16 ([2]) Let (Φ, G) be a G-MS and let $\Gamma : \Phi \longrightarrow \Phi$ be a self-mapping of Φ. Then Γ is said to be a (G-α-ϕ)-contraction of type I if there exist two functions $\alpha : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ and $\phi \in \Psi$ such that for all $r,s,t \in \Phi$,

$$\alpha(r,s,t)G(\Gamma r,\Gamma s,\Gamma t) \leq \phi\bigl(G(r,s,t)\bigr).$$

Corollary 1 (see [[2], Theorem 29]) *Let (Φ, G) be a complete G-MS. Assume that $\Gamma : \Phi \longrightarrow \Phi$ is a(G-α-ϕ)-contraction of type I such that the following conditions are satisfied*:

(i) Γ *is* α-*admissible*;
(ii) *There exists* $r_0 \in \Phi$ *such that* $\alpha(r_0,\Gamma r_0,\Gamma r_0) \geq 1$;
(iii) Γ *is G-continuous.*

Then there exists $u \in \Phi$ *such that* $\Gamma u = u$.

Proof Consider Definition (14) and let $\Gamma s = t$, $\alpha : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be α-admissible mapping, $q > 0$, $\lambda_1 = 0$, and $\lambda_2 = 1$. Then Γ is a (G-α-ϕ)-contractive mapping of type I, and so inequality (10) becomes

$$\alpha(r,s,t)G(\Gamma r,\Gamma s,\Gamma t) \leq \phi\bigl(G(r,s,t)\bigr)$$

for all $r,s,t \in \Phi$ and $\phi \in \Psi$. Hence, the proof follows from Theorem 29 of Alghamdi and Karapınar [2]. □

Corollary 2 (see [[20], Theorem 3.1]) *Let (Φ, G) be a complete G-MS. Suppose that the self-mapping $\Gamma : \Phi \longrightarrow \Phi$ satisfies*

$$G(\Gamma r,\Gamma s,\Gamma t) \leq \phi\bigl(G(r,s,t)\bigr)$$

for all $r,s,t \in \Phi$. Then Γ has a unique FP (say u) and Γ is G-continuous at u.

Proof Consider Definition (14) and let $\alpha(r,s,\Gamma s) = 1$ for all $r,s \in \Phi$, $\Gamma s = t$, $q > 0$, $\lambda_1 = 0$, and $\lambda_2 = 1$. Then inequality (10) becomes

$$G(\Gamma r,\Gamma s,\Gamma t) \leq \phi\bigl(G(r,s,t)\bigr)$$

for all $r,s,t \in \Phi$ and $\phi \in \Psi$. This coincides with Theorem 3.1 due to Shatanawi [20], and so the proof follows in a similar manner. □

Corollary 3 (see [Theorem 1]) *Let (Φ, G) be a complete G-MS and let $\Gamma : \Phi \longrightarrow \Phi$ be a self-mapping satisfying*

$$G(\Gamma r, \Gamma s, \Gamma t) \leq kG(r,s,t)$$

for all $r,s,t \in \Phi$ where $0 \leq k < 1$. Then Γ has a unique FP (say u) and Γ is G-continuous at u.

Proof Consider Definition (14) and let $\alpha(r,s,\Gamma s) = 1$ for all $r,s \in \Phi$, $\Gamma s = t$, $q > 0$, $\lambda_1 = 0$, $\lambda_2 = 1$, and $\phi(p) = kp$ for all $p \geq 0$, $k \in [0,1)$. Then (10) coincides with (6) of Theorem 1. Therefore, it is easy to see that we can find a unique point u in Φ such that $\Gamma u = u$ and Γ is G-continuous at u. □

Remark 3 If, in addition to the assumptions of Remark 2, we specialize the parameters λ_i $(i = 1,2)$ and q, as well as let $\alpha(r,s,\Gamma s) = 1$ for all $r,s \in \Phi$ and $\phi(p) = kp$ for all $t \geq 0$, $k \in (0,1)$, then the following result is also a direct consequence of Theorem 2.

Corollary 4 *Let (Φ, G) be a complete G-MS. If there exists $k \in (0,1)$ such that, for all $r,s \in \Phi$, the mapping $\Gamma : \Phi \longrightarrow \Phi$ satisfies*

$$G(\Gamma r, \Gamma s, \Gamma^2 s) \leq kG(r,s,\Gamma s), \tag{16}$$

then Γ has an FP in Φ.

4 Ulam-type stability

Ulam stability was introduced by Ulam and seen to be a category of data dependence. This idea was further developed by Hyers and other researchers (see [9]). Karapınar and Fulga [9] investigated the general Ulam-type (U_t) stability in the sense of an *FP* problem in MS. Here, we consider the general U_t stability as an *FP* problem in the framework of *G*-MS.

Suppose that $\Gamma : \Phi \longrightarrow \Phi$ is a self-mapping on a *G*-MS (Φ, G). Then we say that the *FP* problem

$$\Gamma r = r \tag{17}$$

has the general U_t stability if and only if there exists a nondecreasing function $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ continuous at 0, $\mu(0) = 0$ in a manner that for every $\epsilon > 0$ and for any $s' \in \Phi$ satisfying the inequality

$$G(s', \Gamma s', \Gamma^2 s') \leq \epsilon, \tag{18}$$

there exists a solution $t \in \Phi$ of (17) such that

$$G(t, s', \Gamma s') \leq \mu(\epsilon). \tag{19}$$

For a positive number C, we take $\mu(p) = Cp$ for all $p \geq 0$. Then the *FP* of (17) is said to be U_t stable.

Let (Φ, G) be a G-MS. Then the *FP* problem (17) is said to be well posed on (Φ, G) if the following assumptions are satisfied:

(i) Γ has a unique *FP* $t \in \Phi$;

(ii) $G(r_x, t, t) = 0$ for any sequence $\{r_x\}_{x\in\mathbb{N}}$ in Φ such that $\lim_{x\to\infty} G(r_x, \Gamma r_x, \Gamma^2 r_x) = 0$.

Theorem 4 *Let (Φ, G) be a complete G-MS. If in addition to the assumptions of Theorem 3 and Remark 2 we have $\lambda_2 \in [0,1)$, then the following hold*:

(i) *FP equation (17) is Ulam–Hyers stable if $\alpha(u, v, \Gamma v) \geq 1$ for each u, v satisfying* (18);

(ii) *FP equation (17) is well posed if $\alpha(t, r_x, \Gamma r_x) \geq 1$ for any $\{r_x\}_{x\in\mathbb{N}}$ in Φ such that $\lim_{x\to\infty} G(r_x, \Gamma r_x, \Gamma^2 r_x) = 0$ and $\operatorname{Fix}(\Gamma) = \{t\}$.*

Proof (i) In Theorem 3, we have shown that there exists unique $t \in \Phi$ such that $\Gamma t = t$. Let $s' \in \Phi$ such that, for any $\epsilon > 0$, we have

$$G(s', \Gamma s', \Gamma^2 s') \leq \epsilon.$$

Then, obviously, t satisfies (18), and so we have $\alpha(t, s', \Gamma s') \geq 1$. Hence, by the rectangle inequality,

$$\begin{aligned}
G(t, s', \Gamma s') &\leq G(t, \Gamma s', \Gamma^2 s') + G(\Gamma^2 s', s', \Gamma s') \\
&= G(\Gamma t, \Gamma s', \Gamma^2 s') + G(s', \Gamma s', \Gamma^2 s') \\
&\leq \alpha(t, s', \Gamma s') G(\Gamma t, \Gamma s', \Gamma^2 s') + G(s', \Gamma s', \Gamma^2 s') \\
&\leq \phi\big(M(t, s', \Gamma s')\big) + G(s', \Gamma s', \Gamma^2 s') \\
&< M(t, s', \Gamma s') + G(s', \Gamma s', \Gamma^2 s') \\
&= \left[\lambda_1 \left(\frac{G(t, \Gamma t, \Gamma^2 t) \cdot G(s', \Gamma s', \Gamma^2 s')}{G(t, s', \Gamma s')}\right)^q + \lambda_2 G(t, s', \Gamma s')^q\right]^{\frac{1}{q}} \\
&\quad + G(s', \Gamma s', \Gamma^2 s') \\
&= \left[\lambda_1 \left(\frac{G(t, t, t) \cdot G(s', \Gamma s', \Gamma^2 s')}{G(t, s', \Gamma s')}\right)^q + \lambda_2 G(t, s', \Gamma s')^q\right]^{\frac{1}{q}} + G(s', \Gamma s', \Gamma^2 s') \\
&= \lambda_2^{\frac{1}{q}} G(t, s', \Gamma s') + G(s', \Gamma s', \Gamma^2 s'),
\end{aligned}$$

from which we obtain

$$\big(1 - \lambda_2^{\frac{1}{q}}\big) G(t, s', \Gamma s') < G(s', \Gamma s', \Gamma^2 s')$$

implying that

$$G(t, s', \Gamma s') < \left(\frac{1}{1 - \lambda_2^{\frac{1}{q}}}\right) G(s', \Gamma s', \Gamma^2 s') \leq C\epsilon,$$

where $C = \frac{1}{1-\lambda_2^{\frac{1}{q}}}$ for any $q > 0$ and $\lambda_2 \in [0,1)$.

(ii) Noting the supplementary condition and since $\mathrm{Fix}(\Gamma) = t$, we have

$$\begin{aligned}
G(t, r_x, \Gamma r_x) &\leq G(t, \Gamma r_x, \Gamma^2 r_x) + G(\Gamma^2 r_x, r_x, \Gamma r_x) \\
&= G(\Gamma t, \Gamma r_x, \Gamma^2 r_x) + G(r_x, \Gamma r_x, \Gamma^2 r_x) \\
&\leq \alpha(t, r_x, \Gamma r_x) G(\Gamma t, \Gamma r_x, \Gamma^2 r_x) + G(r_x, \Gamma r_x, \Gamma^2 r_x) \\
&\leq \phi(M(t, r_x, \Gamma r_x)) + G(r_x, \Gamma r_x, \Gamma^2 r_x) \\
&< M(t, r_x, \Gamma r_x) + G(r_x, \Gamma r_x, \Gamma^2 r_x) \\
&= \left[\lambda_1 \left(\frac{G(t, \Gamma t, \Gamma^2 t) \cdot G(r_x, \Gamma r_x, \Gamma^2 r_x)}{G(t, r_x, \Gamma r_x)}\right)^q + \lambda_2 G(t, r_x, \Gamma r_x)^q\right]^{\frac{1}{q}} \\
&\quad + G(r_x, \Gamma r_x, \Gamma^2 r_x) \\
&= \left[\lambda_1 \left(\frac{G(t, t, t) \cdot G(r_x, \Gamma r_x, \Gamma^2 r_x)}{G(t, r_x, \Gamma r_x)}\right)^q + \lambda_2 G(t, r_x, \Gamma r_x)^q\right]^{\frac{1}{q}} \\
&\quad + G(r_x, \Gamma r_x, \Gamma^2 r_x) \\
&= \lambda_2^{\frac{1}{q}} G(t, r_x, \Gamma r_x) + G(r_x, \Gamma r_x, \Gamma^2 r_x),
\end{aligned}$$

from which we obtain

$$\left(1 - \lambda_2^{\frac{1}{q}}\right) G(t, r_x, \Gamma r_x) < G(r_x, \Gamma r_x, \Gamma^2 r_x)$$

implying that

$$G(t, r_x, \Gamma r_x) < \left(\frac{1}{1 - \lambda_2^{\frac{1}{q}}}\right) G(r_x, \Gamma r_x, \Gamma^2 r_x).$$

Letting $x \to \infty$ and keeping in mind Proposition (1) and $\lim_{x\to\infty} G(r_x, \Gamma r_x, \Gamma^2 r_x) = 0$, we obtain

$$\lim_{x\to\infty} G(r_x, t, t) = \lim_{x\to\infty} G(t, r_x, \Gamma r_x) \leq \lim_{x\to\infty} G(r_x, \Gamma r_x, \Gamma^2 r_x) = 0.$$

□

That is, *FP* equation (17) is well posed.

5 Application to solution of an integral equation

In this section, Corollary 4 is applied to examine the existence criteria for a solution for a class of integral equations. Ideas in this section are motivated by [21].

Consider the integral equation

$$u(t) = h(t) + \int_a^b \mathcal{L}(t, s) f(s, u(s))\, ds, \quad t \in [a, b], \tag{20}$$

where $h : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$, $\mathcal{L} : [a, b] \times [a, b] \longrightarrow \mathbb{R}_+$, and $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ are given continuous functions and u is unknown.

Let $\Phi = C([a, b], \mathbb{R})$ be the set of all continuous real-valued functions defined on $[a, b]$. We equip Φ with the mapping

$$G(u, v, w) = \max_{a\leq t\leq b} \left(|u(t) - v(t)| + |u(t) - w(t)| + |v(t) - w(t)|\right). \tag{21}$$

Then, obviously, (Φ, G) is a complete G-MS. Consider the self-mapping $\Gamma : \Phi \longrightarrow \Phi$ defined by

$$\Gamma u(t) = h(t) + \int_a^b \mathcal{L}(t,s) f\bigl(s, u(s)\bigr)\, ds, \quad t \in [a,b]. \tag{22}$$

One can see that u^* is an *FP* of Γ if and only if u^* is a solution to (20).

Now, we study the existence conditions of integral equation (20) under the following hypotheses.

Theorem 5 *Assume that the following conditions are satisfied*:

(C1) $|f(s,u) - f(s,v)| + |f(s,u) - f(s,w)| + |f(s,v) - f(s,w)| \le |u - v| + |u - w| + |v - w|$ *for all* $s \in [a,b]$, $u, v, w \in \Phi$;

(C2) $\max_{t\in[a,b]} \int_a^b |\mathcal{L}(t,s)|\, ds = \eta < 1$.

Then integral equation (20) *has a solution in* Φ.

Proof Taking (21) into account, we obtain

$$\begin{aligned}
G\bigl(\Gamma u, \Gamma v, \Gamma^2 v\bigr) &= \max_{t\in[a,b]} \bigl|\Gamma u(t) - \Gamma v(t)\bigr| + \max_{t\in[a,b]} \bigl|\Gamma u(t) - \Gamma^2 v(t)\bigr| + \max_{t\in[a,b]} \bigl|\Gamma v(t) - \Gamma^2 v(t)\bigr| \\
&= \max_{t\in[a,b]} \left| \int_a^b \mathcal{L}(t,s)\bigl(f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr)\, ds \right| \\
&\quad + \max_{t\in[a,b]} \left| \int_a^b \mathcal{L}(t,s)\bigl(f\bigl(s,u(s)\bigr) - f\bigl(s,\Gamma v(s)\bigr)\bigr)\, ds \right| \\
&\quad + \max_{t\in[a,b]} \left| \int_a^b \mathcal{L}(t,s)\bigl(f\bigl(s,v(s)\bigr) - f\bigl(s,\Gamma v(s)\bigr)\bigr)\, ds \right| \\
&\le \max_{t\in[a,b]} \int_a^b \bigl|\mathcal{L}(t,s)\bigr| \bigl[\bigl|f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr| + \bigl|f\bigl(s,u(s)\bigr) - f\bigl(s,\Gamma v(s)\bigr)\bigr| \\
&\quad + \bigl|f\bigl(s,v(s)\bigr) - f\bigl(s,\Gamma v(s)\bigr)\bigr|\bigr]\, ds \\
&\le \max_{t\in[a,b]} \int_a^b \bigl|\mathcal{L}(t,s)\bigr| \bigl[\bigl|u(s) - v(s)\bigr| + \bigl|u(s) - \Gamma v(s)\bigr| + \bigl|v(s) - \Gamma v(s)\bigr|\bigr]\, ds \\
&\le \left(\max_{t\in[a,b]} \int_a^b \bigl|\mathcal{L}(t,s)\bigr|\, ds \right) \\
&\quad \left(\max_{t\in[a,b]} \int_a^b \bigl[\bigl|u(s) - v(s)\bigr| + \bigl|u(s) - \Gamma v(s)\bigr| + \bigl|v(s) - \Gamma v(s)\bigr|\bigr]\, ds \right) \\
&= \eta G(u, v, \Gamma v).
\end{aligned}$$

Hence, all the conditions of Corollary 4 are satisfied. It follows that Γ has an *FP* u^* in Φ, which corresponds to a solution of integral equation (20).

Conversely, if u^* is a solution of (20), then u^* is also a solution of (22) so that $\Gamma u^* = u^*$, that is, u^* is an *FP* of Γ. □

6 Open problem

For further research, an open problem is highlighted as follows:

A discretized population balance for continuous systems at steady state can be modeled by the nonlinear integral equation

$$g(t) = \frac{\mu}{2(1+2\mu)} \int_0^t g(t-r)g(r)\,dr + e^{-t}, \quad \mu \in \mathbb{R}. \tag{23}$$

So far, it is still open as to whether or not the existence conditions for solution of (23) can be obtained using any of the results established in this paper.

Remark 4

(i) We can deduce many other corollaries by replacing Γr with s or $\Gamma^2 s$ with t and by particularizing some of the parameters in Definition (14).

(ii) None of the results presented in this work can be expressed in the form $G(r,s,s)$ or $G(r,r,s)$. Hence, they cannot be obtained from their equivalent versions in MS.

7 Conclusion

An extension of MS, called G-MS, was introduced by Mustafa and Sims [14], and several *FP* results were studied in that space. However, Jleli and Samet [7] as well as Samet et al. [19] noted that most *FP* theorems obtained in G-MS are direct consequences of their analogs in MS. With a different opinion to the latter observation, a new general class of contractions, under the name Jaggi–Suzuki-type hybrid (G-α-ϕ)-contraction, is introduced in this manuscript, and some *FP* theorems that cannot be deduced from their corresponding ones in MS are proved. The prevalence of this family of contractions is the fact that its contractive inequality can be specialized in several ways depending on the given parameters. Consequently, a handful of corollaries, including some recently announced results in the literature, are highlighted and analyzed. Nontrivial comparative examples are constructed to validate the assumptions of our obtained theorems. Furthermore, we examined Ulam-type stability and well-posedness for the new contraction proposed herein. In addition, one of our obtained corollaries is applied to setup novel existence conditions for the solution of a class of integral equations. For some future aspects of our results, an open problem concerning discretized population balance model is highlighted, and its solution may be analyzed using the techniques established in this work.

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The authors declare no competing interests.

Author contributions
J. A. Jiddah and M. Noorwali: Conceptualization and Writing, M. S. Shagari and S. Kanwal, H. Aydi: carried out the proof of further consequences, applications and constructed some new special cases in form of corollaries, M. De La Sen and H. Aydi: Review and Editing, J. A. Jiddah and M. S. Shagari: Review and Editing. All authors have read and approved the final manuscript for submission and possible publication. All authors have also agreed to be personally and jointly accountable for their contributions in this manuscript. All authors contributed equally and significantly in writing this article. All authors read and approved the final version.

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