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New stochastic fractional integral and related inequalities of Jensen–Mercer and Hermite–Hadamard–Mercer type for convex stochastic processes

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Abstract

In this investigation, we unfold the Jensen–Mercer ($\mathtt{J}-\mathtt{M}$) inequality for convex stochastic processes via a new fractional integral operator. The incorporation of convex stochastic processes, the $\mathtt{J}-\mathtt{M}$ inequality and a fractional integral operator having an exponential kernel brings a new direction to the theory of inequalities. With this in mind, estimations of Hermite–Hadamard–Mercer ($\mathtt{H}-\mathtt{H}-\mathtt{M}$)-type fractional inequalities involving convex stochastic processes are presented. In the context of the new fractional integral operator, we also investigate a novel identity for differentiable mappings. Then, a new related $\mathtt{H}-\mathtt{H}-\mathtt{M}$-type inequality is presented using this identity as an auxiliary result. Applications to special means and matrices are also presented. These findings are particularly appealing from the perspective of optimization, as they provide a larger context to analyze optimization and mathematical programming problems.

Keywords: Convex stochastic process; Hermite–Hadamard–Mercer inequality; Fractional integral operator; Exponential kernel

1 Introduction

The concept of convexity for stochastic processes has received much attention in recent years because of its usefulness in optimization, optimal designs, and numerical approximations. Guessab *et al.* [1] investigated the error of the barycentric approximation and the convex function. The Jensen-type inequalities on convex polytopes were also presented by Guessab [2, 3], who also looked at the error in the approximation of a convex function. Nikodem [4] proposed convex stochastic processes in 1980 and looked into their regularity characteristics. Skowroǹski [5] derived some more conclusions on convex stochastic processes in 1992, which generalize several known convex functions. More properties of convex and Jensen-convex processes were presented by Pales in [6]. Skowronski [7] investigated wright-convex stochastic processes. Kotrys [8] provided a new generalization of the Hermite–Hadamard ($\mathtt{H}-\mathtt{H}$) inequality for convex stochastic processes. Many mathematicians began researching this notion soon after Kotrys's work was published, and many improvements of $\mathtt{H}-\mathtt{H}$-type inequalities incorporating other types of convexities were established. Kotrys [9] studied strongly convex stochastic processes, Barrez [10] and Saleem [11] investigated h-convex stochastic processes, Iscan *et al.* [12] introduced the p-convex stochastic processes, Maden *et al.* [13] proposed s-convex stochastic processes in the first sense, and derived $\mathtt{H}-\mathtt{H}$-type inequalities for them. Set *et al.* [14] defined s-convex stochastic process in the second sense, and they looked at $\mathtt{H}-\mathtt{H}$-type inequalities for these processes. Furthermore, preinvex stochastic processes have been described in recent studies [15] and Fu *et al.* [16] derived the $\mathtt{H}-\mathtt{H}$-type inequality and its refinements via an n-polynomial convex stochastic process. For some recent generalizations on the stochastic process and corresponding inequalities, see [12, 17, 18] and the references therein. Stochastic processes also have applications in variation principles [19], optimal Robin boundary control problem [20], optimal consumption and equilibrium prices [21], Portfolio optimization [22], gradient iteration [23], and physics [24].

Suppose that $(\Omega, \mathbb{A}, \mathbb{P})$ is an arbitrary probability space. A function $\mathtt{F}: \mathtt{I}\times\Omega \to \mathbb{R}$ is said to be a random variable if it is $\mathbb{A}$-measurable. A function $\mathtt{F}: \mathtt{I}\times\mathbb{R}$ is said to be a stochastic process if for every $\mathtt{u}\in\mathtt{I}$, the function $\mathtt{F}(\mathtt{u},\cdot)$ is a random variable, where $\mathtt{I}\subseteq\mathbb{R}$ is an interval.

The stochastic process $\mathtt{F}$ is said to be:

1. *Stochastically continuous* on $\mathtt{I}$, if

$$\eta - \lim_{\mathtt{u}\to\mathtt{u}_0} \mathtt{F}(\mathtt{u},\cdot) = \mathtt{F}(\mathtt{u}_0,\cdot),$$

for all $\mathtt{u}_0\in\mathtt{I}$, where $\eta-\lim$ denotes the limit in probability.

2. *Mean-square continuous* in $\mathtt{I}$, if

$$\lim_{\mathtt{u}\to\mathtt{u}_0} \mathbb{E}\big[\mathtt{F}(\mathtt{u},\cdot)-\mathtt{F}(\mathtt{u}_0,\cdot)\big]^2 = 0,$$

for all $\mathtt{u}_0\in\mathtt{I}$, where $\mathbb{E}[\mathtt{F}(\mathtt{U},\cdot)]$ denotes the expectation value of the random variable $\mathtt{F}(\mathtt{u},\cdot)$.

Definition 1.1 (see [4, 5, 24]) Let $\mathtt{F}: \mathtt{I}\times\Omega\to\mathbb{R}$ be a stochastic process with $\mathbb{E}[\mathtt{F}(\mathtt{u})^2]\leq\infty$, where $\mathtt{u}\in\mathtt{I}$. Then, the random variable represented by $\mathtt{Z}:\Omega\to\mathbb{R}$ is said to be a mean-square integral of the process $\mathtt{F}$ on $[\eta,\xi]$ if for all sequences of partitions of interval $[\eta,\xi]\subseteq\mathtt{I}$, $\eta=\mathtt{u}_0<\mathtt{u}_1<\cdots<\mathtt{u}_n=\xi$ and for all $\theta_k\in[\mathtt{u}_{k-1};\mathtt{u}_k]$, $k=1,\ldots,n$, we have

$$\lim_{n\to\infty}\mathbb{E}\left[\left(\sum_{k=1}^{n}\mathtt{F}(\theta_k,\cdot)(\mathtt{u}_k-\mathtt{u}_{(k-1)})-\mathtt{Z}(\cdot)\right)^2\right]=0,$$

which can also be written as

$$\int_\eta^\xi \mathtt{F}(\mathtt{u},\cdot)\,d\mathtt{u}=\mathtt{Z}(\cdot).$$

For the existence of the mean-square integral it is enough to assume the mean-square continuity of the stochastic process $\mathtt{F}$.

Definition 1.2 (see [4]) Assume $(\Omega, \mathbb{A}, \mathbb{P})$ is an arbitrary probability space. Then, a stochastic process $\mathtt{F} : \mathtt{I} \times \Omega \to \mathbb{R}$ is said to be a convex stochastic process, if we have

$$\mathtt{F}\big(\mu \mathtt{w} + (1-\mu)\mathtt{y}, \cdot\big) \le \mu \mathtt{F}(\mathtt{w}, \cdot) + (1-\mu)\mathtt{F}(\mathtt{y}, \cdot),$$

for all $\mathtt{w}, \mathtt{y} \in \mathtt{I}$, $\mu \in [0,1]$.

Theorem 1.1 (see [8]) *Let* $\mathtt{F} : \mathtt{I} \times \Omega \to \mathbb{R}$ *be a convex stochastic and mean-square continuous process in the interval* $\mathtt{I} \times \Omega$ *then, the following inequality holds true*:

$$\mathtt{F}\left(\frac{\mathtt{w}+\mathtt{y}}{2}, \cdot\right) \le \frac{1}{\mathtt{y}-\mathtt{w}} \int_{\mathtt{w}}^{\mathtt{y}} \mathtt{F}(x, \cdot)\,dx \le \frac{\mathtt{F}(\mathtt{w}, \cdot) + \mathtt{F}(\mathtt{y}, \cdot)}{2}, \quad (a.e.) \tag{1.1}$$

for $\mathtt{w}, \mathtt{y} \in \mathtt{I}$, $\mathtt{w} < \mathtt{y}$.

The purpose of this study is to establish a counterpart of the $\mathtt{J}-\mathtt{M}$ and $\mathtt{H}-\mathtt{H}-\mathtt{M}$-type inequalities for convex stochastic processes via a new fractional integral operator. To additionally encourage discussion of this article, we present the definition of the R–L fractional operator and related $\mathtt{H}-\mathtt{H}$-type inequalities.

Hafiz [25] introduced the following stochastic mean-square fractional integral operators given as:

Definition 1.3 (see [25]) Let $\mathtt{F} : \mathrm{I} \times \Omega \to \mathbb{R}$ be a stochastic process. Then, the mean-square continuous fractional integrals $\mathrm{I}^{\alpha}_{\eta^+}$ and $\mathrm{I}^{\alpha}_{\xi^-}$ of order $\alpha > 0$ are defined by

$$\mathrm{I}^{\alpha}_{\eta^+}[\mathtt{F}](z) = \frac{1}{\Gamma(\alpha)} \int_{\eta}^{z} (z-x)^{\alpha-1} \mathtt{F}(x, \cdot)\,dx \quad (a.e)$$

and

$$\mathrm{I}^{\alpha}_{\xi^-}[\mathtt{F}](z) = \frac{1}{\Gamma(\alpha)} \int_{z}^{\xi} (x-z)^{\alpha-1} \mathtt{F}(x, \cdot)\,dx \quad (a.e),$$

where $\Gamma(\cdot)$ is the Gamma function.

In [26], the authors employed the stochastic mean-square fractional integrals to obtain fractional stochastic $\mathtt{H}-\mathtt{H}$-type inequalities, given as follows:

Theorem 1.2 ([26]) *Let* $\mathtt{F} : \mathrm{I} \times \Omega \to \mathbb{R}$ *be a convex stochastic process in the interval I. Then, the following stochastic fractional integral inequality holds true*:

$$\mathtt{F}\left(\frac{\eta+\xi}{2}, \cdot\right) \le \frac{\Gamma(\alpha+1)}{2(\xi-\eta)^{\alpha}} \big[\mathrm{I}^{\alpha}_{\eta^+}[\mathtt{F}](\xi) + \mathrm{I}^{\alpha}_{\xi^-}[\mathtt{F}](\eta)\big] \le \frac{\mathtt{F}(\eta, \cdot) + \mathtt{F}(\xi, \cdot)}{2}, \quad (a.e.)$$

for all η, ξ *and* $\alpha > 0$.

During the preceding few decades, researchers have been quite interested in their generalizations and enhancements, as indicated by a large number of publications on the subject. Öğülmüs and Sarikaya [27] introduced the fractional version of the $\mathtt{H}-\mathtt{H}-\mathtt{M}$-type inequality via the Riemann–Liouville fractional operator. After this article, many improved versions of the $\mathtt{H}-\mathtt{H}-\mathtt{M}$-type inequality for different existing fractional operators such as Conformable [28], ψ-Riemann–Liouville [29], Katugampola [30], and Atangana–Baleanu [31] fractional operators have been presented. Motivated by the above-mentioned articles, the results by Agahi and Babakhani [26], and the concept of stochastic processes, the main objective is to generalize the $\mathtt{J}-\mathtt{M}$ and $\mathtt{H}-\mathtt{H}-\mathtt{M}$ inequalities pertaining to a new fractional operator with an exponential function in its kernel.

To generate more generalized results, we introduced a new mean-square fractional integral operator in this study. This is due to the fact that this fractional operator has an exponential kernel. The aforementioned fractional inequalities do not follow our conclusions, which is a distinction between our results and existing generalizations. Many experts have presented extensions of the $\mathtt{H}-\mathtt{H}$ inequality with various fractional integral operators, but there is no exponential characteristic in their results. This research sparked an interest in creating more generalized fractional inequalities with an exponential function as the kernel. Moreover, the applications of convex stochastic processes to the main findings brings a new direction to the field of analytical inequalities. We have incorporated the concepts of stochastic processes with fractional calculus to present new inequalities. Although we can find few studies on the growth of integral inequalities involving convex stochastic processes, there are still numerous gaps to be filled for integral inequalities. As a result, to fill the gap, in this investigation, we put forward our step in establishing $\mathtt{J}-\mathtt{M}$ and $\mathtt{H}-\mathtt{H}-\mathtt{M}$ inequalities for convex stochastic processes employing fractional integral operators. We strongly believe that this article will encourage many researchers to present different improved versions of $\mathtt{J}-\mathtt{M}$-type inequalities via both classical and fractional integrals for various new convex stochastic processes.

2 Jensen–Mercer inequality via convex stochastic processes

Before describing the key conclusions, we introduce the definitions of the new generalized mean-square fractional integrals.

Definition 2.1 Let $\mathtt{F}:\mathrm{I}\times\Omega\to\mathbb{R}$ be a stochastic process. Then, the mean-square continuous fractional integrals $\mathrm{J}^{\alpha}_{\eta^+}$ and $\mathrm{J}^{\alpha}_{\xi^-}$ of order $\alpha>0$ are defined by

$$\mathtt{J}^{\alpha}_{\eta^+}[\mathtt{F}](x) := \frac{1}{\alpha}\int_{\eta}^{x} e^{-\frac{1-\alpha}{\alpha}(x-\mathrm{z})}\mathtt{F}(\mathrm{z},\cdot)\,d\mathrm{z}, \quad (a.e.) \quad (0\le\eta<x<\xi)$$

and

$$\mathtt{J}^{\alpha}_{\xi^-}[\mathtt{F}](x) := \frac{1}{\alpha}\int_{x}^{\xi} e^{-\frac{1-\alpha}{\alpha}(\mathrm{z}-x)}\mathtt{F}(\mathrm{z},\cdot)\,d\mathrm{z}, \quad (a.e.) \quad (0\le\eta<x<\xi),$$

respectively.

For brevity, we denote $\sigma = \frac{1-\alpha}{\alpha}(\mathrm{y}-\mathrm{w})$ throughout the manuscript.

In this section, we derive two new $\mathtt{J}-\mathtt{M}$ inequalities for convex functions, which will be used to present our main results.

Lemma 2.1 *Assume* $\mathtt{F}:\mathrm{I}\times\Omega\longrightarrow\mathbb{R}$ *is a convex stochastic process,* $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_m)$ *and* $0<\mathtt{w}_1\le\mathtt{w}_2\le\cdots\le\mathtt{w}_m$*, such that* $\sum_{k=1}^{m}\alpha_k=1$*, then the following inequality holds true*

$$\mathtt{F}\left(\sum_{k=1}^{m}\alpha_k\mathtt{w}_k,\cdot\right)\le\sum_{k=1}^{m}\alpha_k\mathtt{F}(\mathtt{w}_k,\cdot), \tag{2.1}$$

almost everywhere.

Next, we will prove the $\mathtt{J}-\mathtt{M}$ inequality for convex stochastic process and for that, we need Lemma 2.1.

Lemma 2.2 *Let* $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_m)$ *and* $0<\mathtt{w}_1\le\mathtt{w}_2\le\cdots\le\mathtt{w}_m$*, such that* $\sum_{k=1}^{m}\alpha_k=1$*. If* $\mathtt{F}:\mathrm{I}\times\Omega\longrightarrow\mathbb{R}$ *is a convex stochastic process, then the following inequality*

$$\mathtt{F}(\mathtt{w}_1+\mathtt{w}_m-\mathtt{w}_k,\cdot)\le\mathtt{F}(\mathtt{w}_1,\cdot)+\mathtt{F}(\mathtt{w}_m,\cdot)-\mathtt{F}(\mathtt{w}_k,\cdot), \tag{2.2}$$

holds true almost everywhere.

Proof Let us consider $\mathtt{y}_k=\mathtt{w}_1+\mathtt{w}_m-\mathtt{w}_k$, then it immediately follows that $\mathtt{w}_1+\mathtt{w}_m=\mathtt{w}_k+\mathtt{y}_k$. Therefore, the pairs $\mathtt{w}_1$, $\mathtt{w}_m$ and $\mathtt{w}_k$, $\mathtt{y}_k$ possess the same midpoint. Since this is the case, we have μ such that

$$\mathtt{w}_k=\mu\mathtt{w}_1+(1-\mu)\mathtt{w}_m,$$
$$\mathtt{y}_k=(1-\mu)\mathtt{w}_1+\mu\mathtt{w}_m,$$

where $0\le\mu\le$ and $1\le k\le m$. Now, if we use the general convexity of $\mathtt{F}$, we have

$$\begin{aligned}
\mathtt{F}(\mathtt{y}_k,\cdot)&=\mathtt{F}\big((1-\mu)\mathtt{w}_1+\mu\mathtt{w}_m,\cdot\big)\\
&\le(1-\mu)\mathtt{F}(\mathtt{w}_1,\cdot)+\mu\mathtt{F}(\mathtt{w}_m,\cdot)\\
&=\mathtt{F}(\mathtt{w}_1,\cdot)+\mathtt{F}(\mathtt{w}_m,\cdot)-\big[\mu\mathtt{F}(\mathtt{w}_1,\cdot)+(1-\mu)\mathtt{F}(\mathtt{w}_m,\cdot)\big]\\
&\le\mathtt{F}(\mathtt{w}_1,\cdot)+\mathtt{F}(\mathtt{w}_m,\cdot)-\mathtt{F}\big(\mu\mathtt{w}_1+(1-\mu)\mathtt{w}_m,\cdot\big)\\
&=\mathtt{F}(\mathtt{w}_1,\cdot)+\mathtt{F}(\mathtt{w}_m,\cdot)-\mathtt{F}(\mathtt{w}_k,\cdot)
\end{aligned}$$

and $\mathtt{y}_k=\mathtt{w}_1+\mathtt{w}_m-\mathtt{w}_k$. This yields the desired result. □

Lemma 2.3 *Let* $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_m)$ *and* $0<\mathtt{w}_1\le\mathtt{w}_2\le\cdots\le\mathtt{w}_m$*, such that* $\sum_{k=1}^{m}\alpha_k=1$*. If* $\mathtt{F}:\mathrm{I}\times\Omega\longrightarrow\mathbb{R}$ *is a convex stochastic process, then the following inequality*

$$\mathtt{F}\left(\mathtt{w}_1+\mathtt{w}_m-\sum_{k=1}^{m}\alpha_k\mathtt{w}_k,\cdot\right)\le\mathtt{F}(\mathtt{w}_1,\cdot)+\mathtt{F}(\mathtt{w}_m,\cdot)-\sum_{k=1}^{m}\alpha_k\mathtt{F}(\mathtt{w}_k,\cdot), \tag{2.3}$$

holds true almost everywhere.

Proof The proof follows from Lemma 2.1 and Lemma 2.2 for general convex stochastic processes,

$$\begin{aligned} \mathtt{F}\left(\mathtt{w}_1 + \mathtt{w}_m - \sum_{k=1}^{m} \mu_k \mathtt{w}_k, \cdot\right) &= \mathtt{F}\left(\sum_{k=1}^{m} \mu_k (\mathtt{w}_1 + \mathtt{w}_m - \mathtt{w}_k, \cdot)\right) \\ &\leq \sum_{k=1}^{m} \mu_k \mathtt{F}(\mathtt{w}_1 + \mathtt{w}_m - \mathtt{w}_k, \cdot) \\ &\leq \sum_{k=1}^{m} \mu_k \big[\mathtt{F}(\mathtt{w}_1, \cdot) + \mathtt{F}(\mathtt{w}_m, \cdot) - \mathtt{F}(\mathtt{w}_k, \cdot)\big] \\ &= \mathtt{F}(\mathtt{w}_1, \cdot) + \mathtt{F}(\mathtt{w}_m, \cdot) - \sum_{k=1}^{m} \mu_k \mathtt{F}(\mathtt{w}_k, \cdot). \end{aligned}$$

This yields the desired result. □

3 Hadamard–Jensen–Mercer- and Pachpatte–Mercer-type inequalities via fractional integrals

Theorem 3.1 *Let* $\mathtt{F} : \mathrm{I} \times \Omega \to \mathbb{R}$ *be a convex stochastic process in the interval* I, *such that* $\eta, \xi \in \mathrm{I}$, *with* $0 < \eta < \xi$. *Then, for* $\mathtt{w}, \mathtt{y} > 0$, *the following fractional inequality holds true*:

$$\begin{aligned} \mathtt{F}\left(\eta + \xi - \frac{\mathtt{w} + \mathtt{y}}{2}, \cdot\right) &\leq \big[\mathtt{F}(\eta, \cdot) + \mathtt{F}(\xi, \cdot)\big] - \frac{1 - \alpha}{2(1 - e^{-\sigma})} \big[\mathtt{J}^{\alpha}_{\mathtt{w}^+}[\mathtt{F}](\mathtt{y}) + \mathtt{J}^{\alpha}_{\mathtt{y}^-}[\mathtt{F}](\mathtt{w})\big] \\ &\leq \big[\mathtt{F}(\eta, \cdot) + \mathtt{F}(\xi, \cdot)\big] - \mathtt{F}\left(\frac{\mathtt{w} + \mathtt{y}}{2}, \cdot\right). \quad (a.e.) \end{aligned} \tag{3.1}$$

Proof Since F is a convex function on $[\eta, \xi]$, by the hypotheses of the Jensen–Mercer inequality, we can write

$$\mathtt{F}\left(\eta + \xi - \frac{\mathrm{S}_1 + \mathrm{S}_2}{2}, \cdot\right) \leq \mathtt{F}(\eta, \cdot) + \mathtt{F}(\xi, \cdot) - \mathtt{F}\left(\frac{\mathrm{S}_1 + \mathrm{S}_2}{2}, \cdot\right). \quad (a.e.)$$

Now, if we let $\mathrm{S}_1 = \mu\mathtt{w} + (1 - \mu)\mathtt{y}$ and $\mathrm{S}_2 = \mu\mathtt{y} + (1 - \mu)\mathtt{w}$, we have

$$\mathtt{F}\left(\eta + \xi - \frac{\mathtt{w} + \mathtt{y}}{2}, \cdot\right) \leq \mathtt{F}(\eta, \cdot) + \mathtt{F}(\xi, \cdot) - \frac{\mathtt{F}(\mu\mathtt{w} + (1 - \mu)\mathtt{y}, \cdot) + \mathtt{F}(\mu\mathtt{w} + (1 - \mu)\mathtt{y}, \cdot)}{2}. \quad (a.e.)$$

If we multiply the above inequality by $e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{w})\mu}$ and then integrate the obtained inequality over [0, 1], we find that

$$\begin{aligned} &\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{w})\mu} \mathtt{F}\left(\eta + \xi - \frac{\mathtt{w} + \mathtt{y}}{2}, \cdot\right) d\mu \\ &\quad\leq \mathtt{F}(\eta, \cdot) + \mathtt{F}(\xi, \cdot) \int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{w})\mu} \, d\mu \\ &\qquad - \frac{1}{2} \int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{w})\mu} \big[\mathtt{F}\big(\mu\mathtt{w} + (1 - \mu)\mathtt{y}, \cdot\big) + \mathtt{F}\big(\mu\mathtt{w} + (1 - \mu)\mathtt{y}, \cdot\big)\big] \, d\mu, \end{aligned}$$

consequently,

$$\begin{aligned}&\left(\frac{1-e^{-\sigma}}{\sigma}\right)\mathtt{F}\left(\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right)\\&\quad\leq\left[\mathtt{F}(\eta,\cdot)+\mathtt{F}(\xi,\cdot)\right]\left(\frac{1-e^{-\sigma}}{\sigma}\right)\\&\qquad-\frac{1}{2(\mathtt{y}-\mathtt{w})}\left[\int_{\mathtt{w}}^{\mathtt{y}}e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{u})}\mathtt{F}(\mathtt{u},\cdot)\,d\mathtt{u}+\int_{\mathtt{w}}^{\mathtt{y}}e^{-\frac{1-\alpha}{\alpha}(\mathtt{u}-\mathtt{w})}\mathtt{F}(\mathtt{u},\cdot)\,d\mathtt{u}\right],\end{aligned}$$

which readily yields

$$\begin{aligned}&\mathtt{F}\left(\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right)\\&\quad\leq\left[\mathtt{F}(\eta,\cdot)+\mathtt{F}(\xi,\cdot)\right]-\frac{1-\alpha}{2(1-e^{-\sigma})}\left[\mathtt{J}^{\alpha}_{\mathtt{w}^+}[\mathtt{F}](\mathtt{y})+\mathtt{J}^{\alpha}_{\mathtt{y}^-}[\mathtt{F}](\mathtt{w})\right].\quad(a.e.)\end{aligned}\tag{3.2}$$

This gives us the first part of the desired result. For the second part, we use the convexity of $\mathtt{F}$.

$$\begin{aligned}\mathtt{F}\left(\frac{\mathtt{S}_1+\mathtt{S}_2}{2},\cdot\right)&=\mathtt{F}\left(\frac{(\mu\mathtt{w}+(1-\mu)\mathtt{y})+(\mu\mathtt{y}+(1-\mu)\mathtt{w})}{2},\cdot\right)\\&\leq\frac{\mathtt{F}(\mu\mathtt{w}+(1-\mu)\mathtt{y},\cdot)+\mathtt{F}(\mu\mathtt{y}+(1-\mu)\mathtt{w},\cdot)}{2}.\end{aligned}\tag{3.3}$$

If we multiply the above Equation (3.3) by $e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{w})\mu}$ and then integrate the obtained inequality over $[0, 1]$, we find that

$$\begin{aligned}&\mathtt{F}\left(\frac{\mathtt{S}_1+\mathtt{S}_2}{2},\cdot\right)\int_0^1e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{w})\mu}\,d\mu\\&\quad\leq\frac{1}{2}\left[\int_0^1e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{w})\mu}\mathtt{F}\big(\mu\mathtt{w}+(1-\mu)\mathtt{y},\cdot\big)\,d\mu\right.\\&\qquad\left.+\int_0^1e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{w})\mu}\mathtt{F}\big(\mu\mathtt{y}+(1-\mu)\mathtt{w},\cdot\big)\,d\mu\right].\end{aligned}$$

It follows from the above developments that

$$\left(\frac{1-e^{-\sigma}}{\sigma}\right)\mathtt{F}\left(\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right)\leq\frac{\alpha}{2(\mathtt{y}-\mathtt{w})}\left[\mathtt{J}^{\alpha}_{\mathtt{w}^+}[\mathtt{F}](\mathtt{y})+\mathtt{J}^{\alpha}_{\mathtt{y}^-}[\mathtt{F}](\mathtt{w})\right].$$

This implies

$$-\mathtt{F}\left(\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right)\geqq-\frac{1-\alpha}{2(1-e^{-\sigma})}\left[\mathtt{J}^{\alpha}_{\mathtt{w}^+}[\mathtt{F}](\mathtt{y})+\mathtt{J}^{\alpha}_{\mathtt{y}^-}[\mathtt{F}](\mathtt{w})\right].\quad(a.e.)\tag{3.4}$$

Now, adding $[\mathtt{F}(\eta,\cdot)+\mathtt{F}(\xi,\cdot)]$ to both sides of the above Equation (3.4), we have

$$\begin{aligned}&\left[\mathtt{F}(\eta,\cdot)+\mathtt{F}(\xi,\cdot)\right]-\mathtt{F}\left(\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right)\\&\quad\geqq\left[\mathtt{F}(\eta,\cdot)+\mathtt{F}(\xi,\cdot)\right]-\frac{1-\alpha}{2(1-e^{-\sigma})}\left[\mathtt{J}^{\alpha}_{\mathtt{w}^+}[\mathtt{F}](\mathtt{y})+\mathtt{J}^{\alpha}_{\mathtt{y}^-}[\mathtt{F}](\mathtt{w})\right].\quad(a.e.)\end{aligned}\tag{3.5}$$

From Equations (3.2) and (3.5), the proof is completed. □

Corollary 3.1 *If we put $\eta = \mathtt{w}$ and $\xi = \mathtt{y}$ in Theorem 3.1, we have the following new fractional inequality for convex stochastic processes*:

$$\begin{aligned} \mathtt{F}\left(\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right) &\le \left[\mathtt{F}(\mathtt{w},\cdot)+\mathtt{F}(\mathtt{y},\cdot)\right] - \frac{1-\alpha}{2(1-e^{-\sigma})}\left[\mathtt{J}^{\alpha}_{\mathtt{w}^+}[\mathtt{F}](\mathtt{y}) + \mathtt{J}^{\alpha}_{\mathtt{y}^-}[\mathtt{F}](\mathtt{w})\right] \\ &\le \left[\mathtt{F}(\mathtt{w},\cdot)+\mathtt{F}(\mathtt{y},\cdot)\right] - \mathtt{F}\left(\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right). \quad (a.e.) \end{aligned} \tag{3.6}$$

Corollary 3.2 *If $\alpha \to 1$, we have $\lim_{\alpha\to 1}\frac{1-\alpha}{2(1-e^{-\sigma})} = \frac{1}{2(\mathtt{y}-\mathtt{w})}$. Then, from Theorem 3.1, the following new Hermite–Hadamard–Mercer-type inequality for convex stochastic processes holds true*:

$$\begin{aligned} \mathtt{F}\left(\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right) &\le \left[\mathtt{F}(\eta,\cdot)+\mathtt{F}(\xi,\cdot)\right] - \frac{1}{\mathtt{y}-\mathtt{w}}\int_{\mathtt{w}}^{\mathtt{y}} \mathtt{F}(u)\,du \\ &\le \left[\mathtt{F}(\eta,\cdot)+\mathtt{F}(\xi,\cdot)\right] - \mathtt{F}\left(\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right). \quad (a.e.) \end{aligned} \tag{3.7}$$

Theorem 3.2 *Let $\mathtt{F} : \mathrm{I}\times\Omega \to \mathbb{R}$ be a convex stochastic process in the interval* I, *such that $\eta,\xi \in \mathrm{I}$, with $0 < \eta < \xi$. Then, for $\mathtt{w},\mathtt{y} > 0$, the following fractional inequality holds true*:

$$\begin{aligned} \mathtt{F}\left(\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right) &\le \frac{1-\alpha}{2(1-e^{-\sigma})}\left[\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{w}^-}[\mathtt{F}](\eta+\xi-\mathtt{y}) + \mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{y}^+}[\mathtt{F}](\eta+\xi-\mathtt{w})\right] \\ &\le \mathtt{F}(\eta,\cdot)+\mathtt{F}(\xi,\cdot) - \frac{\mathtt{F}(\mathtt{w},\cdot)+\mathtt{F}(\mathtt{y},\cdot)}{2}. \quad (a.e.) \end{aligned} \tag{3.8}$$

Proof Let $\mathtt{F} : [\eta,\xi] \to \mathbb{R}$ be a convex stochastic process. Then, by hypothesis, we have

$$\begin{aligned} \mathtt{F}\left(\eta+\xi-\frac{\mathrm{S}_1+\mathrm{S}_2}{2},\cdot\right) &= \mathtt{F}\left(\frac{\eta+\xi-\mathrm{S}_1+\eta+\xi-\mathrm{S}_2}{2},\cdot\right) \\ &\le \frac{1}{2}\big(\mathtt{F}(\eta+\xi-\mathrm{S}_1,\cdot)+\mathtt{F}(\eta+\xi-\mathrm{S}_2,\cdot)\big). \quad (a.e.) \end{aligned} \tag{3.9}$$

Now, if we change the variables as

$$\eta+\xi-\mathrm{S}_1 = \mu(\eta+\xi-\mathtt{w}) + (1-\mu)(\eta+\xi-\mathtt{y})$$

and

$$\eta+\xi-\mathrm{S}_2 = \mu(\eta+\xi-\mathtt{y}) + (1-\mu)(\eta+\xi-\mathtt{w})$$

in Equation (3.9), we have

$$\begin{aligned} &\mathtt{F}\left(\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right) \\ &\quad\le \frac{[\mathtt{F}(\mu(\eta+\xi-\mathtt{w})+(1-\mu)(\eta+\xi-\mathtt{y}),\cdot)+\mathtt{F}(\mu(\eta+\xi-\mathtt{y})+(1-\mu)(\eta+\xi-\mathtt{w}),\cdot)]}{2}. \end{aligned} \tag{3.10}$$

Multiplying both sides of the above Equation (3.10) by $e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{w})\mu}$ and then integrating the obtained result over $[0,1]$, we find that

$$\begin{aligned}&\left(\frac{1-e^{-\sigma}}{\sigma}\right)\mathtt{F}\left(\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right)\\&\quad\leq\frac{1}{2(\mathtt{y}-\mathtt{w})}\left[\int_{\eta+\xi-\mathtt{y}}^{\eta+\xi-\mathtt{w}}e^{-\frac{1-\alpha}{\alpha}(\mathtt{u}-(\eta+\xi-\mathtt{y}))}\mathtt{F}(\mathtt{u},\cdot)\,d\mathtt{u}+\int_{\eta+\xi-\mathtt{y}}^{\eta+\xi-\mathtt{w}}e^{-\frac{1-\alpha}{\alpha}((\eta+\xi-\mathtt{w})-\mathtt{u})}\mathtt{F}(\mathtt{u},\cdot)\,d\mathtt{u}\right]\\&\quad=\frac{\alpha}{2(\mathtt{y}-\mathtt{w})}\left[\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{w}^-}[\mathtt{F}](\eta+\xi-\mathtt{y})+\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{y}^+}[\mathtt{F}](\eta+\xi-\mathtt{w})\right],\end{aligned}$$

which readily yields

$$\mathtt{F}\left(\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right)\leq\frac{1-\alpha}{2(1-e^{-\sigma})}\left[\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{w}^-}[\mathtt{F}](\eta+\xi-\mathtt{y})+\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{y}^+}[\mathtt{F}](\eta+\xi-\mathtt{w})\right].\quad(a.e.)$$

This leads us to the first part of the proof. Now, for the second part, we use the convexity of $\mathtt{F}$, given as

$$\mathtt{F}\big(\mu(\eta+\xi-\mathtt{w})+(1-\mu)(\eta+\xi-\mathtt{y}),\cdot\big)\leq\mu\mathtt{F}(\eta+\xi-\mathtt{w},\cdot)+(1-\mu)\mathtt{F}(\eta+\xi-\mathtt{y},\cdot)$$

and

$$\mathtt{F}\big(\mu(\eta+\xi-\mathtt{y})+(1-\mu)(\eta+\xi-\mathtt{w}),\cdot\big)\leq\mu\mathtt{F}(\eta+\xi-\mathtt{y},\cdot)+(1-\mu)\mathtt{F}(\eta+\xi-\mathtt{w},\cdot).$$

Adding both the inequalities, we find that

$$\begin{aligned}&\mathtt{F}\big(\mu(\eta+\xi-\mathtt{w})+(1-\mu)(\eta+\xi-\mathtt{y}),\cdot\big)+\mathtt{F}\big(\mu(\eta+\xi-\mathtt{y})+(1-\mu)(\eta+\xi-\mathtt{w}),\cdot\big)\\&\quad\leq\mathtt{F}(\eta+\xi-\mathtt{w},\cdot)+\mathtt{F}(\eta+\xi-\mathtt{y},\cdot)\\&\quad\leq\mathtt{F}(\eta,\cdot)+\mathtt{F}(\xi,\cdot)-\mathtt{F}(\mathtt{w},\cdot)+\mathtt{F}(\eta,\cdot)+\mathtt{F}(\xi,\cdot)-\mathtt{F}(\mathtt{y},\cdot)\\&\quad=2\big[\mathtt{F}(\eta,\cdot)+\mathtt{F}(\xi,\cdot)\big]-\big[\mathtt{F}(\mathtt{w},\cdot)+\mathtt{F}(\mathtt{y},\cdot)\big].\end{aligned}\tag{3.11}$$

Multiplying both sides of the above Equation (3.11) by $e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{w})\mu}$ and then integrating over $[0,1]$, we have

$$\begin{aligned}&\int_0^1e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{w})\mu}\mathtt{F}\big(\mu(\eta+\xi-\mathtt{w})+(1-\mu)(\eta+\xi-\mathtt{y}),\cdot\big)\,d\mu\\&\qquad+\int_0^1e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{w})\mu}\mathtt{F}\big(\mu(\eta+\xi-\mathtt{y})+(1-\mu)(\eta+\xi-\mathtt{w}),\cdot\big)\,d\mu\\&\quad\leq2\big[\big[\mathtt{F}(\eta,\cdot)+\mathtt{F}(\xi,\cdot)\big]-\big[\mathtt{F}(\mathtt{w},\cdot)+\mathtt{F}(\mathtt{y},\cdot)\big]\big]\int_0^1e^{-\frac{1-\alpha}{\alpha}(\mathtt{y}-\mathtt{w})\mu}\,d\mu.\end{aligned}$$

It follows from the above developments that

$$\begin{aligned}&\frac{\alpha}{2(\mathtt{y}-\mathtt{w})}\left[\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{w}^-}[\mathtt{F}](\eta+\xi-\mathtt{y})+\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{y}^+}[\mathtt{F}](\eta+\xi-\mathtt{w})\right]\\&\quad\leq\left[\mathtt{F}(\eta,\cdot)+\mathtt{F}(\xi,\cdot)-\frac{\mathtt{F}(\mathtt{w},\cdot)+\mathtt{F}(\mathtt{y},\cdot)}{2}\right]\left[\frac{(1-e^{-\sigma})}{\sigma}\right],\end{aligned}$$

which readily yields

$$\frac{1-\alpha}{2(1-e^{-\sigma})}\left[\mathrm{J}^{\alpha}_{\eta+\xi-\mathrm{w}^-}[\mathrm{F}](\eta+\xi-\mathrm{y})+\mathrm{J}^{\alpha}_{\eta+\xi-\mathrm{y}^+}[\mathrm{F}](\eta+\xi-\mathrm{w})\right]$$
$$\leq \mathrm{F}(\eta,\cdot)+\mathrm{F}(\xi,\cdot)-\frac{\mathrm{F}(\mathrm{w},\cdot)+\mathrm{F}(\mathrm{y},\cdot)}{2}. \quad (a.e.)$$

This leads us to the proof of the desired Theorem 3.2. □

Corollary 3.3 *If we put $\eta = \mathrm{w}$ and $\xi = \mathrm{y}$ in Theorem 3.2, we have the following new fractional inequality for convex stochastic processes*:

$$\mathrm{F}\left(\frac{\mathrm{w}+\mathrm{y}}{2},\cdot\right)\leq \frac{1-\alpha}{2(1-e^{-\sigma})}\left[\mathrm{J}^{\alpha}_{\mathrm{w}^+}[\mathrm{F}](\mathrm{y})+\mathrm{J}^{\alpha}_{\mathrm{y}^-}[\mathrm{F}](\mathrm{w})\right]\leq \frac{\mathrm{F}(\mathrm{w},\cdot)+\mathrm{F}(\mathrm{y},\cdot)}{2}. \quad (a.e.)$$

Corollary 3.4 *If $\alpha \to 1$, we have $\lim_{\alpha\to 1}\frac{1-\alpha}{2(1-e^{-\sigma})} = \frac{1}{2(\mathrm{y}-\mathrm{w})}$. Then, from Theorem 3.2, the following new Hermite–Hadamard–Mercer-type inequality for convex stochastic processes holds true*:

$$\mathrm{F}\left(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2},\cdot\right)\leq \frac{1}{\mathrm{y}-\mathrm{w}}\int_{\mathrm{w}}^{\mathrm{y}}\mathrm{F}(\eta+\xi-u,\cdot)\,du$$
$$\leq \left[\mathrm{F}(\eta,\cdot)+\mathrm{F}(\xi,\cdot)\right]-\frac{\mathrm{F}(\mathrm{w},\cdot)+\mathrm{F}(\mathrm{y},\cdot)}{2}. \quad (a.e.) \tag{3.12}$$

Remark 3.1 If we put $\eta = \mathrm{w}$ and $\xi = \mathrm{y}$ in Theorem 3.2, then for $\alpha \to 1$, the Hermite–Hadamard-type inequality (1.1) for convex stochastic processes given by Kotrys [8] is recovered.

Theorem 3.3 *Let $\mathrm{F}:\mathrm{I}\times\Omega\to\mathbb{R}$ be a convex stochastic process in the interval* I, *such that $\eta,\xi\in\mathrm{I}$, with $0<\eta<\xi$. Then, for $\mathrm{w},\mathrm{y}>0$, the following fractional inequality holds true*:

$$\mathrm{F}\left(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2},\cdot\right)$$
$$\leq \frac{1-\alpha}{2(1-e^{-\frac{\sigma}{2}})}\left[\mathrm{J}^{\alpha}_{(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2})^-}[\mathrm{F}](\eta+\xi-\mathrm{y})+\mathrm{J}^{\alpha}_{(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2})^+}[\mathrm{F}](\eta+\xi-\mathrm{w})\right]$$
$$\leq \mathrm{F}(\eta,\cdot)+\mathrm{F}(\xi,\cdot)-\frac{\mathrm{F}(\mathrm{w},\cdot)+\mathrm{F}(\mathrm{y},\cdot)}{2}. \quad (a.e.) \tag{3.13}$$

Proof Using the convexity of F on $[\eta,\xi]$, one has

$$\mathrm{F}\left(\eta+\xi-\frac{\mathrm{S}_1+\mathrm{S}_2}{2},\cdot\right)=\mathrm{F}\left(\frac{\eta+\xi-\mathrm{S}_1+\eta+\xi-\mathrm{S}_2}{2},\cdot\right)$$
$$\leq \frac{\mathrm{F}(\eta+\xi-\mathrm{S}_1,\cdot)+\mathrm{F}(\eta+\xi-\mathrm{S}_2,\cdot)}{2}. \quad (a.e.)$$

If we let $\mathrm{S}_1=\frac{\mu}{2}\mathrm{w}+\frac{2-\mu}{2}\mathrm{y}$ and $\mathrm{S}_1=\frac{\mu}{2}\mathrm{y}+\frac{2-\mu}{2}\mathrm{w}$, we have

$$2\mathrm{F}\left(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2},\cdot\right)$$
$$\leq \mathrm{F}\left(\eta+\xi-\left(\frac{\mu}{2}\mathrm{w}+\frac{2-\mu}{2}\mathrm{y}\right),\cdot\right)+\mathrm{F}\left(\eta+\xi-\left(\frac{\mu}{2}\mathrm{y}+\frac{2-\mu}{2}\mathrm{w}\right),\cdot\right). \tag{3.14}$$

If we multiply the above equation (3.14) by $e^{-\frac{1-\alpha}{2\alpha}(\mathrm{y}-\mathrm{w})\mu}$ and then integrate the obtained inequality over [0, 1] we have

$$\begin{aligned}
&\frac{4(1-e^{-\frac{\sigma}{2}})}{\sigma}\mathrm{F}\left(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2},\cdot\right)\\
&\le\int_0^1 e^{-\frac{1-\alpha}{2\alpha}(\mathrm{y}-\mathrm{w})\mu}\mathrm{F}\left(\eta+\xi-\left(\frac{\mu}{2}\mathrm{w}+\frac{2-\mu}{2}\mathrm{y}\right),\cdot\right)d\mu\\
&\quad+\int_0^1 e^{-\frac{1-\alpha}{2\alpha}(\mathrm{y}-\mathrm{w})\mu}\mathrm{F}\left(\eta+\xi-\left(\frac{\mu}{2}\mathrm{y}+\frac{2-\mu}{2}\mathrm{w}\right),\cdot\right)d\mu\\
&\le\frac{2}{\mathrm{y}-\mathrm{w}}\int_{\eta+\xi-\mathrm{y}}^{\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2}} e^{-\frac{1-\alpha}{\alpha}(\mathrm{u}-(\eta+\xi-\mathrm{y}))}\mathrm{F}(\mathrm{u},\cdot)\,d\mathrm{u}\\
&\quad+\frac{2}{\mathrm{y}-\mathrm{w}}\int_{\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2}}^{\eta+\xi-\mathrm{w}} e^{-\frac{1-\alpha}{\alpha}((\eta+\xi-\mathrm{w})-\mathrm{u})}\mathrm{F}(\mathrm{u},\cdot)\,d\mathrm{u}\\
&=\frac{2\alpha}{\mathrm{y}-\mathrm{w}}\left[\mathrm{J}^{\alpha}_{(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2})^-}[\mathrm{F}](\eta+\xi-\mathrm{y})+\mathrm{J}^{\alpha}_{(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2})^+}[\mathrm{F}](\eta+\xi-\mathrm{w})\right].
\end{aligned}$$

It follows from the above developments that

$$\begin{aligned}
&\mathrm{F}\left(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2},\cdot\right)\\
&\le\frac{1-\alpha}{2(1-e^{-\frac{\sigma}{2}})}\left[\mathrm{J}^{\alpha}_{(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2})^-}[\mathrm{F}](\eta+\xi-\mathrm{y})+\mathrm{J}^{\alpha}_{(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2})^+}[\mathrm{F}](\eta+\xi-\mathrm{w})\right].\quad (a.e.)
\end{aligned}\tag{3.15}$$

This leads us to the first part of the proof. Now, for the second part, we use the J–M inequality

$$\begin{aligned}
&\mathrm{F}\left(\eta+\xi-\left(\frac{\mu}{2}\mathrm{w}+\frac{2-\mu}{2}\mathrm{y}\right),\cdot\right)+\mathrm{F}\left(\eta+\xi-\left(\frac{\mu}{2}\mathrm{y}+\frac{2-\mu}{2}\mathrm{w}\right),\cdot\right)\\
&\le 2\left[\mathrm{F}(\eta,\cdot)+\mathrm{F}(\xi,\cdot)\right]-\mathrm{F}(\mathrm{w},\cdot)+\mathrm{F}(\mathrm{y},\cdot).
\end{aligned}\tag{3.16}$$

If we multiply the above Equation (3.16) by $e^{-\frac{1-\alpha}{2\alpha}(\mathrm{y}-\mathrm{w})\mu}$ and then integrate the obtained result over [0, 1], we find

$$\begin{aligned}
&\frac{2\alpha}{\mathrm{y}-\mathrm{w}}\left[\mathrm{J}^{\alpha}_{(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2})^-}[\mathrm{F}](\eta+\xi-\mathrm{y})+\mathrm{J}^{\alpha}_{(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2})^+}[\mathrm{F}](\eta+\xi-\mathrm{w})\right]\\
&\le\left[2\left[\mathrm{F}(\eta,\cdot)+\mathrm{F}(\xi,\cdot)\right]-\mathrm{F}(\mathrm{w},\cdot)+\mathrm{F}(\mathrm{y},\cdot)\right]\int_0^1 e^{-\frac{1-\alpha}{2\alpha}(\mathrm{y}-\mathrm{w})\mu}\,d\mu.\\
&=\left[2\left[\mathrm{F}(\eta,\cdot)+\mathrm{F}(\xi,\cdot)\right]-\mathrm{F}(\mathrm{w},\cdot)+\mathrm{F}(\mathrm{y},\cdot)\right]\frac{2(1-e^{-\frac{\sigma}{2}})}{\sigma},
\end{aligned}$$

which readily gives

$$\begin{aligned}
&\frac{1-\alpha}{2(1-e^{-\frac{\sigma}{2}})}\left[\mathrm{J}^{\alpha}_{(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2})^-}[\mathrm{F}](\eta+\xi-\mathrm{y})+\mathrm{J}^{\alpha}_{(\eta+\xi-\frac{\mathrm{w}+\mathrm{y}}{2})^+}[\mathrm{F}](\eta+\xi-\mathrm{w})\right]\\
&\le\mathrm{F}(\eta,\cdot)+\mathrm{F}(\xi,\cdot)-\frac{\mathrm{F}(\mathrm{w},\cdot)+\mathrm{F}(\mathrm{y},\cdot)}{2}.\quad (a.e.)
\end{aligned}\tag{3.17}$$

Reorganizing Equations (3.15) and (3.17) completes the proof of Theorem 3.3. □

Corollary 3.5 *If we put $\eta = \mathtt{w}$ and $\xi = \mathtt{y}$ in Theorem 3.3, we have the following new fractional inequality for convex stochastic processes*:

$$\mathtt{F}\left(\frac{\mathtt{w}+\mathtt{y}}{2},\cdot\right) \le \frac{1-\alpha}{2(1-e^{-\frac{\sigma}{2}})}\left[\mathtt{J}^{\alpha}_{(\frac{\mathtt{w}+\mathtt{y}}{2})^-}[\mathtt{F}](\mathtt{w}) + \mathtt{J}^{\alpha}_{(\frac{\mathtt{w}+\mathtt{y}}{2})^+}[\mathtt{F}](\mathtt{y})\right] \le \frac{\mathtt{F}(\mathtt{w},\cdot)+\mathtt{F}(\mathtt{y},\cdot)}{2}. \quad (a.e.)$$

Remark 3.2 If $\alpha \to 1$, then from Theorem 3.3, we recapture (3.12).

Remark 3.3 If we put $\eta = \mathtt{w}$ and $\xi = \mathtt{y}$ in Theorem 3.3, then for $\alpha \to 1$, the Hermite–Hadamard-type inequality (1.1) for convex stochastic processes given by Kotrys [8] is recovered.

4 Further inequalities for differentiable convex stochastic processes

This section focuses on demonstrating a new identity for differentiable stochastic processes via a new fractional integral operator having an exponential kernel. The $\mathtt{H}-\mathtt{H}-\mathtt{M}$-type inequality is then refined by taking this identity into account.

Lemma 4.1 *Let $\mathtt{F} : \mathrm{I} \times \Omega \to \mathbb{R}$ be a convex stochastic process in the interval I, such that $\eta, \xi \in \mathrm{I}$, with $0 < \eta < \xi$. Then, for $\mathtt{w}, \mathtt{y} > 0$,*

$$\begin{aligned}
&\frac{\mathtt{F}(\eta+\xi-\mathtt{w},\cdot)+\mathtt{F}(\eta+\xi-\mathtt{y},\cdot)}{2} \\
&\quad - \frac{1-\alpha}{2(1-e^{-\sigma})}\left[\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{w}^-}[\mathtt{F}](\eta+\xi-\mathtt{y}) + \mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{y}^+}[\mathtt{F}](\eta+\xi-\mathtt{w})\right] \\
&\quad = \frac{\mathtt{y}-\mathtt{w}}{2(1-e^{-\sigma})}\int_0^1 \left[e^{-\sigma(1-\mu)} - e^{-\sigma\mu}\right]\mathtt{F}'\left(\eta+\xi-\left(\mu\mathtt{w}+(1-\mu)\mathtt{y}\right),\cdot\right)d\mu, \qquad (4.1)
\end{aligned}$$

holds true almost everywhere.

Proof

$$\begin{aligned}
\text{Let } I_1 &= \int_0^1 e^{-\sigma\mu}\mathtt{F}'\left(\eta+\xi-\left(\mu\mathtt{w}+(1-\mu)\mathtt{y}\right),\cdot\right)d\mu \\
&= \frac{e^{-\sigma}\mathtt{F}(\eta+\xi-\mathtt{w},\cdot)-\mathtt{F}(\eta+\xi-\mathtt{y},\cdot)}{\mathtt{y}-\mathtt{w}} + \frac{\sigma\alpha}{(\mathtt{y}-\mathtt{w})^2}\left[\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{w}^-}[\mathtt{F}](\eta+\xi-\mathtt{y})\right] \\
&= \frac{e^{-\sigma}\mathtt{F}(\eta+\xi-\mathtt{w},\cdot)-\mathtt{F}(\eta+\xi-\mathtt{y},\cdot)}{\mathtt{y}-\mathtt{w}} + \frac{1-\alpha}{\mathtt{y}-\mathtt{w}}\left[\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{w}^-}[\mathtt{F}](\eta+\xi-\mathtt{y})\right]
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_0^1 e^{-\sigma(1-\mu)}\mathtt{F}'\left(\eta+\xi-\left(\mu\mathtt{w}+(1-\mu)\mathtt{y}\right),\cdot\right)d\mu \\
&= \frac{\mathtt{F}(\eta+\xi-\mathtt{w},\cdot)-e^{-\sigma}\mathtt{F}(\eta+\xi-\mathtt{y},\cdot)}{\mathtt{y}-\mathtt{w}} + \frac{\sigma\alpha}{(\mathtt{y}-\mathtt{w})^2}\left[\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{y}^+}[\mathtt{F}](\eta+\xi-\mathtt{w})\right] \\
&= \frac{\mathtt{F}(\eta+\xi-\mathtt{w},\cdot)-e^{-\sigma}\mathtt{F}(\eta+\xi-\mathtt{y},\cdot)}{\mathtt{y}-\mathtt{w}} - \frac{1-\alpha}{\mathtt{y}-\mathtt{w}}\left[\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{y}^+}[\mathtt{F}](\eta+\xi-\mathtt{w})\right].
\end{aligned}$$

From the above developments, we have

$$\begin{aligned} I_2 - I_1 =& \frac{\mathrm{F}(\eta+\xi-\mathrm{w},\cdot) - e^{-\sigma}\mathrm{F}(\eta+\xi-\mathrm{y},\cdot)}{\mathrm{y}-\mathrm{w}} - \frac{1-\alpha}{\mathrm{y}-\mathrm{w}}\big[\mathrm{J}^{\alpha}_{\eta+\xi-\mathrm{y}^+}[\mathrm{F}](\eta+\xi-\mathrm{w})\big] \\ &- \left[\frac{e^{-\sigma}\mathrm{F}(\eta+\xi-\mathrm{w},\cdot) - \mathrm{F}(\eta+\xi-\mathrm{y},\cdot)}{\mathrm{y}-\mathrm{w}} + \frac{1-\alpha}{\mathrm{y}-\mathrm{w}}\big[\mathrm{J}^{\alpha}_{\eta+\xi-\mathrm{w}^-}[\mathrm{F}](\eta+\xi-\mathrm{y})\big]\right] \\ =& \frac{\mathrm{F}(\eta+\xi-\mathrm{w},\cdot) - e^{-\sigma}\mathrm{F}(\eta+\xi-\mathrm{y},\cdot)}{\mathrm{y}-\mathrm{w}} - \frac{e^{-\sigma}\mathrm{F}(\eta+\xi-\mathrm{w},\cdot) - \mathrm{F}(\eta+\xi-\mathrm{y},\cdot)}{\mathrm{y}-\mathrm{w}} \\ &- \frac{1-\alpha}{\mathrm{y}-\mathrm{w}}\big[\mathrm{J}^{\alpha}_{\eta+\xi-\mathrm{y}^+}[\mathrm{F}](\eta+\xi-\mathrm{w})\big] - \frac{1-\alpha}{\mathrm{y}-\mathrm{w}}\big[\mathrm{J}^{\alpha}_{\eta+\xi-\mathrm{w}^-}[\mathrm{F}](\eta+\xi-\mathrm{y})\big] \\ =& \big(1-e^{-\sigma}\big)\frac{\mathrm{F}(\eta+\xi-\mathrm{w},\cdot) + \mathrm{F}(\eta+\xi-\mathrm{y},\cdot)}{\mathrm{y}-\mathrm{w}} \\ &- \frac{1-\alpha}{\mathrm{y}-\mathrm{w}}\big[\mathrm{J}^{\alpha}_{\eta+\xi-\mathrm{y}^+}[\mathrm{F}](\eta+\xi-\mathrm{w}) + \mathrm{J}^{\alpha}_{\eta+\xi-\mathrm{w}^-}[\mathrm{F}](\eta+\xi-\mathrm{y})\big]. \end{aligned} \tag{4.2}$$

Multiplying both sides of the above equality by $\frac{\mathrm{y}-\mathrm{w}}{2(1-e^{-\sigma})}$, we obtain

$$\begin{aligned} &\frac{\mathrm{F}(\eta+\xi-\mathrm{w},\cdot) + \mathrm{F}(\eta+\xi-\mathrm{y},\cdot)}{2} \\ &\quad - \frac{1-\alpha}{2(1-e^{-\sigma})}\big[\mathrm{J}^{\alpha}_{\eta+\xi-\mathrm{y}^+}[\mathrm{F}](\eta+\xi-\mathrm{w}) + \mathrm{J}^{\alpha}_{\eta+\xi-\mathrm{w}^-}[\mathrm{F}](\eta+\xi-\mathrm{y})\big] \\ &\quad = \frac{\mathrm{y}-\mathrm{w}}{2(1-e^{-\sigma})}[I_2 - I_1]. \end{aligned}$$

This leads us to the proof of Lemma 4.1. □

Corollary 4.1 *If we put $\eta = \mathrm{w}$ and $\xi = \mathrm{y}$ in Lemma 4.1, we have the following new equality for convex stochastic processes*:

$$\begin{aligned} &\frac{\mathrm{F}(\mathrm{w},\cdot) + \mathrm{F}(\mathrm{y},\cdot)}{2} - \frac{1-\alpha}{2(1-e^{-\sigma})}\big[\mathrm{J}^{\alpha}_{\mathrm{y}^-}[\mathrm{F}](\mathrm{w}) + \mathrm{J}^{\alpha}_{\mathrm{w}^+}[\mathrm{F}](\mathrm{y})\big] \\ &\quad = \frac{\mathrm{y}-\mathrm{w}}{2(1-e^{-\sigma})}\int_0^1 \big[e^{-\sigma(1-\mu)} - e^{-\sigma\mu}\big]\mathrm{F}'\big(\mu\mathrm{y} + (1-\mu)\mathrm{w},\cdot\big)\,d\mu. \quad (a.e.) \end{aligned}$$

Theorem 4.1 *Let $\mathrm{F}: \mathrm{I}\times\Omega \to \mathbb{R}$ be a convex stochastic process in the interval I, such that $\eta, \xi \in \mathrm{I}$, with $0 < \eta < \xi$. Then, for $\mathrm{w}, \mathrm{y} > 0$, the following fractional inequality*

$$\begin{aligned} &\bigg|\frac{\mathrm{F}(\eta+\xi-\mathrm{w},\cdot) + \mathrm{F}(\eta+\xi-\mathrm{y},\cdot)}{2} \\ &\quad - \frac{1-\alpha}{2(1-e^{-\sigma})}\big[\mathrm{J}^{\alpha}_{\eta+\xi-\mathrm{y}^+}[\mathrm{F}](\eta+\xi-\mathrm{w}) + \mathrm{J}^{\alpha}_{\eta+\xi-\mathrm{w}^-}[\mathrm{F}](\eta+\xi-\mathrm{y})\big]\bigg| \\ &\quad \le \frac{\mathrm{y}-\mathrm{w}}{2(1-e^{-\sigma})}\left(\frac{1+e^{-\sigma}-2e^{-\frac{\sigma}{2}}}{\sigma}\right)\left[\big(\big|\mathrm{F}'(\eta,\cdot)\big| + \big|\mathrm{F}'(\xi,\cdot)\big|\big) - \frac{|\mathrm{F}'(\mathrm{w},\cdot)| + |\mathrm{F}'(\mathrm{y},\cdot)|}{2}\right], \end{aligned}$$

holds true almost everywhere.

Proof Employing Lemma 4.1, using properties of modulus, the convexity of $|\mathtt{F}|$, and the J–M inequality, we have

$$
\begin{aligned}
&\Bigg|\frac{\mathtt{F}(\eta+\xi-\mathtt{w},\cdot)+\mathtt{F}(\eta+\xi-\mathtt{y},\cdot)}{2} \\
&\quad -\frac{1-\alpha}{2(1-e^{-\sigma})}\left[\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{y}^{+}}[\mathtt{F}](\eta+\xi-\mathtt{w})+\mathtt{J}^{\alpha}_{\eta+\xi-\mathtt{w}^{-}}[\mathtt{F}](\eta+\xi-\mathtt{y})\right]\Bigg| \\
&\leq \frac{\mathtt{y}-\mathtt{w}}{2(1-e^{-\sigma})}\int_0^1 \left|e^{-\sigma(1-\mu)}-e^{-\sigma\mu}\right|\left|\mathtt{F}'\big(\eta+\xi-\big(\mu\mathtt{w}+(1-\mu)\mathtt{y}\big),\cdot\big)\right|d\mu \\
&\leq \frac{\mathtt{y}-\mathtt{w}}{2(1-e^{-\sigma})}\int_0^1 \left|e^{-\sigma(1-\mu)}-e^{-\sigma\mu}\right|\big||\mathtt{F}'(\eta,\cdot)+\mathtt{F}'(\xi,\cdot) \\
&\quad -\big(\mu\mathtt{F}'(\mathtt{w},\cdot)+(1-\mu)\mathtt{F}'(\mathtt{y},\cdot)|\big)\,d\mu \\
&= \frac{\mathtt{y}-\mathtt{w}}{2(1-e^{-\sigma})}\int_0^{\frac{1}{2}}\big(e^{-\sigma\mu}-e^{-\sigma(1-\mu)}\big) \\
&\quad \times\left[\left|\mathtt{F}'(\eta,\cdot)\right|+\left|\mathtt{F}'(\xi,\cdot)\right|-\big(\mu\left|\mathtt{F}'(\mathtt{w},\cdot)\right|+(1-\mu)\left|\mathtt{F}'(\mathtt{y},\cdot)\right|\big)\right]d\mu \\
&\quad +\frac{\mathtt{y}-\mathtt{w}}{2(1-e^{-\sigma})}\int_{\frac{1}{2}}^1\big(e^{-\sigma(1-\mu)}-e^{-\sigma\mu}\big) \\
&\quad \times\left[\left|\mathtt{F}'(\eta,\cdot)\right|+\left|\mathtt{F}'(\xi,\cdot)\right|-\big(\mu\left|\mathtt{F}'(\mathtt{w},\cdot)\right|+(1-\mu)\left|\mathtt{F}'(\mathtt{y},\cdot)\right|\big)\right]d\mu \\
&= \frac{\mathtt{y}-\mathtt{w}}{2(1-e^{-\sigma})}\Bigg\{\big(\left|\mathtt{F}'(\eta,\cdot)\right|+\left|\mathtt{F}'(\xi,\cdot)\right|\big)\int_0^{\frac{1}{2}}\big(e^{-\sigma\mu}-e^{-\sigma(1-\mu)}\big)\,d\mu \\
&\quad -\Bigg\{\left|\mathtt{F}'(\mathtt{w},\cdot)\right|\int_0^{\frac{1}{2}}\big(e^{-\sigma\mu}-e^{-\sigma(1-\mu)}\big)\mu\,d\mu \\
&\quad +\left|\mathtt{F}'(\mathtt{y},\cdot)\right|\int_0^{\frac{1}{2}}\big(e^{-\sigma\mu}-e^{-\sigma(1-\mu)}\big)(1-\mu)\,d\mu\Bigg\}\Bigg\} \\
&\quad +\frac{\mathtt{y}-\mathtt{w}}{2(1-e^{-\sigma})}\Bigg\{\big(\left|\mathtt{F}'(\eta,\cdot)\right|+\left|\mathtt{F}'(\xi,\cdot)\right|\big)\int_{\frac{1}{2}}^1\big(e^{-\sigma(1-\mu)}-e^{-\sigma\mu}\big)\,d\mu \\
&\quad -\Bigg\{\left|\mathtt{F}'(\mathtt{w},\cdot)\right|\int_{\frac{1}{2}}^1\big(e^{-\sigma(1-\mu)}-e^{-\sigma\mu}\big)\mu\,d\mu \\
&\quad +\left|\mathtt{F}'(\mathtt{y},\cdot)\right|\int_{\frac{1}{2}}^1\big(e^{-\sigma(1-\mu)}-e^{-\sigma\mu}\big)(1-\mu)\,d\mu\Bigg\}\Bigg\} \\
&= \frac{\mathtt{y}-\mathtt{w}}{2(1-e^{-\sigma})}\{\big(\left|\mathtt{F}'(\eta,\cdot)\right|+\left|\mathtt{F}'(\xi,\cdot)\right|\big)\left(\frac{e^{-\sigma}(e^{\sigma}+1)-2e^{-\frac{\sigma}{2}}}{\sigma}\right) \\
&\quad -\{\left|\mathtt{F}'(\mathtt{w},\cdot)\right|\frac{e^{-\sigma}(e^{\sigma}-1)-\sigma e^{-\frac{\sigma}{2}}}{\sigma^2} \\
&\quad +\left|\mathtt{F}'(\mathtt{y},\cdot)\right|\left(\frac{e^{-\sigma}((\sigma-1)e^{\sigma}+\sigma+1)-\sigma e^{-\frac{\sigma}{2}}}{\sigma^2}\right)+\big(\left|\mathtt{F}'(\eta,\cdot)\right| \\
&\quad +\left|\mathtt{F}'(\xi,\cdot)\right|\big)\left(\frac{e^{-\sigma}(e^{\sigma}+1)-2e^{-\frac{\sigma}{2}}}{\sigma}\right) \\
&\quad -\left\{\left|\mathtt{F}'(\mathtt{w},\cdot)\right|\left(\frac{e^{-\sigma}((\sigma-1)e^{\sigma}+\sigma+1)-\sigma e^{-\frac{\sigma}{2}}}{\sigma^2}\right)+\left|\mathtt{F}'(\mathtt{y},\cdot)\right|\left(\frac{e^{-\sigma}(e^{\sigma}-1)-\sigma e^{-\frac{\sigma}{2}}}{\sigma^2}\right)\right\}
\end{aligned}
$$

$$= \frac{\mathtt{y}-\mathtt{w}}{2(1-e^{-\sigma})}\left(\frac{1+e^{-\sigma}-2e^{-\frac{\sigma}{2}}}{\sigma}\right)$$
$$\times \left[\left(\left|\mathtt{F}'(\eta,\cdot)\right| + \left|\mathtt{F}'(\xi,\cdot)\right|\right) - \frac{|\mathtt{F}'(\mathtt{w},\cdot)| + |\mathtt{F}'(\mathtt{y},\cdot)|}{2}\right]. \quad (a.e.)$$

This leads us to the desired inequality. □

Corollary 4.2 *If we put $\eta = \mathtt{w}$ and $\xi = \mathtt{y}$ in Theorem 4.1, we have the following inequality for convex stochastic processes*:

$$\left|\frac{\mathtt{F}(\mathtt{w},\cdot)+\mathtt{F}(\mathtt{y},\cdot)}{2} - \frac{1-\alpha}{2(1-e^{-\sigma})}\left[\mathtt{J}^{\alpha}_{\mathtt{y}^-}[\mathtt{F}](\mathtt{w}) + \mathtt{J}^{\alpha}_{\mathtt{w}^+}[\mathtt{F}](\mathtt{y})\right]\right|$$
$$= \frac{\mathtt{y}-\mathtt{w}}{2(1-e^{-\sigma})}\left(\frac{1+e^{-\sigma}-2e^{-\frac{\sigma}{2}}}{\sigma}\right)\frac{|\mathtt{F}'(\mathtt{w},\cdot)| + |\mathtt{F}'(\mathtt{y},\cdot)|}{2}. \quad (a.e.)$$

5 Applications

Here, we present applications of our established results via special means and matrices.

5.1 Applications to special means

1. The arithmetic mean:

$$\mathtt{A} = \mathtt{A}(\mathtt{q}_1,\mathtt{q}_2) = \frac{\mathtt{q}_1+\mathtt{q}_2}{2}, \quad \mathtt{q}_1,\mathtt{q}_2 \geq 0.$$

2. The logarithmic mean:

$$\mathtt{L} = \mathtt{L}(\mathtt{q}_1,\mathtt{q}_2) = \frac{\mathtt{q}_2-\mathtt{q}_1}{\ln \mathtt{q}_2 - \ln \mathtt{q}_1}, \quad \mathtt{q}_1,\mathtt{q}_2 > 0.$$

3. The p-logarithmic mean:

$$\mathtt{L}_p = \mathtt{L}_p(\mathtt{q}_1,\mathtt{q}_2) = \left(\frac{\mathtt{q}_2^{p+1}-\mathtt{q}_1^{p+1}}{(p+1)(\mathtt{q}_2-\mathtt{q}_1)}\right)^{\frac{1}{p}}, \quad \mathtt{q}_1,\mathtt{q}_2 > 0.$$

Proposition 5.1 *Let $\eta,\xi \in \mathbb{R}$ with $\eta < \xi$ then,*

$$\left[2\mathtt{A}(\eta,\xi) - \mathtt{A}(\mathtt{w},\mathtt{y})\right]^n \leq \left[2\mathtt{A}\left(\eta^n,\xi^n\right) - \mathtt{L}^n_n(\mathtt{w},\mathtt{y})\right] \leq \left[2\mathtt{A}\left(\eta^n,\xi^n\right) - \mathtt{A}^n(\mathtt{w},\mathtt{y})\right]. \tag{5.1}$$

Proof Putting $\mathtt{F}(u\cdot,) = u^n$ in Corollary 3.2, then we obtain the inequality (5.1). □

Proposition 5.2 *$\eta,\xi \in \mathbb{R}$ with $\eta < \xi$ then,*

$$\left[2\mathtt{A}(\eta,\xi) - \mathtt{A}(\mathtt{w},\mathtt{y})\right]^n \leq \left[\mathtt{L}^n_n(\eta+\xi-\mathtt{w},\eta+\xi-\mathtt{y})\right] \leq \left[2\mathtt{L}\left(\eta^n,\xi^n\right) - \mathtt{A}\left(\mathtt{w}^n,\mathtt{y}^n\right)\right]. \tag{5.2}$$

Proof Putting $\mathtt{F}(u\cdot,) = u^n$ in Corollary 3.4, then we obtain the inequality (5.2). □

5.2 Applications to matrices

Example 5.1 Let $\mathtt{C}^n$ represent the set of $n \times n$ complex matrices, $\mathtt{M}_n$ represent the algebra of $n \times n$ complex matrices, and $\mathtt{M}^+n$ represent the strictly positive matrices in $\mathtt{M}$. That is, $\mathtt{P} \in \mathtt{M}^+_n$ for every nonzero $u \in \mathtt{C}^n$ if $< \mathtt{P}u, u \gg 0$.

Sababheh [32] proved that $\mathtt{F}(u) = \|\mathtt{P}^x\mathtt{Y}\mathtt{Q}^{1-x} + \mathtt{P}^{1-x}\mathtt{Y}\mathtt{Q}^x\|$, $\mathtt{P},\mathtt{Q} \in \mathbb{M}_n^+$, $\mathtt{Y} \in \mathtt{M}_n$ is convex for all $x \in [0,1]$. If we use the stochastic convex process as $\mathtt{F}(u\cdot,) = \|\mathtt{P}^x\mathtt{Y}\mathtt{Q}^{1-x} + \mathtt{P}^{1-x}\mathtt{Y}\mathtt{Q}^x\|$ then, by using Theorem 3.1, we have

$$\begin{aligned}
&\left\|\mathtt{P}^{\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2}}\mathtt{Y}\mathtt{Q}^{1-(\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2})} + \mathtt{P}^{1-(\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2})}\mathtt{Y}\mathtt{Q}^{\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2}}\right\| \\
&\leqq \left[\left\|\mathtt{P}^{\eta}\mathtt{Y}\mathtt{Q}^{1-\eta} + \mathtt{P}^{1-\eta}\mathtt{Y}\mathtt{Q}^{\eta}\right\| + \left\|\mathtt{P}^{\xi}\mathtt{Y}\mathtt{Q}^{1-\xi} + \mathtt{P}^{1-\xi}\mathtt{Y}\mathtt{Q}^{\xi}\right\|\right] \\
&\quad - \frac{1-\alpha}{2(1-e^{-\sigma})}\left[\mathtt{I}_{\mathtt{w}^+}^{\alpha}\left\|\mathtt{P}^{\mathtt{y}}\mathtt{Y}\mathtt{Q}^{1-\mathtt{y}} + \mathtt{P}^{1-\mathtt{y}}\mathtt{Y}\mathtt{Q}^{\mathtt{y}}\right\| + \mathtt{I}_{\mathtt{y}^-}^{\alpha}\left\|\mathtt{P}^{\mathtt{w}}\mathtt{Y}\mathtt{Q}^{1-\mathtt{w}} + \mathtt{P}^{1-\mathtt{w}}\mathtt{Y}\mathtt{Q}^{\mathtt{w}}\right\|\right] \\
&\leqq \left\|\mathtt{P}^{\eta}\mathtt{Y}\mathtt{Q}^{1-\eta} + \mathtt{P}^{1-\eta}\mathtt{Y}\mathtt{Q}^{\eta}\right\| + \left\|\mathtt{P}^{\xi}\mathtt{Y}\mathtt{Q}^{1-\xi} + \mathtt{P}^{1-\xi}\mathtt{Y}\mathtt{Q}^{\xi}\right\| \\
&\quad - \left\|\mathtt{P}^{\frac{\mathtt{w}+\mathtt{y}}{2}}\mathtt{Y}\mathtt{Q}^{1-\frac{\mathtt{w}+\mathtt{y}}{2}} + \mathtt{P}^{1-\frac{\mathtt{w}+\mathtt{y}}{2}}\mathtt{Y}\mathtt{Q}^{\frac{\mathtt{w}+\mathtt{y}}{2}}\right\|.
\end{aligned}$$

Example 5.2 With the same conditions as in the above example, if we consider Theorem 3.2, we have

$$\begin{aligned}
&\left\|\mathtt{P}^{\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2}}\mathtt{Y}\mathtt{Q}^{1-(\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2})} + \mathtt{P}^{1-(\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2})}\mathtt{Y}\mathtt{Q}^{\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2}}\right\| \\
&\leqq \frac{1-\alpha}{2(1-e^{-\sigma})}\left[\mathtt{I}_{\eta+\xi-\mathtt{w}^-}^{\alpha}\left\|\mathtt{P}^{\eta+\xi-\mathtt{y}}\mathtt{Y}\mathtt{Q}^{1-(\eta+\xi-\mathtt{y})} + \mathtt{P}^{1-(\eta+\xi-\mathtt{y})}\mathtt{Y}\mathtt{Q}^{\eta+\xi-\mathtt{y}}\right\|\right. \\
&\quad \left. + \mathtt{I}_{\eta+\xi-\mathtt{y}^+}^{\alpha}\left\|\mathtt{P}^{\eta+\xi-\mathtt{w}}\mathtt{Y}\mathtt{Q}^{1-(\eta+\xi-\mathtt{w})} + \mathtt{P}^{1-(\eta+\xi-\mathtt{w})}\mathtt{Y}\mathtt{Q}^{\eta+\xi-\mathtt{w}}\right\|\right] \\
&\leqq \left\|\mathtt{P}^{\eta}\mathtt{Y}\mathtt{Q}^{1-\eta} + \mathtt{P}^{1-\eta}\mathtt{Y}\mathtt{Q}^{\eta}\right\| + \left\|\mathtt{P}^{\xi}\mathtt{Y}\mathtt{Q}^{1-\xi} + \mathtt{P}^{1-\xi}\mathtt{Y}\mathtt{Q}^{\xi}\right\| \\
&\quad - \frac{\|\mathtt{P}^{\mathtt{w}}\mathtt{Y}\mathtt{Q}^{1-\mathtt{w}} + \mathtt{P}^{1-\mathtt{w}}\mathtt{Y}\mathtt{Q}^{\mathtt{w}}\| + \|\mathtt{P}^{\mathtt{y}}\mathtt{Y}\mathtt{Q}^{1-\mathtt{y}} + \mathtt{P}^{1-\mathtt{y}}\mathtt{Y}\mathtt{Q}^{\mathtt{y}}\|}{2}.
\end{aligned}$$

Example 5.3 With the same conditions as in the above example, if we consider Theorem 3.3, we have

$$\begin{aligned}
&\left\|\mathtt{P}^{\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2}}\mathtt{Y}\mathtt{Q}^{1-(\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2})} + \mathtt{P}^{1-(\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2})}\mathtt{Y}\mathtt{Q}^{\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2}}\right\| \\
&\leqq \frac{1-\alpha}{2(1-e^{-\frac{\sigma}{2}})}\left[\mathtt{I}_{\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2}^-}^{\alpha}\left\|\mathtt{P}^{\eta+\xi-\mathtt{y}}\mathtt{Y}\mathtt{Q}^{1-(\eta+\xi-\mathtt{y})} + \mathtt{P}^{1-(\eta+\xi-\mathtt{y})}\mathtt{Y}\mathtt{Q}^{\eta+\xi-\mathtt{y}}\right\|\right. \\
&\quad \left. + \mathtt{I}_{\eta+\xi-\frac{\mathtt{w}+\mathtt{y}}{2}^+}^{\alpha}\left\|\mathtt{P}^{\eta+\xi-\mathtt{w}}\mathtt{Y}\mathtt{Q}^{1-(\eta+\xi-\mathtt{w})} + \mathtt{P}^{1-(\eta+\xi-\mathtt{w})}\mathtt{Y}\mathtt{Q}^{\eta+\xi-\mathtt{w}}\right\|\right] \\
&\leqq \left\|\mathtt{P}^{\eta}\mathtt{Y}\mathtt{Q}^{1-\eta} + \mathtt{P}^{1-\eta}\mathtt{Y}\mathtt{Q}^{\eta}\right\| + \left\|\mathtt{P}^{\xi}\mathtt{Y}\mathtt{Q}^{1-\xi} + \mathtt{P}^{1-\xi}\mathtt{Y}\mathtt{Q}^{\xi}\right\| \\
&\quad - \frac{\|\mathtt{P}^{\mathtt{w}}\mathtt{Y}\mathtt{Q}^{1-\mathtt{w}} + \mathtt{P}^{1-\mathtt{w}}\mathtt{Y}\mathtt{Q}^{\mathtt{w}}\| + \|\mathtt{P}^{\mathtt{y}}\mathtt{Y}\mathtt{Q}^{1-\mathtt{y}} + \mathtt{P}^{1-\mathtt{y}}\mathtt{Y}\mathtt{Q}^{\mathtt{y}}\|}{2}.
\end{aligned}$$

6 Conclusion

In this paper, we demonstrate some improved versions of the $\mathtt{J}-\mathtt{M}$ inequality for convex stochastic processes. In addition, for the stochastic process $\mathtt{F}$, new classes of mean-square fractional integrals $\mathtt{J}_{\eta}^{\alpha}[\mathtt{F}](x)$ and $\mathtt{J}_{\xi^-}^{\alpha}[\mathtt{F}](x)$ are introduced. Then, for Jensen-convex stochastic processes pertaining to the introduced stochastic fractional integrals, we proposed the $\mathtt{J}-\mathtt{M}$ and the $\mathtt{H}-\mathtt{H}-\mathtt{M}$ inequalities. In the realm of integral inequalities, all of the conclusions and inequalities derived here are novel, fascinating, and significant. Moreover, we have derived a new identity and related improvements to the $\mathtt{H}-\mathtt{H}-\mathtt{M}$-type inequality for convex stochastic processes in fractional calculus. The presented applications of this manuscript show that inequalities of this type can be applied to means and matrices. Also, in the future, we will check the applicability of $\mathbf{q}$-Digamma functions for these inequalities. In upcoming publications, we will use different fractional operators to investigate Ostrowski–Mercer-, $\mathtt{J}-\mathtt{M}$- and $\mathtt{H}-\mathtt{H}-\mathtt{M}$-type inequalities for some new stochastic processes.

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Abbreviations
The following abbreviations have been used in this manuscript: $\mathtt{H}-\mathtt{H}-\mathtt{M}$, Hermite–Hadamard–Mercer; $\mathtt{H}-\mathtt{H}$, Hermite–Hadamard; $\mathtt{J}-\mathtt{M}$, Jensen–Mercer.

Availability of data and materials
Not data were used to this study.

Declarations

Competing interests
The authors declare no competing interests.

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