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Sharp inequalities related to the volume of the unit ball in $\mathbb{R}^n$

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Abstract

Let $\Omega_n = \pi^{n/2}/\Gamma(\frac{n}{2}+1)$ ($n \in \mathbb{N}$) denote the volume of the unit ball in $\mathbb{R}^n$. In this paper, the logarithmically complete monotonicity of a function involving the ratio of two gamma functions is presented, which yields a sharp double inequality for the quantity $\Omega_n^2/(\Omega_{n-1}\Omega_{n+1})$. Also, we establish new sharp inequalities for the quantity $\Omega_n^2/(\Omega_{n-1}\Omega_{n+1})$.

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1 Introduction

In the recent past, several researchers have established interesting properties of the volume Ω_n of the unit ball in $\mathbb{R}^n$,

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, \quad n \in \mathbb{N} := \{1, 2, \ldots\},$$

including monotonicity properties, inequalities, and asymptotic expansions.

Böhm and Hertel [1, p. 264] pointed out that the sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ is not monotonic. Indeed, we have

$$\Omega_n < \Omega_{n+1} \quad \text{if } 1 \le n \le 4 \quad \text{and} \quad \Omega_n > \Omega_{n+1} \quad \text{if } n \ge 5.$$

Anderson et al. [2] showed that $\{\Omega_n^{1/n}\}_{n\in\mathbb{N}}$ is monotonically decreasing to zero, while Anderson and Qiu [3] proved that the sequence $\{\Omega_n^{1/(n\ln n)}\}_{n\ge 2}$ decreases to $e^{-1/2}$. Guo and Qi [4] proved that the sequence $\{\Omega_n^{1/(n\ln n)}\}_{n\ge 2}$ is logarithmically convex. Klain and Rota [5] proved that the sequence $\{n\Omega_n/\Omega_{n-1}\}_{n\in\mathbb{N}}$ is increasing.

Diverse sharp inequalities for the volume of the unit ball in $\mathbb{R}^n$ have been established [6–18]. For example, Alzer [6] proved that for $n \in \mathbb{N}$,

$$a_1\Omega_{n+1}^{n/(n+1)} \le \Omega_n < b_1\Omega_{n+1}^{n/(n+1)},$$

$$\sqrt{\frac{n+a_2}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} \le \sqrt{\frac{n+b_2}{2\pi}},$$
$$\left(1+\frac{1}{n}\right)^{a_3} \le \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n}\right)^{b_3}, \tag{1.1}$$

with the best possible constants

$$a_1 = \frac{2}{\sqrt{\pi}} = 1.1283\ldots, \qquad b_1 = \sqrt{e} = 1.6487\ldots,$$
$$a_2 = \frac{1}{2}, \qquad b_2 = \frac{\pi}{2} - 1 = 0.5707\ldots,$$
$$a_3 = 2 - \frac{\ln\pi}{\ln 2} = 0.3485\ldots, \qquad b_3 = \frac{1}{2}.$$

Merkle [13] improved the left-hand side of (1.1) and obtained the following result:

$$\left(1+\frac{1}{n+1}\right)^{1/2} \le \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}, \quad n \in \mathbb{N}. \tag{1.2}$$

Chen and Lin [10, Theorem 3.1] developed (1.2) to produce the following symmetric double inequality:

$$\left(1+\frac{1}{n+1}\right)^{\alpha} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \le \left(1+\frac{1}{n+1}\right)^{\beta}, \quad n \in \mathbb{N},$$

with the best possible constants

$$\alpha = \frac{1}{2}, \quad \beta = \frac{2\ln 2 - \ln\pi}{\ln 3 - \ln 2} = 0.5957713\ldots.$$

Ban and Chen [8, Theorem 3.2] proved, for $n \in \mathbb{N}$,

$$\left(1+\frac{1}{n+\theta_1}\right)^{1/2} \le \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n+\theta_2}\right)^{1/2}, \tag{1.3}$$

with the best possible constants

$$\theta_1 = \frac{2\pi^2 - 16}{16 - \pi^2} = 0.60994576\ldots \quad \text{and} \quad \theta_2 = \frac{1}{2}.$$

Recently, Mortici [16] constructed asymptotic series associated with some expressions involving the volume of the n-dimensional unit ball. New refinements and improvements of some old and recent inequalities for Ω_n were also presented. For example, Mortici [16, Theorem 15] presented the following asymptotic expansion for the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$:

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \sim 1 + \frac{1}{2n} - \frac{3}{8n^2} + \frac{3}{16n^3} + \frac{3}{128n^4} - \frac{33}{256n^5} - \frac{39}{1024n^6} + \cdots, \tag{1.4}$$

as $n \to \infty$. Moreover, the author provided a recurrence relation for successively determining the coefficient of $1/n^j$ ($j \in \mathbb{N}$) in expansion (1.4).

Lu and Zhang [12] established a general continued fraction approximation for the nth root of the volume of the unit n-dimensional ball, and then obtained related inequalities. Chen and Paris [11] presented asymptotic expansions and inequalities related to Ω_n and the quantities:

$$\frac{\Omega_{n-1}}{\Omega_n}, \qquad \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}}, \quad \text{and} \quad \frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}}.$$

It is easy to see that

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = \left(\frac{n}{2}+\frac{1}{2}\right)\left(\frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+1)}\right)^2. \tag{1.5}$$

Replacement of $n/2$ by x in (1.5) yields

$$I(x) := \frac{\Omega_{2x}^2}{\Omega_{2x-1}\Omega_{2x+1}} = \left(x+\frac{1}{2}\right)\left(\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right)^2, \tag{1.6}$$

where $\Omega_x = \pi^{x/2}/\Gamma(\frac{x}{2}+1)$.

From (1.5) and (1.6), we see that the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$ is closely related to the ratio of two gamma functions $\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}$. The problem of finding new and sharp inequalities for the gamma function Γ and, in particular, for the Wallis ratio

$$\frac{(2n-1)!!}{(2n)!!} = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(n+1)}, \quad n \in \mathbb{N},$$

has attracted the attention of many researchers (see [19–30] and the references therein). Here, we employ the special double factorial notation as follows:

$$\begin{aligned} &(2n)!! = 2\cdot 4\cdot 6\cdots(2n) = 2^n n!,\\ &(2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-1) = \pi^{-1/2}2^n\Gamma\left(n+\frac{1}{2}\right),\\ &0!! = 1, \qquad (-1)!! = 1. \end{aligned}$$

Chen and Paris [30, Corollary 1(i)] obtained the following double inequality:

$$\begin{aligned} \sqrt{x}\exp\left(\sum_{j=1}^{2m}\left(1-\frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}}\right) &< \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\\ &< \sqrt{x}\exp\left(\sum_{j=1}^{2m+1}\left(1-\frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}}\right) \end{aligned} \tag{1.7}$$

for $x > 0$ and $m \in \mathbb{N}_0$, where B_n ($n \in \mathbb{N}_0$) are the Bernoulli numbers defined by the following generating function:

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi. \tag{1.8}$$

From (1.7), we derive

$$\begin{aligned}&\left(1+\frac{1}{2x}\right)\exp\left(-\sum_{j=1}^{2m}\left(1-\frac{1}{2^{2j}}\right)\frac{2B_{2j}}{j(2j-1)x^{2j-1}}\right)\\&\quad>\frac{\Omega_{2x}^2}{\Omega_{2x-1}\Omega_{2x+1}}=\left(x+\frac{1}{2}\right)\left(\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right)^2\\&\quad>\left(1+\frac{1}{2x}\right)\exp\left(-\sum_{j=1}^{2m+1}\left(1-\frac{1}{2^{2j}}\right)\frac{2B_{2j}}{j(2j-1)x^{2j-1}}\right)\end{aligned}\tag{1.9}$$

for $x>0$ and $m\in\mathbb{N}_0$. Replacing x by $n/2$ in (1.9) yields

$$\begin{aligned}&\left(1+\frac{1}{n}\right)\exp\left(-\sum_{j=1}^{2m}\frac{(2^{2j}-1)B_{2j}}{j(2j-1)n^{2j-1}}\right)\\&\quad>\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}>\left(1+\frac{1}{n}\right)\exp\left(-\sum_{j=1}^{2m+1}\frac{(2^{2j}-1)B_{2j}}{j(2j-1)n^{2j-1}}\right)\end{aligned}$$

for $n\in\mathbb{N}$ and $m\in\mathbb{N}_0$.

In this paper, we prove that the function $G(x)=(1+\frac{1}{2x+\frac{1}{2}})^{1/2}/I(x)$ is logarithmically completely monotonic on $(0,\infty)$ (Theorem 3.1), which yields a sharp double inequality for the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$ (see (3.5)). Also, we establish new sharp inequalities for the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$ (Theorems 4.1 and 4.2).

The numerical values given in this paper have been calculated via the computer program MAPLE 17.

2 Lemmas

Lemma 2.1 ([31]) *Let $-\infty\le a<b\le\infty$. Let f and g be differentiable functions on an interval (a,b). Assume that either $g'>0$ everywhere on (a,b) or $g'<0$ on (a,b). Suppose that $f(a+)=g(a+)=0$ or $f(b-)=g(b-)=0$. Then*

(1) *if $\frac{f'}{g'}$ is increasing on (a,b), then $(\frac{f}{g})'>0$ on (a,b);*
(2) *if $\frac{f'}{g'}$ is decreasing on (a,b), then $(\frac{f}{g})'<0$ on (a,b).*

The gamma function is defined for $x>0$ by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t.$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x)=\Gamma'(x)/\Gamma(x)$, is called psi (or digamma) function, and $\psi^{(k)}(x)$ ($k\in\mathbb{N}$) are called polygamma functions.

Lemma 2.2 ([30]) *Let $m,n\in\mathbb{N}$. Then for $x>0$,*

$$\begin{aligned}&\sum_{j=1}^{2m}\left(1-\frac{1}{2^{2j}}\right)\frac{2B_{2j}}{(2j)!}\frac{(2j+n-2)!}{x^{2j+n-1}}\\&\quad<(-1)^n\left(\psi^{(n-1)}(x+1)-\psi^{(n-1)}\left(x+\frac{1}{2}\right)\right)+\frac{(n-1)!}{2x^n}\end{aligned}$$

$$< \sum_{j=1}^{2m-1} \left(1 - \frac{1}{2^{2j}}\right) \frac{2B_{2j}}{(2j)!} \frac{(2j+n-2)!}{x^{2j+n-1}}, \tag{2.1}$$

where B_n ($n \in \mathbb{N}_0$) are the Bernoulli numbers defined by (1.8).

In particular, we obtain from (2.1) that

$$\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} < \psi(x+1) - \psi\left(x + \frac{1}{2}\right) < \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4}, \quad x > 0, \tag{2.2}$$

$$\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2048x^8} - \frac{31}{2048x^{10}} < \psi(x+1) - \psi\left(x + \frac{1}{2}\right) < \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2048x^8}, \quad x > 0, \tag{2.3}$$

and

$$-\frac{1}{2x^2} + \frac{1}{4x^3} - \frac{1}{16x^5} < \psi'(x+1) - \psi'\left(x + \frac{1}{2}\right), \quad x > 0. \tag{2.4}$$

3 Logarithmically complete monotonicity of the function $(1 + \frac{1}{2x+\frac{1}{2}})^{1/2}/I(x)$

A function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for } x \in I \text{ and } n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{3.1}$$

Dubourdieu [32, p. 98] pointed out that, if a nonconstant function f is completely monotonic on $I = (a, \infty)$, then strict inequality holds true in (3.1). See also [33] for a simpler proof of this result. It is known (Bernstein's theorem) that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\, \mathrm{d}\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. See [34, p. 161].

Recall [35] that a positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k \left[\ln f(x)\right]^{(k)} \geq 0 \quad \text{for } x \in I \text{ and } k \in \mathbb{N}.$$

A logarithmically completely monotonic function f on I must be completely monotonic on I (see, e.g., [36–38]).

Theorem 3.1 *The function*

$$G(x) = \frac{(1 + \frac{1}{2x+\frac{1}{2}})^{1/2}}{I(x)} = \frac{(1 + \frac{1}{2x+\frac{1}{2}})^{1/2}}{(x + \frac{1}{2})} \left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2 \tag{3.2}$$

is logarithmically completely monotonic on $(0, \infty)$.

Proof The logarithm of the gamma function has the following integral representation (see [39, p. 258]):

$$\ln\Gamma(z) = \int_0^\infty \left[(z-1)e^{-t} + \frac{e^{-zt}-e^{-t}}{1-e^{-t}}\right]\frac{\mathrm{d}t}{t}. \tag{3.3}$$

Using (3.3) and

$$\ln x = \int_0^\infty \frac{e^{-t}-e^{-xt}}{t}\,\mathrm{d}t,$$

we obtain

$$\begin{aligned}\ln G(x) &= \frac{1}{2}\ln\frac{x+\frac{3}{4}}{x+\frac{1}{4}} - \ln\left(x+\frac{1}{2}\right) + 2\left[\ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right)\right] \\ &= \int_0^\infty \left(\frac{1}{2}e^{-(x+\frac{1}{4})t} - \frac{1}{2}e^{-(x+\frac{3}{4})t} + e^{-(x+\frac{1}{2})t} + \frac{2[e^{-(x+1)t} - e^{-(x+\frac{1}{2})t}]}{1-e^{-t}}\right)\frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \left(\frac{1}{2e^{t/4}} - \frac{1}{2e^{3t/4}} + \frac{1}{e^{t/2}} - \frac{2}{e^{t/2}+1}\right)\frac{e^{-xt}}{t}\,\mathrm{d}t \\ &= \int_0^\infty q(t)e^{-xt}\,\mathrm{d}t, \end{aligned} \tag{3.4}$$

where

$$q(t) = \frac{(e^{t/4}+1)(e^{t/4}-1)^3}{2te^{3t/4}(e^{t/2}+1)} > 0, \quad t > 0.$$

We conclude from (3.4) that

$$(-1)^n\big(\ln G(x)\big)^{(n)} = \int_0^\infty t^n q(t)e^{-xt}\,\mathrm{d}t > 0 \quad \text{for } x > 0 \text{ and } n \in \mathbb{N}.$$

The proof of Theorem 3.1 is complete. □

Remark 3.1 The function $G(x)$, defined by (3.2), is completely monotonic on $(0,\infty)$. In particular, the sequence $\{G(n/2)\}$ is strictly decreasing for $n \in \mathbb{N}$, and we have

$$1 = G(\infty) < G\left(\frac{n}{2}\right) = \frac{(1+\frac{1}{n+\frac{1}{2}})^{1/2}}{I(\frac{n}{2})} \le G\left(\frac{1}{2}\right) = \frac{\sqrt{15}\pi}{12}, \quad n \in \mathbb{N},$$

which yields the following double inequality for the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$:

$$p\left(1+\frac{1}{n+\frac{1}{2}}\right)^{1/2} \le \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < q\left(1+\frac{1}{n+\frac{1}{2}}\right)^{1/2}, \quad n \in \mathbb{N}, \tag{3.5}$$

with the best possible constants

$$p = \frac{12}{\sqrt{15}\pi} = 0.986247\ldots \quad \text{and} \quad q = 1.$$

4 Sharp inequalities for $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$

Theorem 4.1 *For $n \in \mathbb{N}$, the following double inequality holds*:

$$\left(1+\frac{1}{n+\frac{1}{2}}\right)^{\lambda} \leq \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n+\frac{1}{2}}\right)^{\mu}, \tag{4.1}$$

where the constants

$$\lambda = \frac{2\ln 2 - \ln \pi}{\ln 5 - \ln 3} = 0.47289\ldots \quad \textit{and} \quad \mu = \frac{1}{2}$$

are the best possible.

Proof Inequality (4.1) can be written as

$$\lambda \leq x_n < \mu,$$

where the sequence $\{x_n\}_{n\in\mathbb{N}}$ is defined by

$$x_n = \frac{\ln((\frac{n}{2}+\frac{1}{2})(\frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+1)})^2)}{\ln(1+\frac{1}{n+\frac{1}{2}})}.$$

We are now in a position to show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is strictly increasing. To this end, we consider the function $f(x)$ defined by

$$f(x) = \frac{2\ln\Gamma(x+\frac{1}{2}) - 2\ln\Gamma(x+1) + \ln(x+\frac{1}{2})}{\ln(1+\frac{1}{2x+\frac{1}{2}})} = \frac{f_1(x)}{f_2(x)},$$

where

$$f_1(x) = 2\ln\Gamma\left(x+\frac{1}{2}\right) - 2\ln\Gamma(x+1) + \ln\left(x+\frac{1}{2}\right)$$

and

$$f_2(x) = \ln\left(1+\frac{1}{2x+\frac{1}{2}}\right).$$

We conclude from the asymptotic formula of $\ln\Gamma(z)$ (see [39, p. 257, Eq. (6.1.41)]) that

$$f_1(\infty) = \lim_{x\to\infty} f_1(x) = 0.$$

Elementary calculations show that

$$\frac{4f_1'(x)}{f_2'(x)} = (4x+3)(4x+1)\left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right) - \frac{1}{2x+1}\right] =: f_3(x).$$

By using inequalities (2.2) and (2.4), we obtain, for $x \ge 2$,

$$\begin{aligned}
f_3'(x) &= (32x+16)\left[\psi(x+1)-\psi\left(x+\frac{1}{2}\right)-\frac{1}{2x+1}\right] \\
&\quad + (4x+3)(4x+1)\left[\psi'(x+1)-\psi'\left(x+\frac{1}{2}\right)+\frac{2}{(2x+1)^2}\right] \\
&> (32x+16)\left[\frac{1}{2x}-\frac{1}{8x^2}+\frac{1}{64x^4}-\frac{1}{128x^6}-\frac{1}{2x+1}\right] \\
&\quad + (4x+3)(4x+1)\left[-\frac{1}{2x^2}+\frac{1}{4x^3}-\frac{1}{16x^5}+\frac{2}{(2x+1)^2}\right] \\
&= \frac{352+2001(x-2)+2784(x-2)^2+1656(x-2)^3+456(x-2)^4+48(x-2)^5}{16x^6(2x+1)^2} \\
&> 0.
\end{aligned}$$

Hence, $f_3(x)$ and $\frac{f_1'(x)}{f_2'(x)}$ are both strictly increasing for $x \ge 2$. By Lemma 2.1, the function

$$f(x) = \frac{f_1(x)}{f_2(x)} = \frac{f_1(x)-f_1(\infty)}{f_2(x)-f_2(\infty)}$$

is strictly increasing for $x \ge 2$. Therefore, the sequence $\{x_n\}$ is strictly increasing for $n \ge 4$. Direct computation yields

$$\begin{aligned}
x_1 &= \frac{2\ln 2-\ln\pi}{\ln 5-\ln 3} = 0.47289\ldots, \qquad x_2 = \frac{\ln 3-3\ln 2+\ln\pi}{\ln 7-\ln 5} = 0.48711\ldots, \\
x_3 &= \frac{5\ln 2-2\ln 3-\ln\pi}{2\ln 3-\ln 7} = 0.49253\ldots, \\
x_4 &= \frac{2\ln 3+\ln 5-7\ln 2+\ln\pi}{\ln 11-2\ln 3} = 0.49515\ldots.
\end{aligned}$$

Consequently, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is strictly increasing. This leads to

$$\frac{2\ln 2-\ln\pi}{\ln 5-\ln 3} = x_1 \le x_n < \lim_{n\to\infty} x_n \quad \text{for } n \in \mathbb{N}.$$

It remains to prove that

$$\lim_{n\to\infty} x_n = \frac{1}{2}. \tag{4.2}$$

We conclude from the asymptotic formula of $\ln\Gamma(z)$ (see [39, p. 257, Eq. (6.1.41)]) that

$$x_n = \frac{\frac{1}{2n}-\frac{1}{2n^2}+O(n^{-3})}{\frac{1}{n}-\frac{1}{n^2}+O(n^{-3})} = \frac{\frac{1}{2}+O(n^{-1})}{1+O(n^{-1})} \to \frac{1}{2} \quad \text{as } n \to \infty.$$

Hence, (4.2) holds. This completes the proof of Theorem 4.1. □

Theorem 4.2 *For $n \in \mathbb{N}$, the following double inequality holds*:

$$\left(1+\frac{1}{n+\frac{1}{2}}\right)^{1/2}\left(1-\frac{2}{16n^3+48n^2+60n+a}\right) \le \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n+\frac{1}{2}}\right)^{1/2}\left(1-\frac{2}{16n^3+48n^2+60n+b}\right), \tag{4.3}$$

where the constants

$$a = \frac{2(248\sqrt{15}-305\pi)}{5\pi-4\sqrt{15}} = 21.42398\ldots \quad \textit{and} \quad b = 29$$

are the best possible.

Proof First of all, we show that the double inequality (4.3) with $a = \frac{2(248\sqrt{15}-305\pi)}{5\pi-4\sqrt{15}}$ and $b = 29$ is valid for $n = 1, 2, 3, 4$, and 5. For $n \in \mathbb{N}$, let

$$L_n = \left(1+\frac{1}{n+\frac{1}{2}}\right)^{1/2}\left(1-\frac{2}{16n^3+48n^2+60n+\frac{2(248\sqrt{15}-305\pi)}{5\pi-4\sqrt{15}}}\right),$$
$$U_n = \left(1+\frac{1}{n+\frac{1}{2}}\right)^{1/2}\left(1-\frac{2}{16n^3+48n^2+60n+29}\right).$$

Direct computation yields

$$L_1 = \frac{4}{\pi}, \quad \left[\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right]_{n=1} = \frac{4}{\pi} = 1.2732\ldots, \quad U_1 = 1.2755\ldots,$$
$$L_2 = 1.178064357\ldots, \quad \left[\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right]_{n=2} = 1.17809724510\ldots,$$
$$U_2 = 1.178246681\ldots,$$
$$L_3 = 1.131758795\ldots, \quad \left[\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right]_{n=3} = 1.13176848421\ldots,$$
$$U_3 = 1.131789661\ldots,$$
$$L_4 = 1.104462901\ldots, \quad \left[\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right]_{n=4} = 1.10446616728\ldots,$$
$$U_4 = 1.104470767\ldots,$$
$$L_5 = 1.086496467\ldots, \quad \left[\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right]_{n=5} = 1.08649774484\ldots,$$
$$U_5 = 1.086499056\ldots.$$

Clearly, the double inequality (4.3) with $a = \frac{2(248\sqrt{15}-305\pi)}{5\pi-4\sqrt{15}}$ and $b = 29$ is valid for $n = 1, 2, 3, 4$, and 5. For $n = 1$, the equality on the left-hand side of (4.3) holds.

We now prove that the double inequality (4.3) with $a = \frac{2(248\sqrt{15}-305\pi)}{5\pi-4\sqrt{15}}$ and $b = 29$ is valid for $n \geq 6$. It suffices to show that for $x \geq 3$,

$$\begin{aligned}&\left(1+\frac{1}{2x+\frac{1}{2}}\right)^{1/2}\left(1-\frac{2}{16(2x)^3+48(2x)^2+60(2x)+a}\right)\\&\quad\leq\frac{\Omega_{2x}^2}{\Omega_{2x-1}\Omega_{2x+1}}<\left(1+\frac{1}{2x+\frac{1}{2}}\right)^{1/2}\left(1-\frac{2}{16(2x)^3+48(2x)^2+60(2x)+29}\right),\end{aligned}$$

which can be written as

$$\begin{aligned}&\left(1+\frac{1}{2x+\frac{1}{2}}\right)^{1/2}\left(1-\frac{2}{16(2x)^3+48(2x)^2+60(2x)+a}\right)\\&\quad\leq\left(x+\frac{1}{2}\right)\left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right]^2\\&\quad<\left(1+\frac{1}{2x+\frac{1}{2}}\right)^{1/2}\left(1-\frac{2}{16(2x)^3+48(2x)^2+60(2x)+29}\right).\end{aligned}\tag{4.4}$$

In order to prove the double inequality (4.4) for $x \geq 3$, it suffices to show that

$$f(x) > 0 \quad\text{and}\quad g(x) < 0 \quad\text{for } x \geq 3,$$

where

$$\begin{aligned}f(x)&=2\left[\ln\Gamma\left(x+\frac{1}{2}\right)-\ln\Gamma(x+1)\right]+\ln\left(x+\frac{1}{2}\right)-\frac{1}{2}\ln\left(1+\frac{1}{2x+\frac{1}{2}}\right)\\&\quad-\ln\left(1-\frac{2}{16(2x)^3+48(2x)^2+60(2x)+a}\right),\\g(x)&=2\left[\ln\Gamma\left(x+\frac{1}{2}\right)-\ln\Gamma(x+1)\right]+\ln\left(x+\frac{1}{2}\right)-\frac{1}{2}\ln\left(1+\frac{1}{2x+\frac{1}{2}}\right)\\&\quad-\ln\left(1-\frac{2}{16(2x)^3+48(2x)^2+60(2x)+29}\right).\end{aligned}$$

We conclude from the asymptotic formula of $\ln\Gamma(z)$ (see [39, p. 257, Eq. (6.1.41)]) that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0.$$

Differentiating $f(x)$ and applying the left-hand side of (2.3), and noting that

$$a = \frac{2(248\sqrt{15}-305\pi)}{5\pi-4\sqrt{15}} < \frac{43}{2},$$

we obtain for $x \geq 3$,

$$\begin{aligned}f'(x)&=-2\left[\psi(x+1)-\psi\left(x+\frac{1}{2}\right)\right]+\frac{2(16x^2+20x+5)}{(4x+3)(4x+1)(2x+1)}\\&\quad-\frac{48(16x^2+16x+5)}{(128x^3+192x^2+120x+a-2)(128x^3+192x^2+120x+a)}\end{aligned}$$

$$
\begin{aligned}
&< -2\left(\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2048x^8} - \frac{31}{2048x^{10}}\right) \\
&\quad + \frac{2(16x^2 + 20x + 5)}{(4x+3)(4x+1)(2x+1)} \\
&\quad - \frac{48(16x^2 + 16x + 5)}{(128x^3 + 192x^2 + 120x + \frac{43}{2} - 2)(128x^3 + 192x^2 + 120x + \frac{43}{2})} \\
&= -\frac{P_{12}(x-3)}{1024x^{10}(4x+3)(4x+1)(2x+1)(256x^3 + 384x^2 + 240x + 39)(256x^3 + 384x^2 + 240x + 43)},
\end{aligned}
$$

where

$$
\begin{aligned}
P_{12}(x) &= 2{,}312{,}798{,}031{,}594 + 12{,}277{,}183{,}388{,}658x + 26{,}310{,}509{,}734{,}485x^2 \\
&\quad + 32{,}318{,}240{,}921{,}214x^3 + 26{,}087{,}077{,}081{,}952x^4 + 14{,}780{,}270{,}044{,}224x^5 \\
&\quad + 6{,}067{,}872{,}771{,}744x^6 + 1{,}824{,}299{,}158{,}976x^7 + 399{,}070{,}033{,}152x^8 \\
&\quad + 61{,}948{,}727{,}808x^9 + 6{,}475{,}038{,}720x^{10} + 408{,}944{,}640x^{11} + 11{,}796{,}480x^{12}.
\end{aligned}
$$

Hence, $f'(x) < 0$ for $x \geq 3$. So, $f(x)$ is strictly decreasing for $x \geq 3$, and we have

$$
f(x) > \lim_{t\to\infty} f(t) = 0, \quad x \geq 3.
$$

Therefore, the left-hand side of (4.3) with $a = \frac{2(248\sqrt{15}-305\pi)}{5\pi-4\sqrt{15}}$ is valid for $n \in \mathbb{N}$.

Differentiating $g(x)$ and applying the right-hand side of (2.3), we obtain for $x \geq 3$,

$$
\begin{aligned}
g'(x) &= -2\left[\psi(x+1) - \psi\left(x + \frac{1}{2}\right)\right] + \frac{2(16x^2 + 20x + 5)}{(4x+3)(4x+1)(2x+1)} \\
&\quad - \frac{48(16x^2 + 16x + 5)}{(128x^3 + 192x^2 + 120x + 27)(128x^3 + 192x^2 + 120x + 29)} \\
&> -2\left(\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2048x^8}\right) + \frac{2(16x^2 + 20x + 5)}{(4x+3)(4x+1)(2x+1)} \\
&\quad - \frac{48(16x^2 + 16x + 5)}{(128x^3 + 192x^2 + 120x + 27)(128x^3 + 192x^2 + 120x + 29)} \\
&= -\frac{P_9(x-3)}{1024x^8(4x+3)(4x+1)(2x+1)(128x^3 + 192x^2 + 120x + 27)(128x^3 + 192x^2 + 120x + 29)},
\end{aligned}
$$

where

$$
\begin{aligned}
P_9(x) &= 23{,}529{,}054{,}501 + 184{,}258{,}816{,}470x + 357{,}871{,}998{,}912x^2 \\
&\quad + 340{,}974{,}002{,}496x^3 + 191{,}948{,}408{,}224x^4 + 68{,}526{,}376{,}128x^5 \\
&\quad + 15{,}780{,}445{,}440x^6 + 2{,}282{,}252{,}800x^7 + 189{,}235{,}200x^8 + 6{,}881{,}280x^9.
\end{aligned}
$$

Hence, $g'(x) < 0$ for $x \geq 3$. So, $g(x)$ is strictly increasing for $x \geq 3$, and we have

$$
g(x) < \lim_{t\to\infty} f(t) = 0, \quad x \geq 3.
$$

Therefore, the right-hand side of (4.3) with $b = 29$ is valid for $n \in \mathbb{N}$.

If we write (4.3) as

$$a \leq x_n < b, \quad x_n = \frac{2}{1 - \frac{\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}}{(1+\frac{1}{n+\frac{1}{2}})^{1/2}}} - \left(16n^3 + 48n^2 + 60n\right),$$

we find that

$$x_1 = \frac{2(248\sqrt{15} - 305\pi)}{5\pi - 4\sqrt{15}}$$

and

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \left\{ \frac{2}{1 - \frac{\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}}{(1+\frac{1}{n+\frac{1}{2}})^{1/2}}} - \left(16n^3 + 48n^2 + 60n\right) \right\} \\ &= \lim_{n\to\infty} \left\{ \frac{2}{\frac{1}{8n^3} - \frac{3}{8n^4} + \frac{21}{32n^5} - \frac{101}{128n^6} + O(\frac{1}{n^7})} - \left(16n^3 + 48n^2 + 60n\right) \right\} \\ &= \lim_{n\to\infty} \left\{ 29 + O\left(\frac{1}{n}\right) \right\} = 29.\end{aligned}$$

This limit is obtained by using the asymptotic expansion (1.4).

Hence, the double inequality (4.3) holds for $n \in \mathbb{N}$, and the constants $a = \frac{2(248\sqrt{15}-305\pi)}{5\pi-4\sqrt{15}}$ and $b = 29$ are the best possible. The proof of Theorem 4.2 is complete. □

5 Comparison

It follows form (1.1), (1.2) and (1.3) and (4.3) that

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1 + \frac{1}{n}\right)^{1/2} = u_n \quad \text{(Alzer [6])}, \tag{5.1}$$

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1 + \frac{1}{n+1}\right)^{1/2} = v_n \quad \text{(Merkle [13])}, \tag{5.2}$$

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1 + \frac{1}{n+\frac{1}{2}}\right)^{1/2} = w_n \quad \text{(Ban and Chen [8])}, \tag{5.3}$$

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1 + \frac{1}{n+\frac{1}{2}}\right)^{1/2} \left(1 - \frac{2}{16n^3 + 48n^2 + 60n + 29}\right) = r_n \quad \text{(New)}. \tag{5.4}$$

We here offer some numerical computations (see Table 1) to show the superiority of our sequence $\{r_n\}_{n\geq 1}$ over the sequences $\{u_n\}_{n\geq 1}$, $\{v_n\}_{n\geq 1}$, and $\{w_n\}_{n\geq 1}$.

Table 1 Comparison of approximation formulas (5.1)–(5.4)

n	$\frac{u_n - V_n}{V_n}$	$\frac{V_n - v_n}{V_n}$	$\frac{w_n - V_n}{V_n}$	$\frac{r_n - V_n}{V_n}$
10	2.2651×10^{-3}	1.885×10^{-3}	9.3351×10^{-5}	1.1467×10^{-8}
100	2.4751×10^{-5}	1.885×10^{-5}	1.2131×10^{-7}	2.1845×10^{-15}
1000	2.4975×10^{-7}	2.4925×10^{-7}	1.2462×10^{-10}	2.3273×10^{-22}
10,000	2.4997×10^{-9}	2.4992×10^{-9}	1.2496×10^{-13}	2.3421×10^{-29}

Here $V_n := \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$. In fact, we have, as $n \to \infty$,

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = u_n + O\left(\frac{1}{n^2}\right), \qquad \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = v_n + O\left(\frac{1}{n^2}\right),$$
$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = w_n + O\left(\frac{1}{n^3}\right), \qquad \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = r_n + O\left(\frac{1}{n^7}\right).$$

These formulas are obtained by using the computer program MAPLE 17.

6 Conclusion

Here, in our present investigation, we have first revisited several interesting properties of the volume Ω_n of the unit ball in $\mathbb{R}^n$, including monotonicity properties, inequalities, and asymptotic expansions. We have then shown that the function $G(x) = (1 + \frac{1}{2x+\frac{1}{2}})^{1/2}/I(x)$ is logarithmically completely monotonic on $(0,\infty)$ (Theorem 3.1), which yielded a double inequality for the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$, see (3.5). Also, we have established new sharp inequalities for the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$, see (4.1) and (4.3). We have also considered a number of related developments on the subject of this paper.

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