Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Local limit theorems without assuming finite third moment

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Abstract

One of the most fundamental probabilities is the probability at a particular point. The local limit theorem is the well-known theorem that estimates this probability. In this paper, we estimate this probability by the density function of normal distribution in the case of lattice integer-valued random variables. Our technique is the characteristic function method. We complete to relax the third moment condition of Siripraparat and Neammanee (J. Inequal. Appl. 2021:57, 2021) and the references therein and also obtain explicit constants of the error bound.

MSC: 60F05

Keywords: Local limit theorem; Normal density function; Lattice random variable; Rate of convergence; Characteristic function

1 Introduction and main results

Let X be an integer-valued random variable. One of the most fundamental probabilities is the probability at a particular point, i.e., $P(X = k)$ for some $k \in \mathbb{Z}$. The local limit theorem is one of the theorems that estimate this probability and describe how $P(X = k)$ approaches the normal density, $\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(k-\mu)^2}{2\sigma^2}}$ where μ and σ^2 are mean and variance of X, respectively. There are two well-known techniques for deriving this theorem: the method of characteristic function and the Bernoulli part extraction method. The characteristic function method is to estimate the bound for the characteristic function of a random variable. This method has been used in a number of studies such as in the case of bounded random variables (see [3–6], and [7] for examples) and in the case of lattice random variables (see [7–9], and [10] for examples).

Let $X_1, X_2, \dots, X_n$ be independent integer-valued random variables with mean μ_j and variance σ_j^2 for all $j = 1, 2, \dots, n$. Then let

$$S_n = \sum_{j=1}^{n} X_j, \qquad \mu = \sum_{j=1}^{n} \mu_j, \qquad \sigma^2 = \sum_{j=1}^{n} \sigma_j^2.$$

If $P(X_j = 1) = p_j = 1 - P(X_j = 0)$, then X_j is called a Bernoulli random variable with parameter p_j and S_n is said to be a Poisson binomial random variable. In addition, when we provide $p_1 = p_2 = \cdots = p_n = p$, we call S_n a binomial random variable with parameters n and p and use the notation $S_n \sim B(n,p)$. The first local limit theorem was proved by De Moivre and Laplace ([11], 1754) for a binomial random variable. We call X a lattice random variable with parameter (a,d) if its values belong to $\mathcal{L}(a,d) = \{a + md : m \in \mathbb{Z}\}$, where a and $d > 0$ are integers. In addition, d is said to be maximal if there are no other numbers a' and $d' > d$ for which $P(X \in \mathcal{L}(a',d')) = 1$, and we call X a maximal lattice random variable with parameter (a,d). Observe that the Bernoulli random variable is a maximal lattice random variable with parameter $(0,1)$. In the case that X_js are common lattice $\mathcal{L}(a,d)$ and identically distributed, Ibragimov and Linnik [12] gave the rate of convergence $O(\frac{1}{n^{\frac{1}{2}+\alpha}})$, where $0 < \alpha < \frac{1}{2}$ in 1971. For further information, they showed that if d is maximal and

$$\int_{|x|\geq u} x^2 F(dx) = O\left(\frac{1}{u^{2\alpha}}\right) \quad \text{as } u \to \infty,$$

where F the distribution function of X_1, then

$$\sup_{k\in\mathbb{Z}}\left|\sqrt{n}P(S_n = na + kd) - \frac{d}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\right| = O\left(\frac{1}{n^{\alpha}}\right), \quad 0 < \alpha < \frac{1}{2}. \tag{1.1}$$

A few years later, Petrov [13] proved that if $E|X_1|^3 < \infty$, then (1.1) holds with $\alpha = \frac{1}{2}$. Moreover, for the case that X_js are nonidentically distributed lattice random variables with parameter $(0,1)$ that satisfy the third moment condition and some properties, Petrov [13] gave the following result:

$$\sup_{k\in\mathbb{Z}}\left|P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}}\right| \leq \frac{C}{\sigma^2}.$$

The previous studies had not given explicit constants of the error bound until Korolev and Zhukov ([14], 2000). In 2017, Giuliano and Weber [15] used the Bernoulli part extraction method to give an error bound with explicit constants in the case of nonidentically distributed square integrable random variables taking values in a common lattice $\mathcal{L}(a,d)$. By assuming finite third moment, Sripraparat and Neammanee [1] used the characteristic function technique to illustrate the rate of convergence to $O(\frac{1}{\sigma^2})$ in 2021. For a special case, one can see [2] and [16] in the case of Poisson binomial and binomial, respectively.

In this paper, we relax the third moment condition to find the local limit theorems for sums of independent lattice integer-valued random variables and also give explicit constants of the error bound. Our technique is the characteristic function method inspired by Petrov [13] and Siripraparat and Neammanee [1]. Throughout this paper, let $X_1, X_2, \ldots, X_n$ be independent common lattice random variables with parameter (a,d) such that $E|X_j|^{2+\alpha} < \infty$, where $0 < \alpha < 1$, for $j = 1, 2, \ldots, n$, and let $S_n = \sum_{j=1}^{n} X_j$ with mean μ and variance σ^2. The following are our main results.

Theorem 1.1 *Let* $\beta = \sum_{j=1}^{n} \beta_j$, *where* $\beta_j = 2\sum_{m=-\infty}^{\infty} p_{jm}p_{j(m+1)}$ *and* $p_{jm} = P(X_j = a + md)$. *If* $\beta > 0$ *and* $\sigma^2 > d^2$, *then*

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\left|P(S_n = na + kd) - \frac{d}{\sigma\sqrt{2\pi}}e^{-\frac{(na+kd-\mu)^2}{2\sigma^2}}\right| \\
&\quad\le \frac{0.0020d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(\sum_{j=1}^{n} E|X_j - a|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171\cdot 3^{\frac{1}{\alpha}} d^{\frac{\alpha^2-\alpha+2}{2}} (\sum_{j=1}^{n} E|X_j - a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma^4} \\
&\qquad + \frac{0.3184}{\sigma^2\tau}e^{-\frac{\sigma^2\tau^2}{2}} + \frac{1.5708}{\tau\beta}e^{-\frac{\tau^2\beta}{\pi^2}},
\end{aligned}$$

where $\tau = \frac{d}{3^{\frac{1}{\alpha}}(\sum_{j=1}^{n} E|X_j-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}$.

Furthermore, if $X_1, X_2, \ldots, X_n$ *are identically distributed and* $\beta_1 > 0$, *then*

$$\sup_{k\in\mathbb{Z}}\left|P(S_n = na + kd) - \frac{d}{\sigma_1\sqrt{2n\pi}}e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\right| \le \frac{C_1}{n^{\frac{1+\alpha}{2}}},$$

where

$$\begin{aligned}
C_1 &= \frac{0.0020d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1 - a|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171\cdot 3^{\frac{1}{\alpha}} d^{\frac{\alpha^2-\alpha+2}{2}} (E|X_1 - a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4} \\
&\quad + \frac{0.6368\cdot 3^{\frac{3}{\alpha}} d(E|X_1 - a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{\sigma_1^4} + \frac{15.5032\cdot 3^{\frac{3}{\alpha}} (E|X_1 - a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{d^3\beta_1^2}.
\end{aligned}$$

We note that $\beta_j > 0$ if X_j is a maximal lattice random variable. So, we can apply this result when d is maximal.

Theorem 1.2 *Let* $\upsilon := \min_{1\le j\le n} \upsilon_j$, *where* $\upsilon_j = 2\sum_{m=-\infty}^{\infty} p_{jm}p_{j(m+j)}$. *If* $\upsilon_j > 0$ *for all* $j = 1, 2, \ldots, n$ *and* $\sigma^2 > d^2$, *then*

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\left|P(S_n = na + kd) - \frac{d}{\sigma\sqrt{2\pi}}e^{-\frac{(na+kd-\mu)^2}{2\sigma^2}}\right| \\
&\quad\le \frac{0.0020d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(\sum_{j=1}^{n} E|X_j - a|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171\cdot 3^{\frac{1}{\alpha}} d^{\frac{\alpha^2-\alpha+2}{2}} (\sum_{j=1}^{n} E|X_j - a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma^4} \\
&\qquad + \frac{0.3184}{\sigma^2\tau}e^{-\frac{\sigma^2\tau^2}{2}} + \exp\left(-\frac{n\upsilon}{4}\min\left(1, \left(\frac{n\tau}{2\pi}\right)^2\right)\right),
\end{aligned}$$

where $\tau = \frac{d}{3^{\frac{1}{\alpha}}(\sum_{j=1}^{n} E|X_j-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}$.

Furthermore, if $X_1, X_2, \ldots, X_n$ *are identically distributed and* $\upsilon_j > 0$ *for all* $j = 1, 2, \ldots, n$, *then for* $n \ge (\frac{2\pi\cdot 3^{\frac{1}{\alpha}}(E|X_1-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{d})^{\frac{2+\alpha}{1+\alpha}}$,

$$\sup_{k\in\mathbb{Z}}\left|P(S_n = na + kd) - \frac{d}{\sigma_1\sqrt{2n\pi}}e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\right| \le \frac{C_2}{n^{\frac{1+\alpha}{2}}} + e^{-\frac{n\upsilon}{4}},$$

where

$$C_2 = \frac{0.0020d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1-a|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171\cdot 3^{\frac{1}{\alpha}} d^{\frac{\alpha^2-\alpha+2}{2}}(E|X_1-a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4} + \frac{0.6368\cdot 3^{\frac{3}{\alpha}} d(E|X_1-a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{\sigma_1^4}.$$

We organize this paper as follows: First, we give auxiliary results in Sect. 2 that will be used to prove the main theorems in Sect. 3. Finally, we give some examples in Sect. 4.

2 Auxiliary results

In the following lemmas, we use an idea from [17] to give bounds of a characteristic function to prove Theorem 1.1.

Lemma 2.1 *Let X be any integer-valued random variable with mean μ_X, variance σ_X^2, and characteristic function ψ_X. If $E|X|^{2+\alpha}<\infty$ for some $0<\alpha<1$, then there exists a function g_X such that, for all $|t|\le(\frac{1}{3E|X|^\alpha})^{\frac{1}{\alpha}}$,*

(*i*) $|\psi_X(t)|\ge\frac{1}{3}$ *and*

(*ii*) $\psi_X(t)=\exp\{i\mu_X t-\frac{1}{2}\sigma_X^2t^2+\int_0^t\frac{g_X(s)}{\psi_X(s)}\,ds\}$ *and* $\int_0^t|\frac{g_X(s)}{\psi_X(s)}|\,ds\le 9E|X|^{2+\alpha}|t|^{2+\alpha}$.

Proof (*i*) Using the fact that for $x\in\mathbb{R}$, $e^{ix}=1+2^{1-\alpha}|x|^\alpha\Theta$ for some complex function Θ such that $|\Theta|\le 1$ ([17], p. 359), we get that

$$Ee^{itX}=E\bigl(1+\Theta_1 2^{1-\alpha}|tX|^\alpha\bigr)=1+2^{1-\alpha}E\bigl(\Theta_1|X|^\alpha\bigr)|t|^\alpha, \tag{2.1}$$

where Θ_1 is a complex random variable such that $|\Theta_1|\le 1$. From this fact and the inequality $|z_1+z_2|\ge|z_1|-|z_2|$ for complex numbers z_1 and z_2, we can see that

$$\begin{aligned}\bigl|Ee^{itX}\bigr| &= \bigl|1+2^{1-\alpha}E\bigl(\Theta_1|X|^\alpha\bigr)|t|^\alpha\bigr|\\ &\ge 1-2^{1-\alpha}E\bigl(|\Theta_1||X|^\alpha\bigr)|t|^\alpha\\ &\ge 1-2^{1-\alpha}E|X|^\alpha|t|^\alpha\\ &\ge 1-2E|X|^\alpha|t|^\alpha.\end{aligned}$$

Then, for all $|t|\le(\frac{1}{3E|X|^\alpha})^{\frac{1}{\alpha}}$, we have $|\psi_X(t)|=|Ee^{itX}|\ge\frac{1}{3}$.

(*ii*) Let $t\in\mathbb{R}$ be such that $|t|\le(\frac{1}{3E|X|^\alpha})^{\frac{1}{\alpha}}$. Since $\psi_X(t)=Ee^{itX}$, we obtain $\psi_X'(t)=iE(Xe^{itX})$, which implies that

$$\psi_X'(t)=\biggl(\bigl(i\mu_X-\sigma_X^2t\bigr)+\frac{g_X(t)}{\psi_X(t)}\biggr)\psi_X(t),$$

where

$$g_X(t)=-\bigl(i\mu_X-\sigma_X^2t\bigr)Ee^{itX}+iE\bigl(Xe^{itX}\bigr).$$

Hence

$$\frac{\psi_X'(t)}{\psi_X(t)}=i\mu_X-\sigma_X^2t+\frac{g_X(t)}{\psi_X(t)}$$

and then

$$\ln\psi_X(t) = \int_0^t \frac{\psi_X'(s)}{\psi_X(s)}\,\mathrm{d}s = i\mu_X t - \frac{1}{2}\sigma_X^2 t^2 + \int_0^t \frac{g_X(s)}{\psi_X(s)}\,\mathrm{d}s,$$

that is,

$$\psi_X(t) = \exp\left\{i\mu_X t - \frac{1}{2}\sigma_X^2 t^2 + \int_0^t \frac{g_X(s)}{\psi_X(s)}\,\mathrm{d}s\right\}.$$

From the fact that for $x \in \mathbb{R}$, $e^{ix} = 1 + ix + \frac{2^{1-\alpha}}{1+\alpha}|x|^{1+\alpha}\Theta$ for some complex function Θ such that $|\Theta| \le 1$ ([17], p. 359), we have that

$$Ee^{itX} = 1 + itEX + \frac{2^{1-\alpha}}{1+\alpha}E\bigl(\Theta_2|X|^{1+\alpha}\bigr)|t|^{1+\alpha} \tag{2.2}$$

$$\text{and} \quad iE\bigl(Xe^{itX}\bigr) = i\mu_X - tEX^2 + \frac{2^{1-\alpha}}{1+\alpha}E\bigl(i\Theta_2|X|^{2+\alpha}\bigr)|t|^{1+\alpha}, \tag{2.3}$$

where Θ_2 is a complex random variable such that $|\Theta_2| \le 1$. From (2.1)–(2.3), we have

$$\begin{aligned} g_X(t) &= -i\mu_X Ee^{itX} + \sigma_X^2 tEe^{itX} + iE\bigl(Xe^{itX}\bigr) \\ &= \frac{2^{1-\alpha}}{1+\alpha}\mu_X E\bigl(i\Theta_2|X|^{1+\alpha}\bigr)|t|^{1+\alpha} + 2^{1-\alpha}\sigma_X^2 E\bigl(\Theta_1|X|^{\alpha}\bigr)|t|^{1+\alpha} \\ &\quad + \frac{2^{1-\alpha}}{1+\alpha}E\bigl(i\Theta_2|X|^{2+\alpha}\bigr)|t|^{1+\alpha}. \end{aligned} \tag{2.4}$$

According to Lyapunov's inequality: $(E|X|^r)^{\frac{1}{r}} \le (E|X|^s)^{\frac{1}{s}}$, where $0 < r \le s$, we have that $E|X| \le (E|X|^{2+\alpha})^{\frac{1}{2+\alpha}}$ and $E|X|^{1+\alpha} \le (E|X|^{2+\alpha})^{\frac{1+\alpha}{2+\alpha}}$, which imply that

$$\mu_X E|X| \le E|X|E|X|^{1+\alpha} \le E|X|^{2+\alpha}.$$

We can use the same technique to show that

$$\sigma_X^2 E|X|^{\alpha} \le EX^2 E|X|^{\alpha} \le E|X|^{2+\alpha}.$$

From these facts and (2.4), we have

$$\begin{aligned} \bigl|g_X(t)\bigr| &\le \frac{2^{1-\alpha}}{1+\alpha}\mu_X E\bigl(|i\Theta_2||X|^{1+\alpha}\bigr)|t|^{1+\alpha} + 2^{1-\alpha}\sigma_X^2 E\bigl(|\Theta_1||X|^{\alpha}\bigr)|t|^{1+\alpha} \\ &\quad + \frac{2^{1-\alpha}}{1+\alpha}E\bigl(|i\Theta_2||X|^{2+\alpha}\bigr)|t|^{1+\alpha} \\ &\le \frac{2^{1-\alpha}}{1+\alpha}\mu_X E\bigl(|X|^{1+\alpha}\bigr)|t|^{1+\alpha} + 2^{1-\alpha}\sigma_X^2 E\bigl(|X|^{\alpha}\bigr)|t|^{1+\alpha} \\ &\quad + \frac{2^{1-\alpha}}{1+\alpha}E\bigl(|X|^{2+\alpha}\bigr)|t|^{1+\alpha} \\ &\le \frac{2^{1-\alpha}}{1+\alpha}E\bigl(|X|^{2+\alpha}\bigr)|t|^{1+\alpha} + 2^{1-\alpha}E\bigl(|X|^{2+\alpha}\bigr)|t|^{1+\alpha} \\ &\quad + \frac{2^{1-\alpha}}{1+\alpha}E\bigl(|X|^{2+\alpha}\bigr)|t|^{1+\alpha} \end{aligned}$$

$$= \left(2^{1-\alpha} + \frac{2^{2-\alpha}}{1+\alpha}\right) E|X|^{2+\alpha}|t|^{1+\alpha}$$
$$\leq 6E|X|^{2+\alpha}|t|^{1+\alpha}. \tag{2.5}$$

Hence we can conclude from (i) and (2.5) that for all $|t| \leq (\frac{1}{3E|X|^{\alpha}})^{\frac{1}{\alpha}}$ we have

$$\left|\frac{g_X(t)}{\psi_X(t)}\right| \leq 18E|X|^{2+\alpha}|t|^{1+\alpha},$$

which implies that

$$\int_0^t \left|\frac{g_X(s)}{\psi_X(s)}\right| \mathrm{d}s \leq \frac{18}{2+\alpha} E|X|^{2+\alpha}|t|^{2+\alpha} \leq 9E|X|^{2+\alpha}|t|^{2+\alpha}. \qquad \square$$

Lemma 2.2 *Let* $\tau = \frac{1}{3^{\frac{1}{\alpha}}}(\frac{1}{\sum_{j=1}^n E|X_j|^{2+\alpha}})^{\frac{1}{2+\alpha}}$. *Then*

$$\left|\psi(t) - e^{it\mu - \frac{1}{2}\sigma^2 t^2}\right| \leq 12.5606 \sum_{j=1}^n E|X_j|^{2+\alpha}|t|^{2+\alpha} e^{-\frac{1}{2}\sigma^2 t^2}$$

for all $|t| \leq \tau$.

Proof From Lyapunov's inequality, we have

$$\left(E|X_l|^{\alpha}\right)^{\frac{1}{\alpha}} \leq \left(E|X_l|^{2+\alpha}\right)^{\frac{1}{2+\alpha}},$$

which implies that

$$\left(\frac{1}{\sum_{j=1}^n E|X_j|^{2+\alpha}}\right)^{\frac{1}{2+\alpha}} \leq \left(\frac{1}{E|X_l|^{2+\alpha}}\right)^{\frac{1}{2+\alpha}} \leq \left(\frac{1}{E|X_l|^{\alpha}}\right)^{\frac{1}{\alpha}}$$

for all $l = 1, 2, \ldots, n$. This provides that

$$\left(\frac{1}{3^{\frac{2+\alpha}{\alpha}} \sum_{j=1}^n E|X_j|^{2+\alpha}}\right)^{\frac{1}{2+\alpha}} \leq \left(\frac{1}{3E|X_l|^{\alpha}}\right)^{\frac{1}{\alpha}}$$

for all $l = 1, 2, \ldots, n$. From this fact and Lemma 2.1, we have for all $|t| \leq \tau$,

$$\psi(t) = \exp\left\{i\mu t - \frac{1}{2}\sigma^2 t^2 + \sum_{j=1}^n G_j(t)\right\}, \tag{2.6}$$

where

$$G_j(t) = \int_0^t \frac{g_{X_j}(s)}{\psi_{X_j}(s)} \mathrm{d}s \quad \text{and} \quad \left|G_j(t)\right| \leq 9E|X_j|^{2+\alpha}|t|^{2+\alpha}.$$

From (2.6) and the inequality $|e^z - 1| \leq |z|e^{|z|}$ for a complex number z, we get that for all $|t| \leq \tau$,

$$\left|\psi(t) - e^{i\mu t - \frac{1}{2}\sigma^2 t^2}\right| = \left|e^{i\mu t - \frac{1}{2}\sigma^2 t^2 + \sum_{j=1}^n G_j(t)} - e^{i\mu t - \frac{1}{2}\sigma^2 t^2}\right|$$

$$
\begin{aligned}
&= \left|e^{i\mu t-\frac{1}{2}\sigma^2t^2}\right|\left|e^{\sum_{j=1}^n G_j(t)}-1\right| \\
&\le \left|\sum_{j=1}^n G_j(t)\right| e^{-\frac{1}{2}\sigma^2t^2+|\sum_{j=1}^n G_j(t)|} \\
&\le \sum_{j=1}^n \left|G_j(t)\right| e^{-\frac{1}{2}\sigma^2t^2+\sum_{j=1}^n |G_j(t)|} \\
&\le 9\sum_{j=1}^n E|X_j|^{2+\alpha}|t|^{2+\alpha} \times \exp\left\{-\frac{1}{2}\sigma^2t^2+9\sum_{j=1}^n E|X_j|^{2+\alpha}|t|^{2+\alpha}\right\} \\
&\le 9\sum_{j=1}^n E|X_j|^{2+\alpha}|t|^{2+\alpha} \times \exp\left\{-\frac{1}{2}\sigma^2t^2+\frac{9}{3^{\frac{2+\alpha}{\alpha}}}\right\} \\
&\le 9\sum_{j=1}^n E|X_j|^{2+\alpha}|t|^{2+\alpha} \times \exp\left\{-\frac{1}{2}\sigma^2t^2+\frac{9}{3^3}\right\} \\
&\le 12.5606\sum_{j=1}^n E|X_j|^{2+\alpha}|t|^{2+\alpha}e^{-\frac{1}{2}\sigma^2t^2}.
\end{aligned}
$$

□

3 Proof of the main results

3.1 Proof of Theorem 1.1

Proof First, we will prove the theorem in the case of $a = 0$ and $d = 1$. Let $Y_1, Y_2, \dots, Y_n$ be independent common lattice random variables with parameter $(0,1)$, and let

$$
W_n = Y_1 + Y_2 + \cdots + Y_n
$$

with $E(W_n) = \mu_W$, $Var(W_n) = \sigma_W^2$ and the characteristic function ψ_W. Suppose that $\beta_{Y_j} = 2\sum_{m=-\infty}^{\infty} P(Y_j = m)P(Y_j = m+1) > 0$ for all $j = 1, 2, \dots, n$, and let $\tau = \frac{1}{3^{\frac{1}{\alpha}}}\left(\frac{1}{\sum_{j=1}^n E|Y_j|^{2+\alpha}}\right)^{\frac{1}{2+\alpha}}$. Since $P(W_n = k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\psi_W(t)\,\mathrm{d}t$ ([18], p. 511), we have

$$
\begin{aligned}
&\left|P(W_n = k) - \frac{1}{\sigma_W\sqrt{2\pi}}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\right| \\
&\quad= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\psi_W(t)\,\mathrm{d}t - \frac{1}{\sigma_W\sqrt{2\pi}}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\right| \\
&\quad\le \frac{1}{2\pi}\left|\int_{|t|<\tau} e^{-ikt}\psi_W(t)\,\mathrm{d}t - \int_{|t|<\tau} e^{it(\mu_W-k)-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t\right| \\
&\qquad+ \frac{1}{2\pi}\left|\int_{|t|<\tau} e^{it(\mu_W-k)-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t - \frac{\sqrt{2\pi}}{\sigma_W}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\right| \\
&\qquad+ \frac{1}{2\pi}\left|\int_{\tau\le|t|\le\pi} e^{-ikt}\psi_W(t)\,\mathrm{d}t\right| \\
&\quad:= |A| + |B| + |C|. \qquad (3.1)
\end{aligned}
$$

From Lemma 2.2, we have

$$\begin{aligned}
|A| &\leq \frac{1}{2\pi}\int_{|t|<\tau}\left|e^{-ikt}\right|\left|\psi_W(t)-e^{it\mu_W-\frac{1}{2}\sigma_W^2t^2}\right|\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{|t|<\tau}\left|\psi_W(t)-e^{it\mu_W-\frac{1}{2}\sigma_W^2t^2}\right|\,\mathrm{d}t \\
&\leq \frac{12.5606}{\pi}\sum_{j=1}^{n}E|Y_j|^{2+\alpha}\int_0^{\tau}|t|^{2+\alpha}e^{-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t.
\end{aligned}$$

To bound $\int_0^{\tau}|t|^{2+\alpha}e^{-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t$, we let $\tilde{\tau}=\tau^{\frac{2+\alpha}{2}}$ and note that

$$\begin{aligned}
\int_0^{\tilde{\tau}}t^{2+\alpha}e^{-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t &\leq \int_0^{\tilde{\tau}}t^{2+\alpha}\,\mathrm{d}t \\
&= \frac{\tau^{\frac{(2+\alpha)(3+\alpha)}{2}}}{3+\alpha} \\
&\leq \frac{1}{3^{\frac{\alpha^2+7\alpha+6}{2\alpha}}\left(\sum_{j=1}^{n}E|Y_j|^{2+\alpha}\right)^{\frac{3+\alpha}{2}}} \\
&\leq \frac{0.0005}{\left(\sum_{j=1}^{n}E|Y_j|^{2+\alpha}\right)^{\frac{3+\alpha}{2}}}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\tilde{\tau}}^{\tau}t^{2+\alpha}e^{-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t &= \int_{\tilde{\tau}}^{\tau}\frac{t^3}{t^{1-\alpha}}e^{-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t \\
&\leq \frac{1}{\tau^{\frac{(2+\alpha)(1-\alpha)}{2}}}\int_{\tilde{\tau}}^{\tau}t^3e^{-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t \\
&\leq \frac{1}{\tau^{\frac{(2+\alpha)(1-\alpha)}{2}}}\int_0^{\infty}t^3e^{-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t \\
&= 3^{\frac{(2+\alpha)(1-\alpha)}{2\alpha}}\left(\sum_{j=1}^{n}E|Y_j|^{2+\alpha}\right)^{\frac{1-\alpha}{2}}\left(\frac{2}{\sigma_W^4}\right) \\
&\leq \frac{1.1548\cdot 3^{\frac{1}{\alpha}}}{\sigma_W^4}\left(\sum_{j=1}^{n}E|Y_j|^{2+\alpha}\right)^{\frac{1-\alpha}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|A| &\leq \frac{12.5606}{\pi}\sum_{j=1}^{n}E|Y_j|^{2+\alpha}\left(\int_0^{\tilde{\tau}}t^{2+\alpha}e^{-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t+\int_{\tilde{\tau}}^{\tau}t^{2+\alpha}e^{-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t\right) \\
&\leq \frac{0.0020}{\left(\sum_{j=1}^{n}E|Y_j|^{2+\alpha}\right)^{\frac{1+\alpha}{2}}}+\frac{4.6171\cdot 3^{\frac{1}{\alpha}}}{\sigma_W^4}\left(\sum_{j=1}^{n}E|Y_j|^{2+\alpha}\right)^{\frac{3-\alpha}{2}}. \qquad (3.1)
\end{aligned}$$

By the fact that

$$\int_{|t|<\tau}e^{it(\mu_W-k)-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t=\int_{\mathbb{R}}e^{it(\mu_W-k)-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t-\int_{|t|\geq\tau}e^{it(\mu_W-k)-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t$$

$$
\begin{aligned}
&= \frac{1}{\sigma_W}\int_{\mathbb{R}} e^{\frac{it(\mu_W-k)}{\sigma_W}-\frac{t^2}{2}}\,\mathrm{d}t - \frac{1}{\sigma_W}\int_{|t|\geq\sigma_W\tau} e^{\frac{it(\mu_W-k)}{\sigma_W}-\frac{t^2}{2}}\,\mathrm{d}t \\
&= \frac{\sqrt{2\pi}}{\sigma_W}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}} - \frac{1}{\sigma_W}\int_{|t|\geq\sigma_W\tau} e^{\frac{it(\mu_W-k)}{\sigma_W}-\frac{t^2}{2}}\,\mathrm{d}t,
\end{aligned}
$$

we have

$$
\begin{aligned}
B &= \frac{1}{2\pi}\int_{|t|<\tau} e^{it(\mu_W-k)-\frac{1}{2}\sigma_W^2t^2}\,\mathrm{d}t - \frac{1}{\sigma_W\sqrt{2\pi}}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}} \\
&= -\frac{1}{2\pi\sigma_W}\int_{|t|\geq\sigma_W\tau} e^{\frac{it(\mu_W-k)}{\sigma_W}-\frac{t^2}{2}}\,\mathrm{d}t,
\end{aligned}
$$

and hence,

$$
\begin{aligned}
|B| &\leq \frac{1}{2\pi\sigma_W}\int_{|t|\geq\sigma_W\tau} e^{-\frac{t^2}{2}}\,\mathrm{d}t \\
&\leq \frac{1}{\pi\sigma_W^2\tau}\int_{\sigma_W\tau}^{\infty} te^{-\frac{t^2}{2}}\,\mathrm{d}t \\
&= \frac{0.3184}{\sigma_W^2\tau}e^{\frac{-\sigma_W^2\tau^2}{2}}. \qquad (3.2)
\end{aligned}
$$

Using the fact that $|\psi_W(t)| \leq e^{-\frac{1}{\pi^2}\beta_W t^2}$, where $\beta_W = \sum_{j=1}^n \beta_{Y_j}$, for $t \in [0,\pi)$ ([1], p. 5), we have

$$
\begin{aligned}
|C| &= \left|\frac{1}{2\pi}\int_{\tau\leq|t|\leq\pi} e^{-ikt}\psi_W(t)\,\mathrm{d}t\right| \\
&\leq \frac{1}{2\pi}\int_{\tau\leq|t|\leq\pi} \left|\psi_W(t)\right|\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{\tau}^{\pi} \left|\psi_W(t)\right|\,\mathrm{d}t \\
&\leq \frac{1}{\pi}\int_{\tau}^{\pi} e^{-\frac{1}{\pi^2}\beta_W t^2}\,\mathrm{d}t \\
&\leq \frac{1}{\pi\tau}\int_{\tau}^{\infty} te^{-\frac{1}{\pi^2}\beta_W t^2}\,\mathrm{d}t \\
&= \frac{1}{\pi\tau}\left(\frac{\pi^2 e^{-\frac{\tau^2\beta_W}{\pi^2}}}{2\beta_W}\right) \\
&\leq \frac{1.5708}{\tau\beta_W}e^{-\frac{\tau^2\beta_W}{\pi^2}}. \qquad (3.3)
\end{aligned}
$$

From (3.1)–(3.3), we have

$$
\begin{aligned}
&\left|P(W_n = k) - \frac{1}{\sigma_W\sqrt{2\pi}}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\right| \\
&\quad \leq \frac{0.0020}{(\sum_{j=1}^n E|Y_j|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171\cdot 3^{\frac{1}{\alpha}}}{\sigma_W^4}\left(\sum_{j=1}^n E|Y_j|^{2+\alpha}\right)^{\frac{3-\alpha}{2}}
\end{aligned}
$$

$$+\frac{0.3184}{\sigma_W^2\tau}e^{\frac{-\sigma_W^2\tau^2}{2}}+\frac{1.5708}{\tau\beta_W}e^{-\frac{\tau^2\beta_W}{\pi^2}}. \tag{3.4}$$

In general, let $X_1, X_2, \dots, X_n$ be independent lattice random variables with parameter (a, d). For $j = 1, 2, \dots, n$, let $Y_j = \frac{X_j-a}{d}$ and $W_n = Y_1 + Y_2 + \cdots + Y_n$. Observe that $Y_1, Y_2, \dots, Y_n$ are independent common lattice random variables with parameter $(0, 1)$ and

$$\mu_W = \frac{\mu - na}{d}, \qquad \sigma_W^2 = \frac{\sigma^2}{d^2}, \qquad P(Y_j = m) = P(X_j = a + dm), \tag{3.5}$$

$$E|Y_j|^{2+\alpha} = \frac{E|X_j - a|^{2+\alpha}}{d^{2+\alpha}}, \qquad \tau = \frac{d}{3^{\frac{1}{\alpha}}(\sum_{j=1}^n E|X_j - a|^{2+\alpha})^{\frac{1}{2+\alpha}}}. \tag{3.6}$$

From (3.4)–(3.6), we have

$$\begin{aligned}
&\left|P(W_n = k) - \frac{1}{\sigma_W\sqrt{2\pi}}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\right| \\
&\quad\le \frac{0.0020d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(\sum_{j=1}^n E|X_j - a|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171\cdot 3^{\frac{1}{\alpha}}d^{\frac{\alpha^2-\alpha+2}{2}}(\sum_{j=1}^n E|X_j - a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma^4} \\
&\qquad + \frac{0.3184}{\sigma^2\tau}e^{-\frac{\sigma^2\tau^2}{2}} + \frac{1.5708}{\tau\beta}e^{-\frac{\tau^2\beta}{\pi^2}}.
\end{aligned}$$

From this fact and the fact that

$$\left|P(S_n = na + kd) - \frac{d}{\sigma\sqrt{2\pi}}e^{-\frac{(na+kd-\mu)^2}{2\sigma^2}}\right| = \left|P(W_n = k) - \frac{1}{\sigma_W\sqrt{2\pi}}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\right|,$$

we have the conclusion of the theorem.

Furthermore, if $X_1, X_2, \dots, X_n$ are identically distributed, then

$$\mu = n\mu_1, \qquad \sigma = \sigma_1\sqrt{n}, \qquad \sum_{j=1}^n E|X_j - a|^{2+\alpha} = nE|X_1 - a|^{2+\alpha}, \quad \text{and} \quad \beta = n\beta_1,$$

which imply that

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\left|P(S_n = na + kd) - \frac{d}{\sigma_1\sqrt{2n\pi}}e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\right| \\
&\quad\le \frac{0.0020d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1 - a|^{2+\alpha})^{\frac{1+\alpha}{2}}n^{\frac{1+\alpha}{2}}} + \frac{4.6171\cdot 3^{\frac{1}{\alpha}}d^{\frac{\alpha^2-\alpha+2}{2}}(E|X_1 - a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4 n^{\frac{1+\alpha}{2}}} \\
&\qquad + \frac{0.3184\cdot 3^{\frac{1}{\alpha}}d(E|X_1 - a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{\sigma_1^2 n^{\frac{1+\alpha}{2+\alpha}}}\exp\left(\frac{-\sigma_1^2 n^{\frac{\alpha}{2+\alpha}}}{2\cdot 3^{\frac{2}{\alpha}}(E|X_1 - a|^{2+\alpha})^{\frac{2}{2+\alpha}}}\right) \\
&\qquad + \frac{1.5708\cdot 3^{\frac{1}{\alpha}}(E|X_1 - a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{d\beta_1 n^{\frac{1+\alpha}{2+\alpha}}}\exp\left(\frac{-d^2\beta_1 n^{\frac{\alpha}{2+\alpha}}}{3^{\frac{2}{\alpha}}\pi^2(E|X_1 - a|^{2+\alpha})^{\frac{2}{2+\alpha}}}\right).
\end{aligned} \tag{3.7}$$

Since $\frac{1+2\alpha}{2+\alpha} \geq \frac{1+\alpha}{2}$ and $e^{-x} \leq \frac{1}{x}$ for a real number $x > 0$, we obtain that

$$\begin{aligned}
&\frac{0.3184 \cdot 3^{\frac{1}{\alpha}} d(E|X_1 - a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{\sigma_1^2 n^{\frac{1+\alpha}{2+\alpha}}} \exp\left(\frac{-\sigma_1^2 n^{\frac{\alpha}{2+\alpha}}}{2 \cdot 3^{\frac{2}{\alpha}} (E|X_1 - a|^{2+\alpha})^{\frac{2}{2+\alpha}}}\right) \\
&\quad \leq \frac{0.3184 \cdot 3^{\frac{1}{\alpha}} d(E|X_1 - a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{\sigma_1^2 n^{\frac{1+\alpha}{2+\alpha}}} \left(\frac{2 \cdot 3^{\frac{2}{\alpha}} (E|X_1 - a|^{2+\alpha})^{\frac{2}{2+\alpha}}}{\sigma_1^2 n^{\frac{\alpha}{2+\alpha}}}\right) \\
&\quad \leq \frac{0.6368 \cdot 3^{\frac{3}{\alpha}} d(E|X_1 - a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{\sigma_1^4 n^{\frac{1+\alpha}{2}}}
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\frac{1.5708 \cdot 3^{\frac{1}{\alpha}} (E|X_1 - a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{d\beta_1 n^{\frac{1+\alpha}{2+\alpha}}} \exp\left(\frac{-d^2 \beta_1 n^{\frac{\alpha}{2+\alpha}}}{3^{\frac{2}{\alpha}} \pi^2 (E|X_1 - a|^{2+\alpha})^{\frac{2}{2+\alpha}}}\right) \\
&\quad \leq \frac{1.5708 \cdot 3^{\frac{1}{\alpha}} (E|X_1 - a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{d\beta_1 n^{\frac{1+\alpha}{2+\alpha}}} \left(\frac{3^{\frac{2}{\alpha}} \pi^2 (E|X_1 - a|^{2+\alpha})^{\frac{2}{2+\alpha}}}{d^2 \beta_1 n^{\frac{\alpha}{2+\alpha}}}\right) \\
&\quad \leq \frac{15.5032 \cdot 3^{\frac{3}{\alpha}} (E|X_1 - a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{d^3 \beta_1^2 n^{\frac{1+\alpha}{2}}}.
\end{aligned} \tag{3.9}$$

From (3.7)–(3.9), we have

$$\sup_{k \in \mathbb{Z}} \left| P(S_n = na + kd) - \frac{d}{\sigma_1 \sqrt{2n\pi}} e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}} \right| \leq \frac{C_1}{n^{\frac{1+\alpha}{2}}},$$

where

$$\begin{aligned}
C_1 &= \frac{0.0020 d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1 - a|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171 \cdot 3^{\frac{1}{\alpha}} d^{\frac{\alpha^2 - \alpha + 2}{2}} (E|X_1 - a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4} \\
&\quad + \frac{0.6368 \cdot 3^{\frac{3}{\alpha}} d(E|X_1 - a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{\sigma_1^4} + \frac{15.5032 \cdot 3^{\frac{3}{\alpha}} (E|X_1 - a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{d^3 \beta_1^2}.
\end{aligned}$$

□

3.2 Proof of Theorem 1.2

Proof By the same reason of Theorem 1.1, it suffices to prove the theorem in case $a = 0$ and $d = 1$. Let $Y_1, Y_2, \ldots, Y_n$ be independent common lattice random variables with parameter $(0, 1)$ with the characteristic functions ψ_{Y_j}, and let

$$W_n = Y_1 + Y_2 + \cdots + Y_n$$

with $E(W_n) = \mu_W$, $Var(W_n) = \sigma_W^2$ and the characteristic function ψ_W. Suppose that $\upsilon_{Y_j} = 2\sum_{m=-\infty}^{\infty} P(Y_j = m)P(Y_j = m + j) > 0$ for all $j = 1, 2 \ldots, n$. From (3.1)–(3.2) in Theorem 1.1, we have

$$\begin{aligned}
&\left|P(W_n = k) - \frac{1}{\sigma_W\sqrt{2\pi}} e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\right| \\
&\quad \le \frac{0.0020}{(\sum_{j=1}^n E|Y_j|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171 \cdot 3^{\frac{1}{\alpha}}}{\sigma_W^4}\left(\sum_{j=1}^n E|Y_j|^{2+\alpha}\right)^{\frac{3-\alpha}{2}} \\
&\qquad + \frac{0.3184}{\sigma_W^2\tau} e^{\frac{-\sigma_W^2\tau^2}{2}} + |C|.
\end{aligned} \tag{3.11}$$

Siripraparat and Neammanee ([1], p. 6) showed that

$$\ln\left(\left|\psi_{Y_j}(t)\right|\right) \le -\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} P(Y_j = m)P(Y_j = l)\sin^2\left((m-l)\frac{t}{2}\right).$$

From this fact and the fact that

$$\sum_{j=1}^n \sin^2\left(\frac{jt}{2}\right) \ge \frac{n}{4}\min\left(1, \left(\frac{nt}{2\pi}\right)^2\right)$$

for $|t| \le \pi$ and $n \ge 2$ ([19], p. 399), we have

$$\begin{aligned}
\left|\psi_W(t)\right| &= \prod_{j=1}^n \left|\psi_{Y_j}(t)\right| \\
&\le \prod_{j=1}^n \exp\left(-2\sum_{m=-\infty}^{\infty} P(Y_j = m)P(Y_j = m+j)\sin^2\left(\frac{jt}{2}\right)\right) \\
&\le \exp\left(-\sum_{j=1}^n \upsilon_{Y_j}\sin^2\left(\frac{jt}{2}\right)\right) \\
&\le \exp\left(-\upsilon_W\sum_{j=1}^n \sin^2\left(\frac{jt}{2}\right)\right) \\
&\le \exp\left(-\frac{n\upsilon_W}{4}\min\left(1, \left(\frac{nt}{2\pi}\right)^2\right)\right),
\end{aligned}$$

where $\upsilon_W = \min_{1\le j\le n}\upsilon_{Y_j}$. Hence,

$$\begin{aligned}
|C| &\le \frac{1}{2\pi}\int_{\tau\le|t|\le\pi}\left|\psi_W(t)\right| \,\mathrm{d}t \\
&= \frac{1}{\pi}\int_\tau^\pi \left|\psi_W(t)\right| \,\mathrm{d}t \\
&\le \frac{1}{\pi}\int_\tau^\pi \exp\left(-\frac{n\upsilon_W}{4}\min\left(1, \left(\frac{nt}{2\pi}\right)^2\right)\right)\mathrm{d}t \\
&\le \frac{\pi-\tau}{\pi}\exp\left(-\frac{n\upsilon_W}{4}\min\left(1, \left(\frac{n\tau}{2\pi}\right)^2\right)\right)
\end{aligned}$$

$$\leq \exp\left(-\frac{n\upsilon_W}{4}\min\left(1,\left(\frac{n\tau}{2\pi}\right)^2\right)\right). \tag{3.10}$$

From (3.11) and (3.10),

$$\begin{aligned}
&\left|P(W_n = k) - \frac{1}{\sigma_W\sqrt{2\pi}}e^{-\frac{(k-\mu_W)^2}{2\sigma_W^2}}\right| \\
&\quad \leq \frac{0.0020}{(\sum_{j=1}^n E|Y_j|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171\cdot 3^{\frac{1}{\alpha}}}{\sigma_W^4}\left(\sum_{j=1}^n E|Y_j|^{2+\alpha}\right)^{\frac{3-\alpha}{2}} + \frac{0.3184}{\sigma_W^2\tau}e^{\frac{-\sigma_W^2\tau^2}{2}} \\
&\qquad + e^{-\frac{n\upsilon_W}{4}\min(1,(\frac{n\tau}{2\pi})^2)}.
\end{aligned}$$

Furthermore, if $X_1, X_2, \dots, X_n$ are identically distributed and $\upsilon_j > 0$ for all $j = 1, 2, \dots, n$, then

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\left|P(S_n = na + kd) - \frac{d}{\sigma_1\sqrt{2n\pi}}e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\right| \\
&\quad \leq \frac{0.0020d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1-a|^{2+\alpha})^{\frac{1+\alpha}{2}}n^{\frac{1+\alpha}{2}}} + \frac{4.6171\cdot 3^{\frac{1}{\alpha}}d^{\frac{\alpha^2-\alpha+2}{2}}(E|X_1-a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4 n^{\frac{1+\alpha}{2}}} \\
&\qquad + \frac{0.3184\cdot 3^{\frac{1}{\alpha}}d(E|X_1-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{\sigma_1^2 n^{\frac{1+\alpha}{2+\alpha}}}\exp\left(\frac{-\sigma_1^2 n^{\frac{\alpha}{2+\alpha}}}{2\cdot 3^{\frac{2}{\alpha}}(E|X_1-a|^{2+\alpha})^{\frac{2}{2+\alpha}}}\right) \\
&\qquad + \exp\left(-\frac{n\upsilon}{4}\min\left(1,\left(\frac{n^{\frac{1+\alpha}{2+\alpha}}d}{2\pi\cdot 3^{\frac{1}{\alpha}}(E|X_j-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}\right)^2\right)\right).
\end{aligned}$$

From (3.8) and $n \geq (\frac{2\pi\cdot 3^{\frac{1}{\alpha}}(E|X_1-a|^{2+\alpha})^{\frac{1}{2+\alpha}}}{d})^{\frac{2+\alpha}{1+\alpha}}$, we obtain that

$$\sup_{k\in\mathbb{Z}}\left|P(S_n = na + kd) - \frac{d}{\sigma_1\sqrt{2n\pi}}e^{-\frac{(na+kd-n\mu_1)^2}{2n\sigma_1^2}}\right| \leq \frac{C_2}{n^{\frac{1+\alpha}{2}}} + e^{-\frac{n\upsilon}{4}},$$

where

$$\begin{aligned}
C_2 &= \frac{0.0020d^{\frac{(2+\alpha)(1+\alpha)}{2}}}{(E|X_1-a|^{2+\alpha})^{\frac{1+\alpha}{2}}} + \frac{4.6171\cdot 3^{\frac{1}{\alpha}}d^{\frac{\alpha^2-\alpha+2}{2}}(E|X_1-a|^{2+\alpha})^{\frac{3-\alpha}{2}}}{\sigma_1^4} \\
&\quad + \frac{0.6368\cdot 3^{\frac{3}{\alpha}}d(E|X_1-a|^{2+\alpha})^{\frac{3}{2+\alpha}}}{\sigma_1^4}.
\end{aligned}$$

□

4 Examples

In our work, we relax the condition third moment to almost the second moment. The following example shows that there is an integer-valued random variable where the third moment does not exist but the aim moment exists.

Example 4.1 For $j = 1, 2, \dots, n$, let

$$P(X_j = 0) = P(X_j = 2) = 0.45 \quad \text{and} \quad P(X_j = 2^k) = \frac{5.6}{2^{3k}} \quad \text{for integer } k \geq 2,$$

Table 1 Explicit constants for Example 4.1

α	A_1	A_2	A_3	A_4	A_5
0.1	815,329.3725	30,707.0472	$7.8702 \cdot 10^{-11}$	510,995.2818	$4.7281 \cdot 10^{-12}$
0.2	3830.6039	131.7570	$4.2748 \cdot 10^{-6}$	2192.5659	$2.5681 \cdot 10^{-7}$
0.3	709.0199	22.0903	$1.5207 \cdot 10^{-4}$	367.6043	$9.1360 \cdot 10^{-6}$
0.4	332.9755	9.2986	$8.5826 \cdot 10^{-4}$	154.7385	$5.1561 \cdot 10^{-5}$
0.5	230.5188	5.6875	$2.2941 \cdot 10^{-3}$	94.6455	$1.3782 \cdot 10^{-4}$
0.6	198.0421	4.2260	$4.1553 \cdot 10^{-3}$	70.3248	$2.4963 \cdot 10^{-4}$
0.7	199.3648	3.5533	$5.8776 \cdot 10^{-3}$	59.1302	$3.5310 \cdot 10^{-4}$
0.8	237.4348	3.3064	$6.7879 \cdot 10^{-3}$	55.0225	$4.0779 \cdot 10^{-4}$
0.9	384.3726	3.5196	$5.9908 \cdot 10^{-3}$	58.5688	$3.5990 \cdot 10^{-4}$

and assume that $X_1, X_2, \ldots, X_n$ are independent. Note that $X_1, X_2, \ldots, X_n$ are maximal lattice random variables with parameter $(0,2)$ and $\mu_j = 1.3667$, $\sigma_j^2 = 2.7322$, $\beta_j = 0.2025$,

$$E|X_j|^3 = 3.6 + \sum_{k=2}^{\infty} 2^{3k} P\bigl(X = 2^k\bigr) = 3.6 + \sum_{k=2}^{\infty} 2^{3k}\left(\frac{5.6}{2^{3k}}\right) = \infty,$$

and for $\alpha \in (0,1)$,

$$\begin{aligned} E|X_j|^{2+\alpha} &= 0.45 \cdot 2^{2+\alpha} + \sum_{k=2}^{\infty} 2^{(2+\alpha)k} P\bigl(X = 2^k\bigr) \\ &= 0.45 \cdot 2^{2+\alpha} + \frac{5.6}{2^{2(1-\alpha)} - 2^{1-\alpha}} < \infty \end{aligned}$$

for all $j = 1, 2, \ldots, n$. Let

$$\Delta_n = \sup_{k \in \mathbb{Z}} \left| P(S_n = 2k) - \frac{2}{1.6529\sqrt{2n\pi}} e^{-\frac{(2k-1.3667n)^2}{5.4644n}} \right|.$$

By Theorem 1.1, we have

$$\Delta_n \le \frac{A_1}{n^{\frac{1+\alpha}{2}}} + \frac{A_2}{n^{\frac{1+\alpha}{2+\alpha}}} \exp\bigl(-A_3 n^{\frac{\alpha}{2+\alpha}}\bigr) + \frac{A_4}{n^{\frac{1+\alpha}{2+\alpha}}} \exp\bigl(-A_5 n^{\frac{\alpha}{2+\alpha}}\bigr)$$

and Table 1.

Observe that we cannot apply Theorem 1.2 with Example 4.1 since $\upsilon_j = 0$ for some $j \ge 3$.

Example 4.2 Let $X_1, X_2, \ldots, X_n$ be independent random variables defined by

$$P(X_j = 0) = \frac{7}{8} - \frac{1}{(2j)^6 - (2j)^3}, \qquad P(X_j = 2j) = \frac{1}{8}, \quad \text{and} \quad P\bigl(X_j = (2j)^k\bigr) = (2j)^{-3k}$$

for integer $k \ge 2$. We see that $X_1, X_2, \ldots, X_n$ are common lattice random variables with parameter $(0,2)$ and

$$\begin{aligned} &\mu_j = \frac{j}{4} + \frac{1}{16j^4 - 4j^2}, \qquad \sigma_j^2 = \frac{j^2}{2} + \frac{1}{4j^2 - 2j} - \mu_j^2, \\ &E|X_j|^{2+\alpha} = \frac{(2j)^{2+\alpha}}{8} + \frac{1}{(2j)^{2-2\alpha} - (2j)^{1-\alpha}} \quad \text{and} \quad E|X_j|^3 = \infty. \end{aligned}$$

Table 2 Explicit constants for Example 4.2

α	B_1	B_2	B_3	B_4	B_5
0.1	0.0120	$5.6822 \cdot 10^{8}$	351,302.9406	$1.9714 \cdot 10^{-11}$	$2.5683 \cdot 10^{-12}$
0.2	0.0147	$2.6676 \cdot 10^{6}$	1525.1266	$1.0460 \cdot 10^{-6}$	$1.3627 \cdot 10^{-7}$
0.3	0.0181	$4.9135 \cdot 10^{5}$	257.4494	$3.6709 \cdot 10^{-5}$	$4.7821 \cdot 10^{-6}$
0.4	0.0225	$2.2850 \cdot 10^{5}$	108.5686	$2.0641 \cdot 10^{-4}$	$2.6891 \cdot 10^{-5}$
0.5	0.0282	$1.5570 \cdot 10^{5}$	66.1785	$5.5555 \cdot 10^{-4}$	$7.2372 \cdot 10^{-5}$
0.6	0.0355	$1.3065 \cdot 10^{5}$	48.7213	$1.0249 \cdot 10^{-3}$	$1.3353 \cdot 10^{-4}$
0.7	0.0451	$1.2714 \cdot 10^{5}$	40.3174	$1.4968 \cdot 10^{-3}$	$1.9500 \cdot 10^{-4}$
0.8	0.0576	$1.4431 \cdot 10^{5}$	36.6187	$1.8144 \cdot 10^{-3}$	$2.3638 \cdot 10^{-4}$
0.9	0.0741	$2.1802 \cdot 10^{5}$	37.6321	$1.7180 \cdot 10^{-3}$	$2.2382 \cdot 10^{-4}$

This implies that

$$
\begin{aligned}
&\frac{n^3}{48} \le \sigma^2 \le n^3, \\
&\frac{n^3}{6} \le E|X_j|^{2+\alpha} \le \left(\frac{2^{2+\alpha}}{8} + \frac{2^{1+2\alpha}}{48(2^{1-\alpha}-1)} \right) n^{3+\alpha}.
\end{aligned}
$$

Moreover, we have that

$$
\upsilon = \min_{1\le j\le n} \frac{1}{4}\left(\frac{7}{8} - \frac{1}{(2j)^6-(2j)^3} \right) = \frac{3}{14}.
$$

Let

$$
\Delta_n = \sup_{k\in\mathbb{Z}} \left| P(S_n = 2k) - \frac{2}{\sigma\sqrt{2\pi}} e^{-\frac{(2k-\mu)^2}{2\sigma^2}} \right|.
$$

By Theorem 1.1, we have

$$
\Delta_n \le \frac{B_1}{n^{\frac{3+3\alpha}{2}}} + \frac{B_2}{n^{\frac{\alpha^2+3}{2}}} + \frac{B_3}{n^3} \exp\bigl(-B_4 n^{\frac{\alpha}{2+\alpha}}\bigr) + \exp\bigl(-B_5 n^{\frac{\alpha}{2+\alpha}}\bigr)
$$

and Table 2.

Observe that we cannot apply Theorem 1.1 with Example 4.2 since $\beta_j = 0$ for $j \ge 2$.

Acknowledgements
The authors would like to thank the reviewers for their valuable comments and suggestions.

Funding
This work was supported by the Development and Promotion of Science and Technology Talents Project (DPST).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contribution
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Received: 29 November 2021 Accepted: 19 January 2023 Published online: 07 February 2023

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