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On the degree of approximation of Fourier series based on a certain class of product deferred summability means

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Abstract

In this article, we first introduce and study the basic concepts of deferred Euler and deferred Nörlund product summability means of Fourier series of arbitrary periodic functions. We then estimate the degree of approximation of Fourier series of an arbitrary periodic function in the generalized Zygmund class based upon our proposed product deferred summability means. Moreover, we discuss some important concluding remarks in connection with our findings. Finally, we suggest a direction for future studies on this subject, which are based upon the basic notion of statistical product deferred summability means of Fourier series of arbitrary periodic functions in the generalized Zygmund class.

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1 Introduction, preliminaries, and motivation

The approximation of Fourier series of arbitrary periodic functions is very useful and has substantial importance in various fields of applied mathematics as well as engineering sciences, and in particular in the study of numerical functional analysis. Also, it has been conveying a new direction having wide applications in signal analysis, image processing, and system design in modern telecommunications (see [14] and [15]). Recently, many researchers are working on the degree of approximation of Fourier series and conjugate series of 2π-periodic functions belonging to different kinds of functional sequence spaces such as Lipschitz, Hölder, Besov, and Zygmund spaces via various summability techniques (see, [1–5, 8], and [10]). In fact, here we estimate the degree of approximation of Fourier series of arbitrary periodic functions belonging to the generalized Zygmund class based on a certain class of product deferred summability means.

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Let $s_k(g;x)$ be the nth partial sum of a Fourier series of an arbitrary periodic function g with period $2L$ such that

$$s_k(g;x) = a_0 + \sum_{k=1}^{n}\left(a_k \cos\frac{k\pi}{L}x + b_k \sin\frac{k\pi}{L}x\right) \tag{1.1}$$

$$= \frac{1}{L}\int_{-L}^{L} g(\lambda + x)D_n(\lambda)\,d\lambda, \tag{1.2}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(\mu)\,d\mu,$$

$$a_k = \frac{1}{L}\int_{-L}^{L} g(\mu)\cos\frac{k\pi\mu}{L}\,d\mu \quad (k = 1,2,\ldots),$$

$$b_k = \frac{1}{L}\int_{-L}^{L} g(\mu)\sin\frac{k\pi\mu}{L}\,d\mu \quad (k = 1,2,\ldots),$$

and

$$D_n(\lambda) = \frac{\sin[\frac{(n+1/2)\lambda\pi}{L}]}{2\sin(\frac{\pi\lambda}{2L})} \quad (\because \lambda = \mu - x)$$

is *Dirichlet's kernel*.

We now recall the Zygmund modulus of continuity of $g(x)$ as follows (see [18]):

$$\omega(g,t) = \sup_{0\leqq\eta\leqq t, x\in\mathbb{R}} \left|g(x+\eta) + h(x-\eta) - 2g(x)\right|. \tag{1.3}$$

Let $\mathbb{C}_{2L}$ be the Banach space of all $2L$-periodic continuous functions defined on $[0,2L]$ under the supremum norm.

For $0 < \upsilon \leqq 1$, the function space

$$Z_{(\upsilon)} = \left\{g \in \mathbb{C}_{2L} : \left|g(x+\eta) + g(x-\eta) - 2g(x)\right| = O\left(|\eta|^{\upsilon}\right)\right\}$$

is a Banach space under the norm $\|\cdot\|_{(\upsilon)}$ defined by

$$\|g\|_{(\upsilon)} = \sup_{0\leqq x\leqq 2L}\left|g(x)\right| + \sup_{x,\eta\neq 0}\frac{|g(x+\eta) + g(x-\eta) - 2g(x)|}{|\eta|^{\upsilon}}.$$

Let $g \in \mathcal{L}_\alpha[0,2L]$. Then,

$$\mathcal{L}_\alpha[0,2L] = \left\{g : [0,2L] \to \mathbb{R} \text{ and } \int_0^{2L}\left|g(x)\right|^{\alpha}dx < \infty\right\} \quad (\alpha \geqq 1).$$

The $\mathcal{L}_\alpha[0,2L]$ norm of a function $g(x)$ is defined by

$$\|g\|_\alpha = \begin{cases} (\frac{1}{2L}\int_0^{2L}|g(x)|^\alpha\,dx)^{\frac{1}{\alpha}} & (1 \leqq \alpha < \infty), \\ \operatorname{ess\,sup}_{0<x\leqq 2L}|g(x)| & (\alpha = \infty). \end{cases}$$

For $g \in \mathcal{L}_\alpha[0,2L]$ ($\alpha \geqq 1$), the integral Zygmund modulus of continuity is defined by

$$\omega_\alpha(g,t) = \sup_{0<\eta\leqq t; x\in\mathbb{R}} \left\{ \frac{1}{2L} \int_0^{2L} \left|g(x+\eta) + g(x-\eta) - 2g(x)\right|^\alpha dx \right\}^{\frac{1}{\alpha}},$$

and for $g \in \mathbb{C}_{2L}$ and $\alpha = \infty$, we have

$$\omega_\infty(g,t) = \sup_{0<\eta\leqq t; x\in\mathbb{R}} \max_x \left|g(x+\eta) + g(x-\eta) - 2g(x)\right|.$$

Note that

$$\omega_\alpha(g,t) \to 0 \quad \text{as } \alpha \to 0.$$

We now define

$$Z_{(\upsilon),\alpha} = \left\{ g \in \mathcal{L}_\alpha[0,2L] : \left(\int_0^{2L} \left|g(x+\eta) + g(x-\eta) - 2g(x)\right|^\alpha dx \right)^{\frac{1}{\alpha}} = O\left(|\eta|^\upsilon\right) \right\}.$$

The space $Z_{(\upsilon),\alpha}$ ($\alpha \geqq 1, 0 < \upsilon \leqq 1$) is a Banach space under the norm $\|\cdot\|_{(\upsilon),\alpha}$ defined by

$$\|g\|_{(\upsilon),\alpha} = \|g\|_\alpha + \sup_{\eta\neq 0} \frac{\|g(\cdot+\eta) + g(\cdot-\eta) - 2g(\cdot)\|_\alpha}{|\eta|^\upsilon}.$$

The class $Z^{(\omega)}$ is defined by

$$Z^{(\omega)} = \left\{ g \in \mathbb{C}_{2L} : \left|g(x+\eta) + g(x-\eta) - 2g(x)\right| = O\left(\omega(\eta)\right) \right\},$$

where ω is already mentioned in (1.3). Also, ω is positive, nondecreasing, and a continuous function with the properties:

(i) $\omega(0) = 0$; and

(ii) $\omega(\eta_1 + \eta_2) \leqq \omega(\eta_1) + \omega(\eta_2)$.

Let $\omega : [0,2L] \to \mathbb{R}$ be an arbitrary function with $\omega(\eta) > 0$ ($0 \leqq \eta < 2L$) and let

$$\lim_{\eta\to 0^+} \omega(\eta) = \omega(0) = 0.$$

For $1 \leqq \alpha < \infty$, we define

$$Z_\alpha^{(\omega)} = \left\{ g \in \mathcal{L}_\alpha[0,2L] : \sup_{\eta\neq 0} \frac{\|g(\cdot+\eta) + g(\cdot-\eta) - 2g(\cdot)\|_\alpha}{\omega(\eta)} < \infty \right\}$$

and

$$\|g\|_\alpha^{(\omega)} = \|g\|_\alpha + \sup_{\eta\neq 0} \frac{\|g(\cdot+\eta) + g(\cdot-\eta) - 2g(\cdot)\|_\alpha}{\omega(\eta)} \quad (\alpha \geqq 1),$$

where $\|\cdot\|_\alpha^{(\omega)}$ is a norm on $Z_\alpha^{(\omega)}$.

Remark 1 We know that the completeness of $\mathcal{L}_\alpha[0,2L]$ $(\alpha \geqq 1)$ implies the completeness of $Z_\alpha^{(\omega)}$. Thus, $Z_\alpha^{(\omega)}$ is a Banach space under the norm $\|\cdot\|_\alpha^{(\omega)}$.

Let $(\frac{\omega(\eta)}{\nu(\eta)})$ be positive and nondecreasing. Then,

$$\|g\|_\alpha^{(\upsilon)} \leqq \max\left(1, \frac{\omega(2L)}{\upsilon(2L)}\right)\|g\|_\alpha^{(\omega)} \leqq \infty.$$

Therefore,

$$Z_\alpha^{(\omega)} \subseteq Z_\alpha^{(\nu)} \subseteq \mathcal{L}_\alpha[0,2L] \ (\alpha \geqq 1).$$

Note that,

(i) if $\alpha \to \infty$, then the class $Z_\alpha^{(\omega)}$ reduces to the class $Z^{(\omega)}$;
(ii) if $\omega(\eta) = \eta^\upsilon$, then the class $Z_\alpha^{(\omega)}$ reduces to the class $Z_{(\upsilon),\alpha}$;
(iii) if $\omega(\eta) = \eta^\upsilon$, then the class $Z^{(\omega)}$ reduces to the class $Z_{(\upsilon)}$.

In the above scenario, the generalized Zygmund class $\mathcal{Z}_\alpha^{(\omega)}$ $(\alpha \geqq 1)$ is a generalization of $\mathcal{Z}_{(\upsilon)}$, $\mathcal{Z}_{(\upsilon),\alpha}$, and $\mathcal{Z}^{(\omega)}$ classes, which were earlier investigated by Leindler [7], Moricz [11], and Moricz and Nemeth [12]. In 2013, Lal and Shireen [6] proved the best approximation of Fourier series of a 2π-periodic function belonging to the generalized Zygmund class via matrix-Euler summability means and subsequently, Singh *et al.* [16] established the approximation of functions in the generalized Zygmund class via Hausdörff summability means. Recently, Das *et al.* [2] demonstrated the Euler–Hausdorff product summability means of a Fourier series of 2π-periodic functions for approximation of signals (functions) in the weighted Zygmund class.

Motivated by the above-mentioned investigations and developments, we first introduce and study the concepts of deferred Euler and deferred Nörlund product summability means of Fourier series of arbitrary periodic functions. We then estimate the degree of approximation of Fourier series of an arbitrary periodic function belonging to the generalized Zygmund class based upon our proposed product deferred summability means. Moreover, we highlight some important remarks in connection with our findings in the conclusion section. We also suggest a direction for future studies on this subject that are based upon the basic notion of statistical product deferred summability means of Fourier series of arbitrary periodic functions in the generalized Zygmund class.

Let $\sum u_\tau$ be an infinite series with the sequence of partial sum $\{s_\tau\}$, and let $\{p_\varrho\}$ be a sequence of nonnegative integers such that $p_0 > 0$, and

$$P_\tau = \sum_{\varrho=x_{\tau+1}}^{y_\tau} p_\varrho \to \infty \quad (\tau \to \infty),$$

where (x_τ) and (y_τ) are sequences of nonnegative integers.

Let the sequence-to-sequence transformation

$$\zeta_\tau^N = \frac{1}{P_\tau} \sum_{\varrho=x_\tau+1}^{y_\tau} p_\varrho s_\varrho \quad (\tau = 0,1,2,\ldots), \tag{1.4}$$

define the deferred Nörlund (DN, p_τ) mean of $\{s_\tau\}$ generated by $\{p_\varrho\}$. The series $\sum u_\tau$ is deferred Nörlund (DN, p_τ)-summable to s if

$$\lim_{\tau\to\infty} \zeta_\tau^{DN} = s,$$

which is regular (see [9]).

Next, the sequence-to-sequence transformation

$$DE_\tau^\theta = \frac{1}{(1+\theta)^{y_\tau}} \sum_{\varrho=x_\tau+1}^{y_\tau} \binom{y_\tau}{\varrho} \theta^{y_\tau-\varrho} s_\varrho \tag{1.5}$$

defines the deferred Euler (DE, θ) mean of the sequence $\{s_\tau\}$. The series $\sum u_\tau$ is summable to s with respect to deferred Euler (DE, θ)-mean if

$$\lim_{\tau\to\infty} DE_\tau^\theta = s,$$

which is regular (see [13]).

We now define a new transformation

$$\zeta_\tau^{DEDN} = \frac{1}{(1+\theta)^{y_\tau}} \sum_{\varrho=x_\tau+1}^{y_\tau} \binom{y_\tau}{\varrho} \theta^{y_\tau-\varrho} \left\{ \frac{1}{P_\varrho} \sum_{u=x_\tau+1}^{y_\tau} p_u s_u \right\}, \tag{1.6}$$

which defines the product of deferred Euler and deferred Nörlund $[(DE, \theta)(DN, p_\tau)]$ means of the sequence $\{s_\tau\}$. The series $\sum u_\tau$ is summable to s under the product deferred $[(DE, \theta)(DN, p_\tau)]$ mean if

$$\lim_{\tau\to\infty} \zeta_\tau^{DEDN} = s.$$

Note that (DN, p_τ)- and (DE, θ)-means are regular, so the product deferred $[(DE, \theta)(DN, p_\tau)]$ mean is also regular.

Remark 2 If we substitute $(x_\tau) = 0$ and $(y_\tau) = \tau$ in (1.6), then it yields the usual Euler–Norlund product (or $[(E, \theta)(N, p_\tau)]$) mean of the form

$$\zeta_\tau^{EN} = \frac{1}{(1+\theta)^{\tau}} \sum_{\varrho=1}^{\tau} \binom{\tau}{\varrho} \theta^{\tau-\varrho} \left\{ \frac{1}{P_\varrho} \sum_{u=1}^{\tau} p_u s_u \right\}. \tag{1.7}$$

We use the following notations throughout this paper:

$$\phi(x, \eta) = g(x+\eta) + g(x-\eta) - 2g(x)$$

and

$$K_\tau^{DEDN}(\eta) = \frac{1}{2L(1+\theta)^{y_\tau+1}} \sum_{\varrho=x_\tau+1}^{y_\tau} \binom{y_\tau}{\varrho} \theta^{y_\tau-\varrho} \left\{ \frac{1}{P_\varrho} \sum_{u=x_\tau+1}^{y_\tau} p_u \frac{\sin[\frac{(u+1/2)\pi\eta}{L}]}{2\sin(\frac{\pi\eta}{2L})} \right\}.$$

2 Auxiliary lemmas

In order to prove our main results (below), we need to establish first the following Lemmas.

Lemma 1 $|K_\tau^{DEDN}(\eta)| = O(y_\tau + 1)$ $(0 \leqq \eta \leqq \frac{1}{y_\tau+1})$.

Proof For $0 \leqq \eta \leqq \frac{1}{y_\tau+1}$, $\sin\frac{\eta}{2} \geqq \frac{\eta}{L}$ and $\sin y_\tau\eta \leqq y_\tau\eta$, we have

$$\begin{aligned}
\left|K_\tau^{DEDN}(\eta)\right| &\leqq \frac{1}{2L(1+\theta)^{y_\tau+1}}\left|\sum_{\varrho=x_\tau+1}^{y_\tau}\binom{y_\tau}{\varrho}\theta^{y_\tau-\varrho}\left\{\frac{1}{P_\varrho}\sum_{u=x_\tau+1}^{y_\tau}p_u\frac{\sin[\frac{(u+1/2)\pi\eta}{L}]}{2\sin(\frac{\pi\eta}{2L})}\right\}\right| \\
&= \frac{1}{4L(1+\theta)^{y_\tau+1}}\left|\sum_{\varrho=x_\tau+1}^{y_\tau}\binom{y_\tau}{\varrho}\theta^{y_\tau-\varrho}\left\{\frac{(\varrho+1/2)\pi\eta}{L}\cdot\frac{L^2}{\pi\eta}\frac{1}{P_\varrho}\sum_{u=x_\tau+1}^{y_\tau}p_u\right\}\right| \\
&= \frac{y_\tau+1/2}{4(1+\theta)^{y_\tau+1}}\left|\sum_{\varrho=x_\tau+1}^{y_\tau}\binom{y_\tau}{\varrho}\theta^{y_\tau-\varrho}\right| \\
&= \frac{2y_\tau+1}{8}\left[\therefore (1+\theta)^{y_\tau+1} = \sum_{\varrho=x_\tau+1}^{y_\tau}\binom{y_\tau}{\varrho}\theta^{y_\tau-\varrho}\right] \\
&= O(y_\tau). \qquad (2.1)
\end{aligned}$$

□

Lemma 2 $|K_\tau^{DEDN}(\eta)| = O(\frac{1}{y_\tau})$ $(\frac{1}{y_\tau+1} \leqq \eta \leqq L)$.

Proof For $\frac{1}{y_\tau+1} \leqq \eta \leqq L$ and $\sin y_\tau\eta \leqq y_\tau\sin\eta$, we have

$$\begin{aligned}
\left|K_\tau^{DEDN}(\eta)\right| &= \frac{1}{2L(1+\theta)^{y_\tau+1}}\left|\sum_{\varrho=x_\tau+1}^{y_\tau}\binom{y_\tau}{\varrho}\theta^{y_\tau-\varrho}\left\{\frac{1}{P_\varrho}\sum_{u=x_\tau+1}^{y_\tau}p_u\frac{\sin[\frac{(u+1/2)\pi\eta}{L}]}{2\sin(\frac{\pi\eta}{2L})}\right\}\right| \\
&\leqq \frac{1}{2\eta(1+\theta)^{y_\tau+1}}\left|\sum_{\varrho=x_\tau+1}^{y_\tau}\binom{y_\tau}{\varrho}\theta^{y_\tau-\varrho}\left\{\frac{1}{P_\varrho}\sum_{u=x_\tau+1}^{y_\tau}p_u\frac{(2u+1)\sin(\frac{\pi\eta}{2L})}{2\sin(\frac{\pi\eta}{2L})}\right\}\right| \\
&\leqq \frac{1}{2\eta(1+\theta)^{y_\tau+1}}\left|\sum_{\varrho=x_\tau+1}^{y_\tau}\binom{y_\tau}{\varrho}\theta^{y_\tau-\varrho}\left\{\frac{2\varrho+1}{P_\varrho}\sum_{u=x_\tau+1}^{y_\tau}p_u\right\}\right| \\
&\leqq \frac{2y_\tau+1}{2\eta(1+\theta)^{y_\tau+1}}\left|\sum_{\varrho=x_\tau+1}^{y_\tau}\binom{y_\tau}{\varrho}\theta^{y_\tau-\varrho}\right| \\
&= O\left(\frac{1}{y_\tau}\right). \qquad (2.2)
\end{aligned}$$

□

Lemma 3 (see [6]) *Let $g \in Z_\alpha^{(\omega)}$. Then, for $0 < \eta \leqq L$,*

(i) $\|\phi(\cdot,\eta)\|_\mu = O(\omega(\eta))$;

(ii) $\|\phi(\cdot+t,\eta)+\phi(\cdot-t,\eta)-2\phi(\cdot,\eta)\|_\alpha = \begin{cases} O(\omega(\eta)), \\ O(\omega(t)); \end{cases}$

(iii) *If ω and ν are Zygmund moduli of continuity, then*

$$\left\|\phi(\cdot+t,\eta)+\phi(\cdot-t,\eta)-2\phi(\cdot,\eta)\right\|_\alpha = O\left(u(t)\frac{\omega(\eta)}{u(\eta)}\right),$$

where

$$\phi(x,\eta) = g(x+\eta) + g(x-\eta) - 2g(x).$$

3 Main results

In this section, we state and prove two theorems (that is, Theorem 1 and Theorem 2) via our proposed product deferred (Euler–Nörlund) summability means and accordingly estimate the degree of approximation of g belonging to the generalized Zygmund class.

Theorem 1 *Let (x_τ) and $(y_\tau) \in Z^{0+}$, and let $g \in Z_\alpha^{(\omega)}$ $(\alpha \geqq 1)$ be a real-valued $2L$-periodic Lebesgue integrable function. Then, the degree of approximation of g via the product deferred $[(DE,\theta)(DN,p_\tau)]$-summability mean of Fourier series (1.1) is*

$$E_\tau(g) = \inf_{\zeta_\tau^{DEDN}} \left\| \zeta_\tau^{DEDN} - g \right\|_\alpha^u = O\left(\int_{\frac{1}{y_\tau+1}}^{L} \frac{\omega(\eta)}{\eta u(\eta)} \right), \tag{3.1}$$

where $\omega(\eta)$ and $u(\eta)$ are the same as in (1.3) with $\frac{\omega(\eta)}{u(\eta)}$ positive and increasing.

Theorem 2 *Let (x_τ) and $(y_\tau) \in Z^{0+}$, and let $g \in Z_\alpha^{(\omega)}$ $(\alpha \geqq 1)$ be a real-valued $2L$-periodic Lebesgue integrable function. Then, the degree of approximation of g via the product deferred $[(DE,\theta)(DN,p_\tau)]$-summability mean of Fourier series (1.1) is*

$$E_\tau(g) = \inf_{\zeta_\tau^{DEDN}} \left\| \zeta_\tau^{DEDN} - g \right\|_\alpha^u = O\left(\frac{\omega(\frac{1}{y_\tau+1})}{(y_\tau+1)^2 u(\frac{1}{y_\tau+1})} \big(L(y_\tau+1)-1\big) \right), \tag{3.2}$$

where $w(\eta)$ and $u(\eta)$ are the same as in (1.3) with $\frac{w(\eta)}{\eta u(\eta)}$ positive and decreasing.

4 Proof of Theorem 1

Let $s_\varrho(g;x)$ denote the ϱth partial sum and following [17], we have

$$s_\varrho(g;x) - g(x) = \frac{1}{2L} \int_0^L \phi(x;\eta) \frac{\sin[\frac{(\varrho+1/2)\pi\eta}{L}]}{2\sin(\frac{\pi\eta}{2L})}\, d\eta. \tag{4.1}$$

Therefore, under (1.4), the (DN,p_τ) transform of $s_\varrho(g;x)$ is given by

$$\frac{1}{P_\tau} \sum_{\varrho=x_\tau+1}^{y_\tau} p_\varrho \big[s_\varrho(g;x) - g(x) \big] = \frac{1}{2L} \int_0^L \phi(x;\eta) \frac{1}{P_\tau} \sum_{\varrho=x_\tau+1}^{y_\tau} p_\varrho \frac{\sin[\frac{(\varrho+1/2)\pi\eta}{L}]}{2\sin(\frac{\pi\eta}{2L})}\, d\eta. \tag{4.2}$$

Furthermore, for the product Euler–Norlund (or $[(DE,\theta)(DN,p_\tau)]$) transform of $s_\varrho(g;x)$, we obtain

$$\begin{aligned} &\zeta_\tau^{DEDN} - g(x) \\ &\quad = \frac{1}{\pi(1+\theta)^{y_\tau+1}} \sum_{\varrho=x_\tau+1}^{y_\tau} \int_0^L \phi(x;\eta) \binom{y_\tau}{\varrho} \theta^{y_\tau-\varrho} \left\{ \frac{1}{P_\varrho} \sum_{u=x_\tau+1}^{y_\tau} p_u \frac{\sin[\frac{(u+1/2)\pi\eta}{L}]}{2\sin(\frac{\pi\eta}{2L})} \right\} d\eta, \end{aligned} \tag{4.3}$$

which implies that

$$\begin{aligned}\mathfrak{L}_{y_\tau}(x) &= \zeta_\tau^{DEDN} - g(x)\\ &= \int_0^L \phi(x;\eta)K_\tau^{DEDN}(\eta)\,d\eta.\end{aligned} \tag{4.4}$$

Then,

$$\begin{aligned}&\mathfrak{L}_{y_\tau}(x+t) + \mathfrak{L}_{y_\tau}(x-t) - 2\mathfrak{L}_{y_\tau}(x)\\ &\quad= \int_0^L \bigl[\phi(x+t;\eta) + \phi(x-t;\eta) - 2\phi(x;\eta)\bigr]K_\tau^{DEDN}(\eta)\,d\eta.\end{aligned} \tag{4.5}$$

By using the generalized Minkowski's inequality to equation (4.5), we obtain

$$\begin{aligned}&\bigl\|\mathfrak{L}_{y_\tau}(\cdot+t) + \mathfrak{L}_{y_\tau}(\cdot-t) - 2\mathfrak{L}_{y_\tau}(\cdot)\bigr\|_\alpha\\ &\quad= \left\{\frac{1}{2L}\int_0^{2L}\bigl|\mathfrak{L}_{y_\tau}(x+t) + \mathfrak{L}_{y_\tau}(x-t) - 2\mathfrak{L}_{y_\tau}(x)\bigr|^\alpha\,dx\right\}^{1/\alpha}\\ &\quad= \left\{\frac{1}{2L}\int_0^{2L}\left|\int_0^L \bigl[\phi(x+t;\eta) + \phi(x-t;\eta) - 2\phi(x;\eta)\bigr]K_\tau^{DEDN}(\eta)\,d\eta\right|^\alpha d\eta\right\}^{1/\alpha}\\ &\quad= \int_0^L \left\{\frac{1}{2L}\int_0^{2L}\bigl|\bigl[\phi(x+t;\eta) + \phi(x-t;\eta) - 2\phi(x;\eta)\bigr]K_\tau^{DEDN}(\eta)\,d\eta\bigr|^\alpha\right\}^{1/\alpha} d\eta\\ &\quad= \int_0^L \bigl(\bigl|K_\tau^{DEDN}(\eta)\bigr|\bigr)^{1/\alpha}\\ &\qquad\times \left\{\frac{1}{2L}\int_0^{2L}\bigl|\bigl[\phi(x+t;\eta) + \phi(x-t;\eta) - 2\phi(x;\eta)\bigr]K_\tau^{DEDN}(\eta)\bigr|^\alpha\,d\eta\right\}^{1/\alpha} d\eta\\ &\quad= \int_0^L \bigl\|\phi(\cdot+t;\eta) + \phi(\cdot-t;\eta) - 2\phi(\cdot;\eta)\bigr\|_\alpha \bigl|K_\tau^{DEDN}(\eta)\bigr|\,d\eta\\ &\quad= \int_0^{\frac{1}{y_\tau+1}} \bigl\|\phi(\cdot+t;\eta) + \phi(\cdot-t;\eta) - 2\phi(\cdot;\eta)\bigr\|_\alpha \bigl|K_\tau^{DEDN}(\eta)\bigr|\,d\eta\\ &\qquad+ \int_{\frac{1}{y_\tau+1}}^L \bigl\|\phi(\cdot+t;\eta) + \phi(\cdot-t;\eta) - 2\phi(\cdot;\eta)\bigr\|_\alpha \bigl|K_\tau^{DEDN}(\eta)\bigr|\,d\eta.\\ &\quad= I_1 + I_2 \quad \text{(say)}.\end{aligned} \tag{4.6}$$

Clearly, by Lemma 1, Lemma 3, and the monotonicity of $(\omega(\eta)/u(\eta))$ with respect to η, it yields

$$\begin{aligned}I_1 &= \int_0^{\frac{1}{y_\tau+1}} \bigl\|\phi(\cdot+t;\eta) + \phi(\cdot-t;\eta) - 2\phi(\cdot;\eta)\bigr\|_\alpha \bigl|K_\tau^{DEDN}(\eta)\bigr|\,d\eta\\ &= \int_0^{\frac{1}{y_\tau+1}} O\left(u(t)\frac{\omega(\eta)}{u(\eta)}\right)O(y_\tau)\,d\eta\\ &\leqq O\left(\tau u(t)\int_0^{\frac{1}{y_\tau+1}} \frac{\omega(\eta)}{u(\eta)}\,d\eta\right),\end{aligned}$$

and by using the 2nd mean-value theorem of integral, we have

$$\begin{aligned} I_1 &\leqq O\left(y_\tau u(t)\frac{\omega(\frac{1}{y_\tau+1})}{u(\frac{1}{y_\tau+1})}\int_0^{\frac{1}{y_\tau+1}} d\eta\right) \\ &= O\left(\frac{y_\tau}{y_\tau+1}u(t)\frac{\omega(\frac{1}{y_\tau+1})}{u(\frac{1}{y_\tau+1})}\right) \\ &= O\left(u(t)\frac{\omega(\frac{1}{y_\tau+1})}{u(\frac{1}{y_\tau+1})}\right)\left[\because \frac{y_\tau}{y_\tau+1} = O(1)\right]. \end{aligned} \tag{4.7}$$

Furthermore, by using Lemma 2 and Lemma 3, we obtain

$$\begin{aligned} I_2 &= \int_{\frac{1}{y_\tau+1}}^{L} \left\|\phi(\cdot+t;\eta)+\phi(\cdot-t;\eta)-2\phi(\cdot;\eta)\right\|_\alpha \left|K_\tau^{DEDN}(\eta)\right| d\eta \\ &\leqq \left(\int_{\frac{1}{y_\tau+1}}^{L} u(t)\frac{\omega(\eta)}{u(\eta)}\frac{1}{\eta}\, d\eta\right) \\ &= O\left(u(t)\int_{\frac{1}{y_\tau+1}}^{L}\frac{\omega(\eta)}{\eta u(\eta)}\, d\eta\right). \end{aligned} \tag{4.8}$$

Now, by (4.6), (4.7), and (4.8), we obtain

$$\begin{aligned} &\left\|\mathfrak{L}_{y_\tau}(\cdot+t)+\mathfrak{L}_{y_\tau}(\cdot-t)-2\mathfrak{L}_{y_\tau}(\cdot)\right\|_\alpha \\ &\quad = O\left(u(t)\frac{\omega(\frac{1}{y_\tau+1})}{u(\frac{1}{y_\tau+1})}\right)+O\left(u(t)\int_{\frac{1}{y_\tau+1}}^{L}\frac{\omega(\eta)}{\eta u(\eta)}\, d\eta\right). \end{aligned} \tag{4.9}$$

Thus,

$$\sup_{t\neq 0}\frac{\|\mathfrak{L}_{y_\tau}(\cdot+t)+\mathfrak{L}_{y_\tau}(\cdot-t)-2\mathfrak{L}_{y_\tau}(\cdot)\|_\alpha}{u(t)} = O\left(\frac{\omega(\frac{1}{y_\tau+1})}{u(\frac{1}{y_\tau+1})}\right)+O\left(\int_{\frac{1}{y_\tau+1}}^{L}\frac{\omega(\eta)}{\eta u(\eta)}\, d\eta\right). \tag{4.10}$$

Clearly,

$$\phi(x,\eta) = \left|g(x+\eta)+g(x-\eta)-2g(x)\right|. \tag{4.11}$$

Now, using Minkowski's inequality in (4.11), we obtain

$$\begin{aligned} \left\|\phi(\cdot,\eta)\right\|_\alpha &= \left\|g(x+\eta)+g(x-\eta)-2g(x)\right\|_\alpha \\ &= O\big(\omega(\eta)\big). \end{aligned} \tag{4.12}$$

Using Lemma 1, Lemma 2, (4.12), and (4.4), we obtain

$$\begin{aligned} \left\|\mathfrak{L}_{y_\tau}(\cdot)\right\|_\alpha &\leqq \left(\int_0^{\frac{1}{y_\tau+1}}+\int_{\frac{1}{y_\tau+1}}^{L}\right)\left\|\phi(\cdot,\eta)\right\|_\alpha \left|K_{y_\tau}^{DEDN}(\eta)\right| d\eta \\ &= O\left(y_\tau\int_0^{\frac{1}{y_\tau+1}}\omega(\eta)\, d\eta\right)+O\left(\frac{1}{y_\tau}\int_{\frac{1}{y_\tau+1}}^{L}\frac{\omega(\eta)}{\eta}\, d\eta\right) \end{aligned}$$

$$
\begin{aligned}
&= O\left(y_\tau \omega\left(\frac{1}{y_\tau+1}\right)\int_0^{\frac{1}{y_\tau+1}} \omega(\eta)\,d\eta\right) + O\left(\frac{1}{y_\tau}\int_{\frac{1}{y_\tau+1}}^{\pi} \frac{w(\eta)}{\eta}\,d\eta\right)\\
&= O\left(\frac{y_\tau}{y_\tau+1}\omega\left(\frac{1}{y_\tau+1}\right)\right) + O\left(\frac{1}{y_\tau}\int_{\frac{1}{y_\tau+1}}^{L} \frac{\omega(\eta)}{\eta}\,d\eta\right)\\
&= O\left(\omega\left(\frac{1}{y_\tau+1}\right)\right) + O\left(\frac{1}{y_\tau}\int_{\frac{1}{y_\tau+1}}^{\pi} \frac{\omega(\eta)}{\eta}\,d\eta\right).
\end{aligned}
\tag{4.13}
$$

Next, by using (4.10) and (4.13), we have

$$
\begin{aligned}
\left\|\mathfrak{L}_{y_\tau}(\cdot)\right\|_\alpha^u &= \left\|\mathfrak{L}_{y_\tau}(\cdot)\right\|_\alpha + \sup_{t\neq 0}\frac{\|\mathfrak{L}_{y_\tau}(\cdot+t)+\mathfrak{L}_{y_\tau}(\cdot-t)-2\mathfrak{L}_{y_\tau}(\cdot)\|_\alpha}{u(t)}\\
&= O\left(\omega\left(\frac{1}{y_\tau+1}\right)\right) + O\left(\frac{1}{y_\tau}\int_{\frac{1}{y_\tau+1}}^{L} \frac{\omega(\eta)}{\eta}\,d\eta\right)\\
&\quad + O\left(\frac{\omega(\frac{1}{y_\tau+1})}{u(\frac{1}{y_\tau+1})}\right) + O\left(\int_{\frac{1}{y_\tau+1}}^{\pi} \frac{\omega(\eta)}{\eta u(\eta)}\,d\eta\right).\\
&= \sum_{i=1}^{4} J_i.
\end{aligned}
\tag{4.14}
$$

We now express J_1 in terms of J_3 and further J_2, J_3 in terms of J_4, and by using these facts, we have

$$
\omega(\eta) = \frac{\omega(\eta)}{u(\eta)}.u(\eta) \leqq u(\pi)\frac{\omega(\eta)}{u(\eta)}.u(\eta) = O\left(\frac{\omega(\eta)}{u(\eta)}\right) \quad \left(\because u(\eta) \text{ for } 0<\eta\leqq L\right).
$$

Therefore,

$$
J_1 = O(J_3). \tag{4.15}
$$

Again, by using the monotonicity of $u(\eta)$, we obtain

$$
\begin{aligned}
J_2 &= \int_{\frac{1}{y_\tau+1}}^{L} \frac{\omega(\eta)}{\eta}\,d\eta\\
&= \int_{\frac{1}{y_\tau+1}}^{L} \frac{\omega(\eta)}{\eta u(\eta)}u(\eta)\,d\eta\\
&\leqq u(\pi)\int_{\frac{1}{y_\tau+1}}^{L} \frac{\omega(\eta)}{\eta u(\eta)}\,d\eta\\
&= O(J_4).
\end{aligned}
\tag{4.16}
$$

Also, using the fact that $\frac{\omega(\eta)}{u(\eta)}$ is positive and increasing, it yields

$$
J_4 = \int_{\frac{1}{y_\tau+1}}^{L} \frac{\omega(\eta)}{\eta u(\eta)}\,d\eta = \frac{\omega(\frac{1}{y_\tau+1})}{u(\frac{1}{y_\tau+1})}\int_{\frac{1}{y_\tau+1}}^{L} \frac{d\eta}{\eta} \geqq \frac{\omega(\frac{1}{y_\tau+1})}{u(\frac{1}{y_\tau+1})} \tag{4.17}
$$

and

$$J_3 = O(J_4). \tag{4.18}$$

Now, combining (4.14) with (4.15) to (4.18), we have

$$\left\| \mathfrak{L}_{y_\tau}(\cdot) \right\|_\alpha^u = O(J_4) = O\left(\int_{\frac{1}{y_\tau+1}}^{L} \frac{\omega(\eta)}{\eta u(\eta)} \right). \tag{4.19}$$

Thus,

$$E_\tau(g) = \inf_{y_\tau} \left\| \mathfrak{L}_{y_\tau}(\cdot) \right\|_\alpha^u = O\left(\int_{\frac{1}{y_\tau+1}}^{L} \frac{\omega(\eta)}{\eta u(\eta)} \right). \tag{4.20}$$

5 Proof of Theorem 2

Following the proof of Theorem 1, it yields

$$E_\tau(g) = \inf_{y_\tau} \left\| \mathfrak{L}_{y_\tau}(\cdot) \right\|_\alpha^u = O\left(\int_{\frac{1}{y_\tau+1}}^{L} \frac{\omega(\eta)}{\eta u(\eta)} \, d\eta \right). \tag{5.1}$$

In Theorem 2, let us assume $\frac{\omega(\eta)}{\eta v(\eta)}$ is positive and decreasing. Thus, we have

$$\begin{aligned} E_\tau(g) = \inf_{y_\tau} \left\| \mathfrak{L}_{y_\tau}(\cdot) \right\|_\alpha^u &= O\left(\frac{\omega(\frac{1}{y_\tau+1})}{(y_\tau+1)v(\frac{1}{\tau+1})} \int_{\frac{1}{\tau+1}}^{L} d\eta \right) \\ &= O\left(\frac{\omega(\frac{1}{\tau+1})}{(y_\tau+1)v(\frac{1}{y_\tau+1})} \cdot [\eta]_{\frac{1}{y_\tau+1}}^{L} \right) \\ &= O\left(\frac{\omega(\frac{1}{y_\tau+1})}{(y_\tau+1)^2 u(\frac{1}{y_\tau+1})} \big(L(y_\tau+1) - 1 \big) \right). \end{aligned} \tag{5.2}$$

6 Concluding remarks and observations

In the concluding section of the investigation, we further observe some special cases in view of our main results, that is, Theorem 1 and Theorem 2.

Remark 3 Let $g \in \mathcal{Z}_\alpha^{(\omega)}$ be a real-valued $2L$-periodic Lebesgue integrable function. If we substitute $(x_\tau) = 0$ and $(y_\tau) = \tau$ in Theorem 1, then the degree of approximation of g via the usual product $[(E,\theta)(N,p_\tau)]$-summability means of Fourier series (1.1) is given by

$$E_\tau(g) = \inf_{\zeta_\tau^{EN}} \left\| \zeta_\tau^{EN} - g \right\|_\alpha^u = O\left(\int_{\frac{1}{\tau+1}}^{L} \frac{\omega(\eta)}{\eta u(\eta)} \right), \tag{6.1}$$

where $\frac{\omega(\eta)}{u(\eta)}$ is positive and increasing.

Remark 4 Let $g \in \mathcal{Z}_\alpha^{(\omega)}$ be a real-valued $2L$-periodic Lebesgue integrable function. If we substitute $(x_\tau) = 0$ and $(y_\tau) = \tau$ in Theorem 2, then the degree of approximation of g

via the usual product $[(E,\theta)(N,p_\tau)]$-summability means of Fourier series (1.1) is given by

$$E_\tau(g) = \inf_{\zeta_\tau^{EN}} \left\| \zeta_\tau^{EN} - g \right\|_\alpha^u = O\left(\frac{\omega(\frac{1}{\tau+1})}{(\tau+1)^2 u(\frac{1}{\tau+1})} \left[L(\tau+1) - 1 \right] \right), \tag{6.2}$$

where $\frac{w(\eta)}{\eta u(\eta)}$ is positive and decreasing.

Remark 5 Motivated by a recently published result by Jena *et al.* [3], the interested readers' attention is drawn towards the possibility of investigating the basic notion of statistical product deferred summability means of Fourier series in the generalized Zygmund class.

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The authors declare that they have no competing interests.

Author contribution
BBJ proposed the idea and initiated the writing of the manuscript. SKP analyzed all the results, made necessary improvements, and supervised writing the manuscript. MM followed this with some complementary ideas and methodologies. All the authors read and approved the final manuscript.

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