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On a class of obstacle problem for Hessian equations on Riemannian manifolds

Jinxuan Liu[1*] and Yong Wang[1]

*Correspondence:
16b912021@stu.hit.edu.cn;
liujinxuannikita@163.com
[1]School of Mathematics, Harbin Institute of Technology, Harbin, 150001, China

Abstract

In this paper, we establish the *a priori* C^2 estimates for solutions of a class of obstacle problem for Hessian equations on Riemannian manifolds. Some applications are also discussed. The main contribution of this paper is the boundary estimates for second-order derivatives.

Keywords: Obstacle problem; Hessian equations; Second-order estimates

1 Introduction

Let $(\overline{M}, g)$ be a compact manifold with smooth boundary ∂M. In this paper, we are concerned with the obstacle problem

$$\max\{u-\phi, -(f(\lambda(\nabla^2 u+\chi))-\psi(x,u,\nabla u))\}=0 \quad \text{in } M \tag{1.1}$$

with the boundary condition

$$u=\varphi \quad \text{on } \partial M, \tag{1.2}$$

where f is a smooth, symmetric function defined in an open convex cone $\Gamma \subset \mathbb{R}^n$ with a vertex at the origin and

$$\Gamma_n = \{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n : \text{ each } \lambda_i>0\} \subseteq \Gamma \neq \mathbb{R}^n,$$

$\nabla^2 u$ denotes the Hessian of u, χ is a $(0,2)$-tensor field, $\lambda(h)$ denotes the eigenvalues of a $(0,2)$-tensor field h with respect to the metric g and $\varphi \in C^4(\partial M)$. In this work, we assume the obstacle function $\phi \in C^3(\overline{M})$ satisfies $\phi = \varphi$ on ∂M.

We shall use a penalization technique to establish the *a priori* C^2 estimates for a singular perturbation problem (see (2.1)). A similar problem was studied in [14] and [1], where the obstacle function ϕ is assumed to satisfy $\phi > \varphi$ on ∂M so that near the boundary ∂M, the solution of (2.1) satisfies the Hessian-type equation

$$f(\lambda(\nabla^2 u+\chi)) = \psi(x,u,\nabla u) \tag{1.3}$$

and the second-order boundary estimates follow from studies on Hessian-type equations (see [6], [9], and [10] for examples). In the current paper the obstacle function ϕ is allowed to equal φ on the boundary so that the main difficulty is from the boundary estimates for second-order derivatives.

As in [3], we suppose the function $f \in C^2(\Gamma) \cap C^0(\overline{\Gamma})$ satisfies the structure conditions:

$$f_i = f_{\lambda_i} \equiv \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{in } \Gamma, 1 \le i \le n, \tag{1.4}$$

$$f \text{ is concave in } \Gamma, \tag{1.5}$$

and

$$\begin{cases} f > 0 & \text{in } \Gamma, \\ f = 0 & \text{on } \partial\Gamma. \end{cases} \tag{1.6}$$

In addition, f is also assumed to satisfy that for any positive constants μ_1, μ_2 with $0 < \mu_1 < \mu_2 < \sup_\Gamma f$ there exists a positive constant c_0 depending on μ_1 and μ_2 such that

$$\bigl(f_1(\lambda) \cdot \dots \cdot f_n(\lambda)\bigr)^{1/n} \ge c_0 \tag{1.7}$$

for any $\lambda \in \Gamma_{\mu_1,\mu_2} := \{\lambda \in \Gamma : \mu_1 \le f(\lambda) \le \mu_2\}$ and

$$f_i(\lambda) \ge c_1 \Bigl(1 + \sum_j f_j\Bigr) \quad \text{for any } \lambda \in \Gamma \text{ with } \lambda_i < 0. \tag{1.8}$$

Furthermore, f is supposed to satisfy that for any $A > 0$ and any compact set $K \subset \Gamma$, there exists $R = R(A, K) > 0$ such that

$$f(\lambda_1, \dots, \lambda_{n-1}, \lambda_n + R) \ge A, \quad \text{for all } \lambda \in K \tag{1.9}$$

and

$$f(R\lambda) \ge A, \quad \text{for all } \lambda \in K. \tag{1.10}$$

Following [3], we assume that there exists a large number $R > 0$ such that at each $x \in \partial M$,

$$\bigl(\kappa_1(x), \dots, \kappa_{n-1}(x), R\bigr) \in \Gamma, \tag{1.11}$$

where $(\kappa_1(x), \dots, \kappa_{n-1}(x))$ are the principal curvatures of ∂M at x (relative to the interior normal). Since the function ψ may depend on ∇u, we assume there exists an admissible subsolution $\underline{u} \in C^2(\overline{M})$ satisfying

$$\begin{cases} f(\lambda(\nabla^2 \underline{u} + \chi)) \ge \psi(x, \underline{u}, \nabla \underline{u}) & \text{in } M, \\ \underline{u} = \varphi & \text{on } \partial M, \\ \underline{u} \le \phi & \text{in } M. \end{cases} \tag{1.12}$$

As in [6], the function $\psi(x,z,p) \in C^2(T^*\overline{M} \times \mathbb{R}) > 0$ satisfies

$$\psi(x,z,p) \text{ is convex in } p, \tag{1.13}$$

$$\sup_{(x,z,p)\in T^*\overline{M}\times\mathbb{R}} \frac{-\psi_z(x,z,p)}{\psi(x,z,p)} < \infty \tag{1.14}$$

and the growth condition

$$\begin{aligned} &p \cdot \nabla_p \psi(x,z,p) \le \bar{\psi}(x,z)\big(1 + |p|^{\gamma_1}\big), \\ &p \cdot \nabla_x \psi(x,z,p) + |p|^2 \psi_z(x,z,p) \ge \bar{\psi}(x,z)\big(1 + |p|^{\gamma_2}\big), \end{aligned} \tag{1.15}$$

when $|p|$ is sufficiently large, where $\gamma_1 < 2$, $\gamma_2 < 4$ are positive constants and $\bar{\psi}$ is a positive-continuous function of $(x,z) \in \overline{\Omega} \times \mathbb{R}$.

Definition 1.1 A function $u \in C^2(M)$ is called admissible if $\lambda(\nabla^2 u + \chi) \in \Gamma$ in Ω.

Our main results are stated as follows.

Theorem 1.2 *Suppose f satisfies (1.4)–(1.11) and there exists an admissible subsolution $\underline{u} \in C^2(\overline{M})$ satisfying (1.12). Assume that $\psi > 0$ satisfies (1.13)–(1.15), $\varphi \in C^4(\partial M)$, ϕ is admissible in M and $\phi = \varphi$ on ∂M. Then, there exists an admissible solution $u \in C^{1,1}(\overline{M})$ of (1.1) and (1.2).*

Furthermore, $u \in C^{3,\alpha}(E)$ for any $\alpha \in (0,1)$ and the Hessian equation (1.3) holds in E, where $E := \{x \in M : u(x) < \phi(x)\}$.

Note that in Theorem 1.2, the function ϕ is assumed to be admissible. Under the homogeneous boundary condition, i.e., $\varphi \equiv 0$, and that $\chi \equiv 0$, we can remove this assumption.

Theorem 1.3 *Assume that $\chi \equiv 0$ in (1.1). Suppose (1.4)–(1.11) and there exists an admissible subsolution $\underline{u} \in C^2(\overline{M})$ satisfying (1.12) with $\varphi \equiv 0$. Assume that $\psi > 0$ satisfies (1.13)–(1.15), $\varphi \equiv 0$ and $\phi \equiv 0$ on ∂M. Then, there exists an admissible solution $u \in C^{1,1}(\overline{M})$ of (1.1) and (1.2) and $u \in C^{3,\alpha}(E)$ for any $\alpha \in (0,1)$ and satisfies (1.3) in E.*

Typical examples are given by $f = \sigma_k^{1/k}$, $1 \le k \le n$, defined on the cone $\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1,\dots,k\}$, where $\sigma_k(\lambda)$ are the elementary symmetric functions

$$\sigma_k(\lambda) = \sum_{i_1<\cdots<i_k} \lambda_{i_1}\dots\lambda_{i_k}, \quad k = 1,\dots,n. \tag{1.16}$$

Other interesting examples satisfying (1.4)–(1.11) (see [13]) are

$$f(\lambda) = \sigma_k^{1/k}(\mu_1,\dots,\mu_n), \tag{1.17}$$

defined on the cone $\Gamma = \{\lambda \in \mathbb{R}^n : (\mu_1,\dots,\mu_n) \in \Gamma_k\}$, where μ_i are defined by

$$\mu_i = \sum_{j\ne i} \lambda_j, \quad i = 1,\dots,n.$$

It is an interesting question whether we can establish the *a priori* second-order estimates without the condition (1.13). We note that such a condition is necessary in general (see [11]). It is a longstanding problem of the global C^2 estimates for the k-Hessian equation

$$\sigma_k\big(\lambda\big(D^2u\big)\big) = \psi(x,u,Du)$$

dropping the condition (1.13). The cases $k = 2$, $k = n-1$, and $k = n-2$ were resolved by Guan–Ren–Wang [11], Ren–Wang [20], and Ren–Wang [21], respectively. It is still open for general k. Chu–Jiao [5] considered the case (1.17) and established the curvature estimates without the condition (1.13). Jiao–Liu [13] studied the corresponding Dirichlet problem. It is of interest to ask if the above methods can be applied to the related obstacle problem (1.1).

Given a function $v : \Omega \to \mathbb{R}$, denote $M_v := \{(x, v(x)) : x \in \Omega\}$ to be the graphic hypersurface defined by v. Then, the Gauss curvature of M_v is

$$K(M_v) = \frac{\det D^2v}{(1+|Dv|^2)^{(n+2)/2}}.$$

A classic problem in differential geometry is to find a convex graphic hypersurface with prescribed Gauss curvature K that is equivalent to solving a Monge–Ampère equation

$$\det D^2u = K(x,u)\big(1+|Du|^2\big)^{(n+2)/2}. \tag{1.18}$$

It is also of interest to find hypersurfaces having prescribed Gauss curvature under an obstacle. Such a problem is also equivalent to an obstacle for Monge–Ampère equations. Xiong–Bao [25] proved the $C^{1,1}$ regularity under the condition that the obstacle function is strictly larger than the boundary data. A similar question can be asked if the Gauss curvature is replaced with other kinds of curvatures, such as the mean curvature [4]. The following two theorems can be regarded as applications of Theorem 1.2 and Theorem 1.3.

Theorem 1.4 *Let Ω be a uniformly convex bounded domain in $\mathbb{R}^n$. Given a function $K(x,z) \in C^2(\overline{\Omega}\times\mathbb{R}) > 0$ satisfying that there exists a positive constant A such that*

$$K_z(x,z) \geq -AK(x,z), \quad \textit{for all } (x,z) \in \overline{\Omega}\times\mathbb{R} \tag{1.19}$$

and a piece of uniformly convex graphic hypersurface M_ϕ, suppose there exists a uniformly convex graphic hypersurface $M_{\underline{u}}$ under M_ϕ satisfying the Gauss curvature of $M_{\underline{u}}$,

$$K(M_{\underline{u}})\big(x,\underline{u}(x)\big) \geq K\big(x,\underline{u}(x)\big) \quad \textit{for } x \in \overline{\Omega} \tag{1.20}$$

and $\underline{u} = \phi$ on $\partial\Omega$. Then, there exists a $C^{1,1}$ graphic hypersurface M_u under M_ϕ with the same boundary such that $K(M_u) \geq K(x,u)$ in Ω and $K(M_u) = K(x,u)$ in $E := \{x \in \Omega : u(x) < \phi(x)\}$.

Theorem 1.5 *Suppose $K(x,z) \in C^2(\overline{\Omega}\times\mathbb{R}) > 0$ satisfying (1.19). The graphic hypersurface M_ϕ is of constant boundary, suppose there exist a uniformly convex graphic hypersurface $M_{\underline{u}}$ under M_ϕ satisfying (1.20) and $\underline{u} = \phi$ on $\partial\Omega$. Then, there exists a $C^{1,1}$ graphic hypersurface M_u under M_ϕ with the same constant boundary such that $K(M_u) \geq K(x,u)$ in Ω and $K(M_u) = K(x,u)$ in E.*

Other applications of the obstacle problem for Hessian equations can be found in [2], [4], [15], [19], [22], and so on. The reader is referred to [1] for more applications and background of (1.1).

Similar problems were studied in [14], [1], and [12] under various conditions. In this work, we are mainly concerned with the boundary estimates for second-order estimates. The main difficulty is from the existence of a disturbance term β_ϵ in (2.1). It is also why the conditions (1.9)–(1.11) are needed.

The obstacle problem for Monge–Ampère equations (when $f = \sigma_n^{1/n}$) was studied extensively, see [2], [16], [17], [22], and [25] for examples. For the obstacle problem of Hessian equations on Riemannian manifolds, the reader is referred to [1], [12], and [14]. We refer the reader to [6], [8], [10], [18], and [24] for the study of Hessian-type equations on Riemannian manifolds.

In Sect. 2, we provide the general idea to prove Theorems 1.2 and 1.3 for which we introduce an approximating problem using a penalization technique. Section 3 is devoted to the boundary estimates for second-order estimates for the solution of the approximating problem.

2 Preliminaries

As in [14] and [25], we consider the singular perturbation problem

$$\begin{cases} f(\lambda(\nabla^2 u + \chi) = \psi(x, u, \nabla u) + \beta_\epsilon(u - \phi) & \text{in } M, \\ u = \varphi & \text{on } \partial M, \end{cases} \tag{2.1}$$

where the penalty function β_ϵ is defined by

$$\beta_\epsilon(z) = \begin{cases} 0, & z \le 0, \\ z^3/\epsilon, & z > 0, \end{cases}$$

for $\epsilon \in (0, 1)$. Obviously, $\beta_\epsilon \in C^2(\mathbb{R})$ satisfies

$$\begin{aligned} &\beta_\epsilon, \beta_\epsilon', \beta_\epsilon'' \ge 0; \\ &\beta_\epsilon(z) \to \infty \quad \text{as } \epsilon \to 0^+, \text{ whenever } z > 0; \\ &\beta_\epsilon(z) = 0, \quad \text{whenever } z \le 0. \end{aligned} \tag{2.2}$$

Since $\underline{u} \le \phi$, $\underline{u}$ is also a subsolution to (2.1). Let $u_\epsilon \in C^3(\overline{M}) \cap C^4(M)$ be an admissible solution of (2.1) with $u_\epsilon \ge \underline{u}$. We shall show that there exists a constant C independent of ϵ such that

$$\|u_\epsilon\|_{C^2(\overline{M})} \le C \tag{2.3}$$

for small ϵ.

The C^0 estimates can be easily derived from the fact that $\Gamma \subset \Gamma_1$ and $u \ge \underline{u}$. The following lemma is crucial for our estimates, and its proof can be found in [1] (see [25] for the case of the Monge–Ampère equation). For completeness, we provide a proof here.

Lemma 2.1 *There exists a positive constant c_2 independent of ϵ such that*

$$0 \le \beta_\epsilon(u_\epsilon - \phi) \le c_2 \quad on\ \overline{M}. \tag{2.4}$$

Proof We consider the maximal value of $u_\epsilon - \phi$ on $\overline{M}$. We may assume it is achieved at an interior point $x_0 \in M$ since $u_\epsilon - \phi = \varphi - \phi = 0$ on ∂M. We have, at x_0,

$$\nabla(u_\epsilon - \phi) = 0 \tag{2.5}$$

and

$$\nabla^2 u_\epsilon \le \nabla^2 \phi. \tag{2.6}$$

It follows that, at x_0,

$$\begin{aligned} 0 \le \beta_\epsilon(u_\epsilon - \phi) &= f\big(\lambda\big(\nabla^2 u_\epsilon + \chi\big)\big) - \psi(x, u, \nabla\phi) \\ &\le f\big(\lambda\big(\nabla^2 \phi + \chi\big)\big) - \psi(x, u, \nabla\phi) \le c_2 \end{aligned}$$

for some positive constant c_2 depending only on $\|\phi\|_{C^2(\overline{M})}$ and (2.4) holds. □

After establishing the estimate (2.3), we can find a subsequence u_{ϵ_k} and a function $u \in C^{1,1}(\overline{\Omega})$ such that

$$u_{\epsilon_k} \to u \quad \text{in } C^{1,\alpha}(\bar{M}), \forall\alpha \in (0,1), \text{ as } \epsilon_k \to 0.$$

Then, we see u is an admissible solution of (1.1) and (1.2) as in [25]. The fact that $u \in C^{3,\alpha}(E)$ and satisfies (1.3) in E follows from the Evans–Krylov theory.

The C^1 bound under conditions (1.8) and (1.15) was derived in [14]. It was also shown in [14] how to establish the estimates for second-order derivatives from their bound on the boundary. This paper will focus on the estimates for second-order estimates on the boundary.

Let $u \in C^4(\overline{M})$ be an admissible function. For simplicity we shall use the notation $U = \chi + \nabla^2 u$ and, under an orthonormal local frame $e_1, \dots, e_n$,

$$\begin{aligned} U_{ij} &\equiv U(e_i, e_j) = \chi_{ij} + \nabla_{ij} u, \\ \nabla_k U_{ij} &\equiv \nabla U(e_i, e_j, e_k) = \nabla_k \chi_{ij} + \nabla_{kij} u. \end{aligned} \tag{2.7}$$

Let F be the function defined by

$$F(h) = f\big(\lambda(h)\big)$$

for a $(0,2)$-tensor h on M. Equation (2.1) is therefore written in the form

$$F(U) = \psi(x, u, \nabla u) + \beta_\epsilon(u - \phi). \tag{2.8}$$

Following the literature we denote throughout this paper

$$F^{ij} = \frac{\partial F}{\partial h_{ij}}(U), \qquad F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij}\partial h_{kl}}(U)$$

under an orthonormal local frame $e_1, \ldots, e_n$. The matrix $\{F^{ij}\}$ has eigenvalues $f_1, \ldots, f_n$ and is positive-definite by assumption (1.4), while (1.5) implies that F is a concave function of U_{ij} (see [3]). Moreover, when U_{ij} is diagonal so is $\{F^{ij}\}$. We can derive from (1.4)–(1.6) that

$$\sum_i f_i(\lambda)\lambda_i \geq 0 \quad \text{for any } \lambda \in \Gamma. \tag{2.9}$$

We need the following lemmas that were proved in [7].

Lemma 2.2 *Let $A = \{A_{ij}\} \in \mathcal{S}^{n\times n}$ with $\lambda(A) = (\lambda_1, \ldots, \lambda_n) \in \Gamma$ and $F^{ij} = \frac{\partial F(A)}{\partial A_{ij}}$ with eigenvalues $f_1, \ldots, f_n$, where $\mathcal{S}^{n\times n}$ is the space of all symmetric matrices. There exists an index r such that*

$$\sum_{\beta\leq n-1} F^{ij}A_{i\beta}A_{\beta j} \geq \frac{1}{2}\sum_{i\neq r} f_i\lambda_i^2. \tag{2.10}$$

Lemma 2.3 *For any index r and $\epsilon > 0$, there exists a positive constant C depending only on n such that*

$$\sum f_i|\lambda_i| \leq \epsilon \sum_{i\neq r} f_i\lambda_i^2 + \frac{C}{\epsilon}\sum f_i + Q(r), \tag{2.11}$$

where $Q(r) = f(\lambda) - f(1, \ldots, 1)$ if $\lambda_r \geq 0$ and $Q(r) = 0$ if $\lambda_r < 0$.

In the following section, we will drop the subscript ϵ for convenience.

3 Estimates for second-order derivatives on the boundary

In this section, we establish the boundary estimates for second-order derivatives of the solution of (2.1). Fix an arbitrary point $x_0 \in \partial M$. We choose smooth orthonormal local frames $e_1, \ldots, e_n$ around x_0 such that when restricted to ∂M, e_n is normal to ∂M.

Let $\rho(x)$ denote the distance from x to x_0,

$$\rho(x) \equiv \mathrm{dist}_{M^n}(x, x_0),$$

and $M_\delta = \{x \in M : \rho(x) < \delta\}$. Since ∂M is smooth we may assume the distance function to ∂M

$$d(x) \equiv \mathrm{dist}(x, \partial M)$$

is smooth in M_{δ_0} for fixed $\delta_0 > 0$ sufficiently small (depending only on the curvature of M and the principal curvatures of ∂M). Since $\nabla_{ij}\rho^2(x_0) = 2\delta_{ij}$, we may assume ρ is smooth in M_{δ_0} and

$$\{\delta_{ij}\} \leq \left\{\nabla_{ij}\rho^2\right\} \leq 3\{\delta_{ij}\} \quad \text{in } M_{\delta_0}. \tag{3.1}$$

Since $u - \underline{u} = 0$ on ∂M we have

$$\nabla_{\alpha\beta}(u - \underline{u}) = -\nabla_n(u - \underline{u})\Pi(e_\alpha, e_\beta), \quad \forall 1 \le \alpha, \beta < n \text{ on } \partial M, \tag{3.2}$$

where Π denotes the second fundamental form of ∂M. Therefore,

$$\left|\nabla_{\alpha\beta}u(x_0)\right| \le C, \quad \text{for } 1 \le \alpha, \beta \le n-1. \tag{3.3}$$

Next, we establish the estimate

$$\left|\nabla_{\alpha n}u(x_0)\right| \le C \quad \text{for } \alpha \le n-1. \tag{3.4}$$

Define the linear operator L by

$$Lw := F^{ij}\nabla_{ij}w - \psi_{p_k}\nabla_k w - \beta'_\epsilon(u - \phi)w, \quad \text{for } w \in C^2(M).$$

We first need to construct a barrier as Lemma 6.2 of [6].

Lemma 3.1 *Let*

$$v := u - \underline{u} + td - Nd^2.$$

Then, there exist positive constants t, δ sufficiently small and N sufficiently large such that

$$Lv \le -\epsilon_0\left(1 + \sum_i F^{ii}\right) \tag{3.5}$$

and

$$v \ge 0 \tag{3.6}$$

in M_δ for some uniform constat $\epsilon_0 > 0$.

Proof First, there exists a positive constant θ_0 such that $\underline{u} - \theta_0\rho^2$ is also admissible. By (2.4) and the concavity of F, we have

$$\begin{aligned}
F^{ij}\nabla_{ij}(u - \underline{u}) &\le -\theta_0\sum_i F^{ii} - F\left(\nabla^2\underline{u} - \theta_0 g + \chi\right) + F\left(\nabla^2 u + \chi\right) \\
&= -\theta_0\sum_i F^{ii} - F\left(\nabla^2\underline{u} - \theta_0 g + \chi\right) + \psi + \beta_\epsilon \\
&\le -\theta_0\sum_i F^{ii} + C,
\end{aligned}$$

where the constant C depends on $\|u\|_{C^1(\overline{M})}$ and the constant c_2 in (2.4). Recall that $f_i = \frac{\partial f}{\partial\lambda_i}$, where $\lambda = \lambda(\nabla^2 u + \chi)$ for $i = 1, \dots, n$. Without loss of generality, we may assume $f_n = \min_i\{f_i\}$. Next, since $\nabla d \equiv 1$ on the boundary, we have

$$F^{ij}\nabla_{ij}\left(d^2\right) \ge f_n + 2dF^{ij}\nabla_{ij}d \ge f_n - C\delta\sum_i F^{ii} \quad \text{in } M_\delta,$$

for δ sufficiently small. It follows that

$$Lv + \beta_\epsilon v \le -\theta_0 \sum_i F^{ii} - Nf_n + C(\delta N + t)\left(1 + \sum_i F^{ii}\right)$$

in M_δ. By (1.7), we have

$$\frac{\theta_0}{4} \sum_i F^{ii} + Nf_n \ge \frac{n\theta_0}{4}(Nf_1 \cdot \cdots \cdot f_n)^{1/n} \ge \frac{nc_0\theta_0 N^{1/n}}{4}.$$

Thus, we can choose N sufficiently large and t, δ sufficiently small such that

$$Lv + \beta_\epsilon v \le -\frac{\theta_0}{2} \sum_i F^{ii} - c_3 N^{1/n}.$$

We may further make δ sufficiently small such that $v \ge 0$ in M_δ. Since $\beta'_\epsilon \ge 0$ we obtain (3.5). □

From formula (4.7) in [7] and differentiating the equation (2.1), we have

$$\left|L\nabla_k(u - \phi)\right| \le C\left(1 + \sum_i F^{ii} + \sum_i f_i|\lambda_i|\right), \quad \text{for } 1 \le k \le n, \tag{3.7}$$

where C is a positive constant depending only on $\|u\|_{C^1(\overline{M})}$, $\|\phi\|_{C^3(\overline{M})}$ and $\|\psi\|_{C^1}$. Similar to formula (4.9) in [7], by Lemma 2.2, we find that

$$\begin{aligned} L\left(\sum_{\beta \le n-1} \left(\nabla_\beta(u - \phi)\right)^2\right) &\ge \sum_{\beta \le n-1} F^{ij} U_{\beta i} U_{\beta j} - C\left(1 + \sum_i F^{ii} + \sum_i f_i|\lambda_i|\right) \\ &\quad + \beta'_\epsilon \sum_{\beta \le n-1} \left(\nabla_\beta(u - \phi)\right)^2 \\ &\ge \frac{1}{2} \sum_{i \ne r} f_i \lambda_i^2 - C\left(1 + \sum_i F^{ii} + \sum_i f_i|\lambda_i|\right) \end{aligned} \tag{3.8}$$

for some index $1 \le r \le n$. Let

$$\Psi = A_1 v + A_2 \rho^2 - A_3 \sum_{\beta \le n-1} \left|\nabla_\beta(u - \phi)\right|^2 \tag{3.9}$$

as in [7]. Combining (2.11), (3.7), and (3.8), we can choose $A_1 \gg A_2 \gg A_3 \gg 1$ such that

$$L\left(\Psi \pm \nabla_\alpha(u - \phi)\right) \le 0 \quad \text{in } M_\delta$$

and

$$\Psi \pm \nabla_\alpha(u - \phi) \ge 0 \quad \text{on } \partial M_\delta$$

for any index $1 \le \alpha \le n - 1$. Then, by the maximum principle, we have

$$\Psi \pm \nabla_\alpha(u - \phi) \ge 0 \quad \text{on } \overline{M}_\delta.$$

Since

$$\Psi \pm \nabla_\alpha (u - \phi) = 0 \quad \text{at } x_0$$

we obtain (3.4).

Since $\Delta u + \mathrm{tr}(\chi) > 0$ in M, it suffices to establish the upper bound

$$\nabla_{nn} u(x_0) \le C. \tag{3.10}$$

We first suppose ϕ is admissible in M. As in [7], following an idea of Trudinger [23] we prove that there are uniform constants c_0, R_0 such that for all $R > R_0$, $(\lambda'[\{U_{\alpha\beta}(x_0)\}], R) \in \Gamma$ and

$$f\big(\lambda'\big[\big\{U_{\alpha\beta}(x_0)\big\}\big], R\big) \ge \psi[u](x_0) + \beta_\epsilon(x_0) + c_0, \tag{3.11}$$

which implies (3.10) by Lemma 1.2 in [3], where $\lambda'[\{U_{\alpha\beta}\}] = (\lambda'_1, \dots, \lambda'_{n-1})$ denote the eigenvalues of the $(n-1) \times (n-1)$ matrix $\{U_{\alpha\beta}\}$ $(1 \le \alpha, \beta \le n-1)$. Denote

$$\tilde{m}_R := \min_{x \in \partial M} f\big(\lambda'\big[\big\{U_{\alpha\beta}(x)\big\}\big], R\big).$$

Suppose $\tilde{m}_R$ is achieved at a point $x_0 \in \partial M$. Choose local orthonormal frames $e_1, e_2, \dots, e_n$ around x_0 as before and assume $\nabla_{nn} u(x_0) \ge \nabla_{nn}\phi(x_0)$. Let $\Phi_{ij} := \nabla_{ij}\phi + \chi_{ij}$ and

$$\tilde{c}_R := \min_{x \in \overline{M}_{\delta_0}} f\big(\lambda'\big[\big\{\Phi_{\alpha\beta}(x)\big\}\big], R\big)$$

for δ_0 sufficiently small such that $e_1, \dots, e_n$ are well defined in $\overline{M}_{\delta_0}$. By (1.9) and the fact that ϕ is admissible, we see that

$$\lim_{R \to +\infty} \tilde{c}_R = +\infty. \tag{3.12}$$

We wish to show $\tilde{m}_R \to +\infty$ as $R \to +\infty$. Without loss of generality we assume $\tilde{m}_R < \tilde{c}_R/2$ (otherwise we are done by (3.12)).

For a symmetric $(n-1) \times (n-1)$ matrix $\{r_{\alpha\beta}\}$ such that $(\lambda'[\{r_{\alpha\beta}\}], R) \in \Gamma$, define

$$\tilde{F}[r_{\alpha\beta}] := f\big(\lambda'\big[\{r_{\alpha\beta}\}\big], R\big).$$

Note that $\tilde{F}$ is concave by (1.5). Let

$$\tilde{F}_0^{\alpha\beta} = \frac{\partial \tilde{F}}{\partial r_{\alpha\beta}}\big[U_{\alpha\beta}(x_0)\big].$$

We find

$$\tilde{F}_0^{\alpha\beta} U_{\alpha\beta} - \tilde{F}_0^{\alpha\beta} U_{\alpha\beta}(x_0) \ge \tilde{F}[U_{\alpha\beta}] - \tilde{m}_R \ge 0 \quad \text{on } \partial M \text{ near } x_0. \tag{3.13}$$

By (3.2) we have on ∂M near x_0,

$$U_{\alpha\beta} = \Phi_{\alpha\beta} - \nabla_n(u-\phi)\sigma_{\alpha\beta}, \tag{3.14}$$

where $\sigma_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n\rangle$; note that $\sigma_{\alpha\beta} = \Pi(e_\alpha, e_\beta)$ on ∂M. Define

$$Q = -\eta\nabla_n(u-\phi) + \tilde{F}_0^{\alpha\beta}\Phi_{\alpha\beta} - \tilde{F}_0^{\alpha\beta}U_{\alpha\beta}(x_0),$$

where $\eta = \tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}$. From (3.13) and (3.14) we see that $Q(x_0) = 0$ and $Q \geq 0$ on ∂M near x_0. Furthermore, we have

$$\begin{aligned} Q &\geq -\eta\nabla_n(u-\phi) + \tilde{F}[\Phi_{\alpha\beta}] - \tilde{F}\big[U_{\alpha\beta}(x_0)\big] \\ &\geq -\eta\nabla_n(u-\phi) + \tilde{c}_R - \tilde{m}_R \\ &\geq -\eta\nabla_n(u-\phi) + \frac{\tilde{c}_R}{2} \quad \text{in } \overline{M}_{\delta_0}. \end{aligned} \tag{3.15}$$

By (3.7) and (3.15), we have

$$\begin{aligned} LQ &\leq C\mathcal{F}\Big(1 + \sum F^{ii} + \sum f_i|\lambda_i|\Big) - \frac{\tilde{c}_R}{2}\beta'_\epsilon \\ &\leq C\mathcal{F}\Big(1 + \sum F^{ii} + \sum f_i|\lambda_i|\Big), \end{aligned} \tag{3.16}$$

where

$$\mathcal{F} := \sum_{\alpha\leq n-1}\tilde{F}_0^{\alpha\alpha}.$$

Recall that Ψ is defined in (3.9). Choosing $A_1 \gg A_2 \gg A_3 \gg 1$ as before, we derive

$$\begin{cases} L(\mathcal{F}\Psi + Q) \leq 0 & \text{in } M_\delta, \\ \mathcal{F}\Psi + Q \geq 0 & \text{on } \partial M_\delta. \end{cases} \tag{3.17}$$

By the maximum principle, $\mathcal{F}\Psi + Q \geq 0$ in M_δ. Thus,

$$\nabla_n Q(x_0) \geq -\mathcal{F}\nabla_n\Psi(x_0) \geq -C\mathcal{F}. \tag{3.18}$$

By (1.11), we see, at x_0, $(\lambda'(\sigma_{\alpha\beta}), \sqrt{R_0}) \in \Gamma$ for some R_0 sufficiently large. Thus, there exists a uniform constant $\epsilon_0 > 0$ such that $(\lambda'(\sigma_{\alpha\beta} - \epsilon_0\delta_{\alpha\beta}), \sqrt{R}) \in \Gamma$ for all $R \geq R_0$. From the concavity of $\tilde{F}$ and (1.10) we find, at x_0,

$$\begin{aligned} \sqrt{R}\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta} &= \sqrt{R}\tilde{F}_0^{\alpha\beta}(\sigma_{\alpha\beta} - \epsilon_0\delta_{\alpha\beta}) - \tilde{F}_0^{\alpha\beta}U_{\alpha\beta}(x_0) + \tilde{F}_0^{\alpha\beta}U_{\alpha\beta}(x_0) + \sqrt{R}\epsilon_0\mathcal{F} \\ &\geq \tilde{F}\big[\sqrt{R}(\sigma_{\alpha\beta} - \epsilon_0\delta_{\alpha\beta})\big] - \tilde{F}\big[U_{\alpha\beta}(x_0)\big] + \tilde{F}_0^{\alpha\beta}U_{\alpha\beta}(x_0) + \sqrt{R}\epsilon_0\mathcal{F} \\ &\geq f\big(\sqrt{R}\big(\lambda'(\sigma_{\alpha\beta} - \epsilon_0\delta_{\alpha\beta}), \sqrt{R}\big)\big) - \tilde{F}\big[U_{\alpha\beta}(x_0)\big] + \sqrt{R}\epsilon_0\mathcal{F} - C\mathcal{F} \\ &\geq f\big(\sqrt{R}\big(\lambda'(\sigma_{\alpha\beta} - \epsilon_0\delta_{\alpha\beta}), \sqrt{R_0}\big)\big) - \tilde{m}_R + \frac{\sqrt{R}}{2}\epsilon_0\mathcal{F} \\ &\geq C(R) - \tilde{m}_R + \frac{\sqrt{R}}{2}\epsilon_0\mathcal{F}, \end{aligned}$$

provided R is sufficiently large, where $\lim_{R\to+\infty} C(R) = +\infty$. We may assume $\tilde{m}_R \le C(R)$ for otherwise we are done. It follows that, at x_0,

$$\eta = \tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta} \ge \frac{\epsilon_0}{2}\mathcal{F}. \tag{3.19}$$

Combining (3.18) and (3.19) we obtain

$$\nabla_{nn}u(x_0) \le C.$$

We have established an *a priori* upper bound for all eigenvalues of $\{U_{ij}(x_0)\}$. Consequently, $\lambda[\{U_{ij}(x_0)\}]$ is contained in a compact subset of Γ by (1.6), and therefore

$$\lim_{R\to+\infty} \tilde{m}_R = +\infty$$

by (1.9). This proves (3.11) and the proof of (3.10) is complete.

We now consider the case $\chi \equiv 0$ and $\varphi \equiv 0$ on ∂M to prove Theorem 1.3. By [3] we have

$$\Delta u \ge \delta_0 > 0 \tag{3.20}$$

for some positive constant δ_0 depending only on $\psi_0 = \inf\psi > 0$. Let u_0 be defined by the equation

$$\Delta u_0 = \delta_0 \quad \text{in } M$$

with $u_0 = 0$ on ∂M. By the maximum principle and Hopf's lemma, we see $u_0 < 0$ in M and $(u_0)_\nu < 0$ on ∂M, where ν is the unit interior normal to ∂M. Since ∂M is compact, there exists a uniform constant $\gamma_1 > 0$ such that $(u_0)_\nu \le -\gamma_1$ on ∂M. By (3.20) and the maximum principle, we find that

$$u \le u_0 \quad \text{in } M \text{ and } u = u_0 = 0 \text{ on } \partial M.$$

It follows that

$$\nabla_n u(x_0) \le \nabla_n(u_0)(x_0) \le -\gamma_1. \tag{3.21}$$

We find, at $x_0 \in \partial M$,

$$\nabla_{\alpha\beta}u = -\nabla_n u \Pi(e_\alpha, e_\beta), \quad \text{for } 1 \le \alpha, \beta \le n-1.$$

Since $\underline{u} = 0$, we have, at x_0,

$$\nabla_{\alpha\beta}\underline{u} = -\nabla_n \underline{u} \Pi(e_\alpha, e_\beta), \quad \text{for } 1 \le \alpha, \beta \le n-1.$$

Therefore,

$$\nabla_{\alpha\beta}u = \frac{\nabla_n u}{\nabla_n \underline{u}}\nabla_{\alpha\beta}\underline{u}.$$

By (3.21), we then find the eigenvalues of the $(n-1)\times(n-1)$ matrix $\{\nabla_{\alpha\beta}u(x_0)\}_{\alpha,\beta\le n-1}$ $\lambda'\{\nabla_{\alpha\beta}u(x_0)\}$ belong to a compact subset of Γ', where Γ' denotes the projection of Γ to $\lambda' = (\lambda_1,\dots,\lambda_{n-1})$ of Γ. By (1.9) and Lemma 1.2 of [3], we can prove (3.10).

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Author contribution
JL conceptualized the idea and wrote the first draft. YW reviewed and edited the manuscript. All authors read and approved the final manuscript.

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