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New upper bounds for the dominant eigenvalue of a matrix with Perron–Frobenius property

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Abstract

In this paper, we derive some upper bounds for the dominant eigenvalue of a matrix with some negative entries, which possess the Perron–Frobenius property. Numerical examples are given to illustrate the effectiveness of our new upper bounds.

MSC: 65F10; 15A48

Keywords: Perron–Frobenius property; Nonnegative matrices; Dominant eigenvalue; Upper bound

1 Introduction

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}^{n\times n}$ be the set of $n \times n$ square matrices. If $A \in \mathbb{R}^{n\times n}$ is positive, Perron proved that A has a simple eigenvalue equal to its spectral radius (called the dominant eigenvalue) and that its corresponding eigenvector is also positive [1]. This famous result was extended to nonnegative irreducible matrices by Frobenius in [2]. Based on Geršhgorin's theorem [3–7], the following classical upper bound for the dominant eigenvalue $\rho(A)$ of a nonnegative matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ was presented as follows [1]:

$$\rho(A) \le \max_{i\in N} R_i(A),$$

where $N = \{1, 2, \dots, n\}$, $R_i(A) = \sum_{j\in N} a_{ij}$. Based on Brauer's theorem [3, 4, 7], the authors derived the following improved upper bound [8]:

$$\rho(A) \le \frac{1}{2} \max_{i,j\in N, j\neq i} \Bigl(a_{ii} + a_{jj} + \sqrt{(a_{ii} - a_{jj})^2 + 4R_i'(A)R_j'(A)}\Bigr),$$

where $R_i'(A) = R_i(A) - a_{ii}$.

Consider the dynamics associated with the linear differential system

$$\dot{x}(t) = Ax(t), \quad A \in \mathbb{R}^{n\times n}, x(0) = x_0 \in \mathbb{R}^n, t \ge 0,$$

where the coefficient matrix A is essentially nonnegative, i.e., it has nonnegative offdiagonal entries. Such systems arise frequently in applications in engineering and mathematical biology among others [9]. In 2006, Noutsos extended the Perron–Frobenius theory of nonnegative matrices to the class of matrices with some negative entries, which possess the Perron–Frobenius property, the relationships between eventually positive matrices and the class of matrices with the Perron–Frobenius property are also discussed [10]. First, we give the definition of PF matrices [10].

Definition 1 A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a PF matrix, if its dominant eigenvalue $\rho(A)$ is positive and the corresponding eigenvector x is nonnegative.

The author also presented the following upper bound for the dominant eigenvalue $\rho(A)$ of a matrix with the Perron–Frobenius property [10].

Theorem 1 *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a PF matrix, then*

$$\rho(A) \le \omega = \max_{i\in N} R_i\bigl(A^T\bigr).$$

There exists an extensive literature on bounds for the dominant eigenvalue $\rho(A)$ of a nonnegative matrix, we refer to [1, 8, 11–13] and the references therein. In this paper, we obtain some sharper new upper bounds for the dominant eigenvalue $\rho(A)$ of a PF matrix with some negative entries, these bounds are only dependent on the entries of a tensor, which are easy to check. Two numerical examples are given to show the efficiency of the proposed results in Sect. 3.

2 Main results

For any $i \in N$, we let

$$r_i^+(A) = \sum_{i\in N, j\neq i} [a_{ij}]_+,$$
$$r_i^-(A) = \sum_{i\in N, j\neq i} [a_{ij}]_-,$$

where $[a_{ij}]_+$ is the set of nonnegative entries in the ith row, $[a_{ij}]_-$ is the set of negative entries in the ith row, obviously, $R_i'(A) = r_i^+(A) + r_i^-(A)$. We give the main results as follows.

Theorem 2 *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ possesses the Perron–Frobenius property, then*

$$\rho(A) \le \omega_1 = \max_{i\in N}\bigl(a_{ii} + r_i^+(A)\bigr).$$

Proof Let $\rho(A)$ be the dominant eigenvalue of A with the corresponding eigenvector x, then, $0 \le x \neq 0$. Let $|x_p| = \max_{i\in N} |x_i|$, then, the pth equation of

$$Ax = \rho(A)x$$

is

$$\begin{aligned}\rho(A)x_p &= a_{pp}x_p + \sum_{p\in N, j\neq p} a_{pj}x_j \\ &= a_{pp}x_p + \sum_{p\in N, j\neq p} [a_{pj}]_+x_j + \sum_{p\in N, j\neq p} [a_{pj}]_-x_j \\ &\leq a_{pp}x_p + \sum_{p\in N, j\neq p} [a_{pj}]_+x_j \\ &\leq a_{pp}x_p + \sum_{p\in N, j\neq p} [a_{pj}]_+x_p,\end{aligned}$$

which implies that

$$\rho(A) \leq a_{pp} + r_p^+(A). \qquad \square$$

Theorem 3 *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ possesses the Perron–Frobenius property, then*

$$\rho(A) \leq \omega_2 = \frac{1}{2}\max_{i,j\in N, j\neq i}\left(a_{ii} + a_{jj} + \sqrt{(a_{ii}-a_{jj})^2 + 4r_i^+(A)r_j^+(A)}\right).$$

Proof Let $\rho(A)$ be the dominant eigenvalue of A with the corresponding eigenvector x, then, $0 \leq x \neq 0$. Let $|x_p| = \max_{i\in N}|x_i|$, $|x_q| = \max_{i\in N, i\neq p}|x_i|$. Then, by the pth equation of

$$Ax = \rho(A)x,$$

we have

$$\begin{aligned}\rho(A)x_p - a_{pp}x_p &= \sum_{p\in N, j\neq p} a_{pj}x_j \\ &= \sum_{p\in N, j\neq p} [a_{pj}]_+x_j + \sum_{p\in N, j\neq p} [a_{pj}]_-x_j \\ &\leq \sum_{p\in N, j\neq p} [a_{pj}]_+x_j \\ &\leq \sum_{p\in N, j\neq p} [a_{pj}]_+x_q,\end{aligned}$$

which implies that

$$\bigl(\rho(A) - a_{pp}\bigr)x_p \leq r_p^+(A)x_q. \tag{1}$$

Consider the qth equation of $Ax = \rho(A)x$, we have

$$\begin{aligned}\rho(A)x_q - a_{qq}x_q &= \sum_{q\in N, j\neq q} a_{qj}x_j \\ &= \sum_{q\in N, j\neq q} [a_{qj}]_+x_j + \sum_{q\in N, j\neq q} [a_{qj}]_-x_j\end{aligned}$$

$$\begin{aligned} &\leq \sum_{q\in N, j\neq q} [a_{qj}]_+ x_j \\ &\leq \sum_{q\in N, j\neq q} [a_{qj}]_+ x_p, \end{aligned}$$

which implies that

$$\bigl(\rho(A) - a_{qq}\bigr)x_q \leq r_q^+(A)x_p. \tag{2}$$

If $\rho(A) - a_{qq} \leq 0$ and $\rho(A) - a_{pp} \leq 0$, then,

$$\rho(A) \leq a_{qq}, \qquad \rho(A) \leq a_{pp}.$$

Otherwise, multiplying inequalities (1) with (2), we obtain

$$\bigl(\rho(A) - a_{pp}\bigr)\bigl(\rho(A) - a_{qq}\bigr) \leq r_p^+(A)r_q^+(A), \tag{3}$$

therefore,

$$\rho(A) \leq \frac{1}{2}\Bigl(a_{pp} + a_{qq} + \sqrt{(a_{pp} - a_{qq})^2 + 4r_p^+(A)r_q^+(A)}\Bigr).$$

Furthermore,

$$\begin{aligned} a_{qq} - a_{pp} &\leq \sqrt{(a_{pp} - a_{qq})^2 + 4r_p^+(A)r_q^+(A)}, \\ a_{pp} - a_{qq} &\leq \sqrt{(a_{pp} - a_{qq})^2 + 4r_p^+(A)r_q^+(A)}. \end{aligned}$$

Then, we have

$$a_{qq} \leq \frac{1}{2}\Bigl(a_{pp} + a_{qq} + \sqrt{(a_{pp} - a_{qq})^2 + 4r_p^+(A)r_q^+(A)}\Bigr)$$

and

$$a_{pp} \leq \frac{1}{2}\Bigl(a_{pp} + a_{qq} + \sqrt{(a_{pp} - a_{qq})^2 + 4r_p^+(A)r_q^+(A)}\Bigr),$$

which means that

$$\rho(A) \leq \frac{1}{2}\Bigl(a_{pp} + a_{qq} + \sqrt{(a_{pp} - a_{qq})^2 + 4r_p^+(A)r_q^+(A)}\Bigr)$$

always holds. □

By breaking N into disjoint subsets S and $\bar{S}$, where $\bar{S}$ is the complement of S in N, and let $r_i^{S+}(A) = \sum_{j\in S\setminus\{i\}}[a_{ij}]_+$, $r_i^{\bar{S}+}(A) = \sum_{j\in \bar{S}\setminus\{i\}}[a_{ij}]_+$, we give a new S-type upper bound for the dominant eigenvalue $\rho(A)$ of a matrix with the Perron–Frobenius property.

Theorem 4 *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ possesses the Perron–Frobenius property, then*

$$\rho(A) \le \omega_3 = \frac{1}{2}\min_{S\subseteq N}\max_{i\in S, j\in\bar{S}}\bigl(a_{ii} + r_i^{S+}(A) + a_{jj} + r_j^{\bar{S}+}(A) + \sqrt{\varepsilon}\bigr),$$

where $\varepsilon = (a_{ii} + r_i^{S+}(A) - a_{jj} - r_j^{\bar{S}+}(A))^2 + 4r_i^{\bar{S}+}(A)r_j^{S+}(A)$.

Proof Let $\rho(A)$ be the dominant eigenvalue of A with the corresponding eigenvector x, then, $0 \le x \ne 0$. Let $|x_p| = \max_{i\in S}|x_i|$, $|x_q| = \max_{i\in\bar{S}}|x_i|$. Then, by the pth equation of

$$Ax = \rho(A)x,$$

we have

$$\begin{aligned}
&\rho(A)x_p - a_{pp}x_p \\
&\quad = \sum_{j\in S, j\ne p} a_{pj}x_j + \sum_{j\in\bar{S}} a_{pj}x_j \\
&\quad \le \sum_{j\in S, j\ne p} [a_{pj}]_+x_j + \sum_{j\in\bar{S}} [a_{pj}]_+x_j \\
&\quad \le \sum_{j\in S, j\ne p} [a_{pj}]_+x_p + \sum_{j\in\bar{S}} [a_{pj}]_+x_q,
\end{aligned}$$

which implies that

$$\bigl(\rho(A) - a_{pp} - r_p^{S+}(A)\bigr)x_p \le r_p^{\bar{S}+}(A)x_q. \tag{4}$$

Consider the qth equation of $Ax = \rho(A)x$, similar to the proof of Theorem 3, we have

$$\bigl(\rho(A) - a_{qq} - r_q^{\bar{S}+}(A)\bigr)x_q \le r_q^{S+}(A)x_p. \tag{5}$$

If $\rho(A) - a_{pp} - r_p^{S+}(A) \le 0$ and $\rho(A) - a_{qq} - r_q^{\bar{S}+}(A) \le 0$, then

$$\rho(A) \le a_{pp} + r_p^{S+}(A), \rho(A) \le a_{qq} + r_q^{\bar{S}+}(A).$$

Otherwise, multiplying inequalities (4) with (5), we obtain

$$\bigl(\rho(A) - a_{pp} - r_p^{S+}(A)\bigr)\bigl(\rho(A) - a_{qq} - r_q^{\bar{S}+}(A)\bigr) \le r_p^{\bar{S}+}(A)r_q^{S+}(A), \tag{6}$$

therefore, let $\varepsilon = (a_{pp} + r_p^{S+}(A) - a_{qq} - r_q^{\bar{S}+}(A))^2 + 4r_p^{\bar{S}+}(A)r_q^{S+}(A)$,

$$\rho(A) \le \frac{1}{2}\bigl(a_{pp} + r_p^{S+}(A) + a_{qq} + r_q^{\bar{S}+}(A) + \sqrt{\varepsilon}\bigr).$$

Furthermore,

$$a_{pp} + r_p^{S+}(A) - \bigl(a_{qq} + r_q^{\bar{S}+}(A)\bigr) \le \sqrt{\varepsilon},$$

$$a_{qq} + r_q^{\bar{S}+}(A) - \bigl(a_{pp} + r_p^{S+}(A)\bigr) \le \sqrt{\varepsilon}.$$

Then, we have

$$a_{pp} + r_p^{S+}(A) \le \frac{1}{2}\bigl(a_{pp} + r_p^{S+}(A) + a_{qq} + r_q^{\bar{S}+}(A) + \sqrt{\varepsilon}\bigr)$$

and

$$a_{qq} + r_q^{\bar{S}+}(A) \le \frac{1}{2}\bigl(a_{pp} + r_p^{S+}(A) + a_{qq} + r_q^{\bar{S}+}(A) + \sqrt{\varepsilon}\bigr),$$

which means that

$$\rho(A) \le \frac{1}{2}\bigl(a_{pp} + r_p^{S+}(A) + a_{qq} + r_q^{\bar{S}+}(A) + \sqrt{\varepsilon}\bigr)$$

always holds. Then, the proof is completed by the arbitrary of S. □

The relationships between ω_1, ω_2, and ω_3 are discussed as follows.

Theorem 5 *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ possesses the Perron–Frobenius property, then*

$$\omega_3 \le \omega_2 \le \omega_1.$$

Proof First, we prove $\omega_2 \le \omega_1$. If $\rho(A) \le \omega_2$, from the proof of Theorem 3, we have

$$\bigl(\rho(A) - a_{pp}\bigr)\bigl(\rho(A) - a_{qq}\bigr) \le r_p^+(A) r_q^+(A).$$

If $r_p^+(A) r_q^+(A) = 0$, we obtain

$$\rho(A) \le \max\{a_{pp}, a_{qq}\} \le \max_{i\in N}\bigl\{a_{ii} + r_i^+(A)\bigr\},$$

which implies $\rho(A) \le \omega_1$. If $r_p^+(A) r_q^+(A) > 0$, we obtain

$$\frac{\rho(A) - a_{pp}}{r_p^+(A)} \frac{\rho(A) - a_{qq}}{r_q^+(A)} \le 1,$$

then,

$$\frac{\rho(A) - a_{pp}}{r_p^+(A)} \le 1,$$

or

$$\frac{\rho(A) - a_{qq}}{r_q^+(A)} \le 1,$$

which implies $\rho(A) \le \omega_1$.

Next, we prove $\omega_3 \le \omega_2$. If $\rho(A) \le \omega_3$, from the proof of Theorem 4, we have

$$\bigl(\rho(A) - a_{pp} - r_p^{S+}(A)\bigr)\bigl(\rho(A) - a_{qq} - r_q^{\bar{S}+}(A)\bigr) \le r_p^{\bar{S}+}(A) r_q^{S+}(A),$$

without loss of generality, we assume that $x_p \geq x_q$, from (4), we have

$$\rho(A) - a_{pp} \leq r_p^+(A). \tag{7}$$

Letting $S = \{p\}$, we obtain

$$\big(\rho(A) - a_{pp}\big)\big(\rho(A) - a_{qq} - r_q^{\bar{S}+}(A)\big) \leq r_p^+(A) r_q^{S+}(A),$$

therefore,

$$\begin{aligned}
\big(\rho(A) - a_{pp}\big)\big(\rho(A) - a_{qq}\big) &\leq \big(\rho(A) - a_{pp}\big) r_q^{\bar{S}+}(A) + r_p^+(A) r_q^{S+}(A) \\
&\leq r_p^+(A) r_q^{\bar{S}+}(A) + r_p^+(A) r_q^{S+}(A) \\
&= r_p^+(A) r_q^+(A),
\end{aligned}$$

which implies $\rho(A) \leq \omega_2$. □

3 Numerical examples

In this section, in order to show the efficiency of our results, we give some numerical examples.

Example 3.1 Consider the Example 2.2 in [10]:

$$A_1 = \begin{bmatrix} 1 & -4 & 8 \\ 1 & 1 & 5 \\ -3 & 1 & 8 \end{bmatrix}.$$

Then, A_1 possesses the Perron–Frobenius property with the dominant eigenvalue $\rho(A_1) = 6.868$.

Example 3.2 Consider the Example 2.3 in [10]:

$$A_2 = \begin{bmatrix} 1 & -0.4 & 0.3 \\ 20 & 1 & 5 \\ 20 & 1 & 8 \end{bmatrix}.$$

Then, A_2 possesses the Perron–Frobenius property with the dominant eigenvalue $\rho(A_2) = 8.753$.

The numerical comparison between our results and the result in [10] is given in Table 1. From Table 1, we reveal that our bounds are tighter than the bound in [10].

Table 1 Numerical comparison

	ρ	ω	ω_1	ω_2	ω_3
A_1	6.868	21	9	9	9
A_2	8.753	41	29	27.6787	26.6347

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Jun He and Yanmin Liu wrote the main manuscript text and Wei Lv prepared Examples. All authors reviewed the manuscript.

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