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On new Milne-type inequalities for fractional integrals

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Abstract

In this study, fractional versions of Milne-type inequalities are investigated for differentiable convex functions. We present Milne-type inequalities for bounded functions, Lipschitz functions, functions of bounded variation, etc., found in the literature. New results are established in the area of inequalities. This article is the first to study Milne-type inequalities for fractional integrals.

MSC: 26D07; 26D10; 26D15

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1 Introduction

Studies on numerical integration and error bounds in mathematics have an important place in the literature. Research on inequalities tries to find error bounds for various function classes such as those of bounded functions, Lipschitz functions, functions of bounded variation, etc. In addition, researchers obtained error bounds for differentiable, twice differentiable, or n-times differentiable mappings. Moreover, many authors have also obtained new bounds by utilizing the concepts of fractional calculus. Nowadays, authors have focused on inequalities of the trapezoid, midpoint, and Simpson type. Many authors have contributed to the extension and generalization of these integral inequalities. For instance, Dragomir and Agarwal presented some error estimates for the trapezoidal formula in [9]. Cerone and Dragomir considered trapezoidal-type rules and established explicit bounds through the modern theory of inequalities in [4, p. 93]. The authors examined both Riemann–Stieltjes and Riemann integrals for different states of the boundary. Alomari discussed Lipschitz functions in the context of the generalized trapezoidal inequality [2]. Dragomir studied functions of bounded variation in the context of the trapezoid formula [8]. Sarikaya and Aktan obtained some new inequalities of Simpson and trapezoid type for functions whose second derivative in absolute value is convex [27]. In the articles [28, 32], researchers considered fractional trapezoid-type inequalities. Kırmacı established midpoint-type inequalities for differentiable convex functions [19]. Dragomir presented obtained results for functions of bounded variation in [7]. Sarıkaya et al. obtained several new inequalities for twice differentiable functions in [29]. Fractional analogues of these results [16, 33] have also been discussed. Several mathematicians also established Simpson-type inequalities for differentiable convex mappings [10], s-convex functions [30], extended (s,m)-convex mappings [12], bounded functions [6], twice differentiable convex functions [15, 24, 31], and fractional integrals [5, 14, 17, 20, 22, 23, 25, 26, 34].

A formal definition of a convex function may be stated as follows:

Definition 1 ([11]) Let I be a convex set on $\mathbb{R}$. The function $\mathfrak{F}: I \to \mathbb{R}$ is called convex on I if it satisfies the following inequality:

$$\mathfrak{F}\big(\vartheta\upsilon + (1-\vartheta)\gamma\big) \leq \vartheta\mathfrak{F}(\upsilon) + (1-\vartheta)\mathfrak{F}(\gamma) \tag{1.1}$$

for all $(\upsilon,\gamma) \in I$ and $\vartheta \in [0,1]$. The mapping $\mathfrak{F}$ is concave on I if the inequality (1.1) holds in reversed direction for all $\vartheta \in [0,1]$ and $\upsilon,\gamma \in I$.

In terms of Newton–Cotes formulas, Milne's formula, which is of open type, is parallel to the Simpson's formula, which is of closed type, since they hold under the same conditions. Suppose that $\mathfrak{F}: [\kappa_1,\kappa_2] \to \mathbb{R}$ is a four-times continuously differentiable mapping on (κ_1,κ_2), and let $\|\mathfrak{F}^{(4)}\|_\infty = \sup_{\upsilon\in(\kappa_1,\kappa_2)} |\mathfrak{F}^{(4)}(\upsilon)| < \infty$. Then, one has the inequality [3]

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2)\right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathfrak{F}(\upsilon)\,d\upsilon\right| \\
&\quad \leq \frac{7(\kappa_2-\kappa_1)^4}{23{,}040}\big\|\mathfrak{F}^{(4)}\big\|_\infty.
\end{aligned} \tag{1.2}$$

In this paper we will obtain a fractional version of the left-hand side of (1.2) and will consider several new bounds by using different mapping classes.

The well-known Riemann–Liouville fractional integrals are given as follows:

Definition 2 Let $\mathfrak{F} \in L_1[\kappa_1,\kappa_2]$. The Riemann–Liouville fractional integrals $\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}$ and $\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}$ of order $\alpha > 0$ are defined by

$$\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}(\upsilon) = \frac{1}{\Gamma(\alpha)}\int_{\kappa_1}^{\upsilon} (\upsilon-\vartheta)^{\alpha-1}\mathfrak{F}(\vartheta)\,d\vartheta, \quad \upsilon > \kappa_1,$$

and

$$\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}(\upsilon) = \frac{1}{\Gamma(\alpha)}\int_{\upsilon}^{\kappa_2} (\vartheta-\upsilon)^{\alpha-1}\mathfrak{F}(\vartheta)\,d\vartheta, \quad \upsilon < \kappa_2,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $\mathfrak{I}^0_{\kappa_1+}\mathfrak{F}(\upsilon) = \mathfrak{I}^0_{\kappa_2-}\mathfrak{F}(\upsilon) = \mathfrak{F}(\upsilon)$.

For more information about Riemann–Liouville fractional integrals, please refer to [13, 18, 21].

2 Milne-type inequalities for differentiable convex functions

In this part, we present a few inequalities of Milne-type for differentiable convex mappings.

Lemma 1 *Let $\mathfrak{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a differentiable mapping (κ_1, κ_2) such that $\mathfrak{F}' \in L_1([\kappa_1, \kappa_2])$. Then, the following equality holds*:

$$\begin{aligned}
&\frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2)\right] \\
&\quad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + \mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right] \\
&\quad = \frac{\kappa_2-\kappa_1}{4}\int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)\left[\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1 + \left(\frac{1+\vartheta}{2}\right)\kappa_2\right)\right. \\
&\quad \left. - \mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1 + \left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right] d\vartheta.
\end{aligned}$$

Proof By utilizing integration by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1 + \left(\frac{1-\vartheta}{2}\right)\kappa_2\right) d\vartheta \\
&= -\frac{2}{\kappa_2-\kappa_1}\left(\vartheta^\alpha + \frac{1}{3}\right)\mathfrak{F}\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1 + \left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\Bigg|_0^1 \\
&\quad + \frac{2\alpha}{\kappa_2-\kappa_1}\int_0^1 \vartheta^{\alpha-1}\mathfrak{F}\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1 + \left(\frac{1-\vartheta}{2}\right)\kappa_2\right) d\vartheta \\
&= \frac{2}{3(\kappa_2-\kappa_1)}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{8}{3(\kappa_2-\kappa_1)}\mathfrak{F}(\kappa_1) \\
&\quad + \left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha+1}\alpha\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\left(\frac{\kappa_1+\kappa_2}{2} - \upsilon\right)^{\alpha-1}\mathfrak{F}(\upsilon)\right] d\upsilon \\
&= \frac{2}{3(\kappa_2-\kappa_1)}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{8}{3(\kappa_2-\kappa_1)}\mathfrak{F}(\kappa_1) \\
&\quad + \left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha+1}\Gamma(\alpha+1)\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right).
\end{aligned} \tag{2.1}$$

Similarly, we obtain

$$\begin{aligned}
I_2 &= \int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1 + \left(\frac{1+\vartheta}{2}\right)\kappa_2\right) d\vartheta \\
&= -\frac{2}{3(\kappa_2-\kappa_1)}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + \frac{8}{3(\kappa_2-\kappa_1)}\mathfrak{F}(\kappa_2) \\
&\quad - \left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha+1}\Gamma(\alpha+1)\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right).
\end{aligned} \tag{2.2}$$

From equations (2.1) and (2.2), the following result is obtained:

$$\begin{aligned}
\frac{\kappa_2-\kappa_1}{4}[I_2 - I_1] &= \frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2)\right] \\
&\quad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + \mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right].
\end{aligned}$$

The proof of Lemma 1 is completed. □

Theorem 1 *Assume that the assumptions of Lemma 1 hold. Let $|\mathfrak{F}'|$ be a convex function on $[\kappa_1,\kappa_2]$. Then, we get the following inequality*:

$$\begin{aligned}&\bigg|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]\\&\quad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\bigg|\\&\quad\le\frac{\kappa_2-\kappa_1}{12}\left(\frac{\alpha+4}{\alpha+1}\right)\left(\left|\mathfrak{F}'(\kappa_1)\right|+\left|\mathfrak{F}'(\kappa_2)\right|\right).\end{aligned}\tag{2.3}$$

Proof By taking the absolute value in Lemma 1 and utilizing the convexity of $|\mathfrak{F}'|$, we get

$$\begin{aligned}&\bigg|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]\\&\quad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\bigg|\\&\quad\le\frac{\kappa_2-\kappa_1}{4}\int_0^1\left|\vartheta^\alpha+\frac{1}{3}\right|\left[\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|\right.\\&\quad\left.+\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_2+\left(\frac{1-\vartheta}{2}\right)\kappa_1\right)\right|\right]d\vartheta\\&\quad\le\frac{\kappa_2-\kappa_1}{4}\int_0^1\left(\vartheta^\alpha+\frac{1}{3}\right)\left[\frac{1+\vartheta}{2}\left|\mathfrak{F}'(\kappa_1)\right|+\frac{1-\vartheta}{2}\left|\mathfrak{F}'(\kappa_2)\right|\right.\\&\quad\left.+\frac{1+\vartheta}{2}\left|\mathfrak{F}'(\kappa_2)\right|+\frac{1-\vartheta}{2}\left|\mathfrak{F}'(\kappa_1)\right|\right]d\vartheta\\&\quad\le\frac{\kappa_2-\kappa_1}{4}\left(\frac{1}{3}+\frac{1}{\alpha+1}\right)\left(\left|\mathfrak{F}'(\kappa_1)\right|+\left|\mathfrak{F}'(\kappa_2)\right|\right)\\&\quad=\frac{\kappa_2-\kappa_1}{12}\left(\frac{\alpha+4}{\alpha+1}\right)\left(\left|\mathfrak{F}'(\kappa_1)\right|+\left|\mathfrak{F}'(\kappa_2)\right|\right)\end{aligned}\tag{2.4}$$

which gives inequality (2.3). □

Example 1 Let $[\kappa_1,\kappa_2]=[0,1]$ and define the function $\mathfrak{F}:[0,1]\to\mathbb{R}$ as $\mathfrak{F}(\vartheta)=\frac{\vartheta^3}{3}$ so that $\mathfrak{F}'(\vartheta)=\vartheta^2$ and $|\mathfrak{F}'|$ is convex on $[0,1]$.

Under these assumptions, we have

$$\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]=\frac{5}{24}.$$

By definition of Riemann–Liouville fractional integrals, we obtain

$$\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)=\mathfrak{I}^\alpha_{0+}\mathfrak{F}\left(\frac{1}{2}\right)=\frac{1}{\Gamma(\alpha)}\int_0^{\frac{1}{2}}\left(\frac{1}{2}-\vartheta\right)^{\alpha-1}\frac{\vartheta^3}{3}\,d\vartheta=\frac{1}{2^{\alpha+2}\Gamma(\alpha+4)}$$

and

$$\begin{aligned}\mathfrak{I}^{\alpha}_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) &= \mathfrak{I}^{\alpha}_{1-}\mathfrak{F}\left(\frac{1}{2}\right)\\ &= \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{2}}^{1}\left(\vartheta-\frac{1}{2}\right)^{\alpha-1}\frac{\vartheta^3}{3}\,d\vartheta\\ &= \frac{4\alpha^3+18\alpha^2+20\alpha+3}{3\Gamma(\alpha+4)2^{\alpha+3}}.\end{aligned}$$

Thus we have

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathfrak{I}^{\alpha}_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\ &\quad= 2^{\alpha-1}\Gamma(\alpha+1)\left[\frac{1}{2^{\alpha+2}\Gamma(\alpha+4)}+\frac{4\alpha^3+18\alpha^2+20\alpha+3}{3\Gamma(\alpha+4)2^{\alpha+3}}\right]\\ &\quad= \frac{2\alpha^3+9\alpha^2+10\alpha+3}{12(\alpha+1)(\alpha+2)(\alpha+3)}.\end{aligned}$$

As a result, the left-hand side of inequality (2.3) reduces to

$$\begin{aligned}&\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-}\mathfrak{F}(\kappa_1)+\mathfrak{I}^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+}\mathfrak{F}(\kappa_2)\right]\\ &\quad= \frac{5}{24}-\frac{2\alpha^3+9\alpha^2+10\alpha+3}{12(\alpha+1)(\alpha+2)(\alpha+3)} =: \mathrm{LHS}\end{aligned}$$

and

$$\frac{\kappa_2-\kappa_1}{4}\left(\frac{1}{3}+\frac{1}{\alpha+1}\right)(\kappa_2-\kappa_1)\left(\left|\mathfrak{F}'(\kappa_1)\right|+\left|\mathfrak{F}'(\kappa_2)\right|\right)=\frac{\alpha+4}{12(\alpha+1)} =: \mathrm{RHS}.$$

The results of Example 1 are shown in Fig. 1.

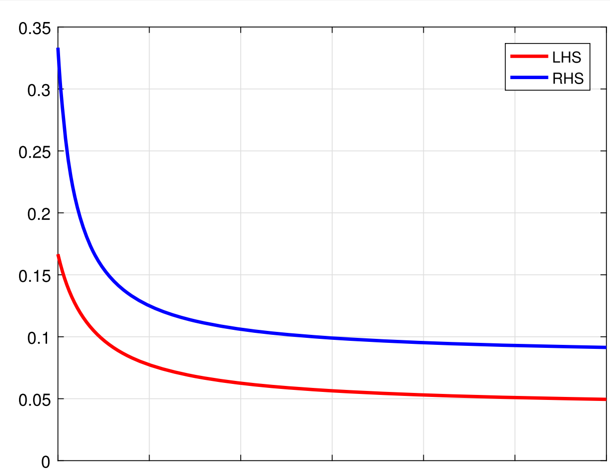

Figure 1 Graph for the Example 1 examined and calculated in MATLAB program

Remark 1 If we choose $\alpha = 1$ in Theorem 1, then we have

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2)\right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathfrak{F}(\vartheta)\,d\vartheta\right| \\ &\quad\leq \frac{5(\kappa_2-\kappa_1)}{24}\left(\left|\mathfrak{F}'(\kappa_1)\right| + \left|\mathfrak{F}'(\kappa_2)\right|\right).\end{aligned}$$

Theorem 2 *Suppose that the assumptions of Lemma 1 hold. Suppose also that the mapping* $|\mathfrak{F}'|^q$, $q > 1$, *is convex on* $[\kappa_1, \kappa_2]$. *Then, the following inequality holds*:

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2)\right]\right. \\ &\qquad\left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{J}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + \mathfrak{J}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\ &\quad\leq \frac{\kappa_2-\kappa_1}{4}\left(\int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)^p d\vartheta\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathfrak{F}'(\kappa_1)|^q + |\mathfrak{F}'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}\right. \\ &\qquad\left. + \left(\frac{|\mathfrak{F}'(\kappa_1)|^q + 3|\mathfrak{F}'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}\right] \\ &\quad\leq \frac{\kappa_2-\kappa_1}{4}\left(4\int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)^p d\vartheta\right)^{\frac{1}{p}}\left(\left|\mathfrak{F}'(\kappa_1)\right| + \left|\mathfrak{F}'(\kappa_2)\right|\right),\end{aligned} \tag{2.5}$$

whenever $\frac{1}{p} + \frac{1}{q} = 1$.

Proof If the absolute value in Lemma 1 is taken, we get

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2)\right]\right. \\ &\qquad\left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{J}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + \mathfrak{J}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\ &\quad\leq \frac{\kappa_2-\kappa_1}{4}\left[\int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1 + \left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right| d\vartheta\right. \\ &\qquad\left. + \int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)\left|\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1 + \left(\frac{1+\vartheta}{2}\right)\kappa_2\right)\right| d\vartheta\right].\end{aligned} \tag{2.6}$$

With the help of Hölder inequality in (2.6) and by utilizing the convexity of $|\mathfrak{F}'|^q$, we get

$$\begin{aligned}&\int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1 + \left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right| d\vartheta \\ &\quad\leq \left(\int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)^p d\vartheta\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1 + \left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|^q d\vartheta\right)^{\frac{1}{q}} \\ &\quad\leq \left(\int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)^p d\vartheta\right)^{\frac{1}{p}}\left[\int_0^1\left(\frac{1+\vartheta}{2}\left|\mathfrak{F}'(\kappa_1)\right|^q + \frac{1-\vartheta}{2}\left|\mathfrak{F}'(\kappa_2)\right|^q\right) d\vartheta\right]^{\frac{1}{q}} \\ &\quad= \left(\int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)^p d\vartheta\right)^{\frac{1}{p}}\left(\frac{3|\mathfrak{F}'(\kappa_1)|^q + |\mathfrak{F}'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}.\end{aligned} \tag{2.7}$$

Similarly, the following inequality is obtained:

$$\begin{aligned}&\int_0^1\left(\vartheta^\alpha+\frac{1}{3}\right)\left|\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)\right|d\vartheta\\&\quad\le\left(\int_0^1\left(\vartheta^\alpha+\frac{1}{3}\right)^p d\vartheta\right)^{\frac{1}{p}}\left(\frac{|\mathfrak{F}'(\kappa_1)|^q+3|\mathfrak{F}'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}.\end{aligned}\tag{2.8}$$

If (2.7) and (2.8) are substituted into (2.6), we have

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]\right.\\&\qquad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\&\quad\le\frac{\kappa_2-\kappa_1}{4}\left(\int_0^1\left(\vartheta^\alpha+\frac{1}{3}\right)d\vartheta\right)\left[\left(\frac{3|\mathfrak{F}'(\kappa_1)|^q+|\mathfrak{F}'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}\right.\\&\qquad\left.+\left(\frac{|\mathfrak{F}'(\kappa_1)|^q+3|\mathfrak{F}'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

The first inequality of (2.5) is proved. For the proof of the second inequality, let $\kappa_{11}=3|\mathfrak{F}'(\kappa_1)|^q$, $\kappa_{21}=|\mathfrak{F}'(\kappa_2)|^q$, $\kappa_{12}=|\mathfrak{F}'(\kappa_1)|^q$, and $\kappa_{22}=3|\mathfrak{F}'(\kappa_2)|^q$. Using the facts that

$$\sum_{k=1}^n(\kappa_{1k}+\kappa_{2k})^s\le\sum_{k=1}^n\kappa_{1k}^s+\sum_{k=1}^n\kappa_{2k}^s,\quad 0\le s<1,$$

and $1+3^{\frac{1}{q}}\le 4$, the required result can be established directly. The proof of Theorem 2 is finished. □

Corollary 1 *If Theorem 2 is written with $\alpha=1$, we get*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathfrak{F}(\vartheta)\,d\vartheta\right|\\&\quad\le\frac{\kappa_2-\kappa_1}{12}\left(\frac{4^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathfrak{F}'(\kappa_1)|^q+|\mathfrak{F}'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{F}'(\kappa_1)|^q+3|\mathfrak{F}'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}\right]\\&\quad\le\frac{\kappa_2-\kappa_1}{12}\left(\frac{4^{p+2}-4}{3(p+1)}\right)^{\frac{1}{p}}\left(\left|\mathfrak{F}'(\kappa_1)\right|+\left|\mathfrak{F}'(\kappa_2)\right|\right).\end{aligned}$$

Theorem 3 *Assume that all the assumptions of Lemma 1 are met. If the mapping $|\mathfrak{F}'|^q$, $q\ge 1$, is convex on $[\kappa_1,\kappa_2]$, then we get the following inequality:*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]\right.\\&\qquad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\&\quad\le\frac{\kappa_2-\kappa_1}{4}\left(\frac{\alpha+4}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\left[\left(\frac{1}{4}+\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}\right)\left|\mathfrak{F}'(\kappa_1)\right|^q\right.\right.\end{aligned}$$

$$+\left(\frac{1}{12}+\frac{1}{2(\alpha+1)(\alpha+2)}\right)\left|\mathfrak{F}'(\kappa_2)\right|^q\Bigg]^{\frac{1}{q}}$$
$$+\left[\left(\frac{1}{12}+\frac{1}{2(\alpha+1)(\alpha+2)}\right)\left|\mathfrak{F}'(\kappa_1)\right|^q+\left(\frac{1}{4}+\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}\right)\left|\mathfrak{F}'(\kappa_2)\right|^q\right]^{\frac{1}{q}}\Bigg).$$

Proof With help of the power-mean inequality in (2.6) and considering the convexity of $|\mathfrak{F}'|^q$, we get

$$\begin{aligned}
&\int_0^1\left(\vartheta^\alpha+\frac{1}{3}\right)\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|d\vartheta\\
&\leq\left(\int_0^1\left(\vartheta^\alpha+\frac{1}{3}\right)d\vartheta\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1\left(\vartheta^\alpha+\frac{1}{3}\right)\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|^q d\vartheta\right)^{\frac{1}{q}}\\
&\leq\left(\frac{\alpha+4}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left[\int_0^1\left(\vartheta^\alpha+\frac{1}{3}\right)\left(\frac{1+\vartheta}{2}\left|\mathfrak{F}'(\kappa_1)\right|^q+\frac{1-\vartheta}{2}\left|\mathfrak{F}'(\kappa_2)\right|^q\right)^{\frac{1}{q}}\right]\\
&=\left(\frac{\alpha+4}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{4}+\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}\right)\left|\mathfrak{F}'(\kappa_1)\right|^q\right.\\
&\quad\left.+\left(\frac{1}{12}+\frac{1}{2(\alpha+1)(\alpha+2)}\right)\left|\mathfrak{F}'(\kappa_2)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}\tag{2.9}$$

Similarly as in getting (2.9), we have

$$\begin{aligned}
&\int_0^1\left(\vartheta^\alpha+\frac{1}{3}\right)\left|\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)\right|d\vartheta\\
&\leq\left(\frac{\alpha+4}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{12}+\frac{1}{2(\alpha+1)(\alpha+2)}\right)\left|\mathfrak{F}'(\kappa_1)\right|^q\right.\\
&\quad\left.+\left(\frac{1}{4}+\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}\right)\left|\mathfrak{F}'(\kappa_2)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}\tag{2.10}$$

Substituting (2.9) and (2.10) into (2.6), we get

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{J}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathfrak{J}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
&\leq\frac{\kappa_2-\kappa_1}{4}\left(\frac{\alpha+4}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\left[\left(\frac{1}{4}+\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}\right)\left|\mathfrak{F}'(\kappa_1)\right|^q\right.\right.\\
&\quad\left.+\left(\frac{1}{12}+\frac{1}{2(\alpha+1)(\alpha+2)}\right)\left|\mathfrak{F}'(\kappa_2)\right|^q\right]^{\frac{1}{q}}\\
&\quad\left.+\left[\left(\frac{1}{12}+\frac{1}{2(\alpha+1)(\alpha+2)}\right)\left|\mathfrak{F}'(\kappa_1)\right|^q+\left(\frac{1}{4}+\frac{2\alpha+3}{2(\alpha+1)(\alpha+2)}\right)\left|\mathfrak{F}'(\kappa_2)\right|^q\right]^{\frac{1}{q}}\right).
\end{aligned}$$

This completes the proof. □

Remark 2 If we consider $\alpha = 1$ in Theorem 3, then we obtain

$$\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2)\right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathfrak{F}(\vartheta)\,d\vartheta\right| \leq \frac{5(\kappa_2-\kappa_1)}{24}\left(\left[\frac{4|\mathfrak{F}'(\kappa_1)|^q + |\mathfrak{F}'(\kappa_2)|^q}{5}\right]^{\frac{1}{q}} + \left[\frac{|\mathfrak{F}'(\kappa_1)|^q + 4|\mathfrak{F}'(\kappa_2)|^q}{5}\right]^{\frac{1}{q}}\right).$$

3 Milne-type inequality for bounded functions involving fractional integrals

Theorem 4 *Assume that the conditions of Lemma 1 hold. If there exist $m, M \in \mathbb{R}$ such that $m \leq \mathfrak{F}'(\vartheta) \leq M$ for $\vartheta \in [\kappa_1, \kappa_2]$, then*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2)\right]\right.\\&\quad\left.- \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + \mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\&\quad\leq \frac{\kappa_2-\kappa_1}{12}\left(\frac{\alpha+4}{\alpha+1}\right)(M-m).\end{aligned}$$

Proof With the help of Lemma 1, we get

$$\begin{aligned}&\frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2)\right]\\&\quad- \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + \mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\&\quad= \frac{\kappa_2-\kappa_1}{4}\int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)\left[\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1 + \left(\frac{1+\vartheta}{2}\right)\kappa_2\right)\right.\\&\quad\left.- \mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1 + \left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right]d\vartheta\\&\quad= \frac{\kappa_2-\kappa_1}{4}\int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)\left[\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1 + \left(\frac{1+\vartheta}{2}\right)\kappa_2\right) - \frac{m+M}{2}\right]d\vartheta\\&\quad+ \int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)\left[\frac{m+M}{2} - \mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1 + \left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right]d\vartheta.\end{aligned} \tag{3.1}$$

Taking the absolute value of (3.1), we have

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2)\right]\right.\\&\quad\left.- \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + \mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\&\quad= \frac{\kappa_2-\kappa_1}{4}\int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)\left|\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1 + \left(\frac{1+\vartheta}{2}\right)\kappa_2\right) - \frac{m+M}{2}\right|d\vartheta\\&\quad+ \int_0^1\left(\vartheta^\alpha + \frac{1}{3}\right)\left|\frac{m+M}{2} - \mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1 + \left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|d\vartheta.\end{aligned}$$

From $m \le \mathfrak{F}(\vartheta) \le M$ for $\vartheta \in [\kappa_1, \kappa_2]$, we get

$$\left| \mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1 + \left(\frac{1+\vartheta}{2}\right)\kappa_2 \right) - \frac{m+M}{2} \right| \le \frac{M-m}{2} \tag{3.2}$$

and

$$\left| \frac{m+M}{2} - \mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1 + \left(\frac{1-\vartheta}{2}\right)\kappa_2 \right) \right| \le \frac{M-m}{2}. \tag{3.3}$$

Using (3.2) and (3.3), we have

$$\begin{aligned}
&\left| \frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2) \right] \right. \\
&\qquad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}} \left[\mathfrak{I}^{\alpha}_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + \mathfrak{I}^{\alpha}_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \right] \right| \\
&\quad \le \frac{\kappa_2-\kappa_1}{4}(M-m) \int_0^1 \left(\vartheta^{\alpha} + \frac{1}{3} \right) d\vartheta \\
&\quad = \frac{\kappa_2-\kappa_1}{12}\left(\frac{\alpha+4}{\alpha+1}\right)(M-m).
\end{aligned}$$

The proof of the theorem is finished. □

Corollary 2 *Considering $\alpha = 1$ in Theorem 4, we obtain*

$$\left| \frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2) \right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathfrak{F}(\vartheta)\, d\vartheta \right| \le \frac{5(\kappa_2-\kappa_1)}{24}(M-m).$$

Corollary 3 *Under the assumptions of Theorem 4, if there exists $M \in \mathbb{R}^+$ such that $|\mathfrak{F}'(\vartheta)| \le M$ for all $\vartheta \in [\kappa_1, \kappa_2]$, then we have*

$$\begin{aligned}
&\left| \frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2) \right] \right. \\
&\qquad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}} \left[\mathfrak{I}^{\alpha}_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + \mathfrak{I}^{\alpha}_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \right] \right| \\
&\quad \le \frac{\kappa_2-\kappa_1}{6}\left(\frac{\alpha+4}{\alpha+1}\right)M.
\end{aligned}$$

Remark 3 If we choose $\alpha = 1$ in Corollary 3, then we get

$$\left| \frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2) \right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathfrak{F}(\vartheta)\, d\vartheta \right| \le \frac{5(\kappa_2-\kappa_1)}{12}M,$$

which was proved by Alomari and Liu [2].

4 Milne-type inequality for Lipschitz functions involving fractional integrals

In this part, we present some fractional Milne-type inequalities for Lipschitz functions.

Theorem 5 *Suppose that the assumptions of Lemma 1 hold. If $\mathfrak{F}'$ is an L-Lipschitz function on $[\kappa_1,\kappa_2]$, then we get the following inequality*:

$$\begin{aligned}
&\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]\\
&\quad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathfrak{I}^{\alpha}_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\
&\quad=\frac{(\kappa_2-\kappa_1)^2}{24}\left(\frac{\alpha+8}{\alpha+2}\right)L
\end{aligned}$$

Proof With help of Lemma 1 and since $\mathfrak{F}'$ is an L-Lipschitz function, we get

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathfrak{I}^{\alpha}_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
&\quad=\left|\frac{\kappa_2-\kappa_1}{4}\int_0^1\left(\vartheta^{\alpha}+\frac{1}{3}\right)\left[\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)\right.\right.\\
&\quad\left.\left.-\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right]d\vartheta\right|\\
&\quad\le\frac{\kappa_2-\kappa_1}{4}\int_0^1\left(\vartheta^{\alpha}+\frac{1}{3}\right)\left|\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)\right.\\
&\quad\left.-\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|d\vartheta\\
&\quad\le\frac{\kappa_2-\kappa_1}{4}\int_0^1\left(\vartheta^{\alpha}+\frac{1}{3}\right)L\vartheta(\kappa_2-\kappa_1)\,d\vartheta\\
&\quad=\frac{(\kappa_2-\kappa_1)^2}{4}L\left(\frac{1}{\alpha+2}+\frac{1}{6}\right).
\end{aligned}$$

The proof of this theorem is completed. □

Corollary 4 *If we consider $\alpha=1$ in Theorem 5, then we get*

$$\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathfrak{F}(\vartheta)\,d\vartheta\right|\le\frac{(\kappa_2-\kappa_1)^2}{8}L.$$

5 Milne-type inequality for functions of bounded variation involving fractional integrals

In this part, we show Milne-type inequality for fractional integrals involving functions of bounded variation.

Theorem 6 *Let $\mathfrak{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a mapping of bounded variation on $[\kappa_1,\kappa_2]$. Then we get*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]\right.\\&\left.\quad-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\le\frac{2}{3}\bigvee_{\kappa_1}^{\kappa_2}(\mathfrak{F}),\end{aligned}$$

where $\bigvee_c^d(\mathfrak{F})$ denotes the total variation of $\mathfrak{F}$ on $[c,d]$.

Proof Define the function $K_\alpha(\upsilon)$ by

$$K_\alpha(\upsilon)=\begin{cases}-(\frac{\kappa_1+\kappa_2}{2}-\upsilon)^\alpha-\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot2^\alpha}, & \kappa_1\le\upsilon\le\frac{\kappa_1+\kappa_2}{2},\\(\upsilon-\frac{\kappa_1+\kappa_2}{2})^\alpha+\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot2^\alpha}, & \frac{\kappa_1+\kappa_2}{2}<\upsilon\le\kappa_2.\end{cases}$$

By utilizing integration by parts, we get

$$\begin{aligned}&\int_{\kappa_1}^{\kappa_2}K_\alpha(\upsilon)\,d\mathfrak{F}(\upsilon)\\&\quad=-\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\left(\frac{\kappa_1+\kappa_2}{2}-\upsilon\right)^\alpha+\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot2^\alpha}\right)d\mathfrak{F}(\upsilon)\\&\qquad+\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\left(\left(\upsilon-\frac{\kappa_1+\kappa_2}{2}\right)^\alpha+\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot2^\alpha}\right)d\mathfrak{F}(\upsilon)\\&\quad=-\left(\left(\frac{\kappa_1+\kappa_2}{2}-\upsilon\right)^\alpha+\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot2^\alpha}\right)\mathfrak{F}(\upsilon)\Big|_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\\&\qquad-\alpha\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-\upsilon\right)^{\alpha-1}\mathfrak{F}(\upsilon)\,d\upsilon\\&\qquad+\left(\left(\upsilon-\frac{\kappa_1+\kappa_2}{2}\right)^\alpha+\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot2^\alpha}\right)\mathfrak{F}(\upsilon)\Big|_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\\&\qquad-\alpha\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\left(\upsilon-\frac{\kappa_1+\kappa_2}{2}\right)^{\alpha-1}\mathfrak{F}(\upsilon)\,d\upsilon\\&\quad=-\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot2^\alpha}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot2^{\alpha-2}}\mathfrak{F}(\kappa_1)-\Gamma(\alpha+1)\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\\&\qquad+\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot2^{\alpha-2}}\mathfrak{F}(\kappa_2)-\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot2^\alpha}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\Gamma(\alpha+1)\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\\&\quad=\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot2^{\alpha-1}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]\\&\qquad-\Gamma(\alpha+1)\left[\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right].\end{aligned}\tag{5.1}$$

That is,

$$\begin{aligned}
&\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right] \\
&\quad -\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right] \\
&\quad =\frac{2^{\alpha-1}}{(\kappa_2-\kappa_1)^\alpha}\int_{\kappa_1}^{\kappa_2}K_\alpha(\upsilon)d\mathfrak{F}(\upsilon).
\end{aligned}$$

It is well known that if $g,\mathfrak{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ are such that g is continuous on $[\kappa_1,\kappa_2]$ and $\mathfrak{F}$ is of bounded variation on $[\kappa_1,\kappa_2]$, then $\int_{\kappa_1}^{\kappa_2}g(\vartheta)\,d\mathfrak{F}(\vartheta)$ exists and

$$\left|\int_{\kappa_1}^{\kappa_2}g(\vartheta)\,d\mathfrak{F}(\vartheta)\right|\le\sup_{\vartheta\in[\kappa_1,\kappa_2]}\left|g(\vartheta)\right|\bigvee_{\kappa_1}^{\kappa_2}(\mathfrak{F}). \tag{5.2}$$

Otherwise, utilizing (5.2), we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]\right. \\
&\quad \left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\quad =\frac{2^{\alpha-1}}{(\kappa_2-\kappa_1)^\alpha}\left|\int_{\kappa_1}^{\kappa_2}K_\alpha(\upsilon)\,d\mathfrak{F}(\upsilon)\right| \\
&\quad \le\frac{2^{\alpha-1}}{(\kappa_2-\kappa_1)^\alpha}\left[\left|\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\left(\frac{\kappa_1+\kappa_2}{2}-\upsilon\right)^\alpha+\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot 2^\alpha}\right)d\mathfrak{F}(\upsilon)\right|\right. \\
&\qquad \left.+\left|\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\left(\left(\upsilon-\frac{\kappa_1+\kappa_2}{2}\right)^\alpha+\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot 2^\alpha}\right)d\mathfrak{F}(\upsilon)\right|\right] \\
&\quad \le\frac{2^{\alpha-1}}{(\kappa_2-\kappa_1)^\alpha}\left[\sup_{\upsilon\in[\kappa_1,\frac{\kappa_1+\kappa_2}{2}]}\left|\left(\frac{\kappa_1+\kappa_2}{2}-\upsilon\right)^\alpha+\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot 2^\alpha}\right|\bigvee_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\mathfrak{F})\right. \\
&\qquad \left.+\sup_{\upsilon\in[\frac{\kappa_1+\kappa_2}{2},\kappa_2]}\left|\left(\upsilon-\frac{\kappa_1+\kappa_2}{2}\right)^\alpha+\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot 2^\alpha}\right|\bigvee_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}(\mathfrak{F})\right] \\
&\quad =\frac{2^{\alpha-1}}{(\kappa_2-\kappa_1)^\alpha}\left[\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot 2^{\alpha-2}}\bigvee_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\mathfrak{F})+\frac{(\kappa_2-\kappa_1)^\alpha}{3\cdot 2^{\alpha-2}}\bigvee_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}(\mathfrak{F})\right] \\
&\quad =\frac{2}{3}\bigvee_{\kappa_1}^{\kappa_2}(\mathfrak{F}).
\end{aligned}$$

□

6 Conclusion

In this article, we established a fractional Milne-type inequality for differentiable mappings. In addition, we considered bounded functions, Lipschitz functions, functions of bounded variation, and obtained Milne-type inequalities for them. Moreover, generalizations of the results of Alomari and Liu [1] were presented. In a future work, curious readers can obtain new versions of Milne-type inequalities for different fractional integrals. What is more, researchers can obtain several new Milne-type inequalities using other notions of convexity.

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