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On some new quantum trapezoid-type inequalities for q-differentiable coordinated convex functions

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Abstract

In this paper, we establish several new inequalities for q-differentiable coordinated convex functions that are related to the right side of Hermite–Hadamard inequalities for coordinated convex functions. We also show that the inequalities proved in this paper generalize the results given in earlier works. Moreover, we give some examples in order to demonstrate our main results.

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1 Introduction

The Hermite–Hadamard inequality is a classical inequality stated as: If $f:[a,b]\to\mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}. \tag{1.1}$$

The double inequality (1.1) was introduced by Hermite [1] in 1883 and was investigated by Hadamard [2] in 1893.

Definition 1.1 ([3]) A function $f:\triangle\to\mathbb{R}$ is said to be coordinated convex, if the partial mappings

$$f_x:[c,d]\ni v\mapsto f(x,v)\in\mathbb{R}\quad\text{and}\quad f_y:[a,b]\ni u\mapsto f(u,y)\in\mathbb{R}$$

are convex for all $x\in(a,b)$ and $y\in(c,d)$.

A formal definition for coordinated convex functions may be stated as follows:

Definition 1.2 A function $f : \triangle \to \mathbb{R}$ is said to be coordinated convex, if

$$f\big(tx + (1-t)z, \lambda y + (1-\lambda)w\big) \le t\lambda f(x,y) + t(1-\lambda)f(x,w) + (1-t)\lambda f(z,y) + (1-t)(1-\lambda)f(z,w) \tag{1.2}$$

holds for all $t, \lambda \in [0,1]$ and $(x,y),(z,w) \in \triangle$.

Dragomir [3] presented the Hermite–Hadamard-type inequalities for coordinated convex functions in 2001 as follows:

Theorem 1.1 *If $f : \triangle \to \mathbb{R}$ is a coordinated convex function, then we have*

$$\begin{aligned} &f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy\right] \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)\,dy\,dx \\ &\le \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c) + f(x,d)\,dx \right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d f(a,y) + f(b,y)\,dy\right] \\ &\le \frac{1}{4}\big[f(a,c) + f(b,c) + f(a,d) + f(b,d)\big]. \end{aligned} \tag{1.3}$$

In 2012, Sarikaya and Set [4] proved some inequalities that give estimations between the middle and the rightmost terms in (1.3).

Theorem 1.2 *Let $f : \triangle \to \mathbb{R}$ be a partially differentiable function on $(a,b) \times (a,b)$. If $|\frac{\partial^2 f}{\partial t \partial s}|$ is coordinated convex on $\triangle$, then the following inequality holds*:

$$\begin{aligned} &\left|\frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)\,ds\,dt - A\right| \\ &\le \frac{(b-a)(d-c)}{16}\left[\frac{|\frac{\partial^2}{\partial t \partial s}f(a,c)| + |\frac{\partial^2}{\partial t \partial s}f(a,b)| + |\frac{\partial^2}{\partial t \partial s}f(b,c)| + |\frac{\partial^2}{\partial t \partial s}f(b,d)|}{4}\right], \end{aligned} \tag{1.4}$$

where

$$A = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,c) + f(x,d)\,dx + \frac{1}{d-c}\int_c^d f(a,y) + f(b,y)\,dx\right].$$

On the other hand, the concept of quantum calculus (sometimes called q-calculus) is known as the study of calculus with no limits. Note that q-calculus can be reduced to ordinary calculus if we take $\lim_{q\to 1}$. In 1910, Jackson [5] introduced the definite q-integral known as the q-Jackson integral. Quantum calculus has many applications in several mathematical areas such as combinatorics, number theory, orthogonal polynomials, basic hypergeometric functions, mechanics, quantum theory, and the theory of relativity, see for instance [6–11] and the references therein. The book by Kac and Cheung [12] covers the fundamental knowledge and also the basic theoretical concepts of quantum calculus.

2 Preliminaries

Throughout this paper, we let $\triangle := [a,b] \times [c,d] \subseteq \mathbb{R}$, $0 < q < 1$ and $0 < q_i < 1$ for $i = 1,2$. Also, here and below, we use the following notation:

$$[n]_q := \begin{cases} \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-2} + q^{n-1}, & \text{if } n \in \mathbb{R}^+, \\ 0, & \text{if } n = 0. \end{cases}$$

In 2013, Tariboon and Ntouyas [13] defined the ${}_aq$-derivative and ${}_aq$-integral of a function on finite intervals and proved some of its properties.

Definition 2.1 ([13]) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then, the ${}_aq$-derivative of f at $x \in (a,b]$ is defined by

$${}_aD_qf(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}.$$

The ${}_aq$-integral is defined by

$$\int_a^x f(t)\,{}_ad_qt = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f\bigl(q^n x + \bigl(1-q^n\bigr)a\bigr).$$

In [14], Alp et al. proved the following quantum Hermite–Hadamard inequality for convex functions using the quantum integrals:

Theorem 2.1 *If $f : [a,b] \to \mathbb{R}$ is a convex function, then we have*

$$f\left(\frac{qa+b}{1+q}\right) \le \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx \le \frac{qf(a)+f(b)}{1+q}. \tag{2.1}$$

In 2020, Bermudo et al. [15] defined the q-derivative and q-integral of a function on finite intervals that are called ${}^bq$-calculus

Definition 2.2 ([15]) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then, the ${}^bq$-derivative of f at $x \in [a,b)$ is defined by

$${}^bD_qf(x) = \frac{f(qx + (1-q)b) - f(x)}{(1-q)(b-x)}.$$

The ${}^bq$-integral is defined by

$$\int_x^b f(t)\,{}^bd_qt = (1-q)(b-x)\sum_{n=0}^{\infty} q^n f\bigl(q^n x + \bigl(1-q^n\bigr)b\bigr).$$

Bermudo et al. also proved the corresponding Hermite–Hadamard inequality for ${}^bq$-integrals, as follows:

Theorem 2.2 *If $f : [a, b] \to \mathbb{R}$ is a convex function, then we have*

$$f\left(\frac{a+qb}{1+q}\right) \le \frac{1}{b-a}\int_a^b f(x)\,{}^b d_q x \le \frac{f(a)+qf(b)}{1+q}. \tag{2.2}$$

In [16] and [17], the authors provide q-integrations by parts as follows:

Lemma 2.1 *For continuous functions $h, f : [a, b] \to \mathbb{R}$, the following equality holds*:

$$\begin{aligned}&\int_0^c h(t)\,{}_aD_q f\big(tb+(1-t)a\big)\,{}_0d_q t\\&\quad= \frac{h(t)f(tb+(1-t)a)}{b-a}\bigg|_0^c - \int_0^c f\big(tb+(1-t)a\big)\,{}_0D_q h(t)\,{}_0d_q t.\end{aligned} \tag{2.3}$$

Lemma 2.2 *For continuous functions $h, f : [a, b] \to \mathbb{R}$, the following equality holds*:

$$\begin{aligned}&\int_0^c h(t)\,{}^bD_q f\big(ta+(1-t)b\big)\,{}_0d_q t\\&\quad= \int_0^c f\big(ta+(1-t)b\big)\,{}_0D_q h(t)\,{}_0d_q t - \frac{h(t)f(ta+(1-t)b)}{b-a}\bigg|_0^c.\end{aligned} \tag{2.4}$$

Several papers were devoted to generalizations and estimations of the left and right sides of the inequalities (2.1) and (2.2). In [18], Noor et al. proved some bounds for the right-hand side of the inequality (2.1), whereas Alp et al. proved some bounds for the left-hand side of the inequality (2.1) in [14]. In [19], by using convex functions, Budak proved some bounds for the left- and right-hand sides of the inequality (2.2).

In [20], Ali et al. proved the following new version of the quantum Hermite–Hadamard inequality involving the ${}_aq$-integral and ${}^bq$-integral. They also proved some inequalities for estimations of the left- and right-hand sides of this inequality.

Theorem 2.3 *If $f : [a, b] \to \mathbb{R}$ is a convex function, then we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\,{}_ad_q x + \int_{\frac{a+b}{2}}^b f(x)\,{}^bd_q x\right] \le \frac{f(a)+f(b)}{2}.$$

Recently, Sitthiwirattham et al. [21] proved some new quantum Hermite–Hadamard inequalities for convex functions by using their new techniques.

Theorem 2.4 *If $f : [a, b] \to \mathbb{R}$ is a convex function, then we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\,{}^{\frac{a+b}{2}}d_q x + \int_{\frac{a+b}{2}}^b f(x)\,{}_{\frac{a+b}{2}}d_q x\right] \le \frac{f(a)+f(b)}{2}. \tag{2.5}$$

On the other hand, in [22] Latif et al. and in [23] Budak et al. introduced the quantum integrals for functions of two variables

Definition 2.3 ([22]) Suppose that $f : \triangle \to \mathbb{R}$ is a function of two variables. Then, the definite integral is given by

$$\int_a^x \int_c^y f(t,s)\,{}_cd_{q_2}s\,{}_ad_{q_1}t = (1-q_1)(1-q_2)(x-a)(y-c) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m f\big(q_1^n x + \big(1-q_1^n\big)a, q_2^m y + \big(1-q_2^m\big)c\big).$$

Definition 2.4 ([23]) Suppose that $f : \triangle \to \mathbb{R}$ is a function of two variables. Then, the definite integrals are given by

$$\int_a^x \int_y^d f(t,s)\,{}^dd_{q_2}s\,{}_ad_{q_1}t = (1-q_1)(1-q_2)(x-a)(d-y) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m f\big(q_1^n x + \big(1-q_1^n\big)a, q_2^m y + \big(1-q_2^m\big)d\big),$$

$$\int_x^b \int_c^y f(t,s)\,{}_cd_{q_2}s\,{}^bd_{q_1}t = (1-q_1)(1-q_2)(b-x)(y-c) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m f\big(q_1^n x + \big(1-q_1^n\big)b, q_2^m y + \big(1-q_2^m\big)c\big)$$

and

$$\int_x^b \int_y^d f(t,s)\,{}^dd_{q_2}s\,{}^bd_{q_1}t = (1-q_1)(1-q_2)(b-x)(d-y) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m f\big(q_1^n x + \big(1-q_1^n\big)b, q_2^m y + \big(1-q_2^m\big)d\big).$$

In the papers [22] and [23], by using these integrals, the authors also proved the Hermite–Hadamard inequalities for coordinated convex functions. By using the concepts given in Definition 2.3 and Definition 2.4, many authors proved several important inequalities [24–28].

In [22] Latif et al. and in [29] Budak et al. introduced the following q_1, q_2-derivatives for functions of two variables

Definition 2.5 ([22]) Let $f : \triangle \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function of two variables. Then, the partial ${}_aq$-, ${}_cq$-, and ${}_{a,c}q$-derivatives at $(x,y) \in \triangle$ can be given as follows:

$$\frac{{}_a\partial_{q_1} f(x,y)}{{}_a\partial_{q_1} x} = \frac{f(q_1x + (1-q_1)a, y) - f(x,y)}{(1-q_1)(x-a)}, \quad x \neq b,$$

$$\frac{{}_c\partial_{q_2} f(x,y)}{{}_c\partial_{q_2} y} = \frac{f(x, q_2y + (1-q_2)c) - f(x,y)}{(1-q_2)(y-c)}, \quad y \neq c,$$

$$\frac{{}_{a,c}\partial^2_{q_1,q_2} f(x,y)}{{}_a\partial_{q_1} x\,{}_c\partial_{q_2} y} = \frac{1}{(x-a)(y-c)(1-q_1)(1-q_2)} \times \big[f\big(q_1x + (1-q_1)a, q_2y + (1-q_2)c\big)$$

$$-f\big(q_1x+(1-q_1)a,y\big)-f\big(x,q_2y+(1-q_2)c\big)+f(x,y)\big],$$
$$x\neq a, y\neq c.$$

Definition 2.6 ([29]) Let $f:\triangle\subseteq\mathbb{R}^2\to\mathbb{R}$ be a continuous function of two variables. Then, the partial ${}^bq$, ${}^dq$, ${}_a^dq$, ${}_c^bq$, and ${}^{b,d}q$-derivatives at $(x,y)\in\triangle$ can be given as follows:

$$\frac{{}^b\partial_{q_1}f(x,y)}{{}^b\partial_{q_1}x}=\frac{f(q_1x+(1-q_1)b,y)-f(x,y)}{(1-q_1)(b-x)},\quad x\neq b,$$
$$\frac{{}^d\partial_{q_2}f(x,y)}{{}^b\partial_{q_2}y}=\frac{f(x,q_2y+(1-q_2)d)-f(x,y)}{(1-q_2)(d-y)},\quad y\neq d,$$
$$\begin{aligned}\frac{{}_a^d\partial^2_{q_1,q_2}f(x,y)}{{}_a\partial_{q_1}x\,{}^d\partial_{q_2}y}&=\frac{1}{(x-a)(d-y)(1-q_1)(1-q_2)}\\&\quad\times\big[f\big(q_1x+(1-q_1)a,q_2y+(1-q_2)d\big)\\&\quad-f\big(q_1x+(1-q_1)a,y\big)-f\big(x,q_2y+(1-q_2)d\big)+f(x,y)\big],\\&\quad x\neq a, y\neq d,\end{aligned}$$
$$\begin{aligned}\frac{{}_c^b\partial^2_{q_1,q_2}f(x,y)}{{}^b\partial_{q_1}x\,{}_c\partial_{q_2}y}&=\frac{1}{(b-x)(y-c)(1-q_1)(1-q_2)}\\&\quad\times\big[f\big(q_1x+(1-q_1)b,q_2y+(1-q_2)c\big)\\&\quad-f\big(q_1x+(1-q_1)b,y\big)-f\big(x,q_2y+(1-q_2)c\big)+f(x,y)\big],\\&\quad x\neq b, y\neq c,\end{aligned}$$
$$\begin{aligned}\frac{{}^{b,d}\partial^2_{q_1,q_2}f(x,y)}{{}^b\partial_{q_1}x\,{}^d\partial_{q_2}y}&=\frac{1}{(b-x)(d-y)(1-q_1)(1-q_2)}\\&\quad\times\big[f\big(q_1x+(1-q_1)b,q_2y+(1-q_2)d\big)\\&\quad-f\big(q_1x+(1-q_1)b,y\big)-f\big(x,q_2y+(1-q_2)d\big)+f(x,y)\big],\\&\quad x\neq b, y\neq d.\end{aligned}$$

Recently, in [30], new versions of q-Hermite–Hadamard-type inequalities for coordinated convex functions have been established.

Theorem 2.5 *Let $f:\triangle\to\mathbb{R}$ be a coordinated convex function. Then, we have*

$$\begin{aligned}&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&\leq\frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}}f\left(x,\frac{c+d}{2}\right){}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^b f\left(x,\frac{c+d}{2}\right){}_{\frac{a+b}{2}}d_{q_1}x\right]\\&\quad+\frac{1}{2(d-c)}\left[\int_c^{\frac{c+d}{2}}f\left(\frac{a+b}{2},y\right){}^{\frac{c+d}{2}}d_{q_2}y+\int_{\frac{c+d}{2}}^d f\left(\frac{a+b}{2},y\right){}_{\frac{c+d}{2}}d_{q_2}y\right]\\&\leq\frac{1}{(b-a)(d-c)}\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x\right.\end{aligned}$$

$$
\begin{aligned}
&+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\\
&+\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg]\\
&\leq\frac{1}{4(b-a)}\Bigg[\int_a^{\frac{a+b}{2}} f(x,c)+f(x,d)\,{}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^b f(x,c)+f(x,d)\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg]\\
&+\frac{1}{4(d-c)}\Bigg[\int_c^{\frac{c+d}{2}} f(a,y)+f(b,y)\,{}^{\frac{c+d}{2}}d_{q_2}y+\int_{\frac{c+d}{2}}^d f(a,y)+f(b,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\Bigg]\\
&\leq\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
$$

In this paper, we study some estimations between the middle and the rightmost terms of the above inequalities.

3 Main results

In this section, we prove several new inequalities for q-differentiable coordinated convex functions that are related to the right side of Hermite–Hadamard inequalities for coordinated convex functions. We may start with some lemmas, which are useful in further considerations.

Lemma 3.1 *Let $f:[a,b]\to\mathbb{R}$. Then, we have*

$$
\begin{aligned}
&\int_0^1 qt\,{}_aD_qf\left(tb+(1-t)\frac{a+b}{2}\right){}_0d_qt\\
&=\frac{2}{b-a}f(b)-\frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^b f(x)\,{}_{\frac{a+b}{2}}d_qx.
\end{aligned}\tag{3.1}
$$

Proof By (2.3) and the definition of a q-integral, we have

$$
\begin{aligned}
&\int_0^1 qt\,{}_aD_qf\left(tb+(1-t)\frac{a+b}{2}\right){}_0d_qt\\
&=\frac{2qtf(tb+(1-t)\frac{a+b}{2})}{b-a}\Bigg|_0^1-\frac{2q}{b-a}\int_0^1 f\left(qtb+(1-qt)\frac{a+b}{2}\right){}_0d_qt\\
&=\frac{2qf(b)}{b-a}-\frac{2q}{b-a}(1-q)\sum_{n=0}^{\infty}q^nf\left(q^{n+1}b+\left(1-q^{n+1}\right)\frac{a+b}{2}\right)\\
&=\frac{2qf(b)}{b-a}-\frac{2}{b-a}(1-q)\sum_{n=0}^{\infty}q^{n+1}f\left(q^{n+1}b+\left(1-q^{n+1}\right)\frac{a+b}{2}\right)\\
&=\frac{2qf(b)}{b-a}-\frac{2}{b-a}(1-q)\sum_{n=1}^{\infty}q^nf\left(q^nb+\left(1-q^n\right)\frac{a+b}{2}\right)\\
&=\frac{2qf(b)}{b-a}-\frac{2}{b-a}(1-q)\left\{\sum_{n=0}^{\infty}q^nf\left(q^nb+\left(1-q^n\right)\frac{a+b}{2}\right)-f(b)\right\}\\
&=\frac{2}{b-a}f(b)-\frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^b f(x)\,{}_{\frac{a+b}{2}}d_qx,
\end{aligned}
$$

which completes the proof. □

Lemma 3.2 *Let $f:[a,b]\to\mathbb{R}$. Then, we have*

$$\begin{aligned}&\int_0^1 qt\,{}^bD_qf\left(ta+(1-t)\frac{a+b}{2}\right)\,{}_0d_qt\\&\quad=-\frac{2}{b-a}f(a)+\frac{4}{(b-a)^2}\int_a^{\frac{a+b}{2}}f(x)\,{}^{\frac{a+b}{2}}d_qx.\end{aligned}\tag{3.2}$$

Proof By (2.4) and the definition of a q-integral, we have

$$\begin{aligned}&\int_0^1 qt\,{}^bD_qf\left(ta+(1-t)\frac{a+b}{2}\right)\,{}_0d_qt\\&\quad=\frac{2q}{b-a}\int_0^1 f\left(qta+(1-qt)\frac{a+b}{2}\right)\,{}_0d_qt-\frac{2qtf(ta+(1-t)\frac{a+b}{2})}{b-a}\bigg|_0^1\\&\quad=\frac{2q}{b-a}(1-q)\sum_{n=0}^{\infty}q^nf\left(q^{n+1}a+\left(1-q^{n+1}\right)\frac{a+b}{2}\right)-\frac{2qf(a)}{b-a}\\&\quad=-\frac{2qf(a)}{b-a}+\frac{2}{b-a}(1-q)\sum_{n=0}^{\infty}q^{n+1}f\left(q^{n+1}a+\left(1-q^{n+1}\right)\frac{a+b}{2}\right)\\&\quad=-\frac{2qf(a)}{b-a}+\frac{2}{b-a}(1-q)\sum_{n=1}^{\infty}q^nf\left(q^na+\left(1-q^n\right)\frac{a+b}{2}\right)\\&\quad=-\frac{2qf(a)}{b-a}+\frac{2}{b-a}(1-q)\left\{\sum_{n=0}^{\infty}q^nf\left(q^na+\left(1-q^n\right)\frac{a+b}{2}\right)-f(a)\right\}\\&\quad=-\frac{2}{b-a}f(a)+\frac{4}{(b-a)^2}\int_a^{\frac{a+b}{2}}f(x)\,{}^{\frac{a+b}{2}}d_qx.\end{aligned}$$

The proof is completed. □

For convenience, we will use the following notations:

$$\Phi(t,s):=\frac{{}_{a,c}\partial^2_{q_1,q_2}f(t,s)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s},\qquad\Theta(t,s):=\frac{{}_a^d\partial^2_{q_1,q_2}f(t,s)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s},$$
$$\Psi(t,s):=\frac{{}_c^b\partial^2_{q_1,q_2}f(t,s)}{{}^b\partial_{q_1}t\,{}_c\partial_{q_2}s}\quad\text{and}\quad\Omega(t,s):=\frac{{}^{b,d}\partial^2_{q_1,q_2}f(t,s)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}.$$

Lemma 3.3 *Let $f:\triangle\to\mathbb{R}$ be a q-partially differential function on $\triangle^\circ$. If $\Phi(t,s)$, $\Theta(t,s)$, $\Psi(t,s)$, and $\Omega(t,s)$ are q-integrable on $\triangle$, then the following identity holds*:

$$\begin{aligned}&\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\\&\quad+\frac{1}{(b-a)(d-c)}\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x\right.\\&\quad+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\end{aligned}$$

$$
\begin{aligned}
&+\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg] \\
&-\frac{1}{2(b-a)}\Bigg[\int_{a}^{\frac{a+b}{2}} f(x,c)+f(x,d)\,{}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^{b} f(x,c)+f(x,d)\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg] \\
&-\frac{1}{2(d-c)}\Bigg[\int_{c}^{\frac{c+d}{2}} f(a,y)+f(b,y)\,{}^{\frac{c+d}{2}}d_{q_2}y+\int_{\frac{c+d}{2}}^{d} f(a,y)+f(b,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\Bigg] \\
=&\ \frac{(b-a)(d-c)}{16}\int_0^1\int_0^1 (q_1t)(q_2s)\Bigg[\Phi\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right) \\
&-\Theta\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right) \\
&-\Psi\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right) \\
&+\Omega\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\Bigg]\,{}_0d_{q_2}s\,{}_0d_{q_1}t.
\end{aligned}
$$

Proof We use Lemma 3.1 for variables s and t, and we obtain

$$
\begin{aligned}
I_1 :=&\int_0^1\int_0^1 (q_1t)(q_2s)\Phi\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right){}_0d_{q_2}s\,{}_0d_{q_1}t \\
=&\int_0^1\int_0^1 (q_1t)(q_2s)\Phi\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right){}_0d_{q_2}s\,{}_0d_{q_1}t \\
=&\int_0^1 q_1t\Bigg[\int_0^1 q_2s\Phi\left(tb+(1-t)\frac{a+b}{2},sd+(1-s)\frac{c+d}{2}\right){}_0d_{q_2}s\Bigg]{}_0d_{q_1}t \\
=&\int_0^1 q_1t\Bigg[\frac{2}{d-c}\frac{{}_a\partial_{q_1}}{{}_a\partial_{q_1}t}f\left(tb+(1-t)\frac{a+b}{2},d\right) \\
&-\frac{4}{(d-c)^2}\int_{\frac{c+d}{2}}^{d}\frac{{}_a\partial_{q_1}}{{}_a\partial_{q_1}t}f\left(tb+(1-t)\frac{a+b}{2},y\right){}_{\frac{c+d}{2}}d_{q_2}y\Bigg]{}_0d_{q_1}t \\
=&\ \frac{2}{d-c}\int_0^1 q_1t\frac{{}_a\partial_{q_1}}{{}_a\partial_{q_1}t}f\left(tb+(1-t)\frac{a+b}{2},d\right){}_0d_{q_1}t \\
&-\frac{4}{(d-c)^2}\int_{\frac{c+d}{2}}^{d}\Bigg[\int_0^1 q_1t\frac{{}_a\partial_{q_1}}{{}_a\partial_{q_1}t}f\left(tb+(1-t)\frac{a+b}{2},y\right){}_0d_{q_1}t\Bigg]{}_{\frac{c+d}{2}}d_{q_2}y \\
=&\ \frac{2}{d-c}\Bigg[\frac{2}{b-a}f(b,d)-\frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^{b}f(x,d)\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg] \\
&-\frac{4}{(d-c)^2}\int_{\frac{c+d}{2}}^{d}\Bigg[\frac{2}{b-a}f(b,y)-\frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^{b}f(x,y)\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg]{}_{\frac{c+d}{2}}d_{q_2}y \\
=&\ \frac{4}{(b-a)(d-c)}f(b,d)-\frac{8}{(b-a)^2(d-c)}\int_{\frac{a+b}{2}}^{b}f(x,d)\,{}_{\frac{a+b}{2}}d_{q_1}x \\
&-\frac{8}{(b-a)(d-c)^2}\int_{\frac{c+d}{2}}^{d}f(b,y)\,{}_{\frac{c+d}{2}}d_{q_2}y \\
&+\frac{16}{(b-a)^2(d-c)^2}\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x.
\end{aligned}
$$

Using Lemma 3.2 and Lemma 3.1 for variables s and t, respectively, we obtain

$$\begin{aligned}
I_2 &:= \int_0^1 \int_0^1 (q_1t)(q_2s)\Theta\left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t \\
&= \int_0^1 \int_0^1 (q_1t)(q_2s)\Theta\left(tb + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t \\
&= \int_0^1 q_1t\left[\int_0^1 q_2s\Theta\left(tb + (1-t)\frac{a+b}{2}, sc + (1-s)\frac{c+d}{2}\right)\,{}_0d_{q_2}s\right]{}_0d_{q_1}t \\
&= \int_0^1 q_1t\left[-\frac{2}{d-c}\frac{{}_a\partial_{q_1}}{{}_a\partial_{q_1}t}f\left(tb + (1-t)\frac{a+b}{2}, c\right)\right. \\
&\quad \left. + \frac{4}{(d-c)^2}\int_c^{\frac{c+d}{2}} \frac{{}_a\partial_{q_1}}{{}_a\partial_{q_1}t}f\left(tb + (1-t)\frac{a+b}{2}, y\right)\,{}^{\frac{c+d}{2}}d_{q_2}y\right]{}_0d_{q_1}t \\
&= -\frac{2}{d-c}\int_0^1 q_1t\frac{{}_a\partial_{q_1}}{{}_a\partial_{q_1}t}f\left(tb + (1-t)\frac{a+b}{2}, c\right){}_0d_{q_1}t \\
&\quad + \frac{4}{(d-c)^2}\int_c^{\frac{c+d}{2}}\left[\int_0^1 q_1t\frac{{}_a\partial_{q_1}}{{}_a\partial_{q_1}t}f\left(tb + (1-t)\frac{a+b}{2}, y\right){}_0d_{q_1}t\right]{}^{\frac{c+d}{2}}d_{q_2}y \\
&= -\frac{2}{d-c}\left[\frac{2}{b-a}f(b,c) - \frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^b f(x,c)\,{}_{\frac{a+b}{2}}d_{q_1}x\right] \\
&\quad + \frac{4}{(d-c)^2}\int_c^{\frac{c+d}{2}}\left[\frac{2}{b-a}f(b,y) - \frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^b f(x,y)\,{}_{\frac{a+b}{2}}d_{q_1}x\right]{}^{\frac{c+d}{2}}d_{q_2}y \\
&= -\frac{4}{(b-a)(d-c)}f(b,c) + \frac{8}{(b-a)^2(d-c)}\int_{\frac{a+b}{2}}^b f(x,c)\,{}_{\frac{a+b}{2}}d_{q_1}x \\
&\quad + \frac{8}{(b-a)(d-c)^2}\int_c^{\frac{c+d}{2}} f(b,y)\,{}^{\frac{c+d}{2}}d_{q_2}y \\
&\quad - \frac{16}{(b-a)^2(d-c)^2}\int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x.
\end{aligned}$$

Similarly, using Lemma 3.1 and Lemma 3.2 for variables s and t, respectively, we obtain

$$\begin{aligned}
I_3 &:= \int_0^1 \int_0^1 (q_1t)(q_2s)\Psi\left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}d\right)\,{}_0d_{q_2}s\,{}_0d_{q_1}t \\
&= -\frac{4}{(b-a)(d-c)}f(a,d) + \frac{8}{(b-a)^2(d-c)}\int_a^{\frac{a+b}{2}} f(x,d)\,{}^{\frac{a+b}{2}}d_{q_1}x \\
&\quad + \frac{8}{(b-a)(d-c)^2}\int_{\frac{c+d}{2}}^d f(a,y)\,{}_{\frac{c+d}{2}}d_{q_2}y \\
&\quad - \frac{16}{(b-a)^2(d-c)^2}\int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x.
\end{aligned}$$

Using Lemma 3.2 for variables s and t, we obtain

$$\begin{aligned}
I_4 &:= \int_0^1 \int_0^1 (q_1 t)(q_2 s)\Omega\left(\frac{1+t}{2}a + \frac{1-t}{2}b, \frac{1+s}{2}c + \frac{1-s}{2}d\right) {}_0d_{q_2}s \, {}_0d_{q_1}t \\
&= \frac{4}{(b-a)(d-c)} f(a,c) - \frac{8}{(b-a)^2(d-c)} \int_a^{\frac{a+b}{2}} f(x,c) \, {}^{\frac{a+b}{2}}d_{q_1}x \\
&\quad - \frac{8}{(b-a)(d-c)^2} \int_c^{\frac{c+d}{2}} f(a,y) \, {}^{\frac{c+d}{2}}d_{q_2}y \\
&\quad + \frac{16}{(b-a)^2(d-c)^2} \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x,y) \, {}^{\frac{c+d}{2}}d_{q_2}y \, {}^{\frac{a+b}{2}}d_{q_1}x.
\end{aligned}$$

Multiplying $(I_1 - I_2 - I_3 + I_4)$ by $\frac{(b-a)(d-c)}{16}$, the proof is completed. □

Theorem 3.1 *Under the assumptions of Lemma 3.3. If $|\Phi(t,s)|$, $|\Theta(t,s)|$, $|\Psi(t,s)|$, and $|\Omega(t,s)|$ are coordinated convex on $\triangle$, then the following inequality holds*:

$$\begin{aligned}
&\Bigg| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&\quad + \frac{1}{(b-a)(d-c)} \Bigg[\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x,y) \, {}^{\frac{c+d}{2}}d_{q_2}y \, {}^{\frac{a+b}{2}}d_{q_1}x \\
&\quad + \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^{d} f(x,y) \, {}_{\frac{c+d}{2}}d_{q_2}y \, {}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} \int_c^{\frac{c+d}{2}} f(x,y) \, {}^{\frac{c+d}{2}}d_{q_2}y \, {}_{\frac{a+b}{2}}d_{q_1}x \\
&\quad + \int_{\frac{a+b}{2}}^{b} \int_{\frac{c+d}{2}}^{d} f(x,y) \, {}_{\frac{c+d}{2}}d_{q_2}y \, {}_{\frac{a+b}{2}}d_{q_1}x \Bigg] \\
&\quad - \frac{1}{2(b-a)} \Bigg[\int_a^{\frac{a+b}{2}} f(x,c)+f(x,d) \, {}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f(x,c)+f(x,d) \, {}_{\frac{a+b}{2}}d_{q_1}x \Bigg] \\
&\quad - \frac{1}{2(d-c)} \Bigg[\int_c^{\frac{c+d}{2}} f(a,y)+f(b,y) \, {}^{\frac{c+d}{2}}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f(a,y)+f(b,y) \, {}_{\frac{c+d}{2}}d_{q_2}y \Bigg] \Bigg| \\
&\leq \frac{(b-a)(d-c)}{64[2]_{q_1}[2]_{q_2}[3]_{q_1}[3]_{q_2}} \\
&\quad \times \big[q_1^3 q_2^3 \big(|\Phi(a,c)| + |\Theta(a,d)| + |\Psi(b,c)| + |\Omega(b,d)| \big) \\
&\quad + q_1^3 \big(2q_2 + 2q_2^2 + q_2^3\big) \big(|\Phi(a,d)| + |\Theta(a,c)| + |\Psi(b,d)| + |\Omega(b,c)| \big) \\
&\quad + q_2^3 \big(2q_1 + 2q_1^2 + q_1^3\big) \big(|\Phi(b,c)| + |\Theta(b,d)| + |\Psi(a,c)| + |\Omega(a,d)| \big) \\
&\quad + \big(2q_1 + 2q_1^2 + q_1^3\big)\big(2q_2 + 2q_2^2 + q_2^3\big) \big(|\Phi(b,d)| + |\Theta(b,c)| + |\Psi(a,d)| + |\Omega(a,c)| \big) \big].
\end{aligned} \tag{3.3}$$

Proof From Lemma 3.3, we have

$$\begin{aligned}
&\Bigg| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&\quad + \frac{1}{(b-a)(d-c)} \Bigg[\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x,y) \, {}^{\frac{c+d}{2}}d_{q_2}y \, {}^{\frac{a+b}{2}}d_{q_1}x
\end{aligned}$$

$$
\begin{aligned}
&+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x \\
&+\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg] \\
&-\frac{1}{2(b-a)}\Bigg[\int_a^{\frac{a+b}{2}} f(x,c)+f(x,d)\,{}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^b f(x,c)+f(x,d)\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg] \\
&-\frac{1}{2(d-c)}\Bigg[\int_c^{\frac{c+d}{2}} f(a,y)+f(b,y)\,{}^{\frac{c+d}{2}}d_{q_2}y+\int_{\frac{c+d}{2}}^d f(a,y)+f(b,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\Bigg]\Bigg| \\
&\leq\frac{(b-a)(d-c)}{16}\int_0^1\int_0^1(q_1t)(q_2s)\Bigg[\left|\Phi\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right| \\
&+\left|\Theta\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right| \\
&+\left|\Psi\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right| \\
&+\left|\Omega\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|\Bigg]\,{}_0d_{q_2}s\,{}_0d_{q_1}t.
\end{aligned}
\tag{3.4}
$$

By the coordinated convexity of $|\Phi(t,s)|$, $|\Theta(t,s)|$, $|\Psi(t,s)|$ and $|\Omega(t,s)|$, we obtain

$$
\begin{aligned}
&\left|\Phi\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right| \\
&\leq\frac{(1-t)(1-s)}{4}\left|\Phi(a,c)\right|+\frac{(1-t)(1+s)}{4}\left|\Phi(a,d)\right| \\
&+\frac{(1+t)(1-s)}{4}\left|\Phi(b,c)\right|+\frac{(1+t)(1+s)}{4}\left|\Phi(b,d)\right|,
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
&\left|\Theta\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right| \\
&\leq\frac{(1-t)(1+s)}{4}\left|\Theta(a,c)\right|+\frac{(1-t)(1-s)}{4}\left|\Theta(a,d)\right| \\
&+\frac{(1+t)(1+s)}{4}\left|\Theta(b,c)\right|+\frac{(1+t)(1-s)}{4}\left|\Theta(b,d)\right|,
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
&\left|\Psi\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right| \\
&\leq\frac{(1+t)(1-s)}{4}\left|\Psi(a,c)\right|+\frac{(1+t)(1+s)}{4}\left|\Psi(a,d)\right| \\
&+\frac{(1-t)(1-s)}{4}\left|\Psi(b,c)\right|+\frac{(1-t)(1+s)}{4}\left|\Psi(b,d)\right|
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
&\left|\Omega\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right| \\
&\leq\frac{(1+t)(1+s)}{4}\left|\Omega(a,c)\right|+\frac{(1+t)(1-s)}{4}\left|\Omega(a,d)\right| \\
&+\frac{(1-t)(1+s)}{4}\left|\Omega(b,c)\right|+\frac{(1-t)(1-s)}{4}\left|\Omega(b,d)\right|.
\end{aligned}
\tag{3.8}
$$

Replacing (3.5)–(3.8) in (3.4), we obtain

$$\begin{aligned}
&\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&\quad + \frac{1}{(b-a)(d-c)}\Bigg[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x \\
&\quad + \int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x \\
&\quad + \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg] \\
&\quad - \frac{1}{2(b-a)}\Bigg[\int_a^{\frac{a+b}{2}} f(x,c)+f(x,d)\,{}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f(x,c)+f(x,d)\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg] \\
&\quad - \frac{1}{2(d-c)}\Bigg[\int_c^{\frac{c+d}{2}} f(a,y)+f(b,y)\,{}^{\frac{c+d}{2}}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f(a,y)+f(b,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\Bigg]\Bigg| \\
&\leq \frac{(b-a)(d-c)}{64}\int_0^1\int_0^1 (q_1t)(q_2s) \\
&\quad \times \big[(1-t)(1-s)\big(\big|\Phi(a,c)\big| + \big|\Theta(a,d)\big| + \big|\Psi(b,c)\big| + \big|\Omega(b,d)\big|\big) \\
&\quad + (1-t)(1+s)\big(\big|\Phi(a,d)\big| + \big|\Theta(a,c)\big| + \big|\Psi(b,d)\big| + \big|\Omega(b,c)\big|\big) \\
&\quad + (1+t)(1-s)\big(\big|\Phi(b,c)\big| + \big|\Theta(b,d)\big| + \big|\Psi(a,c)\big| + \big|\Omega(a,d)\big|\big) \\
&\quad + (1+t)(1+s)\big(\big|\Phi(b,d)\big| + \big|\Theta(b,c)\big| + \big|\Psi(a,d)\big| + \big|\Omega(a,c)\big|\big)\big]\,{}_0d_{q_2}s\,{}_0d_{q_1}t. \qquad (3.9)
\end{aligned}$$

Evaluating each integral in (3.9), we obtain (3.3). The proof is completed. □

Remark 3.1 If we take the limit $q_1, q_2 \to 1$, then (3.3) reduces to (1.4).

Theorem 3.2 *Under the assumptions of Lemma 3.3. If* $|\Phi(t,s)|^r$, $|\Theta(t,s)|^r$, $|\Psi(t,s)|^r$, *and* $|\Omega(t,s)|^r$ *are coordinated convex on* $\triangle$ *and* $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, *then the following inequality holds*:

$$\begin{aligned}
&\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&\quad + \frac{1}{(b-a)(d-c)}\Bigg[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x \\
&\quad + \int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x \\
&\quad + \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg] \\
&\quad - \frac{1}{2(b-a)}\Bigg[\int_a^{\frac{a+b}{2}} f(x,c)+f(x,d)\,{}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f(x,c)+f(x,d)\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg] \\
&\quad - \frac{1}{2(d-c)}\Bigg[\int_c^{\frac{c+d}{2}} f(a,y)+f(b,y)\,{}^{\frac{c+d}{2}}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f(a,y)+f(b,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\Bigg]\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(b-a)(d-c)}{16}\left[\frac{q_1^p q_2^p}{[p+1]_{q_1}[p+1]_{q_2}}\right]^{1/p}\left[\frac{1}{4[2]_{q_1}[2]_{q_2}}\right]^{1/r} \\
&\times \Big[\big(q_1q_2\left|\Phi(a,c)\right|^r + q_1(2+q_2)\left|\Phi(a,d)\right|^r + q_2(2+q_1)\left|\Phi(b,c)\right|^r \\
&+ (2+q_1)(2+q_2)\left|\Phi(b,d)\right|^r\big)^{1/r} \\
&+ \big(q_1(2+q_2)\left|\Theta(a,c)\right|^r + q_1q_2\left|\Theta(a,d)\right|^r + (2+q_1)(2+q_2)\left|\Theta(b,c)\right|^r \\
&+ q_2(2+q_1)\left|\Theta(b,d)\right|^r\big)^{1/r} \\
&+ \big(q_2(2+q_1)\left|\Psi(a,c)\right|^r + (2+q_1)(2+q_2)\left|\Psi(a,d)\right|^r + q_1q_2\left|\Psi(b,c)\right|^r \\
&+ q_1(2+q_2)\left|\Psi(b,d)\right|^r\big)^{1/r} \\
&+ \big((2+q_1)(2+q_2)\left|\Omega(a,c)\right|^r + q_2(2+q_1)\left|\Omega(a,d)\right|^r + q_1(2+q_2)\left|\Omega(b,c)\right|^r \\
&+ q_1q_2\left|\Omega(b,d)\right|^r\big)^{1/r}\Big]. \qquad (3.10)
\end{aligned}$$

Proof From Lemma 3.3, we have

$$\begin{aligned}
&\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&+ \frac{1}{(b-a)(d-c)}\Bigg[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x \\
&+ \int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x \\
&+ \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg] \\
&- \frac{1}{2(b-a)}\Bigg[\int_a^{\frac{a+b}{2}} f(x,c)+f(x,d)\,{}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^{b} f(x,c)+f(x,d)\,{}_{\frac{a+b}{2}}d_{q_1}x\Bigg] \\
&- \frac{1}{2(d-c)}\Bigg[\int_c^{\frac{c+d}{2}} f(a,y)+f(b,y)\,{}^{\frac{c+d}{2}}d_{q_2}y + \int_{\frac{c+d}{2}}^{d} f(a,y)+f(b,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\Bigg]\Bigg| \\
&\leq \frac{(b-a)(d-c)}{16}\int_0^1\int_0^1 (q_1t)(q_2s)\Bigg[\left|\Phi\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right| \\
&+ \left|\Theta\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right| \\
&+ \left|\Psi\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right| \\
&+ \left|\Omega\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|\Bigg]\,{}_0d_{q_2}s\,{}_0d_{q_1}t. \qquad (3.11)
\end{aligned}$$

Now, using the q-Hölder inequality for double integrals, since $|\Phi(t,s)|^r$ is coordinated convex, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1 (q_1t)(q_2s)\left|\Phi\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|\,{}_0d_{q_2}s\,{}_0d_{q_1}t \\
&\leq \left[\int_0^1\int_0^1 (q_1t)^p(q_2s)^p\,{}_0d_{q_2}s\,{}_0d_{q_1}t\right]^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&\times \left[\int_0^1\int_0^1 \left|\Phi\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^r {}_0d_{q_2}s\,{}_0d_{q_1}t\right]^{1/r}\\
&= \left[\frac{q_1^p q_2^p}{[p+1]_{q_1}[p+1]_{q_2}}\right]^{1/p}\\
&\quad\times \left[\int_0^1\int_0^1 \left|\Phi\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^r {}_0d_{q_2}s\,{}_0d_{q_1}t\right]^{1/r}\\
&\le \left[\frac{q_1^p q_2^p}{[p+1]_{q_1}[p+1]_{q_2}}\right]^{1/p}\\
&\quad\times \left[\int_0^1\int_0^1 \frac{(1-t)(1-s)}{4}\left|\Phi(a,c)\right|^r+\frac{(1-t)(1+s)}{4}\left|\Phi(a,d)\right|^r\right.\\
&\quad\left.+\frac{(1+t)(1-s)}{4}\left|\Phi(b,c)\right|^r+\frac{(1+t)(1+s)}{4}\left|\Phi(b,d)\right|^r {}_0d_{q_2}s\,{}_0d_{q_1}t\right]^{1/r}. \qquad (3.12)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_0^1\int_0^1 (q_1t)(q_2s)\left|\Theta\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right| {}_0d_{q_2}s\,{}_0d_{q_1}t\\
&\quad\le \left[\frac{q_1^p q_2^p}{[p+1]_{q_1}[p+1]_{q_2}}\right]^{1/p}\\
&\quad\quad\times \left[\int_0^1\int_0^1 \frac{(1-t)(1+s)}{4}\left|\Theta(a,c)\right|^r+\frac{(1-t)(1-s)}{4}\left|\Theta(a,d)\right|^r\right.\\
&\quad\quad\left.+\frac{(1+t)(1+s)}{4}\left|\Theta(b,c)\right|^r+\frac{(1+t)(1-s)}{4}\left|\Theta(b,d)\right|^r {}_0d_{q_2}s\,{}_0d_{q_1}t\right]^{1/r}, \qquad (3.13)\\
&\int_0^1\int_0^1 (q_1t)(q_2s)\left|\Psi\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right| {}_0d_{q_2}s\,{}_0d_{q_1}t\\
&\quad\le \left[\frac{q_1^p q_2^p}{[p+1]_{q_1}[p+1]_{q_2}}\right]^{1/p}\\
&\quad\quad\times \left[\int_0^1\int_0^1 \frac{(1+t)(1-s)}{4}\left|\Psi(a,c)\right|^r+\frac{(1+t)(1+s)}{4}\left|\Psi(a,d)\right|^r\right.\\
&\quad\quad\left.+\frac{(1-t)(1-s)}{4}\left|\Psi(b,c)\right|^r+\frac{(1-t)(1+s)}{4}\left|\Psi(b,d)\right|^r {}_0d_{q_2}s\,{}_0d_{q_1}t\right]^{1/r} \qquad (3.14)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1\int_0^1 (q_1t)(q_2s)\left|\Omega\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right| {}_0d_{q_2}s\,{}_0d_{q_1}t\\
&\quad\le \left[\frac{q_1^p q_2^p}{[p+1]_{q_1}[p+1]_{q_2}}\right]^{1/p}\\
&\quad\quad\times \left[\int_0^1\int_0^1 \frac{(1+t)(1+s)}{4}\left|\Omega(a,c)\right|^r+\frac{(1+t)(1-s)}{4}\left|\Omega(a,d)\right|^r\right.\\
&\quad\quad\left.+\frac{(1-t)(1+s)}{4}\left|\Omega(b,c)\right|^r+\frac{(1-t)(1-s)}{4}\left|\Omega(b,d)\right|^r {}_0d_{q_2}s\,{}_0d_{q_1}t\right]^{1/r}. \qquad (3.15)
\end{aligned}
$$

Replacing (3.12)–(3.15) in (3.11) and calculating each integral, we obtain

$$\begin{aligned}
&\bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\\
&\quad+\frac{1}{(b-a)(d-c)}\bigg[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x\\
&\quad+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\\
&\quad+\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\bigg]\\
&\quad-\frac{1}{2(b-a)}\bigg[\int_a^{\frac{a+b}{2}} f(x,c)+f(x,d)\,{}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^{b} f(x,c)+f(x,d)\,{}_{\frac{a+b}{2}}d_{q_1}x\bigg]\\
&\quad-\frac{1}{2(d-c)}\bigg[\int_c^{\frac{c+d}{2}} f(a,y)+f(b,y)\,{}^{\frac{c+d}{2}}d_{q_2}y+\int_{\frac{c+d}{2}}^{d} f(a,y)+f(b,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\bigg]\bigg|\\
&\le\frac{(b-a)(d-c)}{16}\bigg[\frac{q_1^p q_2^p}{[p+1]_{q_1}[p+1]_{q_2}}\bigg]^{1/p}\bigg[\frac{1}{4[2]_{q_1}[2]_{q_2}}\bigg]^{1/r}\\
&\quad\times\Big[\big(q_1q_2\big|\Phi(a,c)\big|^r+q_1(2+q_2)\big|\Phi(a,d)\big|^r+q_2(2+q_1)\big|\Phi(b,c)\big|^r\\
&\quad+(2+q_1)(2+q_2)\big|\Phi(b,d)\big|^r\big)^{1/r}\\
&\quad+\big(q_1(2+q_2)\big|\Theta(a,c)\big|^r+q_1q_2\big|\Theta(a,d)\big|^r+(2+q_1)(2+q_2)\big|\Theta(b,c)\big|^r\\
&\quad+q_2(2+q_1)\big|\Theta(b,d)\big|^r\big)^{1/r}\\
&\quad+\big(q_2(2+q_1)\big|\Psi(a,c)\big|^r+(2+q_1)(2+q_2)\big|\Psi(a,d)\big|^r+q_1q_2\big|\Psi(b,c)\big|^r\\
&\quad+q_1(2+q_2)\big|\Psi(b,d)\big|^r\big)^{1/r}\\
&\quad+\big((2+q_1)(2+q_2)\big|\Omega(a,c)\big|^r+q_2(2+q_1)\big|\Omega(a,d)\big|^r+q_1(2+q_2)\big|\Omega(b,c)\big|^r\\
&\quad+q_1q_2\big|\Omega(b,d)\big|^r\big)^{1/r}\Big].
\end{aligned}$$

The proof is completed. □

Theorem 3.3 *Under the assumptions of Lemma 3.3. If* $|\Phi(t,s)|^r$, $|\Theta(t,s)|^r$, $|\Psi(t,s)|^r$, *and* $|\Omega(t,s)|^r$ *are coordinated convex on* $\triangle$ *and* $r\ge 1$, *then the following inequality holds*:

$$\begin{aligned}
&\bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\\
&\quad+\frac{1}{(b-a)(d-c)}\bigg[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x\\
&\quad+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\\
&\quad+\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\bigg]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}} f(x,c)+f(x,d)\,{}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^b f(x,c)+f(x,d)\,{}_{\frac{a+b}{2}}d_{q_1}x\right] \\
&- \frac{1}{2(d-c)}\left[\int_c^{\frac{c+d}{2}} f(a,y)+f(b,y)\,{}^{\frac{c+d}{2}}d_{q_2}y + \int_{\frac{c+d}{2}}^d f(a,y)+f(b,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\right]\Bigg| \\
&\leq \frac{(b-a)(d-c)}{16}\left[\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\right]^{1-1/r}\left[\frac{1}{4[2]_{q_1}[2]_{q_2}[3]_{q_1}[3]_{q_2}}\right]^{1/r} \\
&\times \Big[\big(q_1^3q_2^3\big|\Phi(a,c)\big|^r + q_1^3\big(2q_2+2q_2^2+q_2^3\big)\big|\Phi(a,d)\big|^r \\
&+ q_2^3\big(2q_1+2q_1^2+q_1^3\big)\big|\Phi(b,c)\big|^r + \big(2q_1+2q_1^2+q_1^3\big)\big(2q_2+2q_2^2+q_2^3\big)\big|\Phi(b,d)\big|^r\big)^{1/r} \\
&+ \big(q_1^3\big(2q_2+2q_2^2+q_2^3\big)\big|\Theta(a,c)\big|^r + q_1^3q_2^3\big|\Theta(a,d)\big|^r \\
&+ \big(2q_1+2q_1^2+q_1^3\big)\big(2q_2+2q_2^2+q_2^3\big)\big|\Theta(b,c)\big|^r + q_2^3\big(2q_1+2q_1^2+q_1^3\big)\big|\Theta(b,d)\big|^r\big)^{1/r} \\
&+ \big(q_2^3\big(2q_1+2q_1^2+q_1^3\big)\big|\Psi(a,c)\big|^r + \big(2q_1+2q_1^2+q_1^3\big)\big(2q_2+2q_2^2+q_2^3\big)\big|\Psi(a,d)\big|^r \\
&+ q_1^3q_2^3\big|\Psi(b,c)\big|^r + q_1^3\big(2q_2+2q_2^2+q_2^3\big)\big|\Psi(b,d)\big|^r\big)^{1/r} \\
&+ \big(\big(2q_1+2q_1^2+q_1^3\big)\big(2q_2+2q_2^2+q_2^3\big)\big|\Omega(a,c)\big|^r + q_2^3\big(2q_1+2q_1^2+q_1^3\big)\big|\Omega(a,d)\big|^r \\
&+ q_1^3\big(2q_2+2q_2^2+q_2^3\big)\big|\Omega(b,c)\big|^r + q_1^3q_2^3\big|\Omega(b,d)\big|^r\big)^{1/r}\Big]. \qquad (3.16)
\end{aligned}
$$

Proof From Lemma 3.3, we have

$$
\begin{aligned}
&\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&+ \frac{1}{(b-a)(d-c)}\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x\right. \\
&+ \int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x \\
&\left.+ \int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\right] \\
&- \frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}} f(x,c)+f(x,d)\,{}^{\frac{a+b}{2}}d_{q_1}x + \int_{\frac{a+b}{2}}^b f(x,c)+f(x,d)\,{}_{\frac{a+b}{2}}d_{q_1}x\right] \\
&- \frac{1}{2(d-c)}\left[\int_c^{\frac{c+d}{2}} f(a,y)+f(b,y)\,{}^{\frac{c+d}{2}}d_{q_2}y + \int_{\frac{c+d}{2}}^d f(a,y)+f(b,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\right]\Bigg| \\
&\leq \frac{(b-a)(d-c)}{16}\int_0^1\int_0^1 (q_1t)(q_2s)\left[\left|\Phi\left(\frac{1-t}{2}a+\frac{1+t}{2}b, \frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|\right. \\
&+ \left|\Theta\left(\frac{1-t}{2}a+\frac{1+t}{2}b, \frac{1+s}{2}c+\frac{1-s}{2}d\right)\right| \\
&+ \left|\Psi\left(\frac{1+t}{2}a+\frac{1-t}{2}b, \frac{1-s}{2}c+\frac{1+s}{2}d\right)\right| \\
&\left.+ \left|\Omega\left(\frac{1+t}{2}a+\frac{1-t}{2}b, \frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|\right]\,{}_0d_{q_2}s\,{}_0d_{q_1}t. \qquad (3.17)
\end{aligned}
$$

Now, using the q-power mean inequality for double integrals, since $|\Phi(t,s)|^r$ is coordinated convex, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1 (q_1t)(q_2s)\left|\Phi\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|\,{}_0d_{q_2}s\,{}_0d_{q_1}t\\
&\quad\le\left[\int_0^1\int_0^1 (q_1t)(q_2s)\,{}_0d_{q_2}s\,{}_0d_{q_1}t\right]^{1-1/r}\\
&\qquad\times\left[\int_0^1\int_0^1 (q_1t)(q_2s)\left|\Phi\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^r\,{}_0d_{q_2}s\,{}_0d_{q_1}t\right]^{1/r}\\
&\quad=\left[\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\right]^{1-1/r}\\
&\qquad\times\left[\int_0^1\int_0^1 (q_1t)(q_2s)\left|\Phi\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^r\,{}_0d_{q_2}s\,{}_0d_{q_1}t\right]^{1/r}\\
&\quad\le\left[\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\right]^{1-1/r}\\
&\qquad\times\left[\int_0^1\int_0^1 (q_1t)(q_2s)\left\{\frac{(1-t)(1-s)}{4}\left|\Phi(a,c)\right|^r+\frac{(1-t)(1+s)}{4}\left|\Phi(a,d)\right|^r\right.\right.\\
&\qquad\left.\left.+\frac{(1+t)(1-s)}{4}\left|\Phi(b,c)\right|^r+\frac{(1+t)(1+s)}{4}\left|\Phi(b,d)\right|^r\right\}\,{}_0d_{q_2}s\,{}_0d_{q_1}t\right]^{1/r}\\
&\quad=\left[\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\right]^{1-1/r}\left[\frac{1}{4[2]_{q_1}[2]_{q_2}[3]_{q_1}[3]_{q_2}}\right]^{1/r}\\
&\qquad\times\left[q_1^3q_2^3\left|\Phi(a,c)\right|^r+q_1^3\left(2q_2+2q_2^2+q_2^3\right)\left|\Phi(a,d)\right|^r\right.\\
&\qquad+q_2^3\left(2q_1+2q_1^2+q_1^3\right)\left|\Phi(b,c)\right|^r\\
&\qquad\left.+\left(2q_1+2q_1^2+q_1^3\right)\left(2q_2+2q_2^2+q_2^3\right)\left|\Phi(b,d)\right|^r\right]^{1/r}.
\end{aligned}\tag{3.18}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1\int_0^1 (q_1t)(q_2s)\left|\Theta\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|\,{}_0d_{q_2}s\,{}_0d_{q_1}t\\
&\quad\le\left[\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\right]^{1-1/r}\left[\frac{1}{4[2]_{q_1}[2]_{q_2}[3]_{q_1}[3]_{q_2}}\right]^{1/r}\\
&\qquad\times\left[q_1^3\left(2q_2+2q_2^2+q_2^3\right)\left|\Theta(a,c)\right|^r+q_1^3q_2^3\left|\Theta(a,d)\right|^r\right.\\
&\qquad+\left(2q_1+2q_1^2+q_1^3\right)\left(2q_2+2q_2^2+q_2^3\right)\left|\Theta(b,c)\right|^r\\
&\qquad\left.+q_2^3\left(2q_1+2q_1^2+q_1^3\right)\left|\Theta(b,d)\right|^r\right]^{1/r},
\end{aligned}\tag{3.19}$$

$$\begin{aligned}
&\int_0^1\int_0^1 (q_1t)(q_2s)\left|\Psi\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|\,{}_0d_{q_2}s\,{}_0d_{q_1}t\\
&\quad\le\left[\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\right]^{1-1/r}\left[\frac{1}{4[2]_{q_1}[2]_{q_2}[3]_{q_1}[3]_{q_2}}\right]^{1/r}\\
&\qquad\times\left[q_2^3\left(2q_1+2q_1^2+q_1^3\right)\left|\Psi(a,c)\right|^r+\left(2q_1+2q_1^2+q_1^3\right)\left(2q_2+2q_2^2+q_2^3\right)\left|\Psi(a,d)\right|^r\right.\\
&\qquad\left.+q_1^3q_2^3\left|\Psi(b,c)\right|^r+q_1^3\left(2q_2+2q_2^2+q_2^3\right)\left|\Psi(b,d)\right|^r\right]^{1/r}
\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1 (q_1t)(q_2s)\left|\Omega\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right| {}_0d_{q_2}s\, {}_0d_{q_1}t \\
&\quad\le \left[\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\right]^{1-1/r}\left[\frac{1}{4[2]_{q_1}[2]_{q_2}[3]_{q_1}[3]_{q_2}}\right]^{1/r} \\
&\qquad\times\Big[\big(2q_1+2q_1^2+q_1^3\big)\big(2q_2+2q_2^2+q_2^3\big)\big|\Omega(a,c)\big|^r+q_2^3\big(2q_1+2q_1^2+q_1^3\big)\big|\Omega(a,d)\big|^r \\
&\qquad+q_1^3\big(2q_2+2q_2^2+q_2^3\big)\big|\Omega(b,c)\big|^r+q_1^3q_2^3\big|\Omega(b,d)\big|^r\Big]^{1/r}.
\end{aligned} \tag{3.21}$$

Replacing (3.18)–(3.21) in (3.17), we obtain

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
&\qquad+\frac{1}{(b-a)(d-c)}\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x\right. \\
&\qquad+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}} f(x,y)\,{}^{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x \\
&\qquad\left.+\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d} f(x,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\,{}_{\frac{a+b}{2}}d_{q_1}x\right] \\
&\qquad-\frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}} f(x,c)+f(x,d)\,{}^{\frac{a+b}{2}}d_{q_1}x+\int_{\frac{a+b}{2}}^{b} f(x,c)+f(x,d)\,{}_{\frac{a+b}{2}}d_{q_1}x\right] \\
&\qquad\left.-\frac{1}{2(d-c)}\left[\int_c^{\frac{c+d}{2}} f(a,y)+f(b,y)\,{}^{\frac{c+d}{2}}d_{q_2}y+\int_{\frac{c+d}{2}}^{d} f(a,y)+f(b,y)\,{}_{\frac{c+d}{2}}d_{q_2}y\right]\right| \\
&\quad\le\frac{(b-a)(d-c)}{16}\left[\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\right]^{1-1/r}\left[\frac{1}{4[2]_{q_1}[2]_{q_2}[3]_{q_1}[3]_{q_2}}\right]^{1/r} \\
&\qquad\times\Big[\Big(q_1^3q_2^3\big|\Phi(a,c)\big|^r+q_1^3\big(2q_2+2q_2^2+q_2^3\big)\big|\Phi(a,d)\big|^r \\
&\qquad+q_2^3\big(2q_1+2q_1^2+q_1^3\big)\big|\Phi(b,c)\big|^r+\big(2q_1+2q_1^2+q_1^3\big)\big(2q_2+2q_2^2+q_2^3\big)\big|\Phi(b,d)\big|^r\Big)^{1/r} \\
&\qquad+\Big(q_1^3\big(2q_2+2q_2^2+q_2^3\big)\big|\Theta(a,c)\big|^r+q_1^3q_2^3\big|\Theta(a,d)\big|^r \\
&\qquad+\big(2q_1+2q_1^2+q_1^3\big)\big(2q_2+2q_2^2+q_2^3\big)\big|\Theta(b,c)\big|^r+q_2^3\big(2q_1+2q_1^2+q_1^3\big)\big|\Theta(b,d)\big|^r\Big)^{1/r} \\
&\qquad+\Big(q_2^3\big(2q_1+2q_1^2+q_1^3\big)\big|\Psi(a,c)\big|^r+\big(2q_1+2q_1^2+q_1^3\big)\big(2q_2+2q_2^2+q_2^3\big)\big|\Psi(a,d)\big|^r \\
&\qquad+q_1^3q_2^3\big|\Psi(b,c)\big|^r+q_1^3\big(2q_2+2q_2^2+q_2^3\big)\big|\Psi(b,d)\big|^r\Big)^{1/r} \\
&\qquad+\Big(\big(2q_1+2q_1^2+q_1^3\big)\big(2q_2+2q_2^2+q_2^3\big)\big|\Omega(a,c)\big|^r+q_2^3\big(2q_1+2q_1^2+q_1^3\big)\big|\Omega(a,d)\big|^r \\
&\qquad+q_1^3\big(2q_2+2q_2^2+q_2^3\big)\big|\Omega(b,c)\big|^r+q_1^3q_2^3\big|\Omega(b,d)\big|^r\Big)^{1/r}\Big].
\end{aligned}$$

The proof is completed. □

4 Examples

Now, we give some examples of our main results to demonstrate our theorems.

Example 4.1 Let $f:[0,1]\times[0,1]\to\mathbb{R}$ be a function defined by $f(x,y)=x^2y^2$. Then, f is q-partially differentiable. Moreover, $|\Phi(t,s)|$, $|\Theta(t,s)|$, $|\Psi(t,s)|$, and $|\Omega(t,s)|$ are q-integrable coordinated convex on $[0,1]\times[0,1]$. By applying Theorem 3.1 with $q_1=\frac{1}{4}$ and $q_2=\frac{3}{4}$, we have

$$\begin{aligned}
&\frac{f(0,0)+f(0,1)+f(1,0)+f(1,1)}{4}=\frac{1}{4},\\
&\frac{1}{(1-0)(1-0)}\biggl[\int_0^{\frac{0+1}{2}}\int_0^{\frac{0+1}{2}}x^2y^2\,{}^{\frac{0+1}{2}}d_{\frac{3}{4}}y\,{}^{\frac{0+1}{2}}d_{\frac{1}{4}}x\\
&\quad+\int_0^{\frac{0+1}{2}}\int_{\frac{0+1}{2}}^1x^2y^2\,{}_{\frac{0+1}{2}}d_{\frac{3}{4}}y\,{}^{\frac{0+1}{2}}d_{\frac{1}{4}}x+\int_{\frac{0+1}{2}}^1\int_0^{\frac{0+1}{2}}x^2y^2\,{}^{\frac{0+1}{2}}d_{\frac{3}{4}}y\,{}_{\frac{0+1}{2}}d_{\frac{1}{4}}x\\
&\quad+\int_{\frac{0+1}{2}}^1\int_{\frac{0+1}{2}}^1x^2y^2\,{}_{\frac{0+1}{2}}d_{\frac{3}{4}}y\,{}_{\frac{0+1}{2}}d_{\frac{1}{4}}x\biggr]\\
&=\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}x^2y^2\,{}^{\frac{1}{2}}d_{\frac{3}{4}}y\,{}^{\frac{1}{2}}d_{\frac{1}{4}}x+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1x^2y^2\,{}_{\frac{1}{2}}d_{\frac{3}{4}}y\,{}^{\frac{1}{2}}d_{\frac{1}{4}}x\\
&\quad+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}x^2y^2\,{}^{\frac{1}{2}}d_{\frac{3}{4}}y\,{}_{\frac{1}{2}}d_{\frac{1}{4}}x+\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1x^2y^2\,{}_{\frac{1}{2}}d_{\frac{3}{4}}y\,{}_{\frac{1}{2}}d_{\frac{1}{4}}x\\
&=\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^n\right)^2\left(1-\left(\frac{3}{4}\right)^m\right)^2\\
&\quad+\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^n\right)^2\left(1+\left(\frac{3}{4}\right)^m\right)^2\\
&\quad+\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(1+\left(\frac{1}{4}\right)^n\right)^2\left(1-\left(\frac{3}{4}\right)^m\right)^2\\
&\quad+\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^n\left(\frac{3}{4}\right)^m\frac{1}{4}\cdot\frac{1}{4}\left(1+\left(\frac{1}{4}\right)^n\right)^2\left(1+\left(\frac{3}{4}\right)^m\right)^2\\
&=\frac{85}{116{,}032}+\frac{11{,}339}{1{,}740{,}480}+\frac{1765}{116{,}032}+\frac{235{,}451}{1{,}740{,}480}=\frac{53}{336}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2(1-0)}\biggl[\int_0^{\frac{0+1}{2}}f(x,0)+f(x,1)\,{}^{\frac{0+1}{2}}d_{\frac{1}{4}}x+\int_{\frac{0+1}{2}}^1f(x,0)+f(x,1)\,{}_{\frac{0+1}{2}}d_{\frac{1}{4}}x\biggr]\\
&\quad+\frac{1}{2(1-0)}\biggl[\int_0^{\frac{0+1}{2}}f(0,y)+f(1,y)\,{}^{\frac{0+1}{2}}d_{\frac{3}{4}}y+\int_{\frac{0+1}{2}}^1f(0,y)+f(1,y)\,{}_{\frac{0+1}{2}}d_{\frac{3}{4}}y\biggr]\\
&=\frac{1}{2}\biggl[\int_0^{\frac{1}{2}}x^2\,{}^{\frac{1}{2}}d_{\frac{1}{4}}x+\int_{\frac{1}{2}}^1x^2\,{}_{\frac{1}{2}}d_{\frac{1}{4}}x+\int_0^{\frac{1}{2}}y^2\,{}^{\frac{1}{2}}d_{\frac{3}{4}}y+\int_{\frac{1}{2}}^1y^2\,{}_{\frac{1}{2}}d_{\frac{3}{4}}y\biggr]\\
&=\frac{1}{2}\biggl[\frac{3}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^n\right)^2+\frac{3}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\frac{1}{4}\left(1+\left(\frac{1}{4}\right)^n\right)^2
\end{aligned}$$

$$+\frac{1}{2}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\frac{1}{4}\left(1-\left(\frac{3}{4}\right)^n\right)^2+\frac{1}{4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\frac{1}{4}\left(1+\left(\frac{3}{4}\right)^n\right)^2\Bigg]$$

$$=\frac{1}{2}\left[\frac{68}{3360}+\frac{1412}{3360}+\frac{300}{8288}+\frac{2668}{8288}\right]=\frac{1241}{3108}.$$

Thus, the left-hand side of (3.3) is

$$\left|\frac{1}{4}+\frac{53}{336}-\frac{1241}{3108}\right|=\frac{5}{592}.$$

Next, we consider

$$\begin{aligned}
\Phi(t,s)&=\frac{{}_{0,0}\partial^2{}_{\frac{1}{4},\frac{3}{4}}t^2s^2}{{}_0\partial_{\frac{1}{4}}t\,{}_0\partial_{\frac{3}{4}}s}=\frac{4\cdot 4}{3\cdot 1\cdot ts}\left[\frac{9t^2s^2}{16\cdot 16}-\frac{t^2s^2}{16}-\frac{9t^2s^2}{16}+t^2s^2\right]\\
&=\frac{35ts}{16},\\
\Theta(t,s)&=\frac{{}_0^1\partial^2{}_{\frac{1}{4},\frac{3}{4}}t^2s^2}{{}_0\partial_{\frac{1}{4}}t\,{}^1\partial_{\frac{3}{4}}s}\\
&=\frac{4\cdot 4}{3\cdot 1\cdot t(1-s)}\left[\frac{t^2(3s+1)^2}{16\cdot 16}-\frac{t^2s^2}{16}-\frac{t^2(3s+1)^2}{16}+t^2s^2\right]\\
&=\frac{-5t(7s+1)}{16},\\
\Psi(t,s)&=\frac{{}_0^1\partial^2{}_{\frac{1}{4},\frac{3}{4}}t^2s^2}{{}^1\partial_{\frac{1}{4}}t\,{}_0\partial_{\frac{3}{4}}s}\\
&=\frac{4\cdot 4}{3\cdot 1\cdot(1-t)s}\left[\frac{9(t+3)^2s^2}{16\cdot 16}-\frac{(t+3)^2s^2}{16}-\frac{9t^2s^2}{16}+t^2s^2\right]\\
&=\frac{-7(5t+3)s}{16}
\end{aligned}$$

and

$$\begin{aligned}
\Omega(t,s)&=\frac{{}^{1,1}\partial^2{}_{\frac{1}{4},\frac{3}{4}}t^2s^2}{{}^1\partial_{\frac{1}{4}}t\,{}^1\partial_{\frac{3}{4}}s}\\
&=\frac{4\cdot 4}{3\cdot 1\cdot(1-t)(1-s)}\left[\frac{(t+3)^2(3s+1)^2}{16\cdot 16}-\frac{(t+3)^2s^2}{16}-\frac{t^2(3s+1)^2}{16}+t^2s^2\right]\\
&=\frac{(5t+3)(7s+1)}{16}.
\end{aligned}$$

Hence, the right-hand side of (3.3) is

$$\begin{aligned}
&\frac{(1-0)(1-0)}{64(1+\frac{1}{4})(1+\frac{3}{4})(1+\frac{1}{4}+\frac{1}{16})(1+\frac{3}{4}+\frac{9}{16})}\\
&\quad\times\Bigg[\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^3\left(\left|\Phi(0,0)\right|+\left|\Theta(0,1)\right|+\left|\Psi(1,0)\right|+\left|\Omega(1,1)\right|\right)\\
&\quad+\left(\frac{1}{4}\right)^3\left(\frac{2\cdot 3}{4}+\frac{2\cdot 9}{16}+\frac{27}{64}\right)\left(\left|\Phi(0,1)\right|+\left|\Theta(0,0)\right|+\left|\Psi(1,1)\right|+\left|\Omega(1,0)\right|\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{3}{4}\right)^3\left(\frac{2\cdot 1}{4}+\frac{2\cdot 1}{16}+\frac{1}{64}\right)\left(\left|\Phi(1,0)\right|+\left|\Theta(1,1)\right|+\left|\Psi(0,0)\right|+\left|\Omega(0,1)\right|\right)\\
&+\left(\frac{2\cdot 1}{4}+\frac{2\cdot 1}{16}+\frac{1}{64}\right)\left(\frac{2\cdot 3}{4}+\frac{2\cdot 9}{16}+\frac{27}{64}\right)\left(\left|\Phi(1,1)\right|+\left|\Theta(1,0)\right|\right.\\
&\left.\left.+\left|\Psi(0,1)\right|+\left|\Omega(0,0)\right|\right)\right].\\
={}&\frac{4\cdot 4\cdot 16\cdot 16}{64\cdot 5\cdot 7\cdot 21\cdot 37}\left[\frac{1\cdot 27\cdot 4}{64\cdot 64}+\frac{1\cdot 195\cdot 4}{64\cdot 64}+\frac{27\cdot 41\cdot 4}{64\cdot 64}+\frac{41\cdot 195\cdot 4}{64\cdot 64}\right]\\
={}&\frac{3}{140}.
\end{aligned}
$$

It is clear that

$$
\frac{5}{592}\leq\frac{3}{140},
$$

which demonstrates the result described in Theorem 3.1.

Example 4.2 Let $f:[0,1]\times[0,1]\to\mathbb{R}$ be a function defined by $f(x,y)=x^2y^2$ and $p=r=2$. Then, f is q-partially differentiable. Moreover, $|\Phi(t,s)|^2$, $|\Theta(t,s)|^2$, $|\Psi(t,s)|^2$, and $|\Omega(t,s)|^2$ are q-integrable coordinated convex on $[0,1]\times[0,1]$.

By applying Theorem 3.2 with $q_1=\frac{1}{4}$ and $q_2=\frac{3}{4}$, the left-hand side of (3.10) is similar to the left-hand side of (3.3), which is $\frac{5}{592}$.

Since

$$
\begin{aligned}
&\Phi(t,s)=\frac{35ts}{16},\qquad \Theta(t,s)=\frac{-5t(7s+1)}{16},\\
&\Psi(t,s)=\frac{-7(5t+3)s}{16},\quad\text{and}\quad \Omega(t,s)=\frac{(5t+3)(7s+1)}{16},
\end{aligned}
$$

the right-hand side of (3.10) is

$$
\begin{aligned}
&\frac{(1-0)(1-0)}{16}\left[\frac{1\cdot 9}{16\cdot 16(1+\frac{1}{4}+\frac{1}{16})(1+\frac{3}{4}+\frac{9}{16})}\right]^{1/2}\left[\frac{1}{4(1+\frac{1}{4})(1+\frac{3}{4})}\right]^{1/2}\\
&\times\left[\left(\left(2+\frac{1}{4}\right)\left(2+\frac{3}{4}\right)\left|\Phi(1,1)\right|^2\right)^{1/2}\right.\\
&+\left(\left(2+\frac{1}{4}\right)\left(2+\frac{3}{4}\right)\left|\Theta(1,0)\right|^2+\frac{3}{4}\left(2+\frac{1}{4}\right)\left|\Theta(1,1)\right|^2\right)^{1/2}\\
&+\left(\left(2+\frac{1}{4}\right)\left(2+\frac{3}{4}\right)\left|\Psi(0,1)\right|^2+\frac{1}{4}\left(2+\frac{3}{4}\right)\left|\Psi(1,1)\right|^2\right)^{1/2}\\
&+\left(\left(2+\frac{1}{4}\right)\left(2+\frac{3}{4}\right)\left|\Omega(0,0)\right|^2+\frac{3}{4}\left(2+\frac{1}{4}\right)\left|\Omega(0,1)\right|^2\right.\\
&\left.\left.+\frac{1}{4}\left(2+\frac{3}{4}\right)\left|\Omega(1,0)\right|^2+\frac{1}{4}\cdot\frac{3}{4}\left|\Omega(1,1)\right|^2\right)^{1/2}\right]\\
={}&\frac{1}{16}\left[\frac{3}{7\cdot 37}\right]^{1/2}\left[\frac{4}{5\cdot 7}\right]^{1/2}
\end{aligned}
$$

$$\times \left[\left(\frac{99}{4096}\right)^{1/2} + \left(\frac{43{,}475}{4096}\right)^{1/2} + \left(\frac{78{,}155}{4096}\right)^{1/2} + \left(\frac{38{,}435}{4096}\right)^{1/2} \right]$$
$$= 0.02466\ldots.$$

It is clear that

$$\frac{5}{592} \leq 0.02466\ldots,$$

which demonstrates the result described in Theorem 3.2.

Example 4.3 Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a function defined by $f(x,y) = x^2y^2$ and $r = 2$. Then, f is q-partially differentiable. Moreover, $|\Phi(t,s)|^2$, $|\Theta(t,s)|^2$, $|\Psi(t,s)|^2$, and $|\Omega(t,s)|^2$ are q-integrable coordinated convex on $[0,1] \times [0,1]$.

By applying Theorem 3.3 with $q_1 = \frac{1}{4}$ and $q_2 = \frac{3}{4}$, the left-hand side of (3.16) is similar to the left-hand side of (3.3), which is $\frac{5}{592}$.

Since

$$\Phi(t,s) = \frac{35ts}{16}, \qquad \Theta(t,s) = \frac{-5t(7s+1)}{16},$$
$$\Psi(t,s) = \frac{-7(5t+3)s}{16}, \quad \text{and} \quad \Omega(t,s) = \frac{(5t+3)(7s+1)}{16},$$

the right-hand side of (3.16) is

$$\frac{(1-0)(1-0)}{16}\left[\frac{1\cdot 3}{4\cdot 4\cdot(1+\frac{1}{4})(1+\frac{3}{4})}\right]^{1/2}$$
$$\times \left[\frac{1}{4(1+\frac{1}{4})(1+\frac{3}{4})(1+\frac{1}{4}+\frac{1}{16})(1+\frac{3}{4}+\frac{9}{16})}\right]^{1/2}$$
$$\times \left\{ \left[\left(\frac{2\cdot 1}{4}+\frac{2\cdot 1}{16}+\frac{1}{64}\right)\left(\frac{2\cdot 3}{4}+\frac{2\cdot 9}{16}+\frac{27}{64}\right)\left|\Phi(1,1)\right|^2 \right]^{1/2} \right.$$
$$+ \left[\left(\frac{2\cdot 1}{4}+\frac{2\cdot 1}{16}+\frac{1}{64}\right)\left(\frac{2\cdot 3}{4}+\frac{2\cdot 9}{16}+\frac{27}{64}\right)\left|\Theta(1,0)\right|^2 \right.$$
$$\left. + \left(\frac{3}{4}\right)^3\left(\frac{2\cdot 1}{4}+\frac{2\cdot 1}{16}+\frac{1}{64}\right)\left|\Theta(1,1)\right|^2 \right]^{1/2}$$
$$+ \left(\left(\frac{2\cdot 1}{4}+\frac{2\cdot 1}{16}+\frac{1}{64}\right)\left(\frac{2\cdot 3}{4}+\frac{2\cdot 9}{16}+\frac{27}{64}\right)\left|\Psi(0,1)\right|^2 \right.$$
$$\left. + \left(\frac{1}{4}\right)^3\left(\frac{2\cdot 3}{4}+\frac{2\cdot 9}{16}+\frac{27}{64}\right)\left|\Psi(1,1)\right|^2 \right)^{1/2}$$
$$+ \left(\left(\frac{2\cdot 1}{4}+\frac{2\cdot 1}{16}+\frac{1}{64}\right)\left(\frac{2\cdot 3}{4}+\frac{2\cdot 9}{16}+\frac{27}{64}\right)\left|\Omega(0,0)\right|^2 \right.$$
$$+ \left(\frac{3}{4}\right)^3\left(\frac{2\cdot 1}{4}+\frac{2\cdot 1}{16}+\frac{1}{64}\right)\left|\Omega(0,1)\right|^2$$
$$\left.\left. + \left(\frac{1}{4}\right)^3\left(\frac{2\cdot 3}{4}+\frac{2\cdot 9}{16}+\frac{27}{64}\right)\left|\Omega(1,0)\right|^2 + \left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^3\left|\Omega(1,1)\right|^2 \right)^{1/2} \right\}$$
$$= \frac{1}{16}\left[\frac{1\cdot 3}{5\cdot 7}\right]^{1/2}\left[\frac{4\cdot 16\cdot 16}{5\cdot 7\cdot 21\cdot 37}\right]^{1/2}$$

$$\times\left[\left(\frac{9{,}793{,}875}{1{,}048{,}576}\right)^{1/2}+\left(\frac{1{,}971{,}075}{1{,}048{,}576}\right)^{1/2}+\left(\frac{4{,}137{,}315}{1{,}048{,}576}\right)^{1/2}+\left(\frac{832{,}659}{1{,}048{,}576}\right)^{1/2}\right]$$
$$= 0.02593\ldots.$$

It is clear that

$$\frac{5}{592}\le 0.02593\ldots,$$

which demonstrates the result described in Theorem 3.3.

5 Conclusion

We established several new inequalities for q-differentiable coordinated convex functions that are related to the right side of Hermite–Hadamard inequalities for coordinated convex functions. We also showed that the inequalities proved in this paper generalize the results given in earlier works. Moreover, we gave some examples in order to demonstrate our main results.

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The authors declare no competing interests.

Author contributions
FW reviewed previous work, wrote original draft, edited and corrected the manuscript. KN gave suggestions, revised the manuscript. MZS gave suggestions, revised the manuscript. HB reviewed previous work, gave concepts, gave suggestions, revised the manuscript MAA started the ideas for main results. All authors read and approved the final manuscript.

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