Sumritnorrapong and Suntornchost *Journal of Inequalities and Applications* (2023) 2023:0
https://doi.org/10.1186/s13660-023-02915-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

An improvement of a nonuniform bound for unbounded exchangeable pairs

Patcharee Sumritnorrapong[1] and Jiraphan Suntornchost[2*]

*Correspondence:
Jiraphan.s@chula.ac.th
[2]Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok, Thailand
Full list of author information is available at the end of the article

Abstract

In this paper, we obtain a nonuniform Berry–Esseen bound for a normal approximation via the Stein method and the exchangeable-pair coupling technique where the boundedness condition of the difference between the exchangeable pair is not required. As applications of the result, we obtain nonuniform bounds for the normal approximations in two well-known applications that are the independence test and the quadratic form. Our results suggest that the obtained bounds for the two applications are sharper than other existing bounds.

Keywords: Unbounded Exchangeable Pair; Nonuniform bound; Stein's method; Independence Test; Quadratic forms

1 Introduction

Stein's method [1] for normal approximation was first introduced in 1972 as an alternative approach to the traditional Berry–Esseen normal approximation (see [2, 3]). The method has been intensively studied and generalized to obtain rates of convergences for different distribution approximations. The most common approaches used in Stein's method are the concentration-inequality approach, the inductive approach, and the coupling approach. The exchangeable-pair method introduced by Stein in 1986 [4] is considered as a prominent coupling approach widely studied in the literature. The method is to construct a random variable corresponding to the target variable to make the distribution of the pair commutable. In particular, for a random variable W, we say that the pair (W, W') is an exchangeable pair if for all measurable sets B and B', $P(W \in B, W' \in B') = P(W \in B', W' \in B)$. As in Chen, Goldstein, and Shao [5], the exchangeable pair (W, W') is said to be a λ-Stein pair, for $\lambda \in (0,1)$, if

$$E^{W}\big(W - W'\big) = \lambda(W - R), \tag{1}$$

where R is a random variable of small order. The exchangeable-pair approach has been intensively used as a main tool in constructing bounds of distribution approximation. For instance, Rinott and Rotar [6], Shao and Su [7], and Sumritnorrapong, Neammanee, and Suntornchost [8] applied the exchangeable-pair approach to obtain uniform bounds for a normal approximation of a random variable where the the exchangeable pair exists and the difference between the exchangeable pairs is bounded. That is, $|W - W'| \le A$, for a positive constant A.

Later, in 2019, Shao and Zhang [9] noted that the assumption of bounded distance between the exchangeable pair might lead to a bound that is not optimal if the difference between the exchangeable pair is not sufficiently small. Therefore, they proposed an improvement of the uniform bound for the normal approximation when the boundedness assumption is not required. Their result is stated as follows:

$$\sup_{z\in\mathbb{R}}\bigl|P(W \le z) - \Phi(z)\bigr| \le E\left|1 - \frac{1}{2\lambda}E^W\bigl(\Delta^2\bigr)\right| + E|R| + \frac{1}{\lambda}E\bigl|E^W\bigl(\Delta\Delta^*\bigr)\bigr|,$$

where $\Phi(z)$ is the cumulative distribution function of the standard normal distribution, $\Delta = W - W'$, and $\Delta^* := \Delta^*(W, W')$ is any random variable satisfying $\Delta^*(W, W') = \Delta^*(W', W)$ and $\Delta^* \ge |\Delta|$. As applications of the bound, the authors showed improvements of uniform bounds in many applications such as the quadratic forms.

Later, in 2021, Liu at el. [10] extended the uniform bound of Shao and Zhang [9] to the following nonuniform bound

$$\bigl|P(W \le z) - \Phi(z)\bigr| \le \frac{C}{1 + |z|}\left\{\sqrt{E\left|1 - \frac{1}{2\lambda}E^W\bigl(\Delta^2\bigr)\right|} + E|R| + \frac{1}{\lambda}\sqrt{E\bigl|E^W\bigl(\Delta\Delta^*\bigr)\bigr|^2}\right\},$$

where C is a constant. The result was then applied to extend the uniform bounds for different applications to nonuniform bounds. One application is the extension of the uniform bound for the quadratic forms in Shao and Zhang [9] to a nonuniform bound.

In this paper, with a different approach, we extend the technique in Sumritnorrapong, Neammanee, and Suntornchost [8] to propose a nonuniform bound of the normal approximation when the boundedness of the distance between the exchangeable pair is not required. Moreover, we apply our obtained bound to improve the error bound of the normal approximation in two well-known applications that are the independence test and the quadratic forms. In each application, we show that our bound is sharper than other existing bounds.

The organization of this paper is as follows. First, we state our main theorem. Then, we present an important lemma and the proof of the main theorem in Sect. 2. In Sect. 3, we apply the main theorem to obtain nonuniform bounds of normal approximations in two applications; the independence test and the quadratic forms. Finally, we give a conclusion of our study in Sect. 4.

Theorem 1 *Let* (W, W') *be an exchangeable pair such that* $E^W(W - W') = \lambda(W - R)$ *with some constant* $0 < \lambda < 1$ *and a random variable* R*. Assume that* $E|W - W'|^{2r} = O(\lambda^r)$ *for* $r \in \mathbb{N}$*. Then, for* $z \in \mathbb{R}$ *such that* $|z| \ge 1$,

$$\begin{aligned}\bigl|P(W \le z) - \Phi(z)\bigr| \le{}& \frac{C}{(1 + |z|)^r}\sqrt{E\left(1 - \frac{1}{2\lambda}E^W\bigl(W' - W\bigr)^2\right)^2} \\ &+ \frac{C}{(1 + |z|)^r}O\bigl(\lambda^{\frac{r}{2}}\bigr) + \frac{C}{(1 + |z|)^{r-1}}O(\sqrt{\lambda})\end{aligned}$$

$$+ \frac{C}{(1+|z|)^r}\left[1 + \left(E|R|^2\right)^{\frac{1}{2}} + \left(E|R|^4\right)^{\frac{1}{4}}\right]O(\sqrt{\lambda})$$
$$+ \frac{C}{1+|z|}E|R|,$$

where C is a constant. Furthermore, if $R = 0$, *we have*

$$\left|P(W \le z) - \Phi(z)\right| \le \frac{C}{(1+|z|)^{r-1}}\left[\sqrt{E\left(1 - \frac{1}{2\lambda}E^W\left(W' - W\right)^2\right)^2} + O(\sqrt{\lambda})\right].$$

2 Proof of main result

To prove the main theorem, recall that Stein's equation for the normal approximation is

$$g'(w) - wg(w) = \mathbb{I}(w \le z) - \Phi(z), \tag{2}$$

where $\mathbb{I}(\cdot)$ is the indicator function, Φ is the cumulative distribution function of the standard normal distribution and $g : \mathbb{R} \to \mathbb{R}$ is a continuous and piecewise-differentiable function. The solution g_z of Stein's equation (2) is obtained in [4] as

$$g_z(w) = \begin{cases} \sqrt{2\pi}e^{\frac{w^2}{2}}\Phi(w)[1 - \Phi(z)] & \text{if } w \le z, \\ \sqrt{2\pi}e^{\frac{w^2}{2}}\Phi(z)[1 - \Phi(w)] & \text{if } w > z \end{cases}$$

and

$$g_z'(w) = \begin{cases} (1 - \Phi(z))(1 + \sqrt{2\pi}we^{\frac{w^2}{2}}\Phi(w)), & \text{if } w < z, \\ \Phi(z)(-1 + \sqrt{2\pi}we^{\frac{w^2}{2}}(1 - \Phi(w))), & \text{if } w > z. \end{cases}$$

Applying Stein's solution and the random variable W to (2) and taking an expectation, we obtain

$$\left|P(W \le z) - \Phi(z)\right| = \left|E\left[g_z'(W)\right] - E\left[Wg_z(W)\right]\right|. \tag{3}$$

To obtain a bound of the normal approximation, we construct a bound for the term on the right of (3). Since (W, W') is an exchangeable pair, from Chen, Goldstein, and Shao [5], we obtain

$$E\left(Wg_z(W)\right) = E\int_{-\infty}^{\infty} g_z'(W + t)\widehat{K}(t)\,dt + E\left(Rg_z(W)\right), \tag{4}$$

where

$$\widehat{K}(t) = \frac{1}{2\lambda}\left(W' - W\right)\left[\mathbb{I}\left(0 < t \le W' - W\right) - \mathbb{I}\left(W' - W \le t \le 0\right)\right]. \tag{5}$$

By a direct calculation, we can prove that

$$\int_{-\infty}^{\infty} t^k\widehat{K}(t)\,dt = \frac{(W' - W)^{k+2}}{2\lambda(k+1)}, \tag{6}$$

for any nonnegative integer k. Consequently, we can obtain some moment properties of the exchangeable pair (W, W') as in the following lemma.

Lemma 2 *Let (W, W') be an exchangeable pair satisfying $E|W - W'|^{2r} = O(\lambda^r)$ for $r \in \mathbb{N}$. Then,*

1. $E|W|^{2r} = O(1)$
2. $E|g_z'(W)| \leq \frac{C}{(1+|z|)^{2r}}$.

Proof

1. By the binomial formula and (6), we can show that

$$\begin{aligned}
EW^{2r} &= (2r-1)E\left(\int_{-\infty}^{\infty}(W+t)^{2r-2}\widehat{K}(t)\,dt\right)\\
&= (2r-1)E\left(\int_{-\infty}^{\infty}\sum_{k=0}^{2r-2}\binom{2r-2}{k}W^k t^{2r-k-2}\widehat{K}(t)\,dt\right)\\
&= (2r-1)E\left(\sum_{k=0}^{2r-2}\binom{2r-2}{k}\frac{W^k(W-W')^{2r-k}}{2\lambda(2r-k-1)}\right)\\
&\leq (2r-1)E\left(\sum_{k=0}^{2r-2}\binom{2r-2}{k}\frac{|W|^k|W-W'|^{2r-k}}{\lambda(2r-k-1)}\right).
\end{aligned}$$

Applying the weighted AM-GM inequality, we can show that

$$\begin{aligned}
&\frac{1}{\lambda}E\left(|W|^k\left|W-W'\right|^{2r-k}\right)\\
&\quad\leq\left[\frac{k}{2r}\left(\lambda^{(\frac{2r}{k}-1)(r-1)-1}E|W|^{2r}\right)+\frac{2r-k}{2r}\left(\frac{E|W-W'|^{2r}}{\lambda^r}\right)\right],
\end{aligned}$$

for all $0 \leq k \leq 2r-2$. Consequently,

$$\begin{aligned}
E|W|^{2r} &\leq (2r-1)\left(\frac{2r-2}{2r}\frac{E|W|^{2r}}{(2r+1)^{2r}}+\frac{(2r-1)^{2r-2}}{r\lambda^r}E\left|W-W'\right|^{2r}\right)\\
&\quad+(2r-1)\sum_{k=0}^{2r-3}\binom{2r-2}{k}\frac{1}{2r-k-1}\frac{k}{2r}\lambda^{(\frac{2r}{k}-1)(r-1)-1}E|W|^{2r}\\
&\quad+(2r-1)\sum_{k=0}^{2r-3}\binom{2r-2}{k}\frac{1}{2r-k-1}\frac{2r-k}{2r\lambda^r}E\left|W-W'\right|^{2r}\\
&\leq \frac{2r-2}{2r}E|W|^{2r}+\frac{(2r-1)^{2r-2}}{\lambda^r}E\left|W-W'\right|^{2r}\\
&\quad+(2r-1)\left(\sum_{k=0}^{2r-3}\binom{2r-2}{k}\frac{k}{2r(2r-k-1)}\lambda^{(\frac{2r}{k}-1)(r-1)-1}\right)E|W|^{2r}\\
&\quad+(2r-1)\left(\sum_{k=0}^{2r-3}\binom{2r-2}{k}\frac{2r-k}{2r\lambda^r(2r-k-1)}\right)E\left|W-W'\right|^{2r}.
\end{aligned}$$

Rewriting the inequality, we can show that

$$E|W|^{2r} \leq C\frac{E|W-W'|^{2r}}{\lambda^r}, \tag{7}$$

for some constant C. Since $E|W - W'|^{2r} = O(\lambda^r)$, we can conclude that $E|W|^r = O(1)$.

2. From the fact that

$$1 - \Phi(z) \le \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z} \quad \text{for } z > 0,$$

and

$$\Phi(z) \le \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}|z|} \quad \text{for } z < 0,$$

we can show that

$$0 \le g_z'(w) = \big(1 - \Phi(z)\big)\big(1 + \sqrt{2\pi} w e^{\frac{w^2}{2}} \Phi(w)\big) \le \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}, \quad \text{for } w \le 0,$$

and

$$0 < g_z'(w) \le \big(1 - \Phi(z)\big)\left(1 + \sqrt{2\pi}\frac{z}{2}e^{\frac{z^2}{8}}\right) \le \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z} + \frac{e^{\frac{-3z^2}{8}}}{2}, \quad \text{for } 0 < w \le \frac{z}{2}.$$

Therefore,

$$\left|g_z'(w)\right| \le \begin{cases} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}, & w \le 0, \\ \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z} + \frac{e^{-\frac{3z^2}{8}}}{2}, & 0 < w \le \frac{z}{2}, \\ 1, & w > \frac{z}{2}. \end{cases}$$

Consequently,

$$\begin{aligned} E\left|g_z'(W)\right| &= E\left|g_z'(W)\right|\mathbb{I}(W \le 0) + E\left|g_z'(W)\right|\mathbb{I}\left(0 < W \le \frac{z}{2}\right) \\ &\quad + E\left|g_z'(W)\right|\mathbb{I}\left(W > \frac{z}{2}\right) \\ &\le \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z} + \frac{e^{-\frac{3z^2}{8}}}{2} + P\left(W > \frac{z}{2}\right) \\ &\le \frac{2e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z} + \frac{e^{-\frac{3z^2}{8}}}{2} + \frac{CE|W|^{2r}}{z^{2r}} \\ &= \frac{1}{(1+z)^{2r}}(1+z)^{2r}\left(\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z} + \frac{e^{-\frac{3z^2}{8}}}{2} + \frac{C}{z^{2r}}\right) \qquad (8) \\ &\le \frac{C}{(1+z)^{2r}} \quad \text{for } z > 0, \end{aligned}$$

where we use the property that $E|W|^{2r} = O(1)$ to obtain (8). Therefore, the lemma is proved. □

Having obtained the important moment properties, we applied them to prove the main result as follows.

Proof of Theorem 1 By the property of the standard normal distribution, $\Phi(z) = 1 - \Phi(-z)$, it suffices to assume that $z \geq 1$. From (3) and (4), we obtain

$$
\begin{aligned}
\left|P(W \leq z) - \Phi(z)\right| &\leq \left|Eg_z'(W) - Eg_z'(W)\int_{-\infty}^{\infty}\widehat{K}(t)\,dt\right| \\
&\quad + \left|E\int_{-\infty}^{\infty}\bigl(g_z'(W) - g_z'(W+t)\bigr)\widehat{K}(t)\,dt\right| + ERg_z(W) \\
&= : |T_1| + |T_2| + |T_3|, \qquad (9)
\end{aligned}
$$

where

$$
\begin{aligned}
T_1 &= Eg_z'(W) - Eg_z'(W)\int_{-\infty}^{\infty}\widehat{K}(t)\,dt, \\
T_2 &= E\int_{-\infty}^{\infty}\bigl(g_z'(W) - g_z'(W+t)\bigr)\widehat{K}(t)\,dt,
\end{aligned}
$$

and

$$
T_3 = ERg_z(W).
$$

To obtain the bound in (9), we apply the technique in Sumritnorrapong, Neammanee, and Suntornchost [8] to each of the three terms as follows.

For the bound for $|T_1|$, we note that

$$
\begin{aligned}
|T_1| &\leq \sqrt{E\bigl(g_z'(W)\bigr)^2}\sqrt{E\left(1 - E^W\left(\int_{-\infty}^{\infty}\widehat{K}(t)\,dt\right)\right)^2} \\
&\leq \sqrt{E\bigl(g_z'(W)\bigr)^2}\sqrt{E\left(1 - \frac{1}{2\lambda}E^W\left(W' - W\right)^2\right)^2}.
\end{aligned}
$$

Using the properties of Stein's solution g_z and applying Lemma 2, we obtain

$$
E\bigl(g_z'(W)\bigr)^2 \leq E\bigl|g_z'(W)\bigr| \leq \frac{C}{(1+z)^{2r}}.
$$

Therefore,

$$
|T_1| \leq \frac{C}{(1+z)^r}\sqrt{E\left(1 - \frac{1}{2\lambda}E^W\left(W' - W\right)^2\right)^2}. \qquad (10)
$$

To bound $|T_2|$, we note that

$$
\begin{aligned}
|T_2| &= \left|E\int_{-\infty}^{\infty}\bigl\{g_z'(W) - g_z'(W+t)\bigr\}\widehat{K}(t)\,dt\right| \\
&\leq E\int_{-\infty}^{\infty}\bigl|g_z'(W) - g_z'(W+t)\bigr|\widehat{K}(t)\,dt
\end{aligned}
$$

$$\leq T_{21} + T_{22} + T_{23}, \tag{11}$$

where

$$T_{21} = E\mathbb{I}\left(\delta_W \geq \frac{z}{4}\right)\int_{-\infty}^{\infty}\left|g_z'(W) - g_z'(W+t)\right|\widehat{K}(t)\,dt,$$
$$T_{22} = E\mathbb{I}\left(\delta_W < \frac{z}{4}\right)\int_{-\infty}^{\infty}\mathbb{I}\big(z - \max(0,t) < W < z - \min(0,t)\big)\widehat{K}(t)\,dt,$$
$$T_{23} = E\mathbb{I}\left(\delta_W < \frac{z}{4}\right)\int_{-\infty}^{\infty}\int_{t}^{0} h(W+u)\widehat{K}(t)\,du\,dt,$$

and $h(w) = (wg_z(w))'$.

The bound for the term T_{21} is

$$\begin{aligned}|T_{21}| &\leq E\mathbb{I}\left(\delta_W \geq \frac{z}{4}\right)\int_{-\infty}^{\infty}\widehat{K}(t)\,dt \\ &\leq E\mathbb{I}\left(\delta_W \geq \frac{z}{4}\right)\frac{(W-W')^2}{2\lambda} \\ &\leq \frac{1}{2\lambda}\left(P\left(\delta_W \geq \frac{z}{4}\right)\right)^{\frac{1}{2}}\left(E\left|W-W'\right|^4\right)^{\frac{1}{2}} \\ &\leq \frac{C}{(1+z)^r}O\big(\lambda^{\frac{r}{2}}\big),\end{aligned} \tag{12}$$

where we use Markov's inequality and the assumption that $E|W-W'|^{2r} = O(\lambda^r)$ to obtain the last inequality.

To obtain a bound for T_{22} we define a function $f_{\delta,r}: \mathbb{R} \to \mathbb{R}$, for $\delta > 0$, and $r \in \mathbb{N}$, as

$$f_{\delta,r}(t) = \begin{cases} 2\delta(1+t+\delta)^r & \text{if } t < z - 2\delta, \\ (1+t+\delta)^r(t - z + 4\delta) & \text{if } z - 2\delta \leq t \leq z + 2\delta, \\ 6\delta(1+t+\delta)^r & \text{if } t > z + 2\delta. \end{cases}$$

Then,

$$f_{\delta,r}'(t) = \begin{cases} 2\delta r(1+t+\delta)^{r-1} & \text{if } t < z - 2\delta, \\ (1+t+\delta)^r + (t - z + 2\delta)r(1+t+\delta)^{r-1} & \text{if } z - 2\delta < t < z + 2\delta, \\ 6\delta r(1+t+\delta)^{r-1} & \text{if } t > z + 2\delta, \end{cases}$$

and

$$f_{\delta,r}'(t) \geq \begin{cases} (1+z+\delta)^r & \text{if } z - 2\delta < t < z + 2\delta, \\ 0 & \text{if } t < z - 2\delta \text{ or } t > z - 2\delta. \end{cases} \tag{13}$$

Applying the power mean inequality to $f_{\delta,r}$, we obtain

$$\left|f_{\delta,r}(t)\right| \leq 6 \cdot 3^{r-1}\delta\big(1 + |t|^r + \delta^r\big) \quad \text{for all } t \in \mathbb{R}.$$

Moreover, note that

$$z - 2\delta_W < W + t < z + 2\delta_W$$

for $|t| \leq \delta_W$ and $z - \delta_W < W < z + \delta_W$. Therefore,

$$\begin{aligned}
|T_{22}| &\leq E\int_{|t|\leq\delta_W} \mathbb{I}\big(z - |t| < W < z + |t|\big)\widehat{K}(t)\,dt \\
&\leq E\int_{|t|\leq\delta_W} \frac{(1+z+\delta_W)^r}{(1+z)^r}\mathbb{I}(z-\delta_W < W < z+\delta_W)\widehat{K}(t)\,dt \\
&\leq \frac{1}{(1+z)^r}E\int_{|t|\leq\delta_W} f'_{\delta_W,r}(W+t)\widehat{K}(t)\,dt \\
&= \frac{1}{(1+z)^r}\big(EWf_{\delta_W,r}(W) - ERf_{\delta_W,r}(W)\big) \\
&\leq \frac{C}{(1+z)^r}E\big(|W|\delta_W\big(1+|W|^r+\delta_W^r\big)\big) \\
&\quad + \frac{C}{(1+z)^r}E\big(|R|\delta_W\big(1+|W|^r+\delta_W^r\big)\big).
\end{aligned}$$

Using Hölder's inequality and applying Lemma 2, we obtain

$$\begin{aligned}
E\big(|W|\delta_W\big(1+|W|^r+\delta_W^r\big)\big) &\leq \big(EW^2\big)^{\frac{1}{2}}\big(E\big|W-W'\big|^2\big)^{\frac{1}{2}} + \big(E|W|^{2r}\big)^{\frac{r+1}{2r}}\big(E\big|W-W'\big|^4\big)^{\frac{1}{4}} \\
&\quad + \big(E|W|^4\big)^{\frac{1}{4}}\big(E\big|W-W'\big|^{2r}\big)^{\frac{r+1}{2r}} \\
&\leq O(\sqrt{\lambda})
\end{aligned}$$

and

$$\begin{aligned}
&E\big(|R|\delta_W\big(1+|W|^r+\delta_W^r\big)\big) \\
&\quad\leq \big(E|R|^2\big)^{\frac{1}{2}}\big(E\big|W-W'\big|^2\big)^{\frac{1}{2}} + \big(E|R|^4\big)^{\frac{1}{4}}\big(E|W|^4\big)^{\frac{1}{4}}\big(E\big|W-W'\big|^{2r}\big)^{\frac{1}{2}} \\
&\qquad + \big(E|R|^4\big)^{\frac{1}{4}}\big(E\big|W-W'\big|^{2r}\big)^{\frac{r+1}{2r}} \\
&\quad\leq \big[\big(E|R|^2\big)^{\frac{1}{2}} + \big(E|R|^4\big)^{\frac{1}{4}}\big]O(\sqrt{\lambda}).
\end{aligned}$$

Therefore,

$$|T_{22}| \leq \frac{C}{(1+z)^r}\big[1 + \big(E|R|^2\big)^{\frac{1}{2}} + \big(E|R|^4\big)^{\frac{1}{4}}\big]O(\sqrt{\lambda}). \tag{14}$$

For the bound of T_{23}, we note that

$$\begin{aligned}
T_{23} &\leq E\mathbb{I}\left(\delta_W < \frac{z}{4}\right)\int_{-\infty}^{\infty}\int_t^0 h(W+u)\widehat{K}(t)\mathbb{I}\left(W+u \leq \frac{3z}{4}\right)du\,dt \\
&\quad + E\mathbb{I}\left(\delta_W < \frac{z}{4}\right)\int_{-\infty}^{\infty}\int_t^0 h(W+u)\widehat{K}(t)\mathbb{I}\left(W+u > \frac{3z}{4}\right)du\,dt.
\end{aligned} \tag{15}$$

From the definition of h defined by $h(w) = (wg_z(w))'$, we can obtain a bound of the function h as follows.

If $w \leq \frac{3z}{4}$, then (see Equation (38) in [8]),

$$\begin{aligned} h(w) &\leq \frac{1}{z}\left(1+\frac{9z^2}{16}\right)e^{-\frac{7z^2}{32}} + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \\ &\leq \frac{C}{(1+z)^r}. \end{aligned}$$

For $w > \frac{3z}{4}$, we have

$$h(w) \leq 1.001(1+z).$$

Applying

$$\int_{-\infty}^{\infty}\int_t^0 \widehat{K}(t)\,du\,dt \leq \int_{-\infty}^{\infty} |t|\widehat{K}(t)\,dt = \frac{1}{4\lambda}\left|W'-W\right|^3,$$

and Markov's inequality,

$$P\left(W > \frac{z}{2}\right) \leq \frac{CE|W|^{2r}}{z^{2r}},$$

respectively, to the first and second terms of (15), we obtain

$$\begin{aligned} |T_{23}| &\leq \frac{C}{\lambda(1+z)^r}E\left|W-W'\right|^3 + \frac{C(1+z)}{\lambda}E\left|W-W'\right|^3\mathbb{I}\left(W > \frac{z}{2}\right) \\ &\leq \frac{C}{\lambda(1+z)^r}E\left|W-W'\right|^3 + \frac{C}{\lambda(1+z)^{r-1}}\left(E\left|W-W'\right|^6\right)^{\frac{1}{2}}\left(E|W|^{2r}\right)^{\frac{1}{2}} \\ &\leq \frac{C}{(1+z)^{r-1}}O(\sqrt{\lambda}), \end{aligned} \tag{16}$$

where we use Hölder's inequality to obtain the second inequality.

Using the property of Stein's solution, we obtain

$$|T_3| = \left|ERg_z(W)\right| \leq \frac{1}{z}E|R| \leq \frac{C}{1+z}E|R|. \tag{17}$$

From (9)–(17), we have

$$\begin{aligned} \left|P(W \leq z) - \Phi(z)\right| \leq{}& \frac{C}{(1+|z|)^r}\sqrt{E\left(1-\frac{1}{2\lambda}E^W\left(W'-W\right)^2\right)^2} \\ &+ \frac{C}{(1+|z|)^r}O\left(\lambda^{\frac{r}{2}}\right) + \frac{C}{(1+|z|)^{r-1}}O(\sqrt{\lambda}) \\ &+ \frac{C}{(1+|z|)^r}\left[1+\left(E|R|^2\right)^{\frac{1}{2}} + \left(E|R|^4\right)^{\frac{1}{4}}\right]O(\sqrt{\lambda}) \\ &+ \frac{C}{1+|z|}E|R|. \end{aligned}$$

Therefore, the theorem is proved. □

3 Applications

In this section, we discuss two applications of our main theorem; the independence test and the quadratic forms.

3.1 Independence test

The test of independence is an important test in statistics particularly in multivariate statistics. The main role of the test is to test the independence between different variables in an m-variable population, say $X = (X_1, \dots, X_m)'$ with covariance matrix Σ. Let $\{X_1, X_2, \dots, X_n\}$ be a random sample of size n form the m-variable population represented by a random vector. One well-known test of independence was proposed by Schott [11] where the author considered a test based on the sum of squared sample correlation coefficients. Specifically, let $\mathbf{r} = (r_{ij})_{m\times m}$ be the sample correlation matrix, where

$$r_{ij} = \frac{\sum_{k=1}^{n}(X_{ik} - \overline{X}_i)(X_{jk} - \overline{X}_j)}{\sqrt{\sum_{k=1}^{n}(X_{ik} - \overline{X}_i)^2}\sqrt{\sum_{k=1}^{n}(X_{jk} - \overline{X}_j)^2}},$$

$X_i = (X_{1i}, X_{2i}, \dots, X_{mi})'$ and $\overline{X}_i = \frac{1}{n}\sum_{k=1}^{n} X_{ik}$. Let $t_{n,m}$ be the sum of squared r_{ij}s for $i > j$,

$$t_{n,m} = \sum_{i=2}^{m}\sum_{j=1}^{i-1} r_{ij}^2,$$

and

$$W := W_{n,m} = c_{n,m}\left(t_{n,m} - \frac{m(m-1)}{2(n-1)}\right),$$

where $c_{n,m} = \frac{n\sqrt{n+2}}{\sqrt{m(m-1)(n-1)}}$.

Schott [11] proved that

$$W_{n,m} \xrightarrow{d} N(0,1),$$

where $m = O(n)$.

Later, in 2012, Chen and Shao [12] gave a uniform bound for the independence test with the assumptions $m = O(n)$ and $E(X_{11}^{24}) < \infty$ as

$$\sup_{z\in\mathbb{R}}\left|P(W_{n,m} < z) - \Phi(z)\right| = O\left(\frac{1}{m}\right).$$

Later, in 2019, Rerkruthairat [13] extended the bound to be a nonuniform bound, which is

$$\left|P(W_{n,m} < z) - \Phi(z)\right| \le \frac{1}{0.5 + |z|} \cdot O\left(\frac{1}{m}\right),$$

for every $z \in \mathbb{R}$.

Applying our main result, we can improve the bound of Rerkruthairat [13] by increasing the rate of convergence to $\frac{1}{(1+|z|)^3} \cdot O(\frac{1}{\sqrt{m}})$, as in the following theorem.

Theorem 3 *Let $\{X_{ij}, 1 \le i \le m, 1 \le j \le n\}$ be independent and identically distributed random variables and Φ be the cumulative distribution function of the standard normal distribution. Assume that $m = O(n)$. If $E(X_{11}^{24}) < \infty$, then for $z \in \mathbb{R}$,*

$$\left|P(W_{m,n} \le z) - \Phi(z)\right| \le \frac{1}{(1+|z|)^3} \cdot O\left(\frac{1}{\sqrt{m}}\right).$$

Proof For the case $|z| < 1$, applying the result of uniform bounds for the independence test in [12], we obtain

$$\left|P(W_{m,n} \le z) - \Phi(z)\right| \le \frac{1}{(1+|z|)^3} \cdot O\left(\frac{1}{\sqrt{m}}\right).$$

For the case $|z| \ge 1$, we follow the exchangeable pair constructed in Chen and Shao [12]. In particular, let $X_i^*, 1 \le i \le n$ be an independent copy of $X_i, 1 \le i \le n$, and let I be a uniform distributed random variable on $\{1, 2, \dots, m\}$, which is independent of $\{X_i, X_i^*, 1 \le i \le n\}$.

Define $t_{n,m}^* = t_{n,m} - \sum_{\substack{j=1 \\ j \ne I}}^m r_{Ij}^2 + \sum_{\substack{j=1 \\ j \ne I}}^m r_{I^*j}^2$, where

$$r_{i^*j} = \frac{\sum_{k=1}^n (X_{ik}^* - \overline{X}_i^*)(X_{jk} - \overline{X}_j)}{\sqrt{\sum_{k=1}^n (X_{ik}^* - \overline{X}_i^*)^2}\sqrt{\sum_{k=1}^n (X_{jk} - \overline{X}_j)^2}}.$$

Define $W' := W'_{n,m} = c_{n,m}(t_{n,m}^* - \frac{m(m-1)}{2(n-1)})$. Then, (W, W') is an exchangeable pair such that $E^W(W - W') = \frac{2}{m} W$ and

$$E\left(W - W'\right)^8 = O\left(\frac{1}{m^4}\right). \tag{18}$$

Chen and Shao [12] proved that

$$\left|\frac{1}{2\lambda} E^W\left(W - W'\right)^2 - 1\right| \le \frac{c_{n,m}^2 m}{4}(J_1 + J_2) + O\left(\frac{1}{n}\right),$$

where

$$J_1 = \left|\frac{1}{m}\sum_{i=1}^m \left\{\sum_{\substack{j=1 \\ j \ne i}}^m \left(r_{ij}^2 - \frac{1}{n-1}\right)\right\}^2 - \frac{2(m-1)}{n^2}\right|$$

and

$$J_2 = \left|\frac{1}{m}\sum_{i=1}^m E^W\left\{\sum_{\substack{j=1 \\ j \ne i}}^m \left(r_{i^*j}^2 - \frac{1}{n-1}\right)\right\}^2 - \frac{2(m-1)}{n^2}\right|.$$

Moreover,

$$EJ_1^2 = O\left(\frac{m}{n^4}\right) \quad \text{and} \quad EJ_2^2 = O\left(\frac{m}{n^4}\right).$$

Applying the inequality $(a+b)^2 \leq 2(a^2+b^2)$ and using the property that $m = O(n)$,

$$\begin{aligned} E\left(1 - \frac{1}{2\lambda}E^W\left(W - W'\right)^2\right)^2 &\leq \frac{c_{n,m}^4 m^2}{8} E(J_1 + J_2)^2 + O\left(\frac{1}{n^2}\right) \\ &\leq \frac{c_{n,m}^4 m^2}{4}\left(EJ_1^2 + EJ_2^2\right) + O\left(\frac{1}{n^2}\right) \\ &= O\left(\frac{1}{m}\right). \end{aligned} \tag{19}$$

Substituting $\lambda = \frac{2}{m}$ and $R = 0$ in Theorem 1, and using (18), (19), and the property that $m = O(n)$, we obtain

$$\left|P(W \leq z) - \Phi(z)\right| \leq \frac{1}{(1+z)^3} \cdot O\left(\frac{1}{\sqrt{m}}\right).$$

This completes the proof. □

3.2 Quadratic forms

In this section, we apply our main theorem to the problem of the quadratic forms. In particular, let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with $EX_i = 0$, $\operatorname{Var}(X_i) = 1$ and $EX_i^4 < \infty$ for all $i \in \{1, 2, \ldots, n\}$. Let $A = [a_{ij}]_{n \times n}$ be a real symmetric matrix and

$$\sigma_n^2 = 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2. \tag{20}$$

Therefore, the quadratic forms is defined as

$$W := W_n = \frac{1}{\sigma_n}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}X_iX_j. \tag{21}$$

The problem of quadratic forms has been widely studied in the literature. For example, Jong [14] proved a central limit theorem of the quadratic forms. Götze and Tikhomirov [15] gave an asymptotic distribution of the quadratic forms.

Later, in 2019, Shao and Zhang [9] obtained a uniform bound for the quadratic forms with the assumption that $a_{ii} = 0$ for all $1 \leq i \leq n$ as

$$\sup_{z\in\mathbb{R}}\left|P(W_n \leq z) - \Phi(z)\right| \leq \frac{CEX_1^4}{\sigma_n^2}\left(\sqrt{\sum_i\left(\sum_j a_{ij}^2\right)^2} + \sqrt{\sum_{i,j}\left(\sum_k a_{ik}a_{jk}\right)^2}\right),$$

where C is a positive constant. The bound of normal approximation was extended to a nonuniform bound in Liu et al. [10] as the following:

$$\left|P(W_n \leq z) - \Phi(z)\right| \leq \frac{CEX_1^4}{(1+z)^2\sigma_n^2}\left(\sqrt{\sum_i\left(\sum_j a_{ij}^2\right)^2} + \sqrt{\sum_{i,j}\left(\sum_k a_{ik}a_{jk}\right)^2}\right).$$

In this section, we apply our main result to obtain a sharper nonuniform bound of the normal approximation for the quadratic forms. Our bound is given in the following theorem.

Theorem 4 *Let $X_1, X_2, \dots, X_n$ be independent and identically distributed random variables with $EX_i = 0$, $\operatorname{Var}(X_i) = 1$ and $EX_i^8 < \infty$ for all $i \in \{1, 2, \dots, n\}$, and $A = [a_{ij}]_{n\times n}$ be a real symmetric matrix with $a_{ii} = 0$ for all $1 \le i \le n$. Then, for $z \in \mathbb{R}$,*

$$\begin{aligned}&\bigl|P(W \le z) - \Phi(z)\bigr|\\&\quad\le \frac{C}{(1+|z|)^3}\left(\frac{EX_1^4}{\sigma_n^2}\left(\sqrt{\sum_i\Bigl(\sum_j a_{ij}^2\Bigr)^2} + \sqrt{\sum_{i,j}\Bigl(\sum_k a_{ik}a_{jk}\Bigr)^2}\right) + O\Bigl(\frac{1}{\sqrt{n}}\Bigr)\right).\end{aligned}$$

Proof For the case $|z| < 1$, we apply the result of a uniform bound for quadratic forms in [9] and we obtain

$$\begin{aligned}&\bigl|P(W \le z) - \Phi(z)\bigr|\\&\quad\le \frac{CEX_1^4}{(1+|z|)^3\sigma_n^2}\left(\sqrt{\sum_i\Bigl(\sum_j a_{ij}^2\Bigr)^2} + \sqrt{\sum_{i,j}\Bigl(\sum_k a_{ik}a_{jk}\Bigr)^2}\right)\\&\quad\le \frac{C}{(1+|z|)^3}\left(\frac{EX_1^4}{\sigma_n^2}\left(\sqrt{\sum_i\Bigl(\sum_j a_{ij}^2\Bigr)^2} + \sqrt{\sum_{i,j}\Bigl(\sum_k a_{ik}a_{jk}\Bigr)^2}\right) + O\Bigl(\frac{1}{\sqrt{n}}\Bigr)\right).\end{aligned}$$

For the case $|z| \ge 1$, following Shao and Zhang [9], let X_i' be an independent copy of X_i and I be a random index uniformly distribution over $\{1, 2, \dots, n\}$. Define

$$W' := W_n' = W_n - \frac{2}{\sigma_n}\sum_{j=1}^n a_{Ij}X_j\bigl(X_I - X_I'\bigr). \tag{22}$$

Then, W' is an exchangeable pair of W and

$$W - W' = \frac{2}{\sigma_n}\sum_{j\ne I} a_{Ij}X_j\bigl(X_I - X_I'\bigr). \tag{23}$$

Consequently, $E^W(W - W') = \frac{2}{n}W$, and

$$\begin{aligned}&\sqrt{E\Bigl(1 - \frac{1}{2\lambda}E^W\bigl(W - W'\bigr)^2\Bigr)^2}\\&\quad\le \frac{CEX_1^4}{\sigma_n^2}\left(\sqrt{\sum_i\Bigl(\sum_j a_{ij}^2\Bigr)^2} + \sqrt{\sum_{i,j}\Bigl(\sum_k a_{ik}a_{jk}\Bigr)^2}\right).\end{aligned}\tag{24}$$

Using (23) and the properties of conditional expectation, we obtain

$$\begin{aligned}
E^W\left(W-W'\right)^8 &= \left(\frac{2}{\sigma_n}\right)^8 E^W\left(\sum_{j\neq I} a_{Ij}X_j\left(X_I-X_I'\right)\right)^8 \\
&= \left(\frac{2}{\sigma_n}\right)^8 \frac{1}{n}\sum_{i=1}^n E^W\left(\left(X_i-X_i'\right)^8\left(\sum_{j\neq i} a_{ij}X_j\right)^8\right) \\
&\leq \frac{2^8}{n\sigma_n^8}\sum_{i=1}^n E^W\left(\left(\sum_{k=0}^8\binom{8}{k}|X_i|^k\left|X_i'\right|^{8-k}\right)\left(\sum_{j\neq i} a_{ij}X_j\right)^8\right) \\
&= \frac{2^8}{n\sigma_n^8}\sum_{i=1}^n \left(\sum_{j\neq i} a_{ij}X_j\right)^8\left(\sum_{k=0}^8\binom{8}{k}|X_i|^k E\left|X_i'\right|^{8-k}\right).
\end{aligned} \tag{25}$$

By Hölder's inequality, we have $E|X_i|^k < (E|X_i|^8)^{k/8}$ for $k \leq 8$. Therefore,

$$\begin{aligned}
E\left(\sum_{j\neq i} a_{ij}X_j\right)^8 &= E\left(\sum_{j=1}^n a_{ij}X_j\right)^8 \\
&\leq C\left(EX_1^8\right)\left(\left(\sum_{j=1}^n a_{ij}^3\right)^2\left(\sum_{j=1}^n a_{ij}^2\right)+\left(\sum_{j=1}^n a_{ij}^2\right)^4\right)
\end{aligned} \tag{26}$$

and

$$\sum_{k=0}^8 E|X_i|^k E\left|X_i'\right|^{8-k} \leq CEX_i^8. \tag{27}$$

Using the properties of conditional expectation and (25)–(27), we obtain

$$\begin{aligned}
E\left(W-W'\right)^8 &= E\left(E^W\left(W-W'\right)^8\right) \\
&\leq \frac{C}{n\sigma_n^8}\sum_{i=1}^n E\left(\sum_{j\neq i} a_{ij}X_j\right)^8\left(\sum_{k=0}^8 EX_i^k EX_i'^{8-k}\right) \\
&\leq \frac{C}{n\sigma_n^8}EX_1^8\sum_{i=1}^n\left(\left(\sum_{j=1}^n a_{ij}^3\right)^2\left(\sum_{j=1}^n a_{ij}^2\right)+\left(\sum_{j=1}^n a_{ij}^2\right)^4\right).
\end{aligned}$$

We can see that $\sigma_n^8 = O(n^8)$ and

$$\sum_{i=1}^n\left(\left(\sum_{j=1}^n a_{ij}^3\right)^2\left(\sum_{j=1}^n a_{ij}^2\right)+\left(\sum_{j=1}^n a_{ij}^2\right)^4\right) = O\left(n^5\right).$$

Therefore,

$$E\left(W-W'\right)^8 = O\left(\frac{1}{n^4}\right). \tag{28}$$

Substituting $\lambda = \frac{2}{n}$ and $R = 0$ in Theorem 1, and using (24) and (28), we obtain

$$\begin{aligned}&\big|P(W \le z) - \Phi(z)\big| \\ &\quad \le \frac{C}{(1+|z|)^3}\left(\frac{EX_1^4}{\sigma_n^2}\left(\sqrt{\sum_i\Big(\sum_j a_{ij}^2\Big)^2} + \sqrt{\sum_{i,j}\Big(\sum_k a_{ik}a_{jk}\Big)^2}\right) + O\Big(\frac{1}{\sqrt{n}}\Big)\right).\end{aligned}$$

Therefore, the proof is complete. □

4 Discussions

In this paper, we have constructed a nonuniform bound for normal approximation by using the exchangeable techniques where the distance between the exchangeable pair could be unbounded. Moreover, we have applied our main result to two applications: (1) the independence test where we showed that our bound is sharper than the most recent bound obtained in Rerkruthairat [13]; (2) the quadratic forms where we showed that the obtained bound is smaller than the bound obtained in Liu et al. [10]. Some further extensions of our study can be carried out, for instance, to investigate further applications of the obtained bound, or to adapt the concept to obtain a bound for other coupling techniques.

Funding
This research is supported by the Thailand Science Research and Innovation (Grant Number MRG6180110).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
The authors made the same contributions in preparing the manuscript. All authors reviewed the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Science, Silpakorn University, Nakhon Pathom, Thailand. [2]Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok, Thailand.

Received: 6 November 2022 Accepted: 3 January 2023 Published online: 18 January 2023

References
1. Stein, C.: A bound for the error in the normal approximation to the distribution of a sum of dependent random variable. In: Proc. Sixth Berkeley Symp. Math. Statist. Probab., vol. 2, pp. 583–602 (1972)
2. Berry, A.C.: The accuracy of the Gaussian approximation to the sum of independent variates. Trans. Am. Math. Soc. **49**(1), 122–136 (1941)
3. Esseen, C.: Fourier analysis of distribution functions: a mathematical study of the Laplace-Gaussian law. Acta Math. **77**, 1–125 (1945)
4. Stein, C.: Approximate Computation of Expectations. Lecture Notes-Monograph Series, vol. 7. Institute of Mathematical Statistics, Hayward (1986)
5. Chen, L.H.Y., Goldstein, L., Shao, Q.M.: Normal Approximation by Stein's Method. Springer, New York (2010)
6. Rinott, Y., Rotar, V.: On coupling constructions and rates in the CLT for dependent summands with applications to the antivoter model and weighted U-statistics. Ann. Appl. Probab. **7**, 1080–1105 (1997)
7. Shao, Q.M., Su, Z.G.: The Berry-Esseen bound for character ratios. Proc. Am. Math. Soc. **34**, 2153–2159 (2006)
8. Sumritnorrapong, P., Neammanee, K., Suntornchost, J.: Exponential bounds via Stein's method and exchangeable pairs. ScienceAsia **44**(4), 277–287 (2018)
9. Shao, Q.M., Zhang, Z.S.: Berry-Esseen bounds of normal and nonnormal approximation for unbounded exchangeable pairs. Ann. Probab. **47**(1), 61–108 (2019)

10. Liu, D.L., Li, Z., Wang, H.C., Chen, Z.J.: Non-uniform Berry-Esseen bound by unbounded exchangeable pairs approach. Appl. Math. J. Chin. Univ. **36**(2), 256–268 (2021)
11. Schott, J.R.: Testing for complete independence in high dimensions. Biometrika **92**, 951–956 (2005)
12. Chen, Y., Shao, Q.M.: Berry-Esseen inequality for unbounded exchangeable pairs. In: Probability Approximations and Beyond, pp. 13–30. Springer, New York (2012)
13. Rerkruthairat, N.: A nonuniform bound to an independent test in high dimensional data analysis via Stein's method. J. Probab. Stat. **2019**, 1–10 (2019)
14. Jong, P.D.: A central limit theorem for generalized quadratic forms. Probab. Theory Relat. Fields **75**, 261–277 (1987)
15. Götze, F., Tikhomirov, A.: Asymptotic distribution of quadratic forms. Ann. Probab. Theory **27**, 1072–1098 (1999)