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# On new general inequalities for $s$-convex functions and their applications

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**Abstract**

In this work, we established some new general integral inequalities of Hermite–Hadamard type for $s$-convex functions. To obtain these inequalities, we used the Hölder inequality, power-mean integral inequality, and some generalizations associated with these inequalities. Also we compared some inequalities (e.g., Theorem 6 and Theorem 8). Finally, we gave some applications for special means.

**MSC:** 26A15; 26A51; 26D10

**Keywords:** $s$-Convex function; Hermite–Hadamard inequality; Hölder inequality; Power-mean integral inequality

## 1 Introduction

Let $\zeta : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ ($I \subset \mathbb{R}$) and $\mu, \omega \in I$ with $\mu < \omega$. The following double inequality

$$\zeta\left(\frac{\mu+\omega}{2}\right) \leq \frac{1}{\omega-\mu}\int_{\mu}^{\omega} \zeta(\chi)\,d\chi \leq \frac{\zeta(\mu)+\zeta(\omega)}{2}$$

is known in the literature as the Hermite–Hadamard inequality for convex function.

The Hermite–Hadamard inequality is a significant inequality with many applications for convex functions. Owing to the great importance of this inequality, in the last decade many remarkable refinements, extensions, generalizations, and different forms of Hermite–Hadamard inequality for different classes of convexity, such as $m$-convex, $(\alpha, m)$-convex, $s$-convex, harmonically convex, exponential convex, and co-ordinated convex functions, have been considered in the literature. In addition, with the help of the obtained kernels and identities, many authors have contributed to the development of this field. So, there have been a great number of studies on this subject; we recommend interested readers to read the papers [1–6] and some of the references therein.

**Definition 1** $\zeta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be a convex function if

$$\zeta\big(\tau\chi + (1-\tau)\gamma\big) \leq \tau\zeta(\chi) + (1-\tau)\zeta(\gamma)$$

holds for all $\chi, \gamma \in I$ and $\tau \in [0, 1]$.

The convex functions theory plays an important role in all the fields of pure and applied mathematics. Some noteworthy inequalities have been acquired using the different types of convexity [7–9]. One of these types of convexity is $s$-convexity. In [10], Hudzik and Maligranda investigated the class of $s$-convex functions in the second sense. In this study, various properties (e.g., nonnegative on $[0,\infty)$ and nondecreasing on $(0,\infty)$) of $s$-convex functions are examined and examples are given. Of course, $s$-convexity means the convexity only when $s = 1$. This definition is given as follows.

**Definition 2** $\zeta : [0,\infty) \to \mathbb{R}$ is said to be $s$-convex function in the second sense if the inequality

$$\zeta\left(\tau\chi + (1-\tau)\gamma\right) \le \tau^s \zeta(\chi) + (1-\tau)^s \zeta(\gamma)$$

holds for all $\chi, \gamma \in [0,\infty)$, $\tau \in [0,1]$, and $s \in (0,1]$. The class of $s$-convex functions in the second sense is usually denoted by $K_s^2$.

In [11], Dragomir and Fitzpatrick established a modification of Hermite–Hadamard inequality that holds for the $s$-convex functions in the second sense.

**Theorem 1** *Suppose that $\zeta : [0,\infty) \to [0,\infty)$ is an $s$-convex function in the second sense, where $s \in (0,1]$, and let $\mu, \omega \in [0,\infty)$, $\mu < \omega$. If $\zeta \in L[\mu,\omega]$, then the following inequalities hold*:

$$2^{s-1}\zeta\left(\frac{\mu+\omega}{2}\right) \le \frac{1}{\omega-\mu}\int_\mu^\omega \zeta(\chi)\,d\chi \le \frac{\zeta(\mu)+\zeta(\omega)}{s+1}.$$

*The constant $\xi = \frac{1}{s+1}$ is the best possible in the second inequality. The above inequalities are sharp. For final results and generalizations regarding $s$-convex functions, see* [12–17].

Hölder's inequality, one of the important inequalities of mathematical analysis named after Otto Hölder, a German mathematician, is a fundamental inequality between integrals and an indispensable tool for the study of $L^p$ spaces. Many generalizations and refinements have been obtained in the theory of convex functions using this inequality. However, İşcan obtained a new form of the Hölder inequality using a simple method in [18]. Using the Hölder–Işcan inequality, better upper bounds than those in previous studies are obtained. This new form is as follows.

**Theorem 2** (Hölder–İşcan integral inequality) *Let $\zeta$ and $\vartheta$ be real mappings defined on $[\mu,\omega]$. If $|\zeta|^p$ and $|\vartheta|^q$ are integrable on $[\mu,\omega]$, then*

$$\begin{aligned}\int_\mu^\omega \left|\zeta(\chi)\vartheta(\chi)\right|d\chi \le \frac{1}{\omega-\mu}\Bigg\{&\left(\int_\mu^\omega (\omega-\chi)\left|\zeta(\chi)\right|^p d\chi\right)^{\frac{1}{p}}\left(\int_\mu^\omega (\omega-\chi)\left|\vartheta(\chi)\right|^q d\chi\right)^{\frac{1}{q}} \\ &+ \left(\int_\mu^\omega (\chi-\mu)\left|\zeta(\chi)\right|^p d\chi\right)^{\frac{1}{p}}\left(\int_\mu^\omega (\chi-\mu)\left|\vartheta(\chi)\right|^q d\chi\right)^{\frac{1}{q}}\Bigg\},\end{aligned}$$

*where $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$.*

The power-mean inequality, which is a different version of the Hölder inequality, is well known for its elementary role in many branches of mathematical analysis. In [19], Kadakal *et al.* showed and proved the improved power-mean inequality, which gives better results than the power-mean inequality. This new generalized expression is as follows.

**Theorem 3** (Improved power-mean integral inequality) *Let $\zeta$ and $\vartheta$ be real mappings defined on $[\mu,\omega]$. If $|\zeta|$, $|\zeta||\vartheta|^q$ are integrable on $[\mu,\omega]$, then*

$$\begin{aligned}&\int_\mu^\omega \left|\zeta(\chi)\vartheta(\chi)\right| d\chi\\&\quad\le \frac{1}{\omega-\mu}\left\{\left(\int_\mu^\omega (\omega-\chi)\left|\zeta(\chi)\right| d\chi\right)^{1-\frac{1}{q}}\left(\int_\mu^\omega (\omega-\chi)\left|\zeta(\chi)\right|\left|\vartheta(\chi)\right|^q d\chi\right)^{\frac{1}{q}}\right.\\&\qquad\left.+\left(\int_\mu^\omega (\chi-\mu)\left|\zeta(\chi)\right| d\chi\right)^{1-\frac{1}{q}}\left(\int_\mu^\omega (\chi-\mu)\left|\zeta(\chi)\right|\left|\vartheta(\chi)\right|^q d\chi\right)^{\frac{1}{q}}\right\},\end{aligned}$$

*where $q \ge 1$.*

In [20], İşcan *et al.* developed a new lemma and found new generalizations for convex functions. The point emphasized in this study is that it applied the definition of convex function twice to the obtained inequalities. For the special values of $\varphi$, new inequalities have been obtained and also their relationship with previous studies has been determined. These new identities obtained are as follows.

**Lemma 1** *Let $\zeta : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, where $\mu,\omega \in I^\circ$, with $\mu < \omega$. If $\zeta' \in L[\mu,\omega]$, then the following equality holds*:

$$\begin{aligned}&\left|I_\varphi(\zeta,\mu,\omega)\right|\\&\quad=\sum_{\varepsilon=0}^{\varphi-1}\frac{1}{2\varphi}\left[\zeta\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)+\zeta\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right]-\frac{1}{\omega-\mu}\int_\mu^\omega \zeta(\chi)\,d\chi\\&\quad=\sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^2}\left[\int_0^1 (1-2\tau)\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}+(1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)d\tau\right].\end{aligned}$$

**Theorem 4** *Let $\zeta : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, where $\mu,\omega \in I^\circ$, with $\mu < \omega$. If $|\zeta'|^q$ is convex on $[\mu,\omega]$ for some fixed $q > 1$, then the following inequality is satisfied*:

$$\begin{aligned}&\left|I_\varphi(\zeta,\mu,\omega)\right|\\&\quad\le\sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{\varphi^2 2^{1+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{2\varphi-2\varepsilon-1}{\varphi}\right)\left|\zeta'(\mu)\right|^q+\left(\frac{2\varepsilon+1}{\varphi}\right)\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}},\end{aligned}\tag{1.1}$$

*where $\frac{1}{q}+\frac{1}{p}=1$.*

**Theorem 5** *Let $\zeta : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, where $\mu, \omega \in I^\circ$, with $\mu < \omega$. If $|\zeta'|^q$ is convex on $[\mu, \omega]$ for some fixed $q \geq 1$, then the following inequality is satisfied*:

$$\left|I_\varphi(\zeta,\mu,\omega)\right| \leq \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{4\varphi^2}\left[\left(\frac{2\varphi-2\varepsilon-1}{2\varphi}\right)\left|\zeta'(\mu)\right|^q + \left(\frac{2\varepsilon+1}{2\varphi}\right)\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}. \tag{1.2}$$

Our main contribution in this study is to use $s$-convex functions and to obtain new generalization identities. We also aimed to compare the inequalities obtained by different methods. The fact that these inequalities were conformable with the literature motivated us.

## 2 Main results

**Theorem 6** *Let $\zeta : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^\circ$, where $\mu, \omega \in I^\circ$, with $\mu < \omega$. If $|\zeta'|^q$ is $s$-convex on $[\mu, \omega]$ for some fixed $q > 1$, then the following inequality is satisfied*:

$$\begin{aligned} &\left|I_\varphi(\zeta,\mu,\omega)\right| \\ &\quad\leq \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{2\varphi^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\ &\qquad\times\left[\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q + \left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right]^{\frac{1}{q}}, \end{aligned} \tag{2.1}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof* From Lemma 1 and by using the Hölder inequality, we have

$$\begin{aligned} &\left|I_\varphi(\zeta,\mu,\omega)\right| \\ &\quad\leq \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{2\varphi^2}\left[\int_0^1\left|(1-2\tau)\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}+(1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|d\tau\right] \\ &\quad\leq \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{2\varphi^2}\left(\int_0^1|1-2\tau|^p\,d\tau\right)^{\frac{1}{p}} \\ &\qquad\times\left(\int_0^1\left|\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}+(1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q d\tau\right)^{\frac{1}{q}}. \end{aligned}$$

By using the $s$-convexity of $|\zeta'|^q$, we obtain

$$\begin{aligned} \left|I_\varphi(\zeta,\mu,\omega)\right| \leq{}& \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{2\varphi^2}\left(\int_0^1|1-2\tau|^p\,d\tau\right)^{\frac{1}{p}}\left[\int_0^1\left(\tau^s\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q\right.\right. \\ &\left.\left.+(1-\tau)^s\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right)d\tau\right]^{\frac{1}{q}} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{2\varphi^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\
&\quad\times\left[\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q+\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

Thus, the proof is completed. □

**Corollary 1** *If we choose $s=1$ in Theorem 6, then we obtain*

$$
\begin{aligned}
\left|I_\varphi(\zeta,\mu,\omega)\right| &\le \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{\varphi^2 2^{1+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q\right. \\
&\quad\left.+\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

**Corollary 2** *If we use the s-convexity of $|\zeta'|^q$ once again in Theorem 6, we have*

$$
\begin{aligned}
\left|I_\varphi(\zeta,\mu,\omega)\right| &\le \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{2\varphi^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\
&\quad\times\left[\left(\left(\frac{\varphi-\varepsilon}{\varphi}\right)^s+\left(\frac{\varphi-\varepsilon-1}{\varphi}\right)^s\right)\left|\zeta'(\mu)\right|^q\right. \\
&\quad\left.+\left(\left(\frac{\varepsilon}{\varphi}\right)^s+\left(\frac{\varepsilon+1}{\varphi}\right)^s\right)\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}
\tag{2.2}
$$

*Remark* 1 If we choose $s=1$ in Corollary 2, then inequality (2.2) reduces to inequality (1.1).

**Corollary 3** *If we choose $\varphi=2$ in Corollary 2, then we obtain*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[\frac{\zeta(\mu)+\zeta(\omega)}{2}+\zeta\left(\frac{\mu+\omega}{2}\right)\right]-\frac{1}{\omega-\mu}\int_\mu^\omega \zeta(\chi)\,d\chi\right| \\
&\quad\le \frac{\omega-\mu}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left\{\left(\left[1+\frac{1}{2^s}\right]\left|\zeta'(\mu)\right|^q+\frac{1}{2^s}\left|\zeta'(\omega)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{1}{2^s}\left|\zeta'(\mu)\right|^q+\left[1+\frac{1}{2^s}\right]\left|\zeta'(\omega)\right|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

**Theorem 7** *Let $\zeta: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $I^\circ$, where $\mu,\omega\in I^\circ$, with $\mu<\omega$. If $|\zeta'|^q$ is s-convex on $[\mu,\omega]$ for some fixed $q\ge 1$, then the following inequality is satisfied*:

$$
\begin{aligned}
&\left|I_\varphi(\zeta,\mu,\omega)\right| \\
&\quad\le \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{\varphi^2 2^{2-\frac{1}{q}}}\left(\frac{s}{(s+1)(s+2)}+\frac{1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}} \\
&\quad\times\left[\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q+\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}
\tag{2.3}
$$

*Proof* From Lemma 1 and by using the well-known power-mean inequality, we have

$$\begin{aligned}
\left|I_\varphi(\zeta,\mu,\omega)\right| &\leq \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^2}\left[\int_0^1\left|(1-2\tau)\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right.\right.\right.\\
&\quad\left.\left.\left.+(1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|d\tau\right]\\
&\leq \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^2}\left(\int_0^1|1-2\tau|\,d\tau\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1|1-2\tau|\left|\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right.\right.\right.\\
&\quad\left.\left.\left.+(1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q d\tau\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|\zeta'|^q$ is $s$-convex $[\mu,\omega]$, then

$$\begin{aligned}
\left|I_\varphi(\zeta,\mu,\omega)\right| &\leq \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^2}\left(\int_0^1|1-2\tau|\,d\tau\right)^{1-\frac{1}{q}}\left[\int_0^1|1-2\tau|\left(\tau^s\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q\right.\right.\\
&\quad\left.\left.+(1-\tau)^s\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right)d\tau\right]^{\frac{1}{q}}\\
&= \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{\varphi^2 2^{2-\frac{1}{q}}}\left(\frac{s}{(s+1)(s+2)}+\frac{1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}\\
&\quad\times\left[\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q+\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right]^{\frac{1}{q}},
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
&\int_0^1|1-2\tau|\,d\tau=\frac{1}{2},\\
&\int_0^1|1-2\tau|\tau^s\,d\tau=\int_0^1|1-2\tau|(1-\tau)^s\,d\tau=\frac{s}{(s+1)(s+2)}+\frac{1}{2^s(s+1)(s+2)}.
\end{aligned}$$

Thus, the proof is completed. □

**Corollary 4** *If we choose $s=1$ in Theorem 7, then we obtain*

$$\begin{aligned}
\left|I_\varphi(\zeta,\mu,\omega)\right| &\leq \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{\varphi^2 2^{2+\frac{1}{q}}}\left[\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q\right.\\
&\quad\left.+\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

**Corollary 5** *If we use the s-convexity of $|\zeta'|^q$ once again in Theorem 7, we get*

$$\begin{aligned}\big|I_\varphi(\zeta,\mu,\omega)\big| \leq \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{\varphi^2 2^{2-\frac{1}{q}}}\left(\frac{s}{(s+1)(s+2)}+\frac{1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}} \\ \times\left[\left(\left(\frac{\varphi-\varepsilon}{\varphi}\right)^s+\left(\frac{\varphi-\varepsilon-1}{\varphi}\right)^s\right)\big|\zeta'(\mu)\big|^q \right. \\ \left.+\left(\left(\frac{\varepsilon}{\varphi}\right)^s+\left(\frac{\varepsilon+1}{\varphi}\right)^s\right)\big|\zeta'(\omega)\big|^q\right]^{\frac{1}{q}}.\end{aligned}\tag{2.4}$$

*Remark* 2 If we choose $s=1$ in Corollary 5, then inequality (2.4) reduces to inequality (1.2).

**Corollary 6** *If we choose $\varphi=2$ and $s=1$ in Theorem 7, then we obtain*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{\zeta(\mu)+\zeta(\omega)}{2}+\zeta\left(\frac{\mu+\omega}{2}\right)\right]-\frac{1}{\omega-\mu}\int_\mu^\omega \zeta(\chi)\,d\chi\right| \\ &\quad\leq \frac{\omega-\mu}{2^{2+\frac{1}{q}}}\left[\left(\big|\zeta'(\mu)\big|^q+\left|\zeta'\left(\frac{\mu+\omega}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\zeta'\left(\frac{\mu+\omega}{2}\right)\right|^q+\big|\zeta'(\omega)\big|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

**Theorem 8** *Let $\zeta: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $I^\circ$, where $\mu,\omega\in I^\circ$, with $\mu<\omega$. If $|\zeta'|^q$ is s-convex on $[\mu,\omega]$, then the following inequality is obtained*:

$$\begin{aligned}&\big|I_\varphi(\zeta,\mu,\omega)\big| \\ &\quad\leq \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{s+2}\right)^{\frac{1}{q}} \\ &\qquad\times\left\{\left[\frac{1}{(s+1)}\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q+\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right]^{\frac{1}{q}}\right. \\ &\qquad\left.+\left[\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q+\frac{1}{(s+1)}\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right]^{\frac{1}{q}}\right\},\end{aligned}\tag{2.5}$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

*Proof* From Lemma 1 and by using the Hölder–İşcan inequality, we have

$$\begin{aligned}&\big|I_\varphi(\zeta,\mu,\omega)\big| \\ &\quad\leq \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^2}\left[\int_0^1\left|(1-2\tau)\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right.\right.\right. \\ &\qquad\left.\left.\left.+(1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|\,d\tau\right] \\ &\quad\leq \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^2}\left\{\left(\int_0^1(1-\tau)|1-2\tau|^p\,d\tau\right)^{\frac{1}{p}}\right. \\ &\qquad\times\left(\int_0^1(1-\tau)\left|\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}+(1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\,d\tau\right)^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^1 \tau|1-2\tau|^p\,d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \tau\left|\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right.\right.\right.\\
&\left.\left.\left.+(1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q d\tau\right)^{\frac{1}{q}}\Bigg\}\\
=&\sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\\
&\times\left\{\left(\int_0^1(1-\tau)\left|\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}+(1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q d\tau\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\tau\left|\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}+(1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q d\tau\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

By using the $s$-convexity of $|\zeta'|^q$, we have

$$
\begin{aligned}
&\left|I_\varphi(\zeta,\mu,\omega)\right|\\
&\leq\sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\\
&\quad\times\left\{\left[\int_0^1(1-\tau)\left(\tau^s\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q\right.\right.\right.\\
&\quad\left.\left.+(1-\tau)^s\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right)d\tau\right]^{\frac{1}{q}}\\
&\quad+\left[\int_0^1\tau\left(\tau^s\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q\right.\right.\\
&\quad\left.\left.\left.+(1-\tau)^s\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right)d\tau\right]^{\frac{1}{q}}\right\}\\
&=\sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\\
&\quad\times\left\{\left[\frac{1}{(s+1)(s+2)}\left|\zeta\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q+\frac{1}{s+2}\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\frac{1}{s+2}\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q+\frac{1}{(s+1)(s+2)}\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

That completes the proof. □

**Corollary 7** *If we choose* $s=1$ *in Theorem* 8, *then we obtain*

$$
\begin{aligned}
\left|I_\varphi(\zeta,\mu,\omega)\right|\leq&\sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\\
&\times\left\{\left[\frac{1}{6}\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q+\frac{1}{3}\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right]^{\frac{1}{q}}\right.\\
&\left.+\left[\frac{1}{3}\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q+\frac{1}{6}\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

**Corollary 8** *If we use the s-convexity of* $|\zeta'|^q$ *once again in Theorem* 8, *we have*

$$\begin{aligned}
\left|I_{\varphi}(\zeta,\mu,\omega)\right| \leq{}& \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^{2+\frac{s}{q}}}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{s+2}\right)^{\frac{1}{q}} \\
&\times\left\{\left[\left(\frac{(\varphi-\varepsilon)^s}{s+1}+(\varphi-\varepsilon-1)^s\right)\left|\zeta'(\mu)\right|^q+\left(\frac{\varepsilon^s}{s+1}+(\varepsilon+1)^s\right)\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\left((\varphi-\varepsilon)^s+\frac{(\varphi-\varepsilon-1)^s}{s+1}\right)\left|\zeta'(\mu)\right|^q+\left(\varepsilon^s+\frac{(\varepsilon+1)^s}{s+1}\right)\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

**Corollary 9** *If we choose* $s=1$ *in Corollary* 8, *then we obtain*

$$\begin{aligned}
\left|I_{\varphi}(\zeta,\mu,\omega)\right| \leq{}& \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^{2}}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{3}\right)^{\frac{1}{q}} \\
&\times\left\{\left[\left(\frac{3\varphi-3\varepsilon-2}{2\varphi}\right)\left|\zeta'(\mu)\right|^q+\left(\frac{3\varepsilon+2}{2\varphi}\right)\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\left(\frac{3\varphi-3\varepsilon-1}{2\varphi}\right)\left|\zeta'(\mu)\right|^q+\left(\frac{3\varepsilon+1}{2\varphi}\right)\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

**Corollary 10** *If we choose* $\varphi=1$ *in Theorem* 8, *then we obtain*

$$\begin{aligned}
&\left|\frac{\zeta(\mu)+\zeta(\omega)}{2}-\frac{1}{\omega-\mu}\int_{\mu}^{\omega}\zeta(\chi)\,d\chi\right| \\
&\quad\leq\frac{\omega-\mu}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{s+2}\right)^{\frac{1}{q}} \\
&\quad\times\left\{\left[\frac{1}{s+1}\left|\zeta'(\mu)\right|^q+\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}+\left[\left|\zeta'(\mu)\right|^q+\frac{1}{s+1}\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

**Corollary 11** *In Corollary* 10, *if we choose* $s=1$, *we obtain*

$$\begin{aligned}
&\left|\frac{\zeta(\mu)+\zeta(\omega)}{2}-\frac{1}{\omega-\mu}\int_{\mu}^{\omega}\zeta(\chi)\,d\chi\right| \\
&\quad\leq\frac{\omega-\mu}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left\{\left[\frac{|\zeta'(\mu)|^q+2|\zeta'(\omega)|^q}{6}\right]^{\frac{1}{q}}+\left[\frac{2|\zeta'(\mu)|^q+|\zeta'(\omega)|^q}{6}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

**Corollary 12** *If we choose* $\varphi=2$ *in Theorem* 8, *then we obtain*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{\zeta(\mu)+\zeta(\omega)}{2}+\zeta\left(\frac{\mu+\omega}{2}\right)\right]-\frac{1}{\omega-\mu}\int_{\mu}^{\omega}\zeta(\chi)\,d\chi\right| \\
&\quad\leq\frac{\omega-\mu}{8}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{s+2}\right)^{\frac{1}{q}} \\
&\quad\times\left\{\left[\left(\frac{1}{s+1}\left|\zeta'(\mu)\right|^q+\left|\zeta'\left(\frac{\mu+\omega}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{s+1}\left|\zeta'\left(\frac{\mu+\omega}{2}\right)\right|^q+\left|\zeta'(\omega)\right|^q\right)^{\frac{1}{q}}\right]\right. \\
&\quad\left.+\left[\left(\left|\zeta'(\mu)\right|^q+\frac{1}{s+1}\left|\zeta'\left(\frac{\mu+\omega}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\zeta'\left(\frac{\mu+\omega}{2}\right)\right|^q+\frac{1}{s+1}\left|\zeta'(\omega)\right|^q\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

**Corollary 13** *In Corollary 12, if we choose $s = 1$, we have*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{\zeta(\mu)+\zeta(\omega)}{2}+\zeta\left(\frac{\mu+\omega}{2}\right)\right]-\frac{1}{\omega-\mu}\int_\mu^\omega \zeta(\chi)\,d\chi\right| \\
&\quad\leq \frac{\omega-\mu}{8}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} \\
&\qquad\times\left\{\left[\left(\frac{|\zeta'(\mu)|^q}{6}+\frac{1}{3}\left|\zeta'\left(\frac{\mu+\omega}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{6}\left|\zeta'\left(\frac{\mu+\omega}{2}\right)\right|^q+\frac{1}{3}\left|\zeta'(\omega)\right|^q\right)^{\frac{1}{q}}\right]\right. \\
&\qquad\left.+\left[\left(\frac{1}{3}\left|\zeta'(\mu)\right|^q+\frac{1}{6}\left|\zeta'\left(\frac{\mu+\omega}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{3}\left|\zeta'\left(\frac{\mu+\omega}{2}\right)\right|^q+\frac{|\zeta'(\omega)|^q}{6}\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

*Remark* 3 Inequality (2.5) is better than inequality (2.1). In fact, since the function $\psi : [0,\infty) \to \mathbb{R}$, $\psi(\chi) = \chi^\rho$, $\rho \in (0,1]$ is a concave function, we can write

$$\frac{\theta^\rho+\delta^\rho}{2}=\frac{\psi(\theta)+\psi(\delta)}{2}\leq\psi\left(\frac{\theta+\delta}{2}\right)=\left(\frac{\theta+\delta}{2}\right)^\rho \tag{2.6}$$

for all $\theta, \delta \geq 0$. In inequality (2.6), if we choose

$$\begin{aligned}
\theta &= \frac{|\zeta'(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi})|^q+(s+1)|\zeta'(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi})|^q}{s+2}, \\
\delta &= \frac{(s+1)|\zeta'(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi})|^q+|\zeta'(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi})|^q}{s+2}
\end{aligned}$$

and $\rho = \frac{1}{q}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left[\frac{|\zeta'(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi})|^q+(s+1)|\zeta'(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi})|^q}{s+2}\right]^{\frac{1}{q}} \\
&\qquad+\frac{1}{2}\left[\frac{(s+1)|\zeta'(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi})|^q+|\zeta'(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi})|^q}{s+2}\right]^{\frac{1}{q}} \\
&\quad\leq\left[\frac{|\zeta'(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi})|^q+|\zeta'(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi})|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

So, we have the following inequality:

$$\begin{aligned}
&\sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2\varphi^2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\
&\qquad\times\left\{\left[\frac{|\zeta'(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi})|^q+(s+1)|\zeta'(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi})|^q}{s+2}\right]^{\frac{1}{q}}\right. \\
&\qquad\left.+\left[\frac{(s+1)|\zeta'(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi})|^q+|\zeta'(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi})|^q}{s+2}\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

$$\leq \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{2\varphi^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \times\left[\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q+\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\right]^{\frac{1}{q}}.$$

**Theorem 9** *Let $\zeta : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^\circ$, where $\mu,\omega \in I^\circ$, with $\mu < \omega$. If $|\zeta'|^q$ is s-convex on $[\mu,\omega]$ for some fixed $q \geq 1$, then the following inequality is satisfied*:

$$\begin{aligned}
&\left|I_\varphi(\zeta,\mu,\omega)\right| \\
&\leq \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{2^{3-\frac{2}{q}}\varphi^2}\Bigg\{\Bigg[\left(\frac{(\frac{1}{2})^s-1}{s+1}+\frac{3-3(\frac{1}{2})^{s+1}}{s+2}+\frac{(\frac{1}{2})^{s+1}-2}{s+3}\right)\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q \\
&\quad+\left(\frac{(\frac{1}{2})^{s+1}-1}{s+2}+\frac{2-(\frac{1}{2})^{s+1}}{s+3}\right)\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\Bigg]^{\frac{1}{q}} \\
&\quad+\Bigg[\left(\frac{(\frac{1}{2})^{s+1}-1}{s+2}+\frac{2-(\frac{1}{2})^{s+1}}{s+3}\right)\left|\zeta'\left(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right)\right|^q \\
&\quad+\left(\frac{(\frac{1}{2})^s-1}{s+1}+\frac{3-3(\frac{1}{2})^{s+1}}{s+2}+\frac{(\frac{1}{2})^{s+1}-2}{s+3}\right)\left|\zeta'\left(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q\Bigg]^{\frac{1}{q}}\Bigg\}.
\end{aligned} \tag{2.7}$$

*Proof* From Lemma 1 and by using the improved power-mean inequality, we have

$$\begin{aligned}
&\left|I_\varphi(\zeta,\mu,\omega)\right| \\
&\leq \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{2\varphi^2}\left[\int_0^1\left|(1-2\tau)\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right.\right.\right. \\
&\quad\left.\left.\left.+(1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|\,d\tau\right] \\
&\leq \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{2\varphi^2}\Bigg\{\left(\int_0^1(1-\tau)|1-2\tau|\,d\tau\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1(1-\tau)|1-2\tau|\left|\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right.\right.\right. \\
&\quad\left.\left.\left.+(1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q d\tau\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1\tau|1-2\tau|\,d\tau\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\tau|1-2\tau|\left|\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}+(1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\right)\right|^q d\tau\right)^{\frac{1}{q}}\Bigg\} \\
&= \sum_{\varepsilon=0}^{\varphi-1} \frac{\omega-\mu}{2\varphi^2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}} \\
&\quad\times\Bigg\{\Bigg(\int_0^1(1-\tau)|1-2\tau|\left|\zeta'\left(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\right.\right.
\end{aligned}$$

$$+ (1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\bigg)\bigg|^q d\tau\bigg)^{\frac{1}{q}}$$
$$+ \bigg(\int_0^1 \tau|1-2\tau|\bigg|\zeta'\bigg(\tau\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}$$
$$+ (1-\tau)\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\bigg)\bigg|^q d\tau\bigg)^{\frac{1}{q}}\bigg\}.$$

By using the $s$-convexity of $|\zeta'|^q$, we have

$$\begin{aligned}
&\big|I_\varphi(\zeta,\mu,\omega)\big| \\
&\leq \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2^{3-\frac{2}{q}}\varphi^2}\bigg\{\bigg(\bigg(\int_0^1(1-\tau)|1-2\tau|\bigg(\tau^s\bigg|\zeta'\bigg(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\bigg)\bigg|^q \\
&+ (1-\tau)^s\bigg|\zeta'\bigg(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\bigg)\bigg|^q\bigg)d\tau\bigg)^{\frac{1}{q}} \\
&+ \bigg(\int_0^1\tau|1-2\tau|\bigg[\tau^s\bigg|\zeta'\bigg(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\bigg)\bigg|^q \\
&+ (1-\tau)^s\bigg|\zeta'\bigg(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\bigg)\bigg|^q\bigg]d\tau\bigg)^{\frac{1}{q}}\bigg\} \\
&= \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2^{3-\frac{2}{q}}\varphi^2}\bigg\{\bigg[\bigg(\frac{(\frac{1}{2})^s-1}{s+1}+\frac{3-3(\frac{1}{2})^{s+1}}{s+2}+\frac{(\frac{1}{2})^{s+1}-2}{s+3}\bigg)\bigg|\zeta'\bigg(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\bigg)\bigg|^q \\
&+ \bigg(\frac{(\frac{1}{2})^{s+1}-1}{s+2}+\frac{2-(\frac{1}{2})^{s+1}}{s+3}\bigg)\bigg|\zeta'\bigg(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\bigg)\bigg|^q\bigg]^{\frac{1}{q}} \\
&+ \bigg[\bigg(\frac{(\frac{1}{2})^{s+1}-1}{s+2}+\frac{2-(\frac{1}{2})^{s+1}}{s+3}\bigg)\bigg|\zeta'\bigg(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\bigg)\bigg|^q \\
&+ \bigg(\frac{(\frac{1}{2})^s-1}{s+1}+\frac{3-3(\frac{1}{2})^{s+1}}{s+2}+\frac{(\frac{1}{2})^{s+1}-2}{s+3}\bigg)\bigg|\zeta'\bigg(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\bigg)\bigg|^q\bigg]^{\frac{1}{q}}\bigg\}.
\end{aligned}$$

That completes the proof. □

**Corollary 14** *If we choose $s=1$ in Theorem 9, then we obtain*

$$\begin{aligned}
&\big|I_\varphi(\zeta,\mu,\omega)\big| \\
&\leq \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2^{3+\frac{2}{q}}\varphi^2}\bigg\{\bigg[\bigg|\zeta'\bigg(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\bigg)\bigg|^q+3\bigg|\zeta'\bigg(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\bigg)\bigg|^q\bigg]^{\frac{1}{q}} \\
&+ \bigg[3\bigg|\zeta'\bigg(\frac{(\varphi-\varepsilon)\mu+\varepsilon\omega}{\varphi}\bigg)\bigg|^q+\bigg|\zeta'\bigg(\frac{(\varphi-\varepsilon-1)\mu+(\varepsilon+1)\omega}{\varphi}\bigg)\bigg|^q\bigg]^{\frac{1}{q}}\bigg\}.
\end{aligned}$$

**Corollary 15** *If we use the s-convexity of $|\zeta'|^q$ once again in Theorem 9, we have*

$$\begin{aligned}
&\big|I_\varphi(\zeta,\mu,\omega)\big| \\
&\leq \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2^{3-\frac{2}{q}}\varphi^{2+\frac{s}{q}}}
\end{aligned}$$

$$\times \left\{\left[\left[\left(\frac{(\frac{1}{2})^{s+1}s+5(\frac{1}{2})^{s+1}+s-1}{s^3+6s^2+11s+6}\right)(\varphi-\varepsilon)^s+\left(\frac{(\frac{1}{2})^{s+1}+s+1}{s^2+5s+6}\right)(\varphi-\varepsilon-1)^s\right]\left|\zeta'(\mu)\right|^q\right.\right.$$
$$+\left.\left[\left(\frac{(\frac{1}{2})^{s+1}s+5(\frac{1}{2})^{s+1}+s-1}{s^3+6s^2+11s+6}\right)\varepsilon^s+\left(\frac{(\frac{1}{2})^{s+1}+s+1}{s^2+5s+6}\right)(\varepsilon+1)^s\right]\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}$$
$$+\left[\left[\left(\frac{(\frac{1}{2})^{s+1}s+5(\frac{1}{2})^{s+1}+s-1}{s^3+6s^2+11s+6}\right)(\varphi-\varepsilon-1)^s+\left(\frac{(\frac{1}{2})^{s+1}+s+1}{s^2+5s+6}\right)(\varphi-\varepsilon)^s\right]\left|\zeta'(\mu)\right|^q\right.$$
$$\left.\left.+\left[\left(\frac{(\frac{1}{2})^{s+1}s+5(\frac{1}{2})^{s+1}+s-1}{s^3+6s^2+11s+6}\right)(\varepsilon+1)^s+\left(\frac{(\frac{1}{2})^{s+1}+s+1}{s^2+5s+6}\right)\varepsilon^s\right]\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}\right\}.$$

**Corollary 16** *If we choose $s=1$ in Corollary 15, then we obtain*

$$\left|I_\varphi(\zeta,\mu,\omega)\right| \leq \sum_{\varepsilon=0}^{\varphi-1}\frac{\omega-\mu}{2^{3+\frac{2}{q}}\varphi^{2+\frac{1}{q}}}\left\{\left[(4\varphi-4\varepsilon-3)\left|\zeta'(\mu)\right|^q+(4\varepsilon+3)\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[(4\varphi-4\varepsilon-1)\left|\zeta'(\mu)\right|^q+(4\varepsilon+1)\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}\right\}.$$

**Corollary 17** *If we choose $\varphi=1$ in Corollary 15, then we obtain*

$$\left|\frac{\zeta(\mu)+\zeta(\omega)}{2}-\frac{1}{\omega-\mu}\int_\mu^\omega \zeta(\chi)\,d\chi\right|$$
$$\leq \frac{\omega-\mu}{2^{3-\frac{2}{q}}}\left\{\left[\left(\frac{(\frac{1}{2})^{s+1}s+5(\frac{1}{2})^{s+1}+s-1}{s^3+6s^2+11s+6}\right)\left|\zeta'(\mu)\right|^q+\left(\frac{(\frac{1}{2})^{s+1}+s+1}{s^2+5s+6}\right)\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\left(\frac{(\frac{1}{2})^{s+1}+s+1}{s^2+5s+6}\right)\left|\zeta'(\mu)\right|^q+\left(\frac{(\frac{1}{2})^{s+1}s+5(\frac{1}{2})^{s+1}+s-1}{s^3+6s^2+11s+6}\right)\left|\zeta'(\omega)\right|^q\right]^{\frac{1}{q}}\right\}.$$

**Corollary 18** *In Corollary 17, if we choose $s=1$, we have*

$$\left|\frac{\zeta(\mu)+\zeta(\omega)}{2}-\frac{1}{\omega-\mu}\int_\mu^\omega \zeta(\chi)\,d\chi\right|$$
$$\leq \frac{\omega-\mu}{8}\left\{\left[\frac{|\zeta'(\mu)|^q+3|\zeta'(\omega)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\zeta'(\mu)|^q+|\zeta'(\omega)|^q}{4}\right]^{\frac{1}{q}}\right\}.$$

**Corollary 19** *If we choose $\varphi=2$ and $q=1$ in Corollary 15, then we obtain*

$$\left|\frac{1}{2}\left[\frac{\zeta(\mu)+\zeta(\omega)}{2}+\zeta\left(\frac{\mu+\omega}{2}\right)\right]-\frac{1}{\omega-\mu}\int_\mu^\omega \zeta(\chi)\,d\chi\right|$$
$$\leq \frac{\omega-\mu}{2^{3+s}}\left\{(2^s+2)\left(\frac{(\frac{1}{2})^s s+3(\frac{1}{2})^s+s^2+3s}{s^3+6s^2+11s+6}\right)\left(\left|\zeta'(\mu)\right|+\left|\zeta'(\omega)\right|\right)\right\}.$$

*Remark* 4 Inequality (2.7) in Theorem 9 is better than inequality (2.3) in Theorem 7. The proof can be obtained applying similarly to Remark 3.

## 3 Applications to special means

Hudzik and Maligranda gave the following example in [21]:

Let $\zeta : [0,\infty) \to \mathbb{R}$ be a function as

$$\zeta(\tau) = \begin{cases} \mu, & \tau = 0, \\ \omega\tau^s + \varrho, & \tau > 0. \end{cases}$$

Also, let $s \in (0,1)$ and $\mu, \omega, \varrho \in \mathbb{R}$. If $\omega \geq 0$ and $0 \leq \varrho \leq \mu$, then $\zeta \in K_s^2$. Hence, for $\mu = \varrho = 0$, $\omega = 1$, we have $\zeta : [0,1] \to [0,1]$, $\zeta(\tau) = \tau^s$, $\zeta \in K_s^2$.

Now, using the results of Sect. 2, we consider some special means for which we will have new integral inequalities. Let $\mu, \omega \in \mathbb{R}$,

(1) the arithmetic mean:

$$A = A(\mu,\omega) = \frac{\mu+\omega}{2}, \quad \mu,\omega \geq 0;$$

(2) the logarithmic mean:

$$L = L(\mu,\omega) = \begin{cases} \mu, & \text{if } \mu = \omega, \\ \frac{\omega-\mu}{\ln\omega - \ln\mu}, & \text{if } \mu \neq \omega, \end{cases} \quad \mu,\omega > 0;$$

(3) the $p$-logarithmic mean:

$$L_p = L_p(\mu,\omega) = \begin{cases} \mu, & \text{if } \mu = \omega, \\ [\frac{\omega^{p+1}-\mu^{p+1}}{(p+1)(\omega-\mu)}]^{\frac{1}{p}}, & \text{if } \mu \neq \omega, \end{cases} \quad p \in \mathbb{R} \setminus \{-1,0\}, \mu,\omega > 0.$$

**Proposition 1** *Let $\mu, \omega \in \mathbb{R}$, $0 < \mu < \omega$, and $m \in \mathbb{N}$, $m \geq 2$. Then, for all $q > 1$, the following inequality holds*:

$$\begin{aligned} &\left| \sum_{\varepsilon=0}^{\varphi-1} \frac{1}{\varphi} A\left( \left( \frac{(\varphi-\varepsilon)\mu + \varepsilon\omega}{\varphi} \right)^m, \left( \frac{(\varphi-\varepsilon-1)\mu + (\varepsilon+1)\omega}{\varphi} \right)^m \right) - L_m^m(\mu,\omega) \right| \\ &\quad \leq \sum_{\varepsilon=0}^{\varphi-1} \frac{(\omega-\mu)m}{2\varphi^2} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{s+1} \right)^{\frac{1}{q}} \\ &\quad \times \left[ \left( \left( \frac{\varphi-\varepsilon}{\varphi} \right)^s + \left( \frac{\varphi-\varepsilon-1}{\varphi} \right)^s \right) \mu^{(m-1)q} + \left( \left( \frac{\varepsilon}{\varphi} \right)^s + \left( \frac{\varepsilon+1}{\varphi} \right)^s \right) \omega^{(m-1)q} \right]^{\frac{1}{q}}. \end{aligned}$$

*Proof* The proof is obtained immediately from (2.2) in Corollary 2 with $\zeta(\chi) = \chi^m$, $\chi \in [\mu,\omega]$, $m \in \mathbb{N}$, $m \geq 2$. □

**Proposition 2** *Let $\mu, \omega \in \mathbb{R}$, $0 < \mu < \omega$ and $m \in \mathbb{N}$, $m \geq 2$. Then, for all $q \geq 1$, the following inequality holds*:

$$\begin{aligned} &\left| \sum_{\varepsilon=0}^{\varphi-1} \frac{1}{\varphi} A\left( \left( \frac{(\varphi-\varepsilon)\mu + \varepsilon\omega}{\varphi} \right)^m, \left( \frac{(\varphi-\varepsilon-1)\mu + (\varepsilon+1)\omega}{\varphi} \right)^m \right) - L_m^m(\mu,\omega) \right| \\ &\quad \leq \sum_{\varepsilon=0}^{\varphi-1} \frac{(\omega-\mu)m}{\varphi^2 2^{2-\frac{1}{q}}} \left( \frac{1}{s+1} \left[ \left( \frac{1}{2} \right)^s - 1 \right] - \frac{2}{s+2} \left[ \left( \frac{1}{2} \right)^{s+1} - 1 \right] \right)^{\frac{1}{q}} \end{aligned}$$

$$\times \left[ \left( \left( \frac{\varphi - \varepsilon}{\varphi} \right)^s + \left( \frac{\varphi - \varepsilon - 1}{\varphi} \right)^s \right) \mu^{(m-1)q} + \left( \left( \frac{\varepsilon}{\varphi} \right)^s + \left( \frac{\varepsilon + 1}{\varphi} \right)^s \right) \omega^{(m-1)q} \right]^{\frac{1}{q}}.$$

*Proof* The proof is obtained immediately from (2.4) in Corollary 5 with $\zeta(\chi) = \chi^m$, $\chi \in [\mu, \omega]$, $m \in \mathbb{N}$, $m \geq 2$. □

**Proposition 3** *Let $\mu, \omega \in \mathbb{R}$, $0 < \mu < \omega$ and $m \in \mathbb{N}$, $m \geq 2$. Then, for all $q > 1$, the following inequality holds*:

$$\begin{aligned}
&\left| \sum_{\varepsilon=0}^{\varphi-1} \frac{1}{\varphi} A\left( \left( \frac{(\varphi - \varepsilon)\mu + \varepsilon\omega}{\varphi} \right)^m, \left( \frac{(\varphi - \varepsilon - 1)\mu + (\varepsilon + 1)\omega}{\varphi} \right)^m \right) - L_m^m(\mu, \omega) \right| \\
&\quad \leq \sum_{\varepsilon=0}^{\varphi-1} \frac{(\omega - \mu)m}{2^{1+\frac{1}{p}} \varphi^{2+\frac{s}{q}}} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{s+2} \right)^{\frac{1}{q}} \\
&\quad \times \left\{ \left[ \left( \frac{(\varphi - \varepsilon)^s}{s+1} + (\varphi - \varepsilon - 1)^s \right) \mu^{(m-1)q} + \left( \frac{\varepsilon^s}{s+1} + (\varepsilon + 1)^s \right) \omega^{(m-1)q} \right]^{\frac{1}{q}} \right. \\
&\quad \left. + \left[ \left( (\varphi - \varepsilon)^s + \frac{(\varphi - \varepsilon - 1)^s}{s+1} \right) \mu^{(m-1)q} + \left( \varepsilon^s + \frac{(\varepsilon + 1)^s}{s+1} \right) \omega^{(m-1)q} \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

*Proof* The proof is obtained immediately in Corollary 8 with $\zeta(\chi) = \chi^m$, $\chi \in [\mu, \omega]$, $m \in \mathbb{N}$, $m \geq 2$. □

## 4 Conclusion

In this study, using the generalized identity (Lemma 1), different types of integral inequalities were obtained (different results were found for various values of $n$, $n \in \mathbb{N}$) and comparisons were made between these inequalities. In these comparisons, Theorem 8 yields a better result compared to Theorem 6. Similarly, Theorem 9 yields a better result compared to Theorem 7. In addition to this work, researchers can obtain the new lemma for second sense differentiable functions and can find new results for other convex mappings. We hope that this article will inspire new interesting sequels for researchers working in this field.

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**Competing interests**
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