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Dynamical inertial extragradient techniques for solving equilibrium and fixed-point problems in real Hilbert spaces

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Abstract

In this paper, we propose new methods for finding a common solution to pseudomonotone and Lipschitz-type equilibrium problems, as well as a fixed-point problem for demicontractive mapping in real Hilbert spaces. A novel hybrid technique is used to solve this problem. The method shown here is a hybrid of the extragradient method (a two-step proximal method) and a modified Mann-type iteration. Our methods use a simple step-size rule that is generated by specific computations at each iteration. A strong convergence theorem is established without knowing the operator's Lipschitz constants. The numerical behaviors of the suggested algorithms are described and compared to previously known ones in many numerical experiments.

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Keywords: Equilibrium problem; Subgradient extragradient method; Fixed-point problem; Strong convergence theorems; Demicontractive mapping

1 Introduction

The equilibrium problem (EP) is a broad framework that includes many mathematical models as special cases, such as variational inequality problems, optimization problems, fixed-point problems, complementarity problems, Nash-equilibrium problems, and inverse optimization problems (for more details see [7, 8, 12, 33]). This equilibrium problem can be expressed mathematically as follows.

Suppose that a bifunction $\mathcal{L} : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ together with $\mathcal{L}(\aleph_1, \aleph_1) = 0$, in accordance with $\aleph_1 \in \mathcal{M}$. An *equilibrium problem* for a granted bifunction $\mathcal{L}$ on $\mathcal{M}$ is interpreted as follows: Find $s^* \in \mathcal{M}$ such that

$$\mathcal{L}(s^*, \aleph_1) \geq 0, \quad \forall \aleph_1 \in \mathcal{M}, \tag{1.1}$$

where $\mathcal{Y}$ represents a real Hilbert space and $\mathcal{M}$ represents a nonempty, closed, and convex subset of $\mathcal{Y}$. The study focuses on an iterative strategy for resolving the equilibrium problem. The solution set of problem (1.1) is denoted by $EP(\mathcal{M}, \mathcal{L})$. The problem (1.1) is widely known as the Ky Fan inequality, which has since been studied in [14]. Many authors have

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focused on this topic in recent years, for example, see [10, 11, 13, 19, 21, 23, 32, 35, 50]. This interest comes from the fact that, as observed, it neatly combines all of the above mentioned specific problems. Many writers have established and generalized many conclusions about the presence and nature of an equilibrium problem solution (for more details see [2, 7, 14]). Due to the obvious significance of the equilibrium problem and its implications in both pure and practical sciences, numerous researchers have conducted substantial studies on it in recent years [7, 9, 16]. Let us recall the definition of a Lipschitz-type continuous bifunction. A bifunction $\mathcal{L}$ is said to be *Lipschitz-type continuous* [31] on $\mathcal{M}$ if there exist two constants $c_1, c_2 > 0$, such that

$$\mathcal{L}(\aleph_1,\aleph_3) \le \mathcal{L}(\aleph_1,\aleph_2) + \mathcal{L}(\aleph_2,\aleph_3) + c_1\|\aleph_1-\aleph_2\|^2 + c_2\|\aleph_2-\aleph_3\|^2, \quad \forall \aleph_1,\aleph_2,\aleph_3 \in \mathcal{M}.$$

Flam [15] and Tran et al. [42] generated two sequences $\{s_k\}$ and $\{u_k\}$ in Euclidean spaces in the following manner:

$$\begin{cases} s_1 \in \mathcal{M}, \\ u_k = \arg\min_{u\in\mathcal{M}}\{\delta\mathcal{L}(s_k,u) + \frac{1}{2}\|s_k-u\|^2\}, \\ s_{k+1} = \arg\min_{u\in\mathcal{M}}\{\delta\mathcal{L}(u_k,u) + \frac{1}{2}\|s_k-u\|^2\}, \end{cases} \tag{1.2}$$

where $0 < \delta < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$. Due to Korpelevich's earlier work on the saddle-point problems [25], this approach is often referred to as the two-step extragradient method. It is interesting to note that the method generates a weakly convergent sequence and utilizes a fixed step size that is entirely dependent on bifunctional Lipschitz-type constants. Because Lipschitz-type variables are typically unknown or difficult to discover, this may limit application possibilities. Inertial-type procedures, on the other hand, are two-step iterative procedures wherein the following iteration is derived from the two preceding iterations (see [4, 36] for further details). To increase the numerical efficiency of the iterative sequence, an inertial extrapolation term is usually applied. According to numerical research, inertial phenomena improve numerical performance in terms of execution time and total number of iterations. Several inertial-type techniques have recently been explored for various types of equilibrium problems [3, 5, 17, 19, 47].

In this study, we are interested to find a common solution to an equilibrium problem and a fixed-point problem in a Hilbert space [20, 26, 28, 34, 40]. The motivation and idea for researching such a common solution problem comes from its potential applicability to mathematical models with limitations that may be stated as fixed-point problems. This is especially true in practical scenarios such as signal processing, network-resource allocation, and picture recovery; see, for example, [22, 28, 29]. In this study, we are interested in finding a common solution to an equilibrium problem and a fixed-point problem in a Hilbert space [1, 20, 26, 28, 34, 40, 44–46, 48]. The motivation and idea for researching such a common solution problem come from its potential applicability to mathematical models with limitations that may be stated as fixed-point problems. This is especially true in practical scenarios such as signal processing, network-resource allocation, and picture recovery; see, for example, [22, 28, 29].

Let $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ be a mapping. Then, the fixed-point problem (FPP) for the mapping $\mathcal{T}$ is to determine $s^* \in \mathcal{Y}$ such that

$$\mathcal{T}(s^*) = s^*. \tag{1.3}$$

The solution set of problem (1.3) is known as the fixed-point set of $\mathcal{T}$ and is represented by $\mathrm{Fix}(\mathcal{T})$. The majority of algorithms for addressing problem (1.3) are derived from the basic Mann iteration, in particular from $s_1 \in \mathcal{Y}$, create a sequence $\{s_{k+1}\}$ for all $k \geq 1$ by

$$s_{k+1} = \wp_k s_k + (1 - \wp_k)\mathcal{T} s_k, \tag{1.4}$$

where the random sequence $\{\wp_k\}$ must meet certain conditions in order to achieve weak convergence. The Halpern iteration is yet another formalized iterative mechanism for achieving strong convergence in infinite-dimensional Hilbert spaces. The iterative process can be expressed as follows:

$$s_{k+1} = \wp_k s_1 + (1 - \wp_k)\mathcal{T} s_k, \tag{1.5}$$

where $s_1 \in \mathcal{Y}$ and a sequence $\wp_k \subset (0; 1)$ is slowly diminishing and nonsummable, i.e.,

$$\wp_k \to 0, \quad \text{and} \quad \sum_{k=1}^{\infty} \wp_k = +\infty.$$

In addition to the Halpern iteration, there is a generic variant, namely, the Mann-type algorithm [30], in which the cost mapping $\mathcal{T}$ is combined with such a contraction mapping in the iterates. Furthermore, the hybrid steepest-descent algorithm introduced in [53] is another strategy that yields strong convergence.

Vuong et al. [52] introduced a new numerical algorithm, the extragradient method [15, 43] for trying to solve an equilibrium problem involving a fixed-point problem for a demicontractive mapping using the extragradient method and the hybrid steepest-descent technique in [53]. The authors proved that the proposed algorithm has strong convergence under the premise that the bifunction is pseudomonotone and meets the Lipschitz-type requirement [31]. As stated in [31], this technique has the benefit of being numerically calculated utilizing optimization tools. The extragradient Mann-type approach described in [31] also enables us to eliminate numerous strong criteria in establishing the convergence of previously known extragradient algorithms. Other strongly convergent methods for finding an element in $s^* \in \mathrm{Fix}(\mathcal{T}) \cap EP(\mathcal{M}, \mathcal{L})$ that integrates the extragradient approach with the hybrid or shrinking projection technique may be found in [20, 34, 39].

In this study, inspired and motivated by the findings of Takahashi et al. [40], Maingé [29], and Vuong et al. in [52] and based on the work of [27], we present a new strongly convergent algorithm as a combination of the extragradient method (two-step proximal-like method) and the Mann-type iteration [30] for approximating a common solution of a pseudomonotone and Lipschitz-type equilibrium problem and a fixed-point problem for a demicontractive mapping.

As indicated above, the result in this study is still valid for the more general class of demicontractive mappings when examining a relaxation of a demicontractive mapping. The typical Mann iteration produces weak convergence; however, the approach used in this study, which employs the comparable Mann-type iteration, produces strong convergence. This is especially true in infinite-dimensional Hilbert spaces, where strong norm convergence is more valuable than weak norm convergence. Several numerical experiments in finite- and infinite-dimensional Hilbert spaces demonstrated that the novel strategy is promising and offers competitive advantages over previous approaches.

The paper is organized as follows. Section 2 presented some basic results. Section 3 introduces new methods and validates their convergence analysis, while Sect. 4 describes some applications. Finally, Sect. 5 provides some numerical statistics to demonstrate the practical utility of the techniques presented.

2 Preliminaries

Let $\mathcal{M}$ be a nonempty, closed, and convex subset of $\mathcal{Y}$, the real Hilbert space. The weak convergence is denoted by $s_k \rightharpoonup s$ and the strong convergence by $s_k \to s$. The following information is available for each $\aleph_1, \aleph_2 \in \mathcal{Y}$:

(1) $\|\aleph_1 + \aleph_2\|^2 = \|\aleph_1\|^2 + 2\langle \aleph_1, \aleph_2\rangle + \|\aleph_2\|^2$;

(2) $\|\aleph_1 + \aleph_2\|^2 \le \|\aleph_1\|^2 + 2\langle \aleph_2, \aleph_1 + \aleph_2\rangle$;

(3) $\|a\aleph_1 + (1-a)\aleph_2\|^2 = a\|\aleph_1\|^2 + (1-a)\|\aleph_2\|^2 - a(1-a)\|\aleph_1 - \aleph_2\|^2$.

A *metric projection* $P_{\mathcal{M}}(\aleph_1)$ of an element $\aleph_1 \in \mathcal{Y}$ is defined by:

$$P_{\mathcal{M}}(\aleph_1) = \arg\min\bigl\{\|\aleph_1 - \aleph_2\| : \aleph_2 \in \mathcal{M}\bigr\}.$$

It is generally known that $P_{\mathcal{M}}$ is nonexpansive, and $P_{\mathcal{M}}$ completes the following useful characteristics:

(1) $\langle \aleph_1 - P_{\mathcal{M}}(\aleph_1), \aleph_2 - P_{\mathcal{M}}(\aleph_1)\rangle \le 0, \forall \aleph_2 \in \mathcal{M}$;

(2) $\|P_{\mathcal{M}}(\aleph_1) - P_{\mathcal{M}}(\aleph_2)\|^2 \le \langle P_{\mathcal{M}}(\aleph_1) - P_{\mathcal{M}}(\aleph_2), \aleph_1 - \aleph_2\rangle, \forall \aleph_2 \in \mathcal{M}$.

Definition 2.1 Assume that $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is a nonlinear mapping and $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$. Then, $I - \mathcal{T}$ is called demiclosed at zero if, for each $\{s_k\}$ in $\mathcal{Y}$, the following conclusion remains true:

$$s_k \rightharpoonup s \quad \text{and} \quad (I - \mathcal{T})s_k \to 0 \quad \Rightarrow \quad s \in \mathrm{Fix}(\mathcal{T}).$$

Lemma 2.2 ([37]) *Suppose that there are sequences* $\{g_k\} \subset [0, +\infty)$, $\{h_k\} \subset (0,1)$ *and* $\{r_k\} \subset \mathbb{R}$ *such as those that satisfy the following basic requirements*:

$$g_{k+1} \le (1 - h_k)g_k + h_k r_k, \quad \forall k \in \mathbb{N} \quad \textit{and} \quad \sum_{k=1}^{+\infty} h_k = +\infty.$$

If $\limsup_{j\to+\infty} r_{k_j} \le 0$ *for any subsequence* $\{g_{k_j}\}$ *of* $\{g_k\}$ *meet*

$$\liminf_{j\to+\infty}(g_{k_j+1} - g_{k_j}) \ge 0.$$

Then, $\lim_{k\to+\infty} g_k = 0$.

Definition 2.3 Let $\mathcal{M}$ be a subset of a real Hilbert space $\mathcal{Y}$ and $\digamma : \mathcal{M} \to \mathbb{R}$ a given convex function.

(1) The *normal cone* at $\aleph_1 \in \mathcal{M}$ is defined by

$$N_{\mathcal{M}}(\aleph_1) = \bigl\{\aleph_3 \in \mathcal{Y} : \langle \aleph_3, \aleph_2 - \aleph_1\rangle \le 0, \forall \aleph_2 \in \mathcal{M}\bigr\}. \tag{2.1}$$

(2) The *subdifferential of a function* $\digamma$ at $\aleph_1 \in \mathcal{M}$ is defined by

$$\partial\digamma(\aleph_1) = \bigl\{\aleph_3 \in \mathcal{Y} : \digamma(\aleph_2) - \digamma(\aleph_1) \ge \langle \aleph_3, \aleph_2 - \aleph_1\rangle, \forall \aleph_2 \in \mathcal{M}\bigr\}. \tag{2.2}$$

Lemma 2.4 ([41]) *Let $\digamma : \mathcal{M} \to \mathbb{R}$ be a subdifferentiable and lower semicontinuous function on $\mathcal{M}$. A member $\aleph_1 \in \mathcal{M}$ is called a minimizer of a mapping $\digamma$ if and only if*

$$0 \in \partial \digamma(\aleph_1) + N_{\mathcal{M}}(\aleph_1),$$

where $\partial \digamma(\aleph_1)$ denotes the subdifferential of $\digamma$ at vector $\aleph_1 \in \mathcal{M}$ and $N_{\mathcal{M}}(\aleph_1)$ is the normal cone of $\mathcal{M}$ at vector $\aleph_1$.

3 Main results

In this section, we examine in detail the convergence of several different inertial extragradient algorithms for solving equilibrium and fixed-point problems. First, we consider that our algorithms have distinct characteristics. To justify the strong convergence, the following conditions must be met:

($\mathcal{L}1$) The solution set $\mathrm{Fix}(\mathcal{T}) \cap EP(\mathcal{M}, \mathcal{L}) \neq \emptyset$;

($\mathcal{L}2$) The bifunction $\mathcal{L}$ is said to be *pseudomonotone* [6, 8], i.e.,

$$\mathcal{L}(\aleph_1, \aleph_2) \geq 0 \quad \Longrightarrow \quad \mathcal{L}(\aleph_2, \aleph_1) \leq 0, \quad \forall \aleph_1, \aleph_2 \in \mathcal{M};$$

($\mathcal{L}3$) The bifunction $\mathcal{L}$ is said to be *Lipschitz-type continuous* [31] on $\mathcal{M}$ if there exists two constants $c_1, c_2 > 0$, such that

$$\mathcal{L}(\aleph_1, \aleph_3) \leq \mathcal{L}(\aleph_1, \aleph_2) + \mathcal{L}(\aleph_2, \aleph_3) + c_1 \|\aleph_1 - \aleph_2\|^2 + c_2 \|\aleph_2 - \aleph_3\|^2, \quad \forall \aleph_1, \aleph_2, \aleph_3 \in \mathcal{M};$$

($\mathcal{L}4$) For any sequence $\{\aleph_k\} \subset \mathcal{M}$ satisfying $\aleph_k \rightharpoonup \aleph^*$, then the following inequality holds:

$$\limsup_{k \to +\infty} \mathcal{L}(\aleph_k, \aleph_1) \leq \mathcal{L}(\aleph^*, \aleph_1), \quad \forall \aleph_1 \in \mathcal{M};$$

($\mathcal{L}5$) Assume that $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is a mapping such that $(I - \mathcal{T})$ is demiclosed at zero. A mapping $\mathcal{T}$ is said to be ρ-demicontractive if there exists a constant $0 \leq \rho < 1$ such that

$$\|\mathcal{T}(\aleph_1) - \aleph_2\|^2 \leq \|\aleph_1 - \aleph_2\|^2 + \rho \|(I - \mathcal{T})(\aleph_1)\|^2, \quad \forall \aleph_2 \in \mathrm{Fix}(\mathcal{T}), \aleph_1 \in \mathcal{Y};$$

or equivalently

$$\langle \mathcal{T}(\aleph_1) - \aleph_1, \aleph_1 - \aleph_2 \rangle \leq \frac{\rho - 1}{2} \|\aleph_1 - \mathcal{T}(\aleph_1)\|^2, \quad \forall \aleph_2 \in \mathrm{Fix}(\mathcal{T}), \aleph_1 \in \mathcal{Y}.$$

The first algorithm is described below to find a common solution to an equilibrium and a fixed-point problem. The main advantage of this method is that it employs a monotone step-size rule that is independent of Lipschitz constants. The algorithm employs Mann-type iteration to aid in the solution of a fixed-point problem, and the two-step extragradient approach to solve an equilibrium problem.

Algorithm 1 (Inertial subgradient extragradient method with a monotone step-size rule)

STEP 0: Take $s_0, s_1 \in \mathcal{M}$, $\ell \in (0,1)$, $\tau \in (0,1)$, $\delta_1 > 0$. Choose two positive numbers a, b such that $0 < a, b < 1 - \rho$ and $0 < a, b < 1 - \Im_k$. Moreover, choose $\{\wp_k\} \subset (a,b)$ and $\{\Im_k\} \subset (0,1)$ satisfying the following conditions:

$$\lim_{k \to +\infty} \Im_k = 0 \quad \text{and} \quad \sum_{k=1}^{+\infty} \Im_k = +\infty.$$

STEP 1: Calculate

$$\varkappa_k = s_k + \ell_k (s_k - s_{k-1}),$$

where ℓ_k is taken as follows:

$$0 \le \ell_k \le \hat{\ell}_k \quad \text{and} \quad \hat{\ell}_k = \begin{cases} \min\{\frac{\ell}{2}, \frac{\varrho_k}{\|s_k - s_{k-1}\|}\} & \text{if } s_k \neq s_{k-1}, \\ \frac{\ell}{2} & \text{otherwise.} \end{cases} \tag{3.1}$$

Moreover, a positive sequence $\varrho_k = \circ(\wp_k)$ satisfies $\lim_{k \to +\infty} \frac{\varrho_k}{\wp_k} = 0$.
STEP 2: Calculate

$$u_k = \arg\min_{u \in \mathcal{M}} \left\{ \delta_k \mathcal{L}(\varkappa_k, u) + \frac{1}{2} \|\varkappa_k - u\|^2 \right\}.$$

If $\varkappa_k = u_k$, then STOP. Else, move to *STEP 3*.
STEP 3: Given the current iterates s_k, u_k. First, choose $\omega_k \in \partial_2 \mathcal{L}(\varkappa_k, u_k)$ satisfying $\varkappa_k - \delta_k \omega_k - u_k \in N_{\mathcal{M}}(u_k)$ and generate a half-space

$$\mathcal{Y}_k = \left\{ z \in \mathcal{Y} : \langle \varkappa_k - \delta_k \omega_k - u_k, z - u_k \rangle \le 0 \right\}.$$

Compute

$$v_k = \arg\min_{u \in \mathcal{Y}_k} \left\{ \delta_k \mathcal{L}(u_k, u) + \frac{1}{2} \|\varkappa_k - u\|^2 \right\}.$$

STEP 4: Calculate

$$s_{k+1} = (1 - \wp_k - \Im_k) v_k + \wp_k \mathcal{T}(v_k).$$

STEP 5: Calculate

$$\delta_{k+1} = \begin{cases} \min\{\delta_k, \frac{\tau \|\varkappa_k - u_k\|^2 + \tau \|v_k - u_k\|^2}{2[\mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) - \mathcal{L}(u_k, v_k)]}\} \\ \quad \text{if } \mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) - \mathcal{L}(u_k, v_k) > 0, \\ \delta_k, \quad \text{otherwise.} \end{cases} \tag{3.2}$$

Set $k := k + 1$ and move to *STEP 1*.

The following lemma is used to demonstrate that the monotone step-size sequence generated by equation (3.2) is properly defined and bounded.

Lemma 3.1 *A sequence* $\{\delta_k\}$ *is convergent to* δ *and* $\min\{\frac{\tau}{\max\{2c_1,2c_2\}},\delta_1\} \leq \delta_k \leq \delta_1$.

Proof Let $\mathcal{L}(\varkappa_k,v_k) - \mathcal{L}(\varkappa_k,u_k) - \mathcal{L}(u_k,v_k) > 0$. Thus, we have

$$\begin{aligned} \frac{\tau(\|\varkappa_k - u_k\|^2 + \|v_k - u_k\|^2)}{2[\mathcal{L}(\varkappa_k,v_k) - \mathcal{L}(\varkappa_k,u_k) - \mathcal{L}(u_k,v_k)]} &\geq \frac{\tau(\|\varkappa_k - u_k\|^2 + \|v_k - u_k\|^2)}{2[c_1\|\varkappa_k - u_k\|^2 + c_2\|v_k - u_k\|^2]} \\ &\geq \frac{\tau}{2\max\{c_1,c_2\}}. \end{aligned} \tag{3.3}$$

Thus, we obtain $\lim_{k\to+\infty}\delta_k = \delta$. This completes the proof. □

The second method is described below to find a common solution to an equilibrium and a fixed-point problem. The primary benefit of this method is that it employs a nonmonotone step-size rule that is independent of Lipschitz constants. The algorithm solves a fixed-point problem using Mann-type iteration and an equilibrium problem with the two-step extragradient approach.

Algorithm 2 (Accelerated subgradient extragradient method with a nonmonotone step-size rule)

STEP 0: Take $s_0, s_1 \in \mathcal{M}$, $\ell \in (0,1)$, $\tau \in (0,1)$, $\delta_1 > 0$. Choose two positive numbers a,b such that $0 < a,b < 1-\rho$ and $0 < a,b < 1-\Im_k$. Moreover, choose $\{\wp_k\} \subset (a,b)$ and $\{\Im_k\} \subset (0,1)$ satisfying the following conditions:

$$\lim_{k\to+\infty}\Im_k = 0 \quad \text{and} \quad \sum_{k=1}^{+\infty}\Im_k = +\infty.$$

STEP 1: Calculate

$$\varkappa_k = s_k + \ell_k(s_k - s_{k-1}),$$

where ℓ_k is taken as follows:

$$0 \leq \ell_k \leq \hat{\ell}_k \quad \text{and} \quad \hat{\ell}_k = \begin{cases} \min\{\frac{\ell}{2}, \frac{\varrho_k}{\|s_k - s_{k-1}\|}\} & \text{if } s_k \neq s_{k-1}, \\ \frac{\ell}{2} & \text{otherwise.} \end{cases} \tag{3.4}$$

Moreover, a positive sequence $\varrho_k = \circ(\wp_k)$ satisfies $\lim_{k\to+\infty}\frac{\varrho_k}{\wp_k} = 0$.

STEP 2: Calculate

$$u_k = \arg\min_{u\in\mathcal{M}}\left\{\delta_k\mathcal{L}(\varkappa_k,u) + \frac{1}{2}\|\varkappa_k - u\|^2\right\}.$$

If $\varkappa_k = u_k$, then STOP. Else, move to *STEP 3*.

STEP 3: Given the current iterates s_k, u_k. First, choose $\omega_k \in \partial_2 \mathcal{L}(\varkappa_k, u_k)$ satisfying $\varkappa_k - \delta_k \omega_k - u_k \in N_{\mathcal{M}}(u_k)$ and generate a half-space

$$\mathcal{Y}_k = \left\{ z \in \mathcal{Y} : \langle \varkappa_k - \delta_k \omega_k - u_k, z - u_k \rangle \leq 0 \right\}.$$

Compute

$$v_k = \arg\min_{u \in \mathcal{Y}_k} \left\{ \delta_k \mathcal{L}(u_k, u) + \frac{1}{2} \|\varkappa_k - u\|^2 \right\}.$$

STEP 4: Calculate

$$s_{k+1} = (1 - \wp_k - \Im_k) v_k + \wp_k \mathcal{T}(v_k).$$

STEP 5: Moreover, choose a nonnegative real sequence $\{\chi_k\}$ such that $\sum_{k=1}^{+\infty} \chi_k < +\infty$. Calculate

$$\delta_{k+1} = \begin{cases} \min\{\delta_k + \chi_k, \frac{\tau \|\varkappa_k - u_k\|^2 + \tau \|v_k - u_k\|^2}{2[\mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) - \mathcal{L}(u_k, v_k)]}\} \\ \quad \text{if } \mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) - \mathcal{L}(u_k, v_k) > 0, \\ \delta_k + \chi_k, \quad \text{otherwise.} \end{cases} \tag{3.5}$$

Set $k := k + 1$ and move to *STEP 1.*

The following lemma is employed to establish that the nonmonotone step-size sequence created by equation (3.5) is properly defined and bounded. We give a proof that completely establishes the boundedness and convergence of a step-size sequence.

Lemma 3.2 *A sequence* $\{\delta_k\}$ *is convergent to* δ *and* $\min\{\frac{\tau}{\max\{2c_1, 2c_2\}}, \delta_1\} \leq \delta_k \leq \delta_1 + P$ *along with* $P = \sum_{k=1}^{+\infty} \chi_k$.

Proof Let $\mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) - \mathcal{L}(u_k, v_k) > 0$. Thus, we have

$$\begin{aligned} \frac{\tau(\|\varkappa_k - u_k\|^2 + \|v_k - u_k\|^2)}{2[\mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) - \mathcal{L}(u_k, v_k)]} &\geq \frac{\tau(\|\varkappa_k - u_k\|^2 + \|v_k - u_k\|^2)}{2[c_1 \|\varkappa_k - u_k\|^2 + c_2 \|v_k - u_k\|^2]} \\ &\geq \frac{\tau}{2\max\{c_1, c_2\}}. \end{aligned} \tag{3.6}$$

The idea of δ_{k+1} may be deduced through mathematical induction.

$$\min\left\{ \frac{\tau}{\max\{2c_1, 2c_2\}}, \delta_1 \right\} \leq \delta_k \leq \delta_1 + P.$$

Assume that $[\delta_{k+1} - \delta_k]^+ = \max\{0, \delta_{k+1} - \delta_k\}$ and

$$[\delta_{k+1} - \delta_k]^- = \max\left\{0, -(\delta_{k+1} - \delta_k)\right\}.$$

We receive $\{\delta_k\}$ because of the definition

$$\sum_{k=1}^{+\infty}(\delta_{k+1}-\delta_k)^+ = \sum_{k=1}^{+\infty}\max\{0,\delta_{k+1}-\delta_k\} \le P < +\infty. \tag{3.7}$$

That is, the series $\sum_{k=1}^{+\infty}(\delta_{k+1}-\delta_k)^+$ is convergent. The convergence must now be proven of $\sum_{k=1}^{+\infty}(\delta_{k+1}-\delta_k)^-$. Let $\sum_{k=1}^{+\infty}(\delta_{k+1}-\delta_k)^- = +\infty$. Due to the fact that

$$\delta_{k+1}-\delta_k = (\delta_{k+1}-\delta_k)^+ - (\delta_{k+1}-\delta_k)^-,$$

we might be able to obtain

$$\delta_{k+1}-\delta_1 = \sum_{k=0}^{k}(\delta_{k+1}-\delta_k) = \sum_{k=0}^{k}(\delta_{k+1}-\delta_k)^+ - \sum_{k=0}^{k}(\delta_{k+1}-\delta_k)^-. \tag{3.8}$$

Letting $k \to +\infty$ in (3.8), we have $\delta_k \to -\infty$ as $k \to +\infty$. This is an absurdity. As a result of the series convergence $\sum_{k=0}^{k}(\delta_{k+1}-\delta_k)^+$ and $\sum_{k=0}^{k}(\delta_{k+1}-\delta_k)^-$ taking $k \to +\infty$ in (3.8), we obtain $\lim_{k\to+\infty}\delta_k = \delta$. This concludes the proof. □

The following lemma can be used to verify the boundedness of an iterative sequence. It is critical in terms of proving the boundedness of a sequence and proving the strong convergence of a proposed sequence to find a common solution.

Lemma 3.3 *Suppose that $\{s_k\}$ is a sequence generated by Algorithm 1 that meets the conditions ($\mathcal{L}1$)–($\mathcal{L}5$). Then, we have*

$$\|\nu_k - s^*\|^2 \le \|\varkappa_k - s^*\|^2 - \left(1-\frac{\tau\delta_k}{\delta_{k+1}}\right)\|\varkappa_k - u_k\|^2 - \left(1-\frac{\tau\delta_k}{\delta_{k+1}}\right)\|\nu_k - u_k\|^2.$$

Proof By the use of Lemma 2.4, we have

$$0 \in \partial_2\left\{\delta_k\mathcal{L}(u_k,\cdot) + \frac{1}{2}\|\varkappa_k - \cdot\|^2\right\}(\nu_k) + N_{\mathcal{Y}_k}(\nu_k).$$

There is a vector $\omega \in \partial\mathcal{L}(u_k,\nu_k)$ and there exists a vector $\overline{\omega} \in N_{\mathcal{Y}_k}(\nu_k)$ in order that

$$\delta_k\omega + \nu_k - \varkappa_k + \overline{\omega} = 0.$$

The preceding phrase suggests that

$$\langle\varkappa_k - \nu_k, u - \nu_k\rangle = \delta_k\langle\omega, u-\nu_k\rangle + \langle\overline{\omega}, u-\nu_k\rangle, \quad \forall u \in \mathcal{Y}_k.$$

Since $\overline{\omega} \in N_{\mathcal{Y}_k}(\nu_k)$ implies that $\langle\overline{\omega}, u-\nu_k\rangle \le 0$, for all $u \in \mathcal{Y}_k$. As a result, we acquire

$$\langle\varkappa_k - \nu_k, u-\nu_k\rangle \le \delta_k\langle\omega, u-\nu_k\rangle, \quad \forall u \in \mathcal{Y}_k. \tag{3.9}$$

Furthermore, $\omega \in \partial\mathcal{L}(u_k,\nu_k)$ and because of the concept of subdifferential, we obtain

$$\mathcal{L}(u_k,u) - \mathcal{L}(u_k,\nu_k) \ge \langle\omega, u-\nu_k\rangle, \quad \forall u \in \mathcal{Y}. \tag{3.10}$$

We obtain by combining the formulas (3.9) and (3.10)

$$\delta_k \mathcal{L}(u_k, u) - \delta_k \mathcal{L}(u_k, v_k) \geq \langle \varkappa_k - v_k, u - v_k \rangle, \quad \forall u \in \mathcal{Y}_k. \tag{3.11}$$

Due to the concept of a half-space $\mathcal{Y}_k$, we have

$$\delta_k \langle \omega_k, v_k - u_k \rangle \geq \langle \varkappa_k - u_k, v_k - u_k \rangle. \tag{3.12}$$

Due to $\omega_k \in \partial \mathcal{L}(\varkappa_k, u_k)$ this indicates that

$$\mathcal{L}(\varkappa_k, u) - \mathcal{L}(\varkappa_k, u_k) \geq \langle \omega_k, u - u_k \rangle, \quad \forall u \in \mathcal{Y}.$$

By inserting $u = v_k$, we derive

$$\mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) \geq \langle \omega_k, v_k - u_k \rangle. \tag{3.13}$$

From (3.12) and (3.13), we derive

$$\delta_k \bigl\{ \mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) \bigr\} \geq \langle \varkappa_k - u_k, v_k - u_k \rangle. \tag{3.14}$$

By inserting $u = s^*$ into formula (3.11), we obtain

$$\delta_k \mathcal{L}\bigl(u_k, s^*\bigr) - \delta_k \mathcal{L}(u_k, v_k) \geq \bigl\langle \varkappa_k - v_k, s^* - v_k \bigr\rangle. \tag{3.15}$$

Given $s^* \in EP(\mathcal{L}, \mathcal{M})$, we conclude that $\mathcal{L}(s^*, u_k) \geq 0$. Due to the pseudomonotonicity of the bifunction $\mathcal{L}$, we derive $\mathcal{L}(u_k, s^*) \leq 0$. We have achieved this by using equation (3.15) such that

$$\bigl\langle \varkappa_k - v_k, v_k - s^* \bigr\rangle \geq \delta_k \mathcal{L}(u_k, v_k). \tag{3.16}$$

By using the definition of δ_{k+1}, we obtain

$$\mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) - \mathcal{L}(u_k, v_k) \leq \frac{\tau \|\varkappa_k - u_k\|^2 + \tau \|v_k - u_k\|^2}{2\delta_{k+1}}. \tag{3.17}$$

Due to the expressions (3.16) and (3.17), we obtain

$$\begin{aligned} \bigl\langle \varkappa_k - v_k, v_k - s^* \bigr\rangle \geq{} & \delta_k \bigl\{ \mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) \bigr\} \\ & - \frac{\tau \delta_k}{2\delta_{k+1}} \|\varkappa_k - u_k\|^2 - \frac{\tau \delta_k}{2\delta_{k+1}} \|v_k - u_k\|^2. \end{aligned} \tag{3.18}$$

Integrating the formulas (3.14) and (3.18), we obtain

$$\begin{aligned} \bigl\langle \varkappa_k - v_k, v_k - s^* \bigr\rangle \geq{} & \langle \varkappa_k - u_k, v_k - u_k \rangle \\ & - \frac{\tau \delta_k}{2\delta_{k+1}} \|\varkappa_k - u_k\|^2 - \frac{\tau \delta_k}{2\delta_{k+1}} \|v_k - u_k\|^2. \end{aligned} \tag{3.19}$$

We have the following identities that are valuable to us:

$$-2\langle \varkappa_k - \nu_k, \nu_k - s^* \rangle = -\|\varkappa_k - s^*\|^2 + \|\nu_k - \varkappa_k\|^2 + \|\nu_k - s^*\|^2, \tag{3.20}$$

$$2\langle u_k - \varkappa_k, u_k - \nu_k \rangle = \|\varkappa_k - u_k\|^2 + \|\nu_k - u_k\|^2 - \|\varkappa_k - \nu_k\|^2. \tag{3.21}$$

By using expressions (3.19), (3.20), and (3.21), we obtain

$$\begin{aligned}\|\nu_k - s^*\|^2 \le \|\varkappa_k - s^*\|^2 &- \left(1 - \frac{\tau\delta_k}{\delta_{k+1}}\right)\|\varkappa_k - u_k\|^2 \\ &- \left(1 - \frac{\tau\delta_k}{\delta_{k+1}}\right)\|\nu_k - u_k\|^2.\end{aligned} \tag{3.22}$$

□

The following theorem is the main theorem that is used to establish the strong convergence of an iterative sequence. This theorem proves the boundedness of a sequence and the strong convergence of a suggested sequence to a common solution. This is the key theorem, and it proves that the suggested sequence strongly converges to a solution in the case of monotone and nonmonotone step-size criteria.

Theorem 3.4 *Suppose that $\mathcal{L} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ satisfies the conditions ($\mathcal{L}1$)–($\mathcal{L}5$). Then, sequence $\{s_k\}$ generated by Algorithm 1 strongly converges to $s^* \in \mathrm{Fix}(\mathcal{T}) \cap EP(\mathcal{M}, \mathcal{L})$, where $s^* = P_{\mathrm{Fix}(\mathcal{T})\cap EP(\mathcal{M},\mathcal{L})}(0)$.*

Proof Claim 1: *The sequence $\{s_k\}$ is bounded.*

It is worth noting that $EP(\mathcal{M}, \mathcal{L})$ and $\mathrm{Fix}(\mathcal{T})$ are both closed, convex subsets. It is given that

$$s^* = P_{EP(\mathcal{M},\mathcal{L})\cap \mathrm{Fix}(\mathcal{T})}(0).$$

Namely, $s^* \in EP(\mathcal{M}, \mathcal{L}) \cap \mathrm{Fix}(\mathcal{T})$, as well as

$$\langle 0 - s^*, u - s^* \rangle \le 0, \quad \forall u \in EP(\mathcal{M}, \mathcal{L}) \cap \mathrm{Fix}(\mathcal{T}). \tag{3.23}$$

As $s^* \in \Omega$ and based on the description of s_{k+1}, we have

$$\begin{aligned}\|s_{k+1} - s^*\| &= \|(1 - \wp_k - \Im_k)\nu_k + \wp_k \mathcal{T}(\nu_k) - s^*\| \\ &= \|(1 - \wp_k - \Im_k)(\nu_k - s^*) + \wp_k(\mathcal{T}(\nu_k) - s^*) - \Im_k s^*\| \\ &\le \|(1 - \wp_k - \Im_k)(\nu_k - s^*) + \wp_k(\mathcal{T}(\nu_k) - s^*)\| + \Im_k\|s^*\|.\end{aligned} \tag{3.24}$$

Then, we must compute the following:

$$\begin{aligned}&\|(1 - \wp_k - \Im_k)(\nu_k - s^*) + \wp_k(\mathcal{T}(\nu_k) - s^*)\|^2 \\ &\quad= (1 - \wp_k - \Im_k)^2\|\nu_k - s^*\|^2 + \wp_k^2\|\mathcal{T}(\nu_k) - s^*\|^2 \\ &\qquad+ 2\langle (1 - \wp_k - \Im_k)(\nu_k - s^*), \wp_k(\mathcal{T}(\nu_k) - s^*)\rangle \\ &\quad\le (1 - \wp_k - \Im_k)^2\|\nu_k - s^*\|^2 + \wp_k^2\left[\|\nu_k - s^*\|^2 + \rho\|\nu_k - \mathcal{T}(\nu_k)\|^2\right]\end{aligned}$$

$$
\begin{aligned}
&+2\wp_k(1-\wp_k-\Im_k)\left[\left\|\nu_k-s^*\right\|^2+\frac{\rho-1}{2}\left\|\mathcal{T}(\nu_k)-\nu_k\right\|^2\right] \\
&\le (1-\Im_k)^2\|\nu_k-s^*\|^2+\wp_k\big[\wp_k-(1-\rho)(1-\Im_k)\big]\left\|\mathcal{T}(\nu_k)-\nu_k\right\|^2 \\
&\le (1-\Im_k)^2\|\nu_k-s^*\|^2.
\end{aligned}
\tag{3.25}
$$

As a result, the previous expression implies that

$$
\left\|(1-\wp_k-\Im_k)\big(\nu_k-s^*\big)+\wp_k\big(\mathcal{T}(\nu_k)-s^*\big)\right\| \le (1-\Im_k)\|\nu_k-s^*\|. \tag{3.26}
$$

From expressions (3.24) and (3.26), we have

$$
\left\|s_{k+1}-s^*\right\| \le (1-\Im_k)\left\|\nu_k-s^*\right\|+\Im_k\left\|s^*\right\|. \tag{3.27}
$$

In the context of Lemma 3.3, we derive

$$
\left\|\nu_k-s^*\right\|^2 \le \left\|\varkappa_k-s^*\right\|^2-\left(1-\frac{\tau\delta_k}{\delta_{k+1}}\right)\|\varkappa_k-u_k\|^2-\left(1-\frac{\tau\delta_k}{\delta_{k+1}}\right)\|\nu_k-u_k\|^2. \tag{3.28}
$$

Due to Lemma 3.1, we obtain

$$
\lim_{k\to+\infty}\left(1-\frac{\tau\delta_k}{\delta_{k+1}}\right)=1-\tau>0. \tag{3.29}
$$

Thus, this means that there exists $N_1\in\mathbb{N}$ such that

$$
\lim_{k\to+\infty}\left(1-\frac{\tau\delta_k}{\delta_{k+1}}\right)>0, \quad \forall k\ge N_1. \tag{3.30}
$$

According to expressions (3.28) and (3.30), we have

$$
\left\|\nu_k-s^*\right\|^2 \le \left\|\varkappa_k-s^*\right\|^2. \tag{3.31}
$$

From expression (3.1), we have

$$
\ell_k\|s_k-s_{k-1}\| \le \varrho_k, \quad \text{for all } k\in\mathbb{N}
$$

and

$$
\lim_{k\to+\infty}\left(\frac{\varrho_k}{\Im_k}\right)=0.
$$

As a result, this indicates that

$$
\lim_{k\to+\infty}\frac{\ell_k}{\Im_k}\|s_k-s_{k-1}\| \le \lim_{k+\to\infty}\frac{\varrho_k}{\Im_k}=0. \tag{3.32}
$$

From the formulas (3.31) and (3.32) with the definition of $\{\varkappa_k\}$, we obtain

$$
\begin{aligned}
\left\|\nu_k-s^*\right\| \le \left\|\varkappa_k-s^*\right\| &= \left\|s_k+\ell_k(s_k-s_{k-1})-s^*\right\| \\
&\le \left\|s_k-s^*\right\|+\ell_k\|s_k-s_{k-1}\|
\end{aligned}
$$

$$
\begin{aligned}
&= \|s_k - s^*\| + \Im_k \frac{\ell_k}{\Im_k}\|s_k - s_{k-1}\| \\
&\le \|s_k - s^*\| + \Im_k K_1,
\end{aligned} \tag{3.33}
$$

where $K_1 > 0$ is a constant

$$
\frac{\ell_k}{\Im_k}\|s_k - s_{k-1}\| \le K_1, \quad \forall k \ge 1. \tag{3.34}
$$

Considering the formulas (3.31) and (3.33), we obtain

$$
\|v_k - s^*\| \le \|\varkappa_k - s^*\| \le \|s_k - s^*\| + \Im_k K_1, \quad \forall k \ge N_1. \tag{3.35}
$$

Combining (3.26) and (3.35), we obtain

$$
\begin{aligned}
\|s_{k+1} - s^*\| &\le (1-\Im_k)\|v_k - s^*\| + \Im_k\|s^*\| \\
&\le (1-\Im_k)\|s_k - s^*\| + (1-\Im_k)\Im_k K_1 + \Im_k\|s^*\| \\
&\le (1-\Im_k)\|s_k - s^*\| + \Im_k\big(K_1 + s^*\big) \\
&\le \max\big\{\|s_k - s^*\|, K_1 + s^*\big\} \\
&\le \max\big\{\|s_{N_1} - s^*\|, K_1 + s^*\big\}.
\end{aligned} \tag{3.36}
$$

As a result, we infer that the sequence $\{s_k\}$ is bounded.

Claim 2:

$$
\begin{aligned}
&\left(1 - \frac{\tau\delta_k}{\delta_{k+1}}\right)\|\varkappa_k - u_k\|^2 + \left(1 - \frac{\tau\delta_k}{\delta_{k+1}}\right)\|v_k - u_k\|^2 + \wp_k[1 - \rho - \wp_k]\|\mathcal{T}(v_k) - v_k\|^2 \\
&\quad \le \|s_k - s^*\|^2 - \|s_{k+1} - s^*\|^2 + \Im_k K_4
\end{aligned} \tag{3.37}
$$

for some $K_4 > 0$. Indeed, it follows from relation (3.35) that

$$
\begin{aligned}
\|\varkappa_k - s^*\|^2 &\le \big(\|s_k - s^*\| + \Im_k K_1\big)^2 \\
&= \|s_k - s^*\|^2 + \Im_k\big(2K_1\|s_k - s^*\| + \Im_k K_1^2\big) \\
&\le \|s_k - s^*\|^2 + \Im_k K_2,
\end{aligned} \tag{3.38}
$$

for some $K_2 > 0$. In addition, we have

$$
\begin{aligned}
\|s_{k+1} - s^*\|^2 &= \|(1 - \wp_k - \Im_k)v_k + \wp_k\mathcal{T}(v_k) - s^*\|^2 \\
&= \|\big(v_k - s^*\big) + \wp_k\big(\mathcal{T}(v_k) - v_k\big) - \Im_k v_k\|^2 \\
&\le \|\big(v_k - s^*\big) + \wp_k\big(\mathcal{T}(v_k) - v_k\big)\|^2 - 2\Im_k\langle v_k, s_{k+1} - s^*\rangle \\
&= \|v_k - s^*\|^2 + \wp_k^2\|\mathcal{T}(v_k) - v_k\|^2 + 2\wp_k\langle\mathcal{T}(v_k) - v_k, v_k - s^*\rangle \\
&\quad + 2\Im_k\langle v_k, s^* - s_{k+1}\rangle \\
&\le \|v_k - s^*\|^2 + \wp_k^2\|\mathcal{T}(v_k) - v_k\|^2 + \wp_k(\rho - 1)\|v_k - \mathcal{T}(v_k)\|^2 + \Im_k K_3
\end{aligned}
$$

$$\begin{aligned}
&\leq \|s_k - s^*\|^2 + \Im_k K_4 - \wp_k\big[(1-\rho) - \wp_k\big]\|\mathcal{T}(\nu_k) - \nu_k\|^2 \\
&\quad - \left(1 - \frac{\tau\delta_k}{\delta_{k+1}}\right)\|\varkappa_k - u_k\|^2 - \left(1 - \frac{\tau\delta_k}{\delta_{k+1}}\right)\|\nu_k - u_k\|^2,
\end{aligned} \tag{3.39}$$

where $K_4 = K_2 + K_3$. Finally, we have

$$\begin{aligned}
&\left(1 - \frac{\tau\delta_k}{\delta_{k+1}}\right)\|\varkappa_k - u_k\|^2 + \left(1 - \frac{\tau\delta_k}{\delta_{k+1}}\right)\|\nu_k - u_k\|^2 + \wp_k[1-\rho-\wp_k]\|\mathcal{T}(\nu_k) - \nu_k\|^2 \\
&\leq \|s_k - s^*\|^2 - \|s_{k+1} - s^*\|^2 + \Im_k K_4.
\end{aligned} \tag{3.40}$$

Claim 3:

$$\begin{aligned}
\|s_{k+1} - s^*\|^2 \leq (1-\Im_k)\|s_k - s^*\|^2 + \Im_k\bigg[&2\wp_k\|\mathcal{T}(\nu_k) - \nu_k\|\|s_{k+1} - s^*\| \\
&+ \frac{3K\eth_k}{\Im_k}\|s_k - s_{k-1}\| + 2\langle s^*, s^* - s_{k+1}\rangle\bigg].
\end{aligned} \tag{3.41}$$

By setting the following value

$$t_k = (1-\wp_k)\nu_k + \wp_k\mathcal{T}(\nu_k),$$

we have

$$s_{k+1} = t_k - \Im_k\nu_k = (1-\Im_k)t_k - \Im_k(\nu_k - t_k) = (1-\Im_k)t_k - \Im_k\wp_k\big(\nu_k - \mathcal{T}(\nu_k)\big), \tag{3.42}$$

where

$$\nu_k - t_k = \nu_k - (1-\wp_k)\nu_k - \wp_k\mathcal{T}(\nu_k) = \wp_k\big(\nu_k - \mathcal{T}(\nu_k)\big).$$

By definition of s_{k+1}, we can write

$$\begin{aligned}
&\|s_{k+1} - s^*\|^2 \\
&\quad = \|(1-\Im_k)t_k + \wp_k\Im_k\big(\mathcal{T}(\nu_k) - \nu_k\big) - s^*\|^2 \\
&\quad = \|(1-\Im_k)\big(t_k - s^*\big) + \big[\wp_k\Im_k\big(\mathcal{T}(\nu_k) - \nu_k\big) - \Im_k s^*\big]\|^2 \\
&\quad \leq (1-\Im_k)^2\|t_k - s^*\|^2 \\
&\qquad + 2\big\langle\wp_k\Im_k\big(\mathcal{T}(\nu_k) - \nu_k\big) - \Im_k s^*, (1-\Im_k)\big(t_k - s^*\big) + \wp_k\Im_k\big(\mathcal{T}(\nu_k) - \nu_k\big) - \Im_k s^*\big\rangle \\
&\quad = (1-\Im_k)^2\|t_k - s^*\|^2 + 2\Im_k\big\langle\wp_k\big(\mathcal{T}(\nu_k) - \nu_k\big) - s^*, s_{k+1} - s^*\big\rangle \\
&\quad \leq (1-\Im_k)\|t_k - s^*\|^2 + 2\wp_k\Im_k\big\langle\mathcal{T}(\nu_k) - \nu_k, s_{k+1} - s^*\big\rangle + 2\Im_k\big\langle s^*, s^* - s_{k+1}\big\rangle.
\end{aligned} \tag{3.43}$$

Next, we have to evaluate

$$\begin{aligned}
&\|t_k - s^*\|^2 \\
&\quad = \|(1-\wp_k)\nu_k + \wp_k\mathcal{T}(\nu_k) - s^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \left\|(1-\wp_k)\left(v_k - s^*\right) + \wp_k\left(\mathcal{T}(v_k) - s^*\right)\right\|^2 \\
&= (1-\wp_k)^2\left\|v_k - s^*\right\|^2 + \wp_k^2\left\|\mathcal{T}(v_k) - s^*\right\|^2 + 2\left\langle(1-\wp_k)\left(v_k - s^*\right), \wp_k\left(\mathcal{T}(v_k) - s^*\right)\right\rangle \\
&\leq (1-\wp_k)^2\left\|v_k - s^*\right\|^2 + \wp_k^2\left\|v_k - s^*\right\|^2 + \wp_k^2\rho\left\|\mathcal{T}(v_k) - v_k\right\|^2 \\
&\quad + 2(1-\wp_k)\wp_k\left[\left\|v_k - s^*\right\|^2 - \frac{1-\rho}{2}\left\|\mathcal{T}(v_k) - s^*\right\|^2\right] \\
&\leq \left\|v_k - s^*\right\|^2 + \wp_k[\wp_k - 1 + \rho]\left\|\mathcal{T}(v_k) - s^*\right\|^2.
\end{aligned} \tag{3.44}
$$

It is given that $\wp_k \subset (0, 1-\rho)$ and using the expression (3.31), we obtain

$$
\left\|t_k - s^*\right\|^2 \leq \left\|\varkappa_k - s^*\right\|^2. \tag{3.45}
$$

According to the definition of $\varkappa_k$, one obtains

$$
\begin{aligned}
\left\|\varkappa_k - s^*\right\|^2 &= \left\|s_k + \ell_k(s_k - s_{k-1}) - s^*\right\|^2 \\
&= \left\|s_k - s^* + \ell_k(s_k - s_{k-1})\right\|^2 \\
&= \left\|s_k - s^*\right\|^2 + \ell_k^2\|s_k - s_{k-1}\|^2 + 2\left\langle s_k - s^*, \ell_k(s_k - s_{k-1})\right\rangle \\
&\leq \left\|s_k - s^*\right\|^2 + \ell_k^2\|s_k - s_{k-1}\|^2 + 2\ell_k\left\|s_k - s^*\right\|\|s_k - s_{k-1}\| \\
&= \left\|s_k - s^*\right\|^2 + \ell_k\|s_k - s_{k-1}\|\left[2\left\|s_k - s^*\right\| + \ell_k\|s_k - s_{k-1}\|\right] \\
&\leq \left\|s_k - s^*\right\|^2 + 3\ell_k K\|s_k - s_{k-1}\|,
\end{aligned} \tag{3.46}
$$

where

$$
K = \sup_{k\in\mathbb{N}}\left\{\left\|s_k - s^*\right\|, \ell_k\|s_k - s_{k-1}\|\right\}.
$$

Combining expressions (3.43), (3.44), and (3.46), we obtain

$$
\begin{aligned}
&\left\|s_{k+1} - s^*\right\|^2 \\
&\quad\leq (1-\Im_k)\|t_k - s^*\|^2 + 2\wp_k\Im_k\left\langle\mathcal{T}(v_k) - v_k, s_{k+1} - s^*\right\rangle + 2\Im_k\left\langle s^*, s^* - s_{k+1}\right\rangle \\
&\quad\leq (1-\Im_k)\|s_k - s^*\|^2 + \Im_k\left[2\wp_k\|\mathcal{T}(v_k) - v_k\|\|s_{k+1} - s^*\|\right. \\
&\quad\quad\left. + 2\left\langle s^*, s^* - s_{k+1}\right\rangle + \frac{3\ell_k K}{\Im_k}\|s_k - s_{k-1}\|\right].
\end{aligned} \tag{3.47}
$$

Claim 4: *The sequence* $\|s_k - s^*\|^2$ *converges to zero.*
Set

$$
p_k := \left\|s_k - s^*\right\|^2
$$

and

$$
r_k := \left[2\wp_k\|\mathcal{T}(v_k) - v_k\|\|s_{k+1} - s^*\| + 2\left\langle s^*, s^* - s_{k+1}\right\rangle + \frac{3\ell_k K}{\Im_k}\|s_k - s_{k-1}\|\right].
$$

Then, *Claim* 4 can be rewritten as follows:

$$p_{k+1} \leq (1-\Im_k)p_k + \Im_k r_k.$$

Indeed, from Lemma 2.2, it suffices to show that $\limsup_{j\to\infty} r_{k_j} \leq 0$ for every subsequence $\{p_{k_j}\}$ of $\{p_k\}$ satisfying

$$\liminf_{j\to+\infty}(p_{k_j+1} - p_{k_j}) \geq 0.$$

This is equivalent to the need to show that

$$\limsup_{j\to\infty}\langle s^*, s^* - s_{k_j+1}\rangle \leq 0$$

for every subsequence $\{\|s_{k_j} - s^*\|\}$ of $\{\|s_k - s^*\|\}$ satisfying

$$\liminf_{j\to+\infty}\big(\|s_{k_j+1} - s^*\| - \|s_{k_j} - s^*\|\big) \geq 0.$$

Assume that $\{\|s_{k_j} - s^*\|\}$ is a subsequence of $\{\|s_k - s^*\|\}$ satisfying

$$\liminf_{j\to+\infty}\big(\|s_{k_j+1} - s^*\| - \|s_{k_j} - s^*\|\big) \geq 0.$$

Then,

$$\begin{aligned}
&\liminf_{j\to+\infty}\big(\|s_{k_j+1} - s^*\|^2 - \|s_{k_j} - s^*\|^2\big) \\
&\quad = \liminf_{j\to+\infty}\big(\|s_{k_j+1} - s^*\| - \|s_{k_j} - s^*\|\big)\big(\|s_{k_j+1} - s^*\| + \|s_{k_j} - s^*\|\big) \geq 0.
\end{aligned} \tag{3.48}$$

It follows from Claim 2 that

$$\begin{aligned}
&\limsup_{j\to\infty}\bigg[\bigg(1 - \frac{\tau\delta_{k_j}}{\delta_{k_j+1}}\bigg)\|\varkappa_{k_j} - u_{k_j}\|^2 + \bigg(1 - \frac{\tau\delta_{k_j}}{\delta_{k_j+1}}\bigg)\|v_{k_j} - u_{k_j}\|^2 \\
&\qquad + \wp_k[1-\rho-\wp_k]\big\|\mathcal{T}(v_{k_j}) - v_{k_j}\big\|^2\bigg] \\
&\quad \leq \limsup_{j\to\infty}\big[\|s_{k_j} - s^*\|^2 - \|s_{k_j+1} - s^*\|^2 + \Im_{k_j}K_4\big] \\
&\quad = -\liminf_{j\to\infty}\big[\|s_{k_j+1} - s^*\|^2 - \|s_{k_j} - s^*\|^2\big] \\
&\quad \leq 0.
\end{aligned} \tag{3.49}$$

The above relation implies that

$$\begin{aligned}
&\lim_{j\to\infty}\|\varkappa_{k_j} - u_{k_j}\| = 0, \\
&\lim_{j\to\infty}\|v_{k_j} - u_{k_j}\| = 0, \\
&\lim_{j\to\infty}\big\|\mathcal{T}(v_{k_j}) - v_{k_j}\big\| = 0.
\end{aligned} \tag{3.50}$$

Therefore, we obtain

$$\lim_{j\to\infty} \|v_{k_j} - \varkappa_{k_j}\| = 0. \tag{3.51}$$

According to the definition of $\varkappa_k$ one has

$$\begin{aligned} \|\varkappa_{k_j} - s_{k_j}\| &= \ell_{k_j}\|s_{k_j} - s_{k_j-1}\| \\ &= \wp_{k_j}\frac{\ell_{k_j}}{\wp_{k_j}}\|s_{k_j} - s_{k_j-1}\| \to 0, \quad \text{as } k \to +\infty. \end{aligned} \tag{3.52}$$

This, together with $\lim_{j\to\infty} \|v_{k_j} - \varkappa_{k_j}\| = 0$, yields that

$$\lim_{j\to\infty} \|v_{k_j} - s_{k_j}\| = 0. \tag{3.53}$$

From expressions (3.50) and (3.53), we deduce that

$$\|s_{k_j+1} - s_{k_j}\| \le \|v_{k_j} - s_{k_j}\| + \Im_{k_j}\|v_{k_j}\| + \wp_{k_j}\big\|\mathcal{T}(v_{k_j}) - v_{k_j}\big\|. \tag{3.54}$$

Taking limit $j \to \infty$ on both sides of the equation, we have

$$\lim_{j\to\infty} \|s_{k_j+1} - s_{k_j}\| = 0. \tag{3.55}$$

The following phrase suggests that

$$\lim_{j\to\infty} \|\varkappa_{k_j} - s_{k_j+1}\| \le \lim_{j\to\infty} \|\varkappa_{k_j} - s_{k_j}\| + \lim_{j\to\infty} \|s_{k_j} - s_{k_j+1}\| = 0. \tag{3.56}$$

Due to expression (3.11), we have

$$\delta_{k_j}\mathcal{L}(u_{k_j}, u) \ge \delta_{k_j}\mathcal{L}(u_{k_j}, v_{k_j}) + \langle \varkappa_{k_j} - v_{k_j}, u - v_{k_j}\rangle. \tag{3.57}$$

By expression (3.17), we obtain

$$\begin{aligned} \delta_{k_j}\mathcal{L}(u_{k_j}, v_{k_j}) &\ge \delta_{k_j}\mathcal{L}(\varkappa_{k_j}, v_{k_j}) - \delta_{k_j}\mathcal{L}(\varkappa_{k_j}, u_{k_j}) \\ &\quad - \frac{\delta_{k_j}\tau(\|\varkappa_{k_j} - u_{k_j}\|^2 + \|v_{k_j} - u_{k_j}\|^2)}{2\delta_{k_j+1}}. \end{aligned} \tag{3.58}$$

Combining relations (3.57), (3.58), and (3.14) we write

$$\begin{aligned} \delta_{k_j}\mathcal{L}(u_{k_j}, u) &\ge \langle \varkappa_{k_j} - u_{k_j}, v_{k_j} - u_{k_j}\rangle - \frac{\tau\delta_{k_j}}{2\delta_{k_j+1}}\|\varkappa_{k_j} - u_{k_j}\|^2 \\ &\quad - \frac{\tau\delta_{k_j}}{2\delta_{k_j+1}}\|u_{k_j} - v_{k_j}\|^2 + \langle \varkappa_{k_j} - v_{k_j}, u - v_{k_j}\rangle, \end{aligned} \tag{3.59}$$

where u is an arbitrary element in $\mathcal{Y}_k$. By using the boundedness of the sequence and expression (3.50), that right-hand side of the last inequality goes to zero. By the use of

$\delta_{k_j} \geq \delta > 0$, we obtain

$$0 \leq \limsup_{j\to\infty} \mathcal{L}(u_{k_j}, u) \leq \mathcal{L}(\hat{s}, u), \quad \forall u \in \mathcal{Y}_k.$$

It is given that $\mathcal{M} \subset \mathcal{Y}_k$, that is $\mathcal{L}(\hat{s}, u) \geq 0$, for all $u \in \mathcal{M}$. This gives that $\hat{s} \in EP(\mathcal{L}, \mathcal{M})$. By the demiclosedness of $(I - \mathcal{T})$, we obtain that $\hat{s} \in Fix(\mathcal{T})$. Since the sequence $\{s_k\}$ is bounded, this implies that there exists a subsequence $\{s_{k_j}\}$ of $\{s_k\}$ such that $s_{k_j} \rightharpoonup \hat{s}$. It is given that

$$s^* = P_{EP(\mathcal{M},\mathcal{L})\cap \mathrm{Fix}(\mathcal{T})}(0).$$

Namely, $s^* \in EP(\mathcal{M}, \mathcal{L}) \cap \mathrm{Fix}(\mathcal{T})$ as well as

$$\langle 0 - s^*, u - s^* \rangle \leq 0, \quad \forall u \in EP(\mathcal{M}, \mathcal{L}) \cap \mathrm{Fix}(\mathcal{T}).$$

It is given that $\hat{s} \in EP(\mathcal{M}, \mathcal{A}) \cap Fix(\mathcal{T})$. Thus, we have

$$\begin{aligned}&\limsup_{k\to\infty}\langle s^*, s^* - s_k\rangle \\ &\quad = \lim_{j\to\infty}\langle s^*, s^* - s_{k_j}\rangle = \langle s^*, s^* - \hat{s}\rangle \leq 0. \end{aligned}\tag{3.60}$$

By using the fact $\lim_{j\to\infty}\|s_{k_j+1} - s_{k_j}\| = 0$. Thus, we have

$$\begin{aligned}&\limsup_{k\to\infty}\langle s^*, s^* - s_{k+1}\rangle \\ &\quad \leq \limsup_{j\to\infty}\langle s^*, s_{k_j} - s_{k_{j+1}}\rangle + \limsup_{j\to\infty}\langle s^*, s^* - s_{k_j}\rangle \\ &\quad = \langle s^*, \hat{s} - s^*\rangle \leq 0. \end{aligned}\tag{3.61}$$

Combining Claim 3 and in the light of Lemma 2.2, we observe that $s_k \to s^*$ as $k \to \infty$. The proof of Theorem 3.4 is completed. □

The third method does not involve subgradient techniques and is effective in some situations. Its proof is the same as that of Algorithm 1. The third strategy is discussed below to obtain a common solution to an equilibrium and a fixed-point problem without using the subgradient technique. The key feature of this method is that it adopts a monotone step-size rule that is independent of Lipschitz constants. The algorithm uses Mann-type iteration to solve a fixed-point problem and the two-step extragradient technique to solve an equilibrium problem.

Algorithm 3 (Inertial extragradient method with a monotone step-size rule)

STEP 0: Take $s_0, s_1 \in \mathcal{M}$, $\ell \in (0,1)$, $\tau \in (0,1)$, $\delta_1 > 0$. Choose two positive numbers a, b such that $0 < a, b < 1 - \rho$ and $0 < a, b < 1 - \Im_k$. Moreover, choose $\{\wp_k\} \subset (a, b)$ and $\{\Im_k\} \subset (0,1)$ satisfying the following conditions:

$$\lim_{k\to+\infty} \Im_k = 0 \quad \text{and} \quad \sum_{k=1}^{+\infty} \Im_k = +\infty.$$

STEP 1: Calculate

$$\varkappa_k = s_k + \ell_k(s_k - s_{k-1}),$$

while ℓ_k is taken as follows:

$$0 \le \ell_k \le \hat{\ell}_k \quad \text{and} \quad \hat{\ell}_k = \begin{cases} \min\{\frac{\ell}{2}, \frac{\varrho_k}{\|s_k - s_{k-1}\|}\} & \text{if } s_k \neq s_{k-1}, \\ \frac{\ell}{2} & \text{otherwise.} \end{cases} \tag{3.62}$$

Moreover, a positive sequence $\varrho_k = \circ(\wp_k)$ satisfies $\lim_{k\to+\infty} \frac{\varrho_k}{\wp_k} = 0$.
STEP 2: Calculate

$$u_k = \arg\min_{u\in\mathcal{M}} \left\{ \delta_k \mathcal{L}(\varkappa_k, u) + \frac{1}{2}\|\varkappa_k - u\|^2 \right\}.$$

If $\varkappa_k = u_k$, then STOP. Else, move to *STEP 3*.
STEP 3: Calculate

$$v_k = \arg\min_{u\in\mathcal{M}} \left\{ \delta_k \mathcal{L}(u_k, u) + \frac{1}{2}\|\varkappa_k - u\|^2 \right\}.$$

STEP 4: Calculate

$$s_{k+1} = (1 - \wp_k - \Im_k)v_k + \wp_k \mathcal{T}(v_k).$$

STEP 5: Calculate

$$\delta_{k+1} = \begin{cases} \min\{\delta_k, \frac{\tau\|\varkappa_k - u_k\|^2 + \tau\|v_k - u_k\|^2}{2[\mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) - \mathcal{L}(u_k, v_k)]}\} \\ \quad \text{if } \mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) - \mathcal{L}(u_k, v_k) > 0, \\ \delta_k, \quad \text{otherwise.} \end{cases} \tag{3.63}$$

Set $k := k + 1$ and move to *STEP 1*.

The fourth method, which does not use a subgradient method, is successful in some scenarios. Its proof is the same as that of Algorithm 1. The key feature of this technique is that it uses a nonmonotone step-size rule that is independent of Lipschitz constants.

Algorithm 4 (Accelerated extragradient method with a nonmonotone step-size rule)

STEP 0: Take $s_0, s_1 \in \mathcal{M}$, $\ell \in (0,1)$, $\tau \in (0,1)$, $\delta_1 > 0$. Choose two positive numbers a, b such that $0 < a, b < 1 - \rho$ and $0 < a, b < 1 - \Im_k$. Moreover, choose $\{\wp_k\} \subset (a, b)$ and $\{\Im_k\} \subset (0,1)$ satisfying the following conditions:

$$\lim_{k\to+\infty} \Im_k = 0 \quad \text{and} \quad \sum_{k=1}^{+\infty} \Im_k = +\infty.$$

STEP 1: Calculate

$$\varkappa_k = s_k + \ell_k(s_k - s_{k-1}),$$

while ℓ_k is taken as follows:

$$0 \le \ell_k \le \hat{\ell}_k \quad \text{and} \quad \hat{\ell}_k = \begin{cases} \min\{\frac{\ell}{2}, \frac{\varrho_k}{\|s_k - s_{k-1}\|}\} & \text{if } s_k \neq s_{k-1}, \\ \frac{\ell}{2} & \text{otherwise.} \end{cases} \tag{3.64}$$

Moreover, a positive sequence $\varrho_k = \circ(\wp_k)$ satisfies $\lim_{k\to+\infty} \frac{\varrho_k}{\wp_k} = 0$.
STEP 2: Calculate

$$u_k = \arg\min_{u\in\mathcal{M}} \left\{ \delta_k \mathcal{L}(\varkappa_k, u) + \frac{1}{2}\|\varkappa_k - u\|^2 \right\}.$$

If $\varkappa_k = u_k$, then STOP. Else, move to *STEP 3*.
STEP 3: Calculate

$$v_k = \arg\min_{u\in\mathcal{M}} \left\{ \delta_k \mathcal{L}(u_k, u) + \frac{1}{2}\|\varkappa_k - u\|^2 \right\}.$$

STEP 4: Calculate

$$s_{k+1} = (1 - \wp_k - \Im_k)v_k + \wp_k \mathcal{T}(v_k).$$

STEP 5: Moreover, choose a nonnegative real sequence $\{\chi_k\}$ such that $\sum_{k=1}^{+\infty} \chi_k < +\infty$. Calculate

$$\delta_{k+1} = \begin{cases} \min\{\delta_k + \chi_k, \frac{\tau\|\varkappa_k - u_k\|^2 + \tau\|v_k - u_k\|^2}{2[\mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) - \mathcal{L}(u_k, v_k)]}\} \\ \quad \text{if } \mathcal{L}(\varkappa_k, v_k) - \mathcal{L}(\varkappa_k, u_k) - \mathcal{L}(u_k, v_k) > 0, \\ \delta_k + \chi_k, \quad \text{otherwise.} \end{cases} \tag{3.65}$$

Set $k := k + 1$ and move to *STEP 1*.

4 Applications

In this section, we need to find a common solution of the variational inequalities and fixed-point problems using the results from our main results. The expression (4.2) is employed to obtain the following conclusions. All the methods are based on our main findings, which are interpreted below.

Let $\mathcal{A} : \mathcal{M} \to \mathcal{Y}$ be an operator. First, we look at the classic variational inequality problem [24, 38], which is expressed as follows:

$$\langle \mathcal{A}(s^*), \aleph_1 - s^* \rangle \ge 0, \quad \forall \aleph_1 \in \mathcal{M}. \tag{4.1}$$

Let us define a bifunction $\mathcal{F}$ defined as follows:

$$\mathcal{F}(\aleph_1, \aleph_2) := \langle \mathcal{A}(\aleph_1), \aleph_2 - \aleph_1 \rangle, \quad \forall \aleph_1, \aleph_2 \in \mathcal{M}. \tag{4.2}$$

Then, the equilibrium problem converts into the problem of variational inequalities defined in (4.1) and the Lipschitz constant of the mapping $\mathcal{A}$ is $L = 2c_1 = 2c_2$.

The following corollary is derived from the proposed Algorithm 1 and the minimization problem for solving equilibrium problems that transform into projections on a convex set. This result helps in the finding of a common solution to a variational inequality problem and a fixed-point problem.

Corollary 4.1 *Suppose that $\mathcal{A} : \mathcal{M} \to \mathcal{Y}$ is a weakly continuous, pseudomonotone, and L-Lipschitz continuous mapping and the solution set* $\mathrm{Fix}(\mathcal{T}) \cap VI(\mathcal{M}, \mathcal{A})$ *is nonempty. Take* $s_0, s_1 \in \mathcal{M}$, $\ell \in (0,1)$, $\tau \in (0,1)$, $\delta_1 > 0$. *Choose two positive numbers a, b such that* $0 < a, b < 1 - \rho$ *and* $0 < a, b < 1 - \Im_k$. *Moreover, choose* $\{\wp_k\} \subset (a, b)$ *and* $\{\Im_k\} \subset (0,1)$ *satisfying the following conditions*:

$$\lim_{k \to +\infty} \Im_k = 0 \quad \textit{and} \quad \sum_{k=1}^{+\infty} \Im_k = +\infty.$$

Calculate

$$\varkappa_k = s_k + \ell_k (s_k - s_{k-1}),$$

while ℓ_k is taken as follows:

$$0 \leq \ell_k \leq \hat{\ell}_k \quad \textit{and} \quad \hat{\ell}_k = \begin{cases} \min\{\frac{\ell}{2}, \frac{\varrho_k}{\|s_k - s_{k-1}\|}\} & \textit{if } s_k \neq s_{k-1}, \\ \frac{\ell}{2} & \textit{otherwise.} \end{cases} \tag{4.3}$$

Moreover, a positive sequence $\varrho_k = \circ(\wp_k)$ *satisfies* $\lim_{k \to +\infty} \frac{\varrho_k}{\wp_k} = 0$. *First, we have to compute*

$$\begin{cases} u_k = P_{\mathcal{M}}(\varkappa_k - \delta_k \mathcal{A}(\varkappa_k)), \\ v_k = P_{\mathcal{Y}_k}(\varkappa_k - \delta_k \mathcal{A}(u_k)), \end{cases}$$

where

$$\mathcal{Y}_k = \{ z \in \mathcal{Y} : \langle \varkappa_k - \delta_k \mathcal{A}(\varkappa_k) - u_k, z - u_k \rangle \leq 0 \} \quad \textit{for each } k \geq 0.$$

Calculate

$$s_{k+1} = (1 - \wp_k - \Im_k) v_k + \wp_k \mathcal{T}(v_k).$$

The following step size should be updated:

$$\delta_{k+1} = \begin{cases} \min\{\delta_k, \frac{\tau\|\varkappa_k-u_k\|^2+\tau\|v_k-u_k\|^2}{2\langle\mathcal{A}(\varkappa_k)-\mathcal{A}(u_k),v_k-u_k\rangle}\} \\ \quad \text{if } \langle\mathcal{A}(\varkappa_k)-\mathcal{A}(u_k), v_k-u_k\rangle > 0, \\ \delta_k, \quad \text{otherwise.} \end{cases}$$

Then, the sequence $\{s_k\}$ *converges strongly to* $\mathrm{Fix}(\mathcal{T}) \cap VI(\mathcal{M}, \mathcal{A})$.

The following corollary comes from the proposed Algorithm 2 and the minimization problem for resolving equilibrium problems that transform into projections on a convex set.

Corollary 4.2 *Suppose that* $\mathcal{A}: \mathcal{M} \to \mathcal{Y}$ *is a weakly continuous, pseudomonotone, and L-Lipschitz continuous mapping and the solution set* $\mathrm{Fix}(\mathcal{T}) \cap VI(\mathcal{M}, \mathcal{A})$ *is nonempty. Take* $s_0, s_1 \in \mathcal{M}$, $\ell \in (0,1)$, $\tau \in (0,1)$, $\delta_1 > 0$. *Choose two positive numbers* a, b *such that* $0 < a, b < 1 - \rho$ *and* $0 < a, b < 1 - \Im_k$. *Moreover, choose* $\{\wp_k\} \subset (a,b)$ *and* $\{\Im_k\} \subset (0,1)$ *satisfying the following conditions*:

$$\lim_{k\to+\infty} \Im_k = 0 \quad \text{and} \quad \sum_{k=1}^{+\infty} \Im_k = +\infty.$$

Calculate

$$\varkappa_k = s_k + \ell_k(s_k - s_{k-1}),$$

while ℓ_k *is taken as follows*:

$$0 \le \ell_k \le \hat{\ell}_k \quad \text{and} \quad \hat{\ell}_k = \begin{cases} \min\{\frac{\ell}{2}, \frac{\varrho_k}{\|s_k-s_{k-1}\|}\} & \text{if } s_k \ne s_{k-1}, \\ \frac{\ell}{2} & \text{otherwise.} \end{cases} \tag{4.4}$$

Moreover, a positive sequence $\varrho_k = \circ(\wp_k)$ *satisfies* $\lim_{k\to+\infty} \frac{\varrho_k}{\wp_k} = 0$. *First, we have to compute*

$$\begin{cases} u_k = P_{\mathcal{M}}(\varkappa_k - \delta_k\mathcal{A}(\varkappa_k)), \\ v_k = P_{\mathcal{Y}_k}(\varkappa_k - \delta_k\mathcal{A}(u_k)), \end{cases}$$

where

$$\mathcal{Y}_k = \{z \in \mathcal{Y} : \langle\varkappa_k - \delta_k\mathcal{A}(\varkappa_k) - u_k, z - u_k\rangle \le 0\} \quad \text{for each } k \ge 0.$$

Calculate

$$s_{k+1} = (1 - \wp_k - \Im_k)v_k + \wp_k\mathcal{T}(v_k).$$

Moreover, choose a nonnegative real sequence $\{\chi_k\}$ *such that* $\sum_{k=1}^{+\infty}\chi_k < +\infty$. *The following step size should be updated*:

$$\delta_{k+1} = \begin{cases} \min\{\delta_k + \chi_k, \frac{\tau\|\varkappa_k - u_k\|^2 + \tau\|v_k - u_k\|^2}{2\langle \mathcal{A}(\varkappa_k) - \mathcal{A}(u_k), v_k - u_k\rangle}\} \\ \quad \textit{if } \langle \mathcal{A}(\varkappa_k) - \mathcal{A}(u_k), v_k - u_k\rangle > 0, \\ \delta_k + \chi_k, \quad \textit{otherwise.} \end{cases}$$

Then, the sequence $\{s_k\}$ *converges strongly to* $\mathrm{Fix}(\mathcal{T}) \cap VI(\mathcal{M}, \mathcal{A})$.

The following corollary comes from the proposed Algorithm 3 and the minimization problem for resolving equilibrium problems that transform into projections on a convex set.

Corollary 4.3 *Suppose that* $\mathcal{A} : \mathcal{M} \to \mathcal{Y}$ *is a weakly continuous, pseudomonotone, and L-Lipschitz continuous mapping and the solution set* $\mathrm{Fix}(\mathcal{T}) \cap VI(\mathcal{M}, \mathcal{A})$ *is nonempty. Take* $s_0, s_1 \in \mathcal{M}$, $\ell \in (0,1)$, $\tau \in (0,1)$, $\delta_1 > 0$. *Choose two positive numbers* a, b *such that* $0 < a, b < 1 - \rho$ *and* $0 < a, b < 1 - \Im_k$. *Moreover, choose* $\{\wp_k\} \subset (a,b)$ *and* $\{\Im_k\} \subset (0,1)$ *satisfying the following conditions*:

$$\lim_{k\to+\infty} \Im_k = 0 \quad \textit{and} \quad \sum_{k=1}^{+\infty} \Im_k = +\infty.$$

Calculate

$$\varkappa_k = s_k + \ell_k(s_k - s_{k-1}),$$

while ℓ_k *is taken as follows*:

$$0 \le \ell_k \le \hat{\ell}_k \quad \textit{and} \quad \hat{\ell}_k = \begin{cases} \min\{\frac{\ell}{2}, \frac{\varrho_k}{\|s_k - s_{k-1}\|}\} & \textit{if } s_k \ne s_{k-1}, \\ \frac{\ell}{2} & \textit{otherwise.} \end{cases} \tag{4.5}$$

Moreover, a positive sequence $\varrho_k = \circ(\wp_k)$ *satisfies* $\lim_{k\to+\infty}\frac{\varrho_k}{\wp_k} = 0$. *First, we have to compute*

$$\begin{cases} u_k = P_{\mathcal{M}}(\varkappa_k - \delta_k \mathcal{A}(\varkappa_k)), \\ v_k = P_{\mathcal{M}}(\varkappa_k - \delta_k \mathcal{A}(u_k)). \end{cases}$$

Calculate

$$s_{k+1} = (1 - \wp_k - \Im_k)v_k + \wp_k \mathcal{T}(v_k).$$

The following step size should be updated:

$$\delta_{k+1} = \begin{cases} \min\{\delta_k, \frac{\tau\|\varkappa_k - u_k\|^2 + \tau\|v_k - u_k\|^2}{2\langle \mathcal{A}(\varkappa_k) - \mathcal{A}(u_k), v_k - u_k\rangle}\} \\ \quad \textit{if } \langle \mathcal{A}(\varkappa_k) - \mathcal{A}(u_k), v_k - u_k\rangle > 0, \\ \delta_k, \quad \textit{otherwise.} \end{cases}$$

Then, the sequence $\{s_k\}$ *converges strongly to* $\mathrm{Fix}(\mathcal{T}) \cap VI(\mathcal{M}, \mathcal{A})$.

The proposed Algorithm 4 and the minimization problem for resolving equilibrium problems that transform into projections on a convex set lead to the following corollary.

Corollary 4.4 *Suppose that* $\mathcal{A}:\mathcal{M}\to\mathcal{Y}$ *is a weakly continuous, pseudomonotone, and L-Lipschitz continuous mapping and the solution set* $\mathrm{Fix}(\mathcal{T})\cap VI(\mathcal{M},\mathcal{A})$ *is nonempty. Take* $s_0,s_1\in\mathcal{M}$, $\ell\in(0,1)$, $\tau\in(0,1)$, $\delta_1>0$. *Choose two positive numbers* a,b *such that* $0<a,b<1-\rho$ *and* $0<a,b<1-\Im_k$. *Moreover, choose* $\{\wp_k\}\subset(a,b)$ *and* $\{\Im_k\}\subset(0,1)$ *satisfying the following conditions*:

$$\lim_{k\to+\infty}\Im_k=0\quad\textit{and}\quad\sum_{k=1}^{+\infty}\Im_k=+\infty.$$

Calculate

$$\varkappa_k=s_k+\ell_k(s_k-s_{k-1}),$$

while ℓ_k *is taken as follows*:

$$0\le\ell_k\le\hat{\ell}_k\quad\textit{and}\quad\hat{\ell}_k=\begin{cases}\min\{\frac{\ell}{2},\frac{\varrho_k}{\|s_k-s_{k-1}\|}\} & \textit{if } s_k\neq s_{k-1},\\ \frac{\ell}{2} & \textit{otherwise.}\end{cases}\tag{4.6}$$

Moreover, a positive sequence $\varrho_k=\circ(\wp_k)$ *satisfies* $\lim_{k\to+\infty}\frac{\varrho_k}{\wp_k}=0$. *First, we have to compute*

$$\begin{cases}u_k=P_{\mathcal{M}}(\varkappa_k-\delta_k\mathcal{A}(\varkappa_k)),\\ v_k=P_{\mathcal{M}}(\varkappa_k-\delta_k\mathcal{A}(u_k)).\end{cases}$$

Calculate

$$s_{k+1}=(1-\wp_k-\Im_k)v_k+\wp_k\mathcal{T}(v_k).$$

Moreover, choose a nonnegative real sequence $\{\chi_k\}$ *such that* $\sum_{k=1}^{+\infty}\chi_k<+\infty$. *The following step size should be updated*:

$$\delta_{k+1}=\begin{cases}\min\{\delta_k+\chi_k,\frac{\tau\|\varkappa_k-u_k\|^2+\tau\|v_k-u_k\|^2}{2\langle\mathcal{A}(\varkappa_k)-\mathcal{A}(u_k),v_k-u_k\rangle}\}\\ \quad\textit{if } \langle\mathcal{A}(\varkappa_k)-\mathcal{A}(u_k),v_k-u_k\rangle>0,\\ \delta_k+\chi_k, \quad \textit{otherwise.}\end{cases}$$

Then, the sequence $\{s_k\}$ *converges strongly to* $\mathrm{Fix}(\mathcal{T})\cap VI(\mathcal{M},\mathcal{A})$.

5 Numerical illustrations

This section covers the computational consequences of the presented methodologies, as well as an examination of how variations in control settings impact the numerical efficacy of the suggested algorithms. All computations are run in MATLAB R2018b on an HP i5 Core (TM) i5-6200 laptop with 8.00 GB (7.78 GB useable) RAM.

Example 5.1 The first sample problem here is taken from the Nash–Cournot Oligopolistic Equilibrium model in [43]. Suppose that a function $q : \mathcal{Y} \to \mathbb{R}$ is described through

$$lev_{\leq q} := \left\{ s \in \mathcal{Y} : q(s) \leq 0 \right\} \neq \emptyset.$$

The subgradient projection is a mapping that is characterized as follows:

$$\mathcal{T}(s) = \begin{cases} s - \frac{q(s)}{\|r(s)\|^2} r(s), & \text{if } q(s) \geq 0, \\ s, & \text{otherwise,} \end{cases}$$

wherein $r(s) \in \partial q(s)$. In such instance, $\mathcal{T}$ is quasinonexpansive, demiclosed at zero, and $\mathrm{Fix}(\mathcal{T}) = lev_{\leq q}$. In this instance, the bifunction $\mathcal{F}$ can be expressed as follows:

$$\mathcal{F}(s, u) = \langle Ps + Qu + c, u - s \rangle,$$

wherein $c \in \mathbb{R}^M$ and P, Q are matrices of order M. The matrix $Q - P$ is symmetric negative-semidefinite, while the matrix P is symmetric positive-semidefinite, through Lipschitz-like parameters $c_1 = c_2 = \frac{1}{2}\|P - Q\|$ (for additional information, see [43]). The starting point for this study is $s_0 = s_1 = (2, 2, \ldots, 2)$ and the size of the space is chosen differently with the stopping condition $D_k = \|\varkappa_k - u_k\| \leq 10^{-3}$. Figures 1–10 depict numerical observations for Example 5.1. The following control criteria are in use:

(1) Algorithm 1 in [52] (briefly, *EGM*):

$$\wp_k = \frac{1}{(10k + 4)}, \qquad \Im_k = \frac{1 - \Im}{5}, \qquad \delta_k = \min\left\{ \frac{1}{4c_1}, \frac{1}{4c_2} \right\};$$

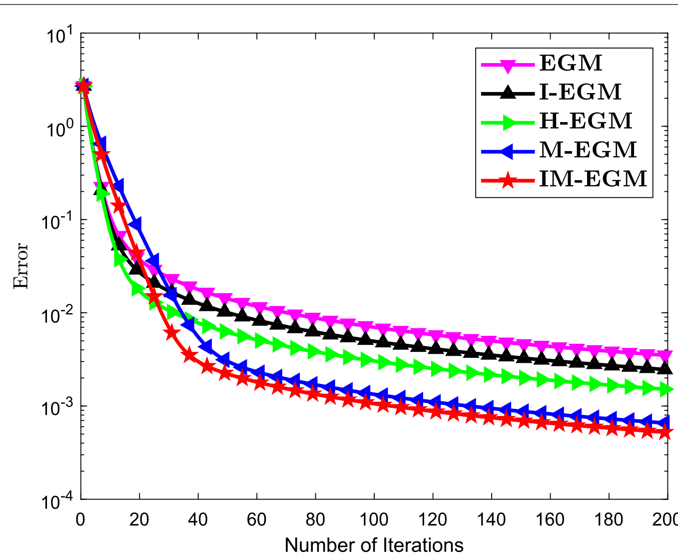

Figure 1 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $N = 10$ for the first 100 iterations

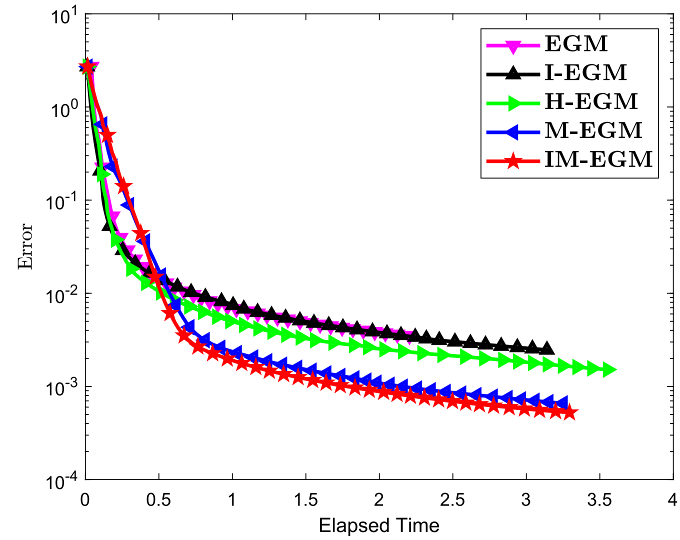

Figure 2 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $N = 10$ for the first 100 iterations

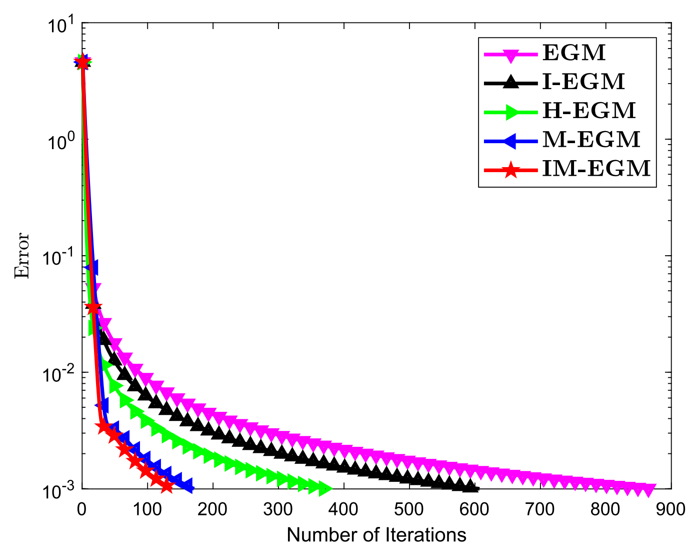

Figure 3 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $N = 20$ for the error term 10^{-3}

(2) Algorithm 2 in [51] (briefly, *I-EGM*):

$$\wp_k = \frac{1}{(10k+4)}, \qquad \Im_k = \frac{1-\Im}{5}, \qquad \delta_k = \min\left\{\frac{1}{4c_1}, \frac{1}{4c_2}\right\};$$

(3) Algorithm 1 in [18] (briefly, *H-EGM*):

$$\wp_k = \frac{1}{(10k+4)}, \qquad \Im_k = \frac{1}{5}, \qquad \delta_k = \min\left\{\frac{1}{4c_1}, \frac{1}{4c_2}\right\};$$

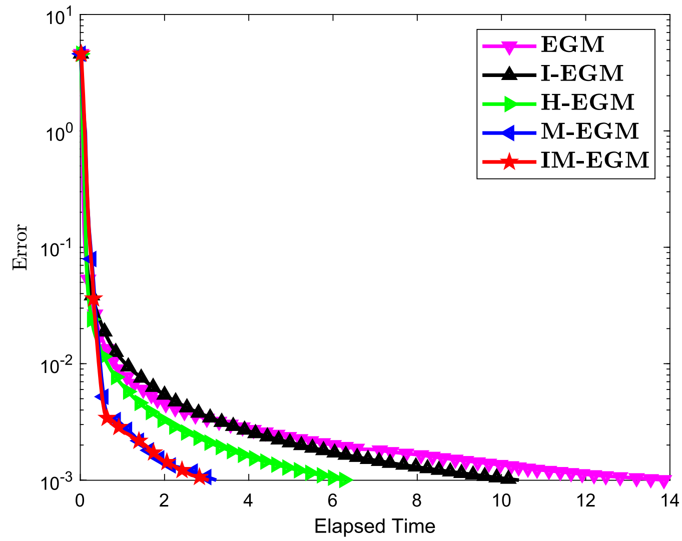

Figure 4 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $N = 20$ for the error term 10^{-3}

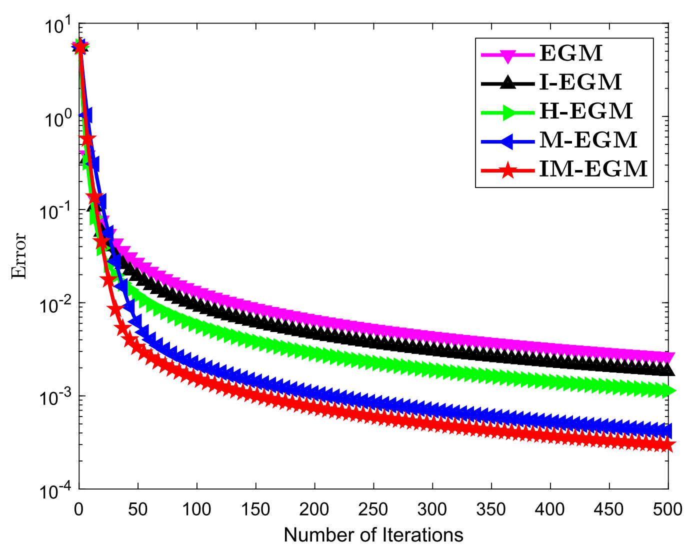

Figure 5 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $N = 30$ for the first 500 iterations

(4) Algorithm 1 (briefly, *M-EGM*):

$$\delta_1 = 0.36, \qquad \ell = 0.57, \qquad \tau = 0.264, \qquad \varrho_k = \frac{10}{(1+k)^3},$$
$$\Im_k = \frac{1-\Im}{5}, \qquad \wp_k = \frac{1}{(10k+4)}, \qquad g(s) = \frac{s}{5};$$

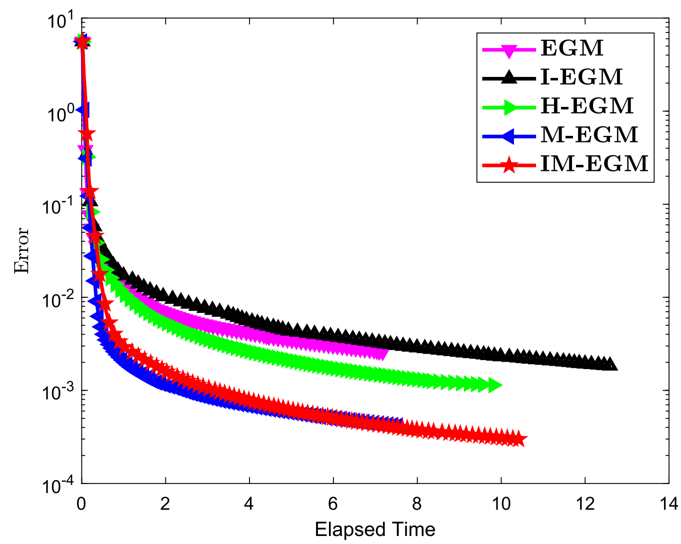

Figure 6 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $N = 30$ for the first 500 iterations

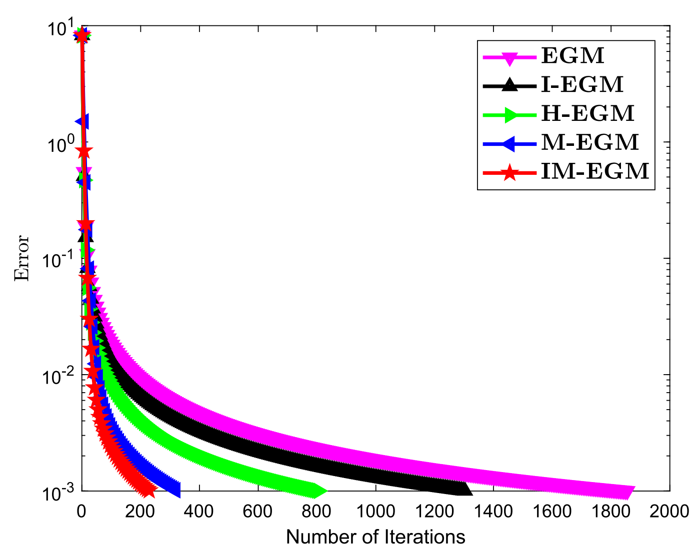

Figure 7 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $N = 40$ for the error term 10^{-3}

(5) Algorithm 2 (briefly, *IM-EGM*):

$$\delta_1 = 0.36, \qquad \ell = 0.57, \qquad \tau = 0.264, \qquad \varrho_k = \frac{10}{(1+k)^2},$$
$$\Im_k = \frac{1-\Im}{5}, \qquad \wp_k = \frac{1}{(10k+4)}, \qquad g(s) = \frac{s}{5}, \qquad \chi_k = \frac{20}{(1+k)^2}.$$

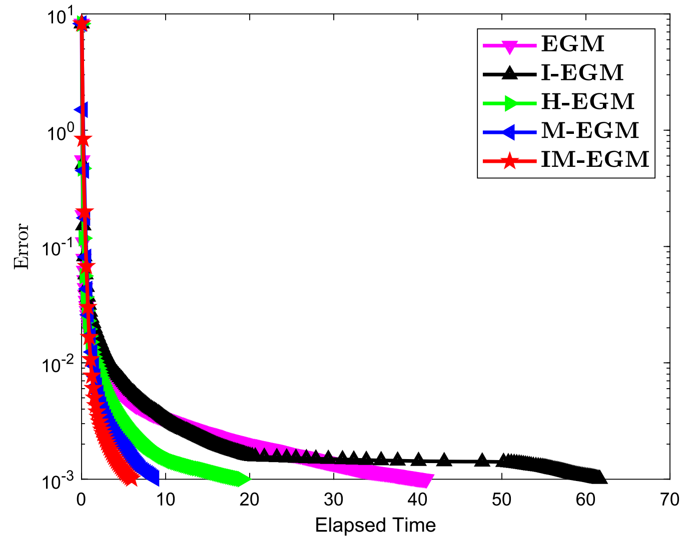

Figure 8 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $N = 40$ for the error term 10^{-3}

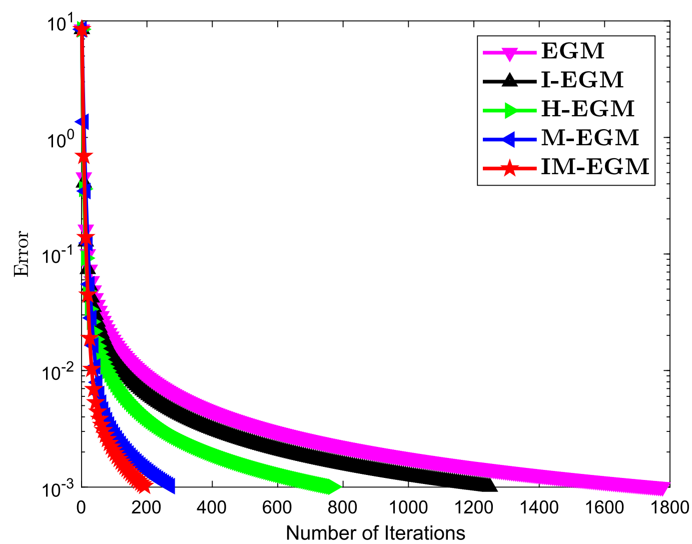

Figure 9 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $N = 50$ for the error term 10^{-3}

Example 5.2 Consider the fact that $\mathcal{Y} = L^2([0,1])$ is a real Hilbert space through an inner product $\langle s,u\rangle = \int_0^1 s(t)u(t)\,dt$, $\forall s,u \in \mathcal{Y}$, in which the induced norm obtains

$$\|s\| = \sqrt{\int_0^1 \left|s(t)\right|^2 dt}.$$

Assume an operator $\mathcal{A} : \mathcal{M} \to \mathcal{Y}$ is specified by

$$\mathcal{A}(s)(t) = \int_0^1 \bigl(s(t) - H(t,s)f\bigl(s(t)\bigr)\bigr)\,ds + g(t),$$

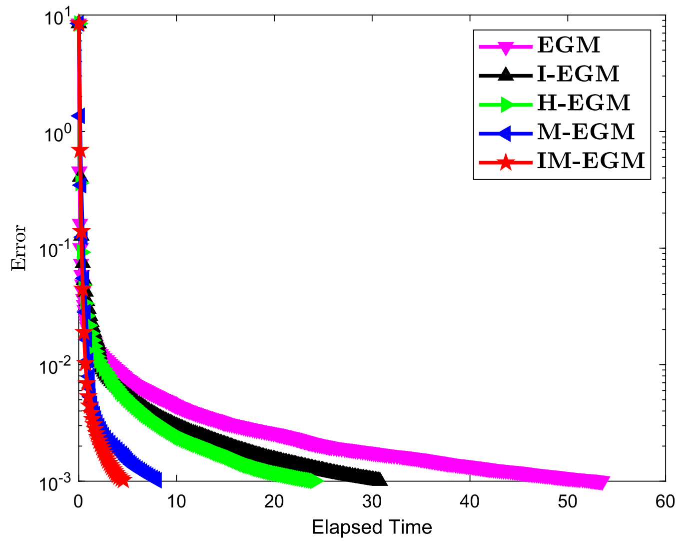

Figure 10 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $N = 50$ for the error term 10^{-3}

where $\mathcal{M} := \{s \in L^2([0,1]) : \|s\| \leq 1\}$ is the unit ball and

$$H(t,s) = \frac{2tse^{(t+s)}}{e\sqrt{e^2-1}}, \qquad f(s) = \cos s, \qquad g(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

The bifunction is stated as follows:

$$\mathcal{F}(s,u) := \langle \mathcal{A}(s), u - s \rangle, \quad \forall s, u \in \mathcal{M}.$$

Moreover, $\mathcal{F}$ is clearly a Lipschitz-type continuous bifunction with the Lipschitz constant $c_1 = c_2 = 1$ and the monotone [49]. A metric projection on $\mathcal{M}$ is evaluated as follows:

$$P_{\mathcal{M}}(s) = \begin{cases} \frac{s}{\|s\|} & \text{if } \|s\| > 1, \\ s, & \|s\| \leq 1. \end{cases}$$

A $\mathcal{T} : L^2([0,1]) \to L^2([0,1])$ is written as follows:

$$\mathcal{T}(s)(t) = \int_0^1 ts(s)\,ds, \quad t \in [0,1].$$

A simple calculation shows that $\mathcal{T}$ is 0-demicontractive. The solution to the problem is $s^*(t) = 0$. Figures 11–18 depict numerical observations for Example 5.2. The following control criteria are in use:

(1) Algorithm 1 in [52] (briefly, *EGM*):

$$\wp_k = \frac{1}{(5k+10)}, \qquad \Im_k = \frac{1-\Im}{4}, \qquad \delta_k = \min\left\{\frac{1}{4c_1}, \frac{1}{4c_2}\right\};$$

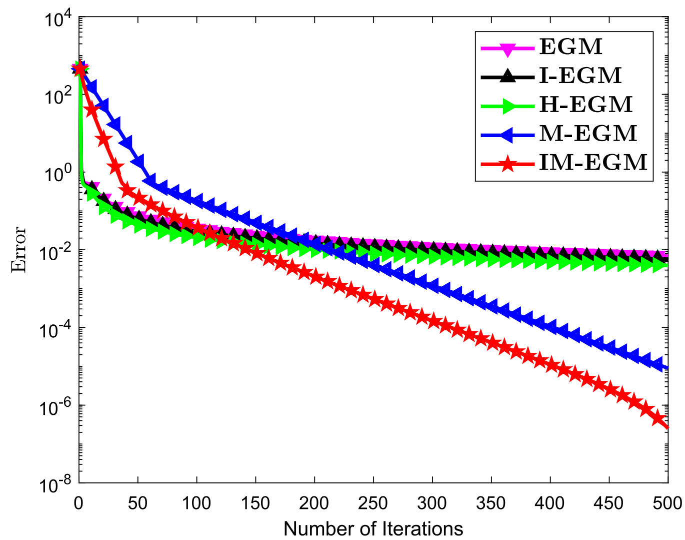

Figure 11 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $s_0 = s_1 = t$ for the first 500 iterations

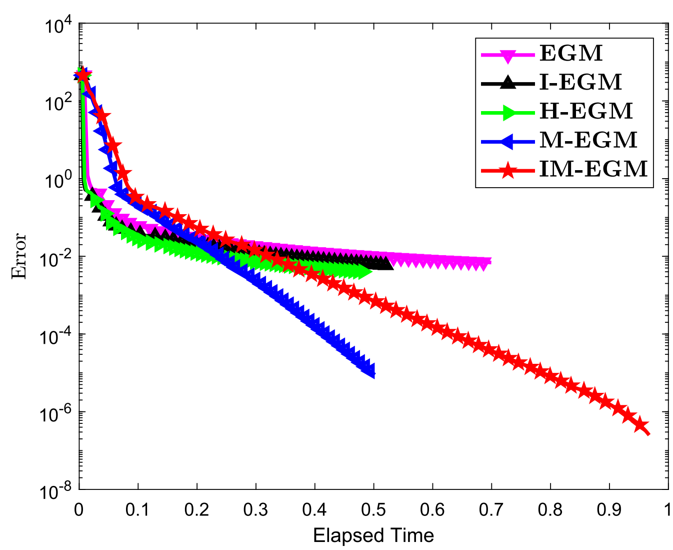

Figure 12 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $s_0 = s_1 = t$ for the first 500 iterations

(2) Algorithm 2 in [51] (briefly, *I-EGM*):

$$\wp_k = \frac{1}{(5k+10)}, \qquad \Im_k = \frac{1-\Im}{4}, \qquad \delta_k = \min\left\{\frac{1}{4c_1}, \frac{1}{4c_2}\right\};$$

(3) Algorithm 1 in [18] (briefly, *H-EGM*):

$$\wp_k = \frac{1}{(5k+10)}, \qquad \Im_k = \frac{1}{5}, \qquad \delta_k = \min\left\{\frac{1}{4c_1}, \frac{1}{4c_2}\right\};$$

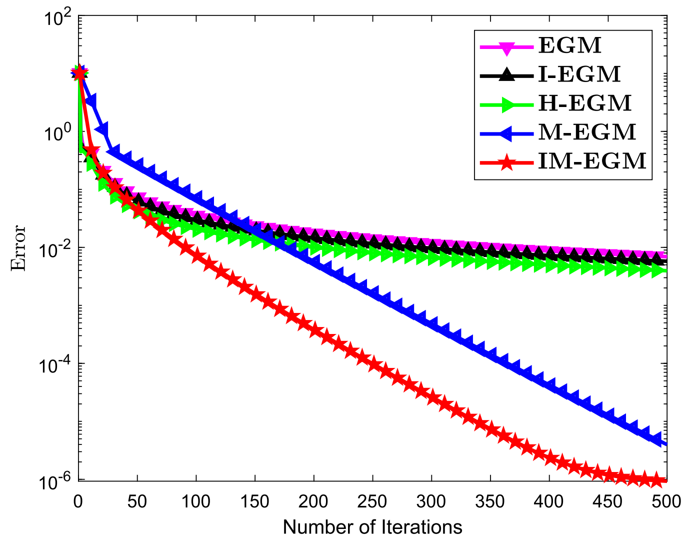

Figure 13 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $s_0 = s_1 = \sin(t)$ for the first 500 iterations

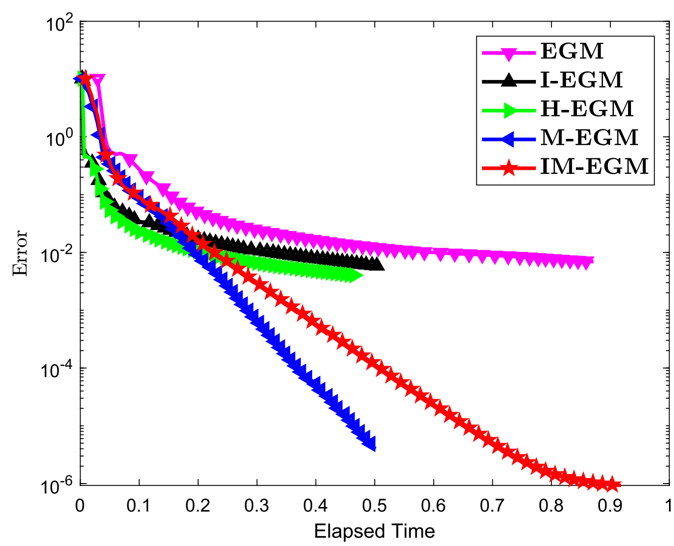

Figure 14 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $s_0 = s_1 = \sin(t)$ for the first 500 iterations

(4) Algorithm 1 (briefly, *M-EGM*):

$$\delta_1 = 0.42, \qquad \ell = 0.67, \qquad \tau = 0.33, \qquad \varrho_k = \frac{10}{(1+k)^2},$$

$$\Im_k = \frac{1-\Im}{3}, \qquad \wp_k = \frac{1}{(5k+10)}, \qquad g(s) = \frac{s}{3};$$

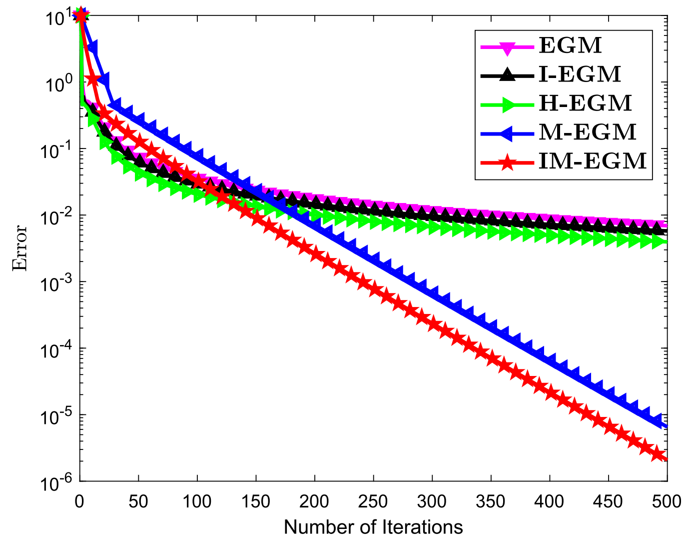

Figure 15 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $s_0 = s_1 = \cos(t)$ for the first 500 iterations

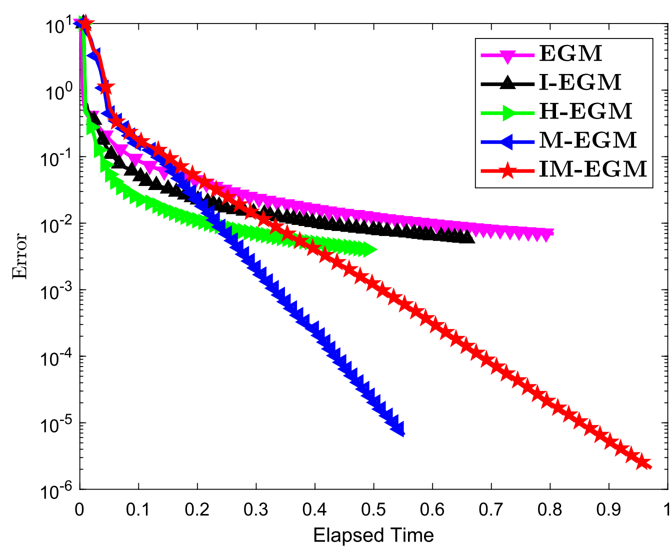

Figure 16 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $s_0 = s_1 = \cos(t)$ for the first 500 iterations

(5) Algorithm 2 (briefly, *IM-EGM*):

$$\delta_1 = 0.42, \qquad \ell = 0.67, \qquad \tau = 0.33, \qquad \varrho_k = \frac{10}{(1+k)^2},$$

$$\Im_k = \frac{1-\Im}{3}, \qquad \wp_k = \frac{1}{(5k+10)}, \qquad g(s) = \frac{s}{3}, \qquad \chi_k = \frac{10}{(1+k)^2}.$$

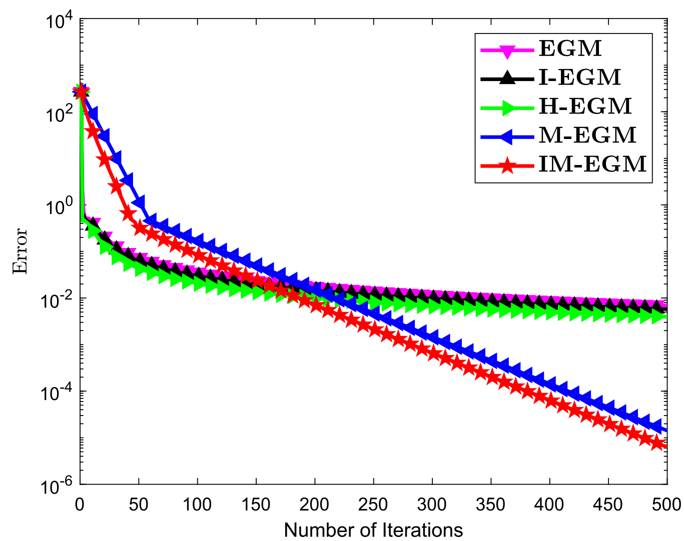

Figure 17 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $s_0 = s_1 = \cos(t)$ for the first 500 iterations

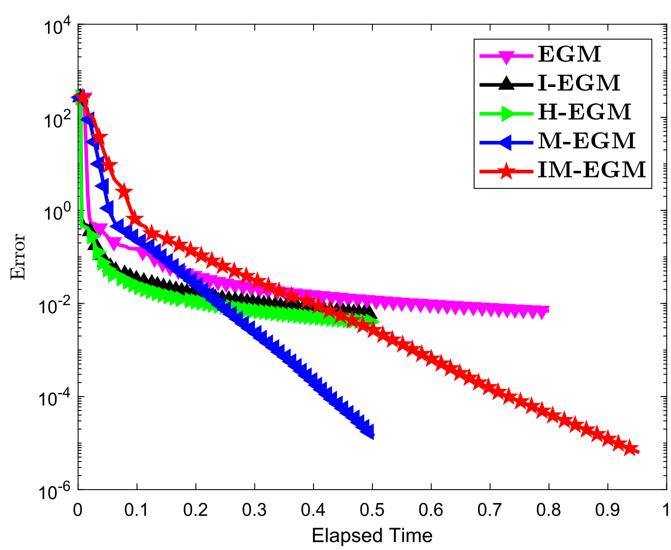

Figure 18 Numerical comparison of Algorithm 1 and Algorithm 2 using Algorithm 1 in [52], Algorithm 2 in [51], and Algorithm 1 in [18], while $s_0 = s_1 = \cos(t)$ for the first 500 iterations

6 Conclusion

The paper provides two explicit extragradient-like approaches for finding a common solution to an equilibrium problem containing a pseudomonotone and Lipschitz-type bifunction with such a fixed-point problem needing a ρ-demicontractive mapping in a real Hilbert space. A new step-size criterion that is not reliant on Lipschitz-type constant information has been developed. Under certain standard conditions, strong convergence theorems for the proposed algorithms are established. The computational data was studied to confirm the suggested approaches' arithmetic superiority over current methods.

These computational findings show that the nonmonotone variable step-size rule continues to improve the iterative sequence's performance in this case.

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Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
BP, CK, NP(N. Pholasa) and NP (N. Pakkranang): Conceptualization, Methodology, Supervision, Writing and Editing manuscript preparation. CK, NP(N. Pholasa) and NP(N. Pakkranang): Investigation, Formal Analysis, Investigation, Review and Validation. BP, NP(N. Pholasa) and NP(N. Pakkranang): Investigation, Funding Acquisition and Validation. All authors have read and approved with the final of this manuscript.

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