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L^∞-error estimate of a generalized parallel Schwarz algorithm for elliptic quasi-variational inequalities related to impulse control problem

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Abstract

The generalized Schwarz algorithm for a class of elliptic quasi-variational inequalities related to impulse control problems is studied in this paper. The principal result is to prove the error estimate in L^∞-norm for m subdomains with overlapping nonmatching grids. This approach combines the geometrical convergence and the uniform convergence.

Keywords: Quasivariational inequalities; Schwarz algorithm; Finite-element method; L^∞ error estimate

1 Introduction

In the present paper, we are concerned with the L^∞-convergence of the standard finite-element approximation for the impulse control problem associated with the elliptic quasi-variational inequality (QVI):

$$\begin{cases} \text{find} \quad u \in K_g(u) \quad \text{such that} \\ a(u, v-u) \geq (f, v-u), \quad \forall v \in K_g(u). \end{cases} \tag{1.1}$$

Here, f is a right hande side in $L^\infty(\Omega)$, such that $f \geq 0$, $K_g(u)$ is the implicit convex set defined by

$$K_g(u) = \left\{ v \in H^1(\Omega) / v = g \text{ on } \partial\Omega, 0 \leq v \leq Mu \text{ in } \Omega \right\}, \tag{1.2}$$

where Ω is a bounded convex domain of $\mathbb{R}^N$ with suffciently smooth boundary $\partial\Omega$ and M is a nonlinear operator from $L^\infty(\Omega)$ into itself defined by

$$Mu(x) = k + \inf u(x+\xi), \quad x \in \Omega, \xi \geq 0, x+\xi \in \Omega, k > 0. \tag{1.3}$$

The function Mu is called the obstacle of impulse control, see [1].

$(\cdot,\cdot)$ is the scalar product in $L^2(\Omega)$, and $a(\cdot,\cdot)$ is the bilinear form assumed to be continuous and strongly coercive

$$a(u,v) = \int_\Omega \left(\sum_{1\le l,k\le N} a_{lk}(x)\frac{\partial u}{\partial x_l}\frac{\partial v}{\partial x_k} + \sum_{1\le k\le N} a_k(x)\frac{\partial u}{\partial x_k} + a_0(x)uv \right) dx \tag{1.4}$$

Let V_h be the finite-element space consisting of continuous piecewise-linear functions and r_h be the usual interpolation operator. We define the discrete counterpart of (1.1) by

$$\begin{cases} \text{find} \quad u_h \in K_{gh}(u_h) \quad \text{satisfying} \\ a(u_h, v_h - u_h) \ge (f, v_h - u_h), \quad \forall v_h \in K_{gh}(u_h), \end{cases} \tag{1.5}$$

where

$$K_{gh}(u_h) = \{v_h \in V_h / v_h = \pi_h g \text{ on } \partial\Omega,\ 0 \le v_h \le r_h M u_h \text{ in } \Omega\}.. \tag{1.6}$$

The existence, uniqueness and regularity of the continuous solution{(1.1) and the discrete solution (1.5)} have been studied and established in the past years (see [1]).

Naturally, the structure of problem (1.1) is analogous to that of the classical obstacle problem where the obstacle is replaced by an implicit one depending upon the solution sought. The terminology "quasivariational inequality" being chosen is a result of this remark. This QVI arises in impulse-control problems: an introduction to impulse control with numerous examples and applications can be found in [1].

To estimate an error of the solution, we apply the generalized parallel Schwarz algorithm. We consider a domain that is the union of m overlapping subdomains where each subdomain has its own generated triangulation, under a discrete maximum principle [7], we show that the discretization on each subdomain converges quasioptimally in the L^∞-norm. This approach has already been proved for variational and quasivariational inequalities when the domain was split into two subdomains using the alternating Schwarz algorithm we refer the reader to [2, 3, 6, 8–10]

The paper consists of two parts. In the first we show the monotonicity and stability properties of the discrete solution, then we state the continuous and the discrete Schwarz sequence for quasivariational inequalities and define their respective finite-element counterparts in the context of overlapping nonmatching grids in the second part we prove a fundamental lemma for m auxiliary sequences and we establish a main result concerning the error estimate of solution in L^∞-norm, taking into account the combination of geometrical convergence and the error estimate of Cortey-Dumont [5].

2 Schwarz algorithm for quasivariational inequalities

2.1 Assumptions and notations

Let u_h be the discrete solution of QVI

$$\begin{cases} a(u_h, v - u_h) \ge (f, v - u_h), \forall v \in V_h \\ u_h = \pi_h g \quad \text{on} \quad \partial\Omega, u_h \le r_h M u_h, \text{ in } \Omega \\ v_h = \pi_h g \quad \text{on} \quad \partial\Omega, v_h \le r_h M u_h, \text{ in } \Omega \end{cases} \tag{2.1}$$

and let $\tilde{u}_h$ be the discrete solution of QVI

$$\begin{cases} a(\tilde{u}_h, v - \tilde{u}_h) \geq (f, v - \tilde{u}_h), \quad \forall v \in V_h, \\ \tilde{u}_h = \pi_h \tilde{g} \quad \text{on } \partial\Omega, \qquad \tilde{u}_h \leq r_h M u_h, \quad \text{in } \Omega, \\ v = \pi_h \tilde{g} \quad \text{on } \partial\Omega, \qquad v \leq r_h M u_h, \quad \text{in } \Omega, \end{cases} \tag{2.2}$$

where $\tilde{g}$ is a regular function defined on $\partial\Omega$.

Let us write $\sigma_h(g, Mu_h)$ the solution of the problem (2.1), where σ_h is a mapping $L^\infty(\Omega)$ into itself. We establish the monotonicity and stability properties of the solution.

Lemma 2.1 *Let g and $\tilde{g}$ be two given functions and $u_h = \sigma_h(g, Mu_h)$, $\tilde{u}_h = \sigma_h(\tilde{g}, Mu_h)$ the corresponding discrete solutions of (2.1) (resp. (2.2)). If $g \geq \tilde{g}$, then $\sigma_h(g, Mu_h) \geq \sigma_h(\tilde{g}, Mu_h)$.*

Proof let $v_h = min(0, u_h - \tilde{u}_h)$. In the region where v_h is negative ($v_h < 0$), we have

$$0 \leq u_h < \tilde{u}_h \leq r_h M u_h$$

which means that the obstacle $r_h M u_h$ is not active for u_h.

So, for that v_h, we have

$$a(u_h, v_h) = (f, v_h) \tag{2.3}$$

we suppose $w_h = \tilde{u}_h + v_h$, so $w_h \leq r_h M u_h$, then

$$a(\tilde{u}_h, v_h) \geq (f, v_h) \tag{2.4}$$

Subtracting (2.3) and (2.4) from each other, we obtain

$$a(\tilde{u}_h - u_h, v_h) \geq 0$$

or

$$a(v_h, v_h) = a(u_h - \tilde{u}_h, v_h) = -a(\tilde{u}_h - u_h, v_h) \leq 0$$

so

$$a(v_h, v_h) \leq 0$$

as $a(.,.)$ is strongly coercive, then $v_h = 0$, so

$$u_h \geq \tilde{u}_h$$

This completes the demonstration. □

Proposition 2.2 *Under the notations and conditions of the preceding lemma, we have*

$$\|u_h - \tilde{u}_h\|_{L^\infty(\Omega)} \leq \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}. \tag{2.5}$$

Proof Setting

$$\phi = \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}.$$

We have

$$g - \tilde{g} \leq \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}$$

thus,

$$g \leq \tilde{g} + \phi.$$

By Lemma 2.1, it follows that

$$\sigma_h(g, Mu_h) = \sigma_h(\tilde{g} + \phi, Mu_h + \phi),$$

however,

$$\sigma_h(\tilde{g} + \phi, Mu_h + \phi) = \sigma_h(\tilde{g}, Mu_h) + \phi,$$

from where

$$\sigma_h(g, Mu_h) - \sigma_h(\tilde{g}, Mu_h) \leq \phi.$$

Similarly, by interchanging the roles of g and $\tilde{g}$, we also obtain

$$\sigma_h(\tilde{g}, Mu_h) - \sigma_h(g, Mu_h) \leq \phi.$$

This complete the proof. □

Theorem 2.3 ([5]) *Under the preceding notations and conditions, there exists a constant c independent of h such that*

$$\|u - u_h\|_{L^\infty(\Omega)} \leq ch^2 |\log h|^2. \tag{2.6}$$

2.2 The continuous Schwarz sequences

We consider the problem: find $u \in K_0(u)$ such that

$$a(u, v - u) \geq (f, v - u) \quad \forall v \in K_0(u), \tag{2.7}$$

where $K_0(u)$ is defined in (1.2) with $g = 0$.

We split Ω into m overlapping subdomains such that

$$\begin{cases} \text{for all distinct } i, j, k \in \{1, \dots, m\}, \text{ if } \Omega_i \cap \Omega_j \neq \emptyset \\ \text{and } \Omega_i \cap \Omega_k \neq \emptyset, \text{ then } \Omega_j \cap \Omega_k = \emptyset \end{cases}$$

and u satisfies the local regularity condition

$$u/_{\Omega_i} \in W^{2,p}(\Omega_i); \quad 2 \le p < \infty. \tag{2.8}$$

We set $\Gamma_{ij} = \partial\Omega_i \cap \Omega_j$, where $\partial\Omega_i$ denotes the boundary of Ω_i.

The intersection of Γ_{ij} and Γ_{ji} $(i \neq j)$ is assumed to be empty.

Let

$$V_{ij} = \{v \in H^1(\Omega_i)/v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega\}, \quad i = \overline{1,m}, j = \overline{1,m}, (i \neq j).$$

For $w \in C^0(\overline{\Gamma}_{ij})$, we define

$$V_{ij}^{(w)}(\Omega_i) = \{v \in V_{ij}(\Omega_i)/v = w \text{ on } \Gamma_{ij}\}, \quad i = \overline{1,m}, j = \overline{1,m}, (i \neq j).$$

We associate with problem (2.7) the following system: Find $u_i \in V_{ij}^{(u_j)}$, a solution of

$$\begin{cases} a_i(u_i, v - u_i) \ge (f_i, v - u_i), & \forall v \in V_{ij}^{(u_j)}, \\ u_i = u_j, & \text{on } \Gamma_{ij}. \end{cases} \tag{2.9}$$

For $u_i^0, u_j^0 \in C^0(\overline{\Omega})$ the initial values, we define the Schwarz sequences (u_i^{n+1}) on Ω_i such that $u_i^{n+1} \in V_{ij}^{(u_j^n)}$ solves

$$\begin{cases} a_i(u_i^{n+1}, v - u_i^{n+1}) \ge (f_i, v - u_i^{n+1}) & \forall v \text{ in } V_{ij}^{(u_j^n)} \\ u_i^{n+1} \le Mu_i^n \quad \text{in } \Omega_i, & v \le Mu_i^n \quad \text{in } \Omega_i, \end{cases} \tag{2.10}$$

where

$$a_i(u,v) = \int_{\Omega_i} \left(\sum_{1 \le l,k \le N} a_{lk}(x) \frac{\partial u}{\partial x_l} \frac{\partial v}{\partial x_k} + \sum_{1 \le k \le N} a_k(x) \frac{\partial u}{\partial x_k} v + a_0(x)uv \right) dx, \quad i = 1, \dots, m.$$

$u_i^0 = u^0$ in Ω_i, $u_i^{n+1} = 0$ in $\overline{\Omega}/\overline{\Omega_i}$.

2.3 Geometrical convergence

Theorem 2.4 *The sequences* $(u_1^{n+1}, u_2^{n+1}, \dots, u_m^{n+1})$, $n \ge 0$ *produced by the generalized Schwarz algorithm converge geometrically to the solution* $(u_1, u_2, \dots, u_m)$ *of the problem* (2.9). *More precisely, there exist* m *constants* $k_1, k_2, \dots, k_m \in (0,1)$, $\forall i = \overline{1, m-1}$, $j = \overline{2, m}$ *and* $i < j$ *such that*

$$\begin{aligned} \|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} &\le k_i^n k_j^n \|u - u^0\|_{L^\infty(\Gamma_{ij})}, \\ \|u_j - u_j^{n+1}\|_{L^\infty(\Omega_j)} &\le k_i^n k_j^n \|u - u^0\|_{L^\infty(\Gamma_{ji})} \end{aligned} \tag{2.11}$$

and we consider a continuous function $w_i \in L^\infty(\Omega_i)$ *in* $\overline{\Omega_i} \setminus (\overline{\Gamma_i} \cap \partial\Omega)$

such that

$$\Delta w_i = 0, \quad \text{in } \Omega_i,$$

where

$$w_i = \begin{cases} 0, & on\ \partial\Omega_i/\overline{\partial\Omega_i \cap \Omega}, \\ 1, & on\ \partial\Omega_i \cap \Omega, \end{cases}$$

and

$$k_i = \sup\{w_j(x)/x \in \partial\Omega_i \cap \Omega, i \neq j\} \in (0,1), \quad \forall i,j = \overline{1,m}. \tag{2.12}$$

Proof From the maximum principle, we have

$$\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} \le \|u_i - u_j^{n+1}\|_{L^\infty(\Gamma_{ij})}$$

and

$$\begin{aligned}
\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} &\le \|u_j - u_j^{n}\|_{L^\infty(\Gamma_{ij})} \le \|w_i u_j - w_i u_j^{n}\|_{L^\infty(\Gamma_{ij})} \\
&\le \|w_i u_j - w_i u_j^{n}\|_{L^\infty(\Omega_j)} \le \|w_i u_j - w_i u_j^{n}\|_{L^\infty(\Gamma_{ji})} \\
&\le \|w_i\|_{L^\infty(\Gamma_{ji})} \|u_j - u_j^{n}\|_{L^\infty(\Gamma_{ji})} \le \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_j - w_j u_j^{n}\|_{L^\infty(\Gamma_{ji})} \\
&\le \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_i - w_j u_i^{n-1}\|_{L^\infty(\Gamma_{ji})} \\
&\le \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_i - w_i u_i^{n}\|_{L^\infty(\Omega_i)} \\
&\le \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j u_i - w_j u_i^{n-1}\|_{L^\infty(\Gamma_{ij})} \\
&\le \|w_i\|_{L^\infty(\Gamma_{ji})} \|w_j\|_{L^\infty(\Gamma_{ij})} \|u_i - u_i^{n-1}\|_{L^\infty(\Gamma_{ij})},
\end{aligned}$$

Using (2.12), hence

$$\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} \le k_i k_j \|u_i - u_i^{n-1}\|_{L^\infty(\Gamma_{ij})},$$

By induction, we obtain

$$\begin{aligned}
\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} &\le k_i^n k_j^n \|u_i - u_i^{0}\|_{L^\infty(\Gamma_{ij})} \\
&\le k_i^n k_j^n \|u - u^{0}\|_{L^\infty(\Gamma_{ij})},
\end{aligned}$$

where $u_i^0 = u^0$ on Γ_{ij}, $u_i^0 = 0$ on $\partial\Omega_i \cap \partial\Omega$.

Similary, we have

$$\begin{aligned}
\|u_j - u_j^{n}\|_{L^\infty(\Omega_j)} &\le \|u_j - u_j^{n}\|_{L^\infty(\Gamma_{ji})} \\
&\le \|u_i - u_i^{n}\|_{L^\infty(\Gamma_{ji})} \le \|w_j u_i - w_j u_i^{n}\|_{L^\infty(\Gamma_{ji})} \\
&\le \|w_j u_i - w_j u_i^{n}\|_{L^\infty(\Omega_i)} \le \|w_j u_i - w_j u_i^{n}\|_{L^\infty(\Gamma_{ij})} \\
&\le \|w_j\|_{L^\infty(\Gamma_{ij})} \|u_i - u_i^{n}\|_{L^\infty(\Gamma_{ij})} \le \|w_j\|_{L^\infty(\Gamma_{ij})} \|w_i u_j - w_i u_j^{n-1}\|_{L^\infty(\Gamma_{ij})} \\
&\le \|w_j\|_{L^\infty(\Gamma_{ij})} \|w_i u_j - w_i u_j^{n-1}\|_{L^\infty(\Omega_j)}
\end{aligned}$$

$$\begin{aligned}&\leq \|w_j\|_{L^\infty(\Gamma_{ij})}\left\|w_iu_j - w_iu_j^{n-1}\right\|_{L^\infty(\Gamma_{ji})}\\&\leq \|w_i\|_{L^\infty(\Gamma_{ji})}\|w_j\|_{L^\infty(\Gamma_{ij})}\left\|u_j - u_j^{n-1}\right\|_{L^\infty(\Gamma_{ji})} \leq k_ik_j\left\|u_i - u_j^n\right\|_{L^\infty(\Gamma_{ji})},\end{aligned}$$

then,

$$\begin{aligned}\left\|u_j - u_j^{n+1}\right\|_{L^\infty(\Omega_j)} &\leq k_i^nk_j^n\left\|u_j - u_j^0\right\|_{L^\infty(\Gamma_{ji})}\\&\leq k_i^nk_j^n\left\|u - u^0\right\|_{L^\infty(\Gamma_{ji})},\end{aligned}$$

where $u_j^0 = u^0$ on Γ_{ji}, $u_j^0 = 0$ on $\partial\Omega_j \cap \partial\Omega$. □

2.4 The discretization

Let $\tau^{h_{ij}}$ be a standard regular and quasiuniform finite-element triangulation in Ω_i, h_{ij} being the meshsizes.

We assume that every two triangulations are mutually independent on $\Omega_i \cap \Omega_j$, in the sense that a triangle belonging to one triangulation does not necessarily belong to the other, $i = \overline{1,m}$, $j = \overline{1,m}$, $(i \neq j)$

Let $V_{h_{ij}} = V_{h_{ij}}(\Omega_i)$ be the space of continuous piecewise-linear functions on $\tau^{h_{ij}}$ that vanish on $\partial\Omega \cap \partial\Omega_i$. For given $\omega \in C(\overline{\Gamma}_{ij})$, we set

$$V_{h_{ij}}^{(w)} = \left\{v \in V_{h_{ij}}(\Omega_i) : v = 0 \text{ on } \partial\Omega \cap \partial\Omega_i; v = \pi_{h_{ij}}(w) \text{ on } \Gamma_{ij}\right\},$$

where $\pi_{h_{ij}}$ denotes a suitable interpolation operator on Γ_{ij}.

Now, we define the discrete Schwarz sequences and we suppose that the matrices of discretizations of problem (2.10) are *M-matrices* (see [4]).

Let $u_{h_i}^0 = r_{h_{ij}}u^0$, $u_{ih_{ij}}^{n+1} \in V_{h_{ij}}^{(u_{jh_{ij}}^n)}$ such that

$$\begin{cases} a_i(u_{ih_{ij}}^{n+1}, v - u_{ih_{ij}}^{n+1}) \geq (f_i, v - u_{ih_{ij}}^{n+1}) \quad \forall v \in V_{h_{ij}}^{(u_{jh_{ji}}^n)}, \\ u_{ih_{ij}}^{n+1} \leq r_{h_{ij}}Mu_{ih_{ij}}^n \quad \text{in } \tau^{h_{ij}}, \qquad v \leq r_{h_{ij}}Mu_{ih_{ij}}^n \quad \text{in } \tau^{h_{ij}}, \end{cases} \tag{2.13}$$

where $r_{h_{ij}}$ is a usual restriction operator in Ω_i and $u_{ih_{ij}}^0 = u_{h_{ij}}^0$ in Ω_i, $i = \overline{1,m}$, $j = \overline{1,m}$, $(i \neq j)$.

3 Error analysis

The aim of this section is to show the main result of this paper. To that end, we start by introducing two discrete auxiliary sequences and prove a fundamental lemma.

3.1 Auxiliary Schwarz sequences

For $\omega_{h_{ij}}^0 = u_{h_{ij}}^0$, we define the sequences $\omega_{ih_{ij}}^{n+1} \in V_{h_{ij}}^{(u_j^n)}$ such that

$$\begin{cases} a_i(\omega_{ih_{ij}}^{n+1}, v - \omega_{ih_{ij}}^{n+1}) \geq (f_i, v - \omega_{ih_{ij}}^{n+1}) \quad \forall v V_{h_{ij}}^{(u_j^n)}, \\ \omega_{ih_{ij}}^{n+1} \leq r_{h_{ij}}Mu_{ih_{ij}}^n \quad \text{in } \tau^{h_{ij}}, \qquad v \leq r_{h_{ij}}Mu_{ih_{ij}}^n \quad \text{in } \tau^{h_{ij}}. \end{cases} \tag{3.1}$$

Lemma 3.1 *For $i = \overline{1, m-1}$, $j = \overline{2, m}$ and $i < j$*
for $n \in \mathbb{N}$ is an even number such that $n = 2q$

$$\begin{aligned} \|u_i^{2q+1} - u_{ih}^{2q+1}\|_i &\le \sum_{p=0}^{q} \|u_i^{2p+1} - \omega_{ih}^{2p+1}\|_i + \sum_{p=0}^{q} \|u_j^{2p} - \omega_{jh}^{2p}\|_j, \\ \|u_j^{2q+1} - u_{jh}^{2q+1}\|_j &\le \sum_{p=0}^{q} \|u_j^{2p+1} - \omega_{jh}^{2p+1}\|_j + \sum_{p=0}^{q} \|u_i^{2p} - \omega_{ih}^{2p}\|_i \end{aligned} \tag{3.2}$$

for $n \in \mathbb{N}$ is an odd number such that $n = 2q + 1$

$$\begin{aligned} \|u_i^{2q+2} - u_{ih}^{2q+2}\|_i &\le \sum_{p=0}^{q+1} \|u_i^{2p} - \omega_{ih}^{2p}\|_i + \sum_{p=0}^{q} \|u_j^{2p+1} - \omega_{jh}^{2p+1}\|_j, \\ \|u_j^{2q+2} - u_{jh}^{2q+2}\|_j &\le \sum_{p=0}^{q+1} \|u_j^{2p} - \omega_{jh}^{2p}\|_j + \sum_{p=0}^{q} \|u_i^{2p+1} - \omega_{ih}^{2p+1}\|_i. \end{aligned} \tag{3.3}$$

Proof Let us reason by recurrence. For $n = 0$, $(q = 0)$: according to Proposition 2.2, we have

$$\begin{aligned} \|u_i^1 - u_{ih}^1\|_i &\le \|u_i^1 - \omega_{ih}^1\|_1 + \|\omega_{ih}^1 - u_{ih}^1\|_i \\ &\le \|u_i^1 - \omega_{ih}^1\|_i + |\pi_h u_j^0 - \pi_h u_{jh}^0|_{ij} \\ &\le \|u_i^1 - \omega_{ih}^1\|_i + |u_j^0 - u_{jh}^0|_{ij} \\ &\le \|u_i^1 - \omega_{ih}^1\|_i + \|u_j^0 - u_{jh}^0\|_j, \\ \|u_j^1 - u_{jh}^1\|_j &\le \|u_j^1 - \omega_{jh}^1\|_j + \|\omega_{jh}^1 - u_{jh}^1\|_j \\ &\le \|u_j^1 - \omega_{jh}^1\|_j + |\pi_h u_i^0 - \pi_h u_{ih}^0|_{ji} \\ &\le \|u_j^1 - \omega_{jh}^1\|_j + |u_i^0 - u_{ih}^0|_{ji} \\ &\le \|u_j^1 - \omega_{jh}^1\|_j + \|u_i^0 - u_{ih}^0\|_i, \end{aligned}$$

hence,

$$\begin{aligned} \|u_i^1 - u_{ih}^1\|_i &\le \sum_{p=0}^{0} \|u_i^{2p+1} - \omega_{ih}^{2p+1}\|_i + \sum_{p=0}^{0} \|u_j^{2p} - \omega_{jh}^{2p}\|_j, \\ \|u_j^1 - u_{jh}^1\|_j &\le \sum_{p=0}^{0} \|u_j^{2p+1} - \omega_{jh}^{2p+1}\|_j + \sum_{p=0}^{0} \|u_i^{2p} - \omega_{ih}^{2p}\|_i \end{aligned}$$

by recurrence. For $n = 1$, $(q = 0)$: using proposition 2.2, we have

$$\begin{aligned} \|u_i^2 - u_{ih}^2\|_i &\le \|u_i^2 - \omega_{ih}^2\|_i + \|\omega_{ih}^2 - u_{ih}^2\|_i \\ &\le \|u_i^2 - \omega_{ih}^2\|_i + |\pi_h u_j^1 - \pi_h u_{jh}^1|_{ij} \\ &\le \|u_i^2 - \omega_{ih}^2\|_i + |u_j^1 - u_{jh}^1|_{ij} \\ &\le \|u_i^2 - \omega_{ih}^2\|_i + \|u_j^1 - u_{jh}^1\|_j \end{aligned}$$

$$
\begin{aligned}
&\le \left\| u_i^2 - \omega_{ih}^2 \right\|_i + \left\| u_j^1 - \omega_{jh}^1 \right\|_j + \left\| u_i^0 - u_{ih}^0 \right\|_i, \\
\left\| u_j^2 - u_{jh}^2 \right\|_j &\le \left\| u_j^2 - \omega_{jh}^2 \right\|_j + \left\| \omega_{jh}^2 - u_{jh}^2 \right\|_j \\
&\le \left\| u_j^2 - \omega_{jh}^2 \right\|_j + \left| \pi_h u_i^1 - \pi_h u_{ih}^1 \right|_{ji} \\
&\le \left\| u_j^2 - \omega_{jh}^2 \right\|_{ij} + \left| u_i^1 - u_{ih}^1 \right|_{ji} \\
&\le \left\| u_j^2 - \omega_{jh}^2 \right\|_j + \left\| u_i^1 - u_{ih}^1 \right\|_i \\
&\le \left\| u_j^2 - \omega_{jh}^2 \right\|_j + \left\| u_i^1 - \omega_{ih}^1 \right\|_i + \left\| u_j^0 - \omega_{jh}^0 \right\|_j,
\end{aligned}
$$

hence,

$$
\begin{aligned}
\left\| u_i^1 - u_{ih}^1 \right\|_i &\le \sum_{p=0}^{1} \left\| u_i^{2p} - \omega_{ih}^{2p} \right\|_i + \sum_{p=0}^{0} \left\| u_j^{2p+1} - \omega_{jh}^{2p+1} \right\|_j, \\
\left\| u_j^1 - u_{jh}^1 \right\|_j &\le \sum_{p=0}^{1} \left\| u_j^{2p} - \omega_{jh}^{2p} \right\|_j + \sum_{p=0}^{0} \left\| u_i^{2p+1} - \omega_{ih}^{2p+1} \right\|_i.
\end{aligned}
$$

We assume that

$$
\begin{aligned}
\left\| u_i^{2q+1} - u_{ih}^{2q+1} \right\|_i &\le \sum_{p=0}^{q} \left\| u_i^{2p+1} - \omega_{ih}^{2p+1} \right\|_i + \sum_{p=0}^{q} \left\| u_j^{2p} - \omega_{jh}^{2p} \right\|_j, \\
\left\| u_j^{2q+1} - u_{jh}^{2q+1} \right\|_j &\le \sum_{p=0}^{q} \left\| u_j^{2p+1} - \omega_{jh}^{2p+1} \right\|_j + \sum_{p=0}^{q} \left\| u_i^{2p} - \omega_{ih}^{2p} \right\|_i,
\end{aligned}
$$

then, using Proposition 2.2 again, we obtain

$$
\begin{aligned}
\left\| u_i^{2q+2} - u_{ih}^{2q+2} \right\|_i &\le \left\| u_i^{2q+2} - \omega_{ih}^{2q+2} \right\|_i + \left\| \omega_{ih}^{2q+2} - u_{ih}^{2q+2} \right\|_i \\
&\le \left\| u_i^{2q+2} - \omega_{ih}^{2q+2} \right\|_i + \left| \pi_h u_j^{2q+1} - \pi_h u_{jh}^{2q+1} \right|_{ij} \\
&\le \left\| u_i^{2q+2} - \omega_{ih}^{2q+2} \right\|_i + \left| u_j^{2q+1} - u_{jh}^{2q+1} \right|_{ij} \\
&\le \left\| u_i^{2q+2} - \omega_{ih}^{2q+2} \right\|_i + \left\| u_j^{q+1} - u_{jh}^{2q+1} \right\|_j \\
&\le \left\| u_i^{2(q+1)} - u_{ih}^{2(q+1)} \right\|_i + \sum_{p=0}^{q} \left\| u_j^{2p+1} - \omega_{jh}^{2p+1} \right\|_j + \sum_{p=0}^{q} \left\| u_i^{2p} - \omega_{ih}^{2p} \right\|_i.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\left\| u_i^{2q+1} - u_{ih}^{2q+1} \right\|_i &\le \sum_{p=0}^{q+1} \left\| u_i^{2p} - \omega_{ih}^{2p} \right\|_i + \sum_{p=0}^{q} \left\| u_j^{2p+1} - \omega_{jh}^{2p+1} \right\|_j, \\
\left\| u_j^{2q+2} - u_{jh}^{2q+2} \right\|_j &\le \left\| u_j^{2q+2} - \omega_{jh}^{2q+2} \right\|_j + \left\| \omega_{jh}^{2q+2} - u_{jh}^{2q+2} \right\|_j \\
&\le \left\| u_j^{2q+2} - \omega_{jh}^{2q+2} \right\|_j + \left| \pi_h u_i^{2q+1} - \pi_h u_{ih}^{2q+1} \right|_{ji} \\
&\le \left\| u_j^{2q+2} - \omega_{jh}^{2q+2} \right\|_j + \left| u_i^{2q+1} - u_{ih}^{2q+1} \right|_{ji} \\
&\le \left\| u_j^{2q+2} - \omega_{jh}^{2q+2} \right\|_j + \left\| u_i^{2q+1} - u_{ih}^{2q+1} \right\|_i
\end{aligned}
$$

$$\leq \left\| u_j^{2(q+1)} - u_{jh}^{2(q+1)} \right\|_j + \sum_{p=0}^{q} \left\| u_i^{2p+1} - \omega_{ih}^{2p+1} \right\|_i + \sum_{p=0}^{q} \left\| u_j^{2p} - \omega_{jh}^{2p} \right\|_j.$$

Then,

$$\left\| u_j^{2q+2} - u_{jh}^{2q+2} \right\|_j \leq \sum_{p=0}^{q+1} \left\| u_j^{2p} - \omega_{jh}^{2p} \right\|_j + \sum_{p=0}^{q} \left\| u_i^{2p+1} - \omega_{ih}^{2p+1} \right\|_i.$$

Now, we suppose that

$$\left\| u_i^{2q} - u_{ih}^{2q} \right\|_i \leq \sum_{p=0}^{q} \left\| u_i^{2p} - \omega_{ih}^{2p} \right\|_i + \sum_{p=0}^{q-1} \left\| u_j^{2p+1} - \omega_{jh}^{2p+1} \right\|_j,$$

$$\left\| u_j^{2q} - u_{jh}^{2q} \right\|_j \leq \sum_{p=0}^{q} \left\| u_j^{2p} - \omega_{jh}^{2p} \right\|_j + \sum_{p=0}^{q-1} \left\| u_i^{2p+1} - \omega_{ih}^{2p+1} \right\|_i$$

and using Proposition 2.2, we obtain

$$\begin{aligned}
\left\| u_i^{2q+1} - u_{ih}^{2q+1} \right\|_i &\leq \left\| u_i^{2q+1} - \omega_{ih}^{2q+1} \right\|_i + \left\| \omega_{ih}^{2q+1} - u_{ih}^{2q+1} \right\|_i \\
&\leq \left\| u_i^{2q+1} - \omega_{ih}^{2q+1} \right\|_i + \left| \pi_h u_j^{2q} - \pi_h u_{jh}^{2q} \right|_{ij} \\
&\leq \left\| u_i^{2q+1} - \omega_{ih}^{2q+1} \right\|_i + \left| u_j^{2q} - u_{jh}^{2q} \right|_{ij} \\
&\leq \left\| u_i^{2q+1} - \omega_{ih}^{2q+1} \right\|_i + \left\| u_j^{q} - u_{jh}^{2q} \right\|_j \\
&\leq \left\| u_i^{2q+1} - u_{ih}^{2q+1} \right\|_i + \sum_{p=0}^{q} \left\| u_j^{2p} - \omega_{jh}^{2p} \right\|_j + \sum_{p=0}^{q-1} \left\| u_i^{2p+1} - \omega_{ih}^{2p+1} \right\|_i.
\end{aligned}$$

Then,

$$\left\| u_i^{2q+1} - u_{ih}^{2q+1} \right\|_i \leq \sum_{p=0}^{q} \left\| u_i^{2p+1} - \omega_{ih}^{2p+1} \right\|_i + \sum_{p=0}^{q} \left\| u_j^{2p} - \omega_{jh}^{2p} \right\|_j,$$

$$\begin{aligned}
\left\| u_j^{2q+1} - u_{jh}^{2q+1} \right\|_j &\leq \left\| u_j^{2q+1} - \omega_{jh}^{2q+1} \right\|_j + \left\| \omega_{jh}^{2q+1} - u_{jh}^{2q+1} \right\|_j \\
&\leq \left\| u_j^{2q+1} - \omega_{jh}^{2q+1} \right\|_j + \left| \pi_h u_i^{2q} - \pi_h u_{ih}^{2q} \right|_{ji} \\
&\leq \left\| u_j^{2q+1} - \omega_{jh}^{2q+1} \right\|_j + \left| u_i^{2q} - u_{ih}^{2q} \right|_{ji} \\
&\leq \left\| u_j^{2q+1} - \omega_{jh}^{2q+1} \right\|_j + \left\| u_i^{2q} - u_{ih}^{2q} \right\|_i \\
&\leq \left\| u_j^{2q+1} - u_{jh}^{2q+1} \right\|_j + \sum_{p=0}^{q} \left\| u_i^{2p} - \omega_{ih}^{2p} \right\|_i + \sum_{p=0}^{q-1} \left\| u_j^{2p+1} - \omega_{jh}^{2p+1} \right\|_j.
\end{aligned}$$

Then,

$$\left\| u_j^{2q+1} - u_{jh}^{2q+1} \right\|_j \leq \sum_{p=0}^{q} \left\| u_j^{2p+1} - \omega_{jh}^{2p+1} \right\|_j + \sum_{p=0}^{q} \left\| u_i^{2p} - \omega_{ih}^{2p} \right\|_i.$$

□

3.2 L^∞ error estimate

Theorem 3.2 *Let $h = \max(h_i, h_j)$, $i = \overline{1, m-1}; j = \overline{2, m}$ and $i < j$. Then, there exists a constant c independent of both h and n such that*

$$\left\|u_M - u_{Mh}^{n+1}\right\|_{L^\infty(\Omega_M)} \le ch^2|\log h|^3; \quad M = \overline{i, j}. \tag{3.4}$$

Proof For $M = i$, let $k = \max(k_i, k_j)$ using Theorem 2.4, Lemma 3.1, and Theorem 2.3 we obtain:

For $n \in \mathbb{N}$ is an even number such that $n = 2q$

$$\begin{aligned}\left\|u_i - u_{ih}^{n+1}\right\|_i &\le \left\|u_i - u_i^{n+1}\right\|_i + \left\|u_i^{n+1} - u_{ih}^{n+1}\right\|_i \\ &\le k^{2n}\left|u - u^0\right|_{ij} + \sum_{p=0}^{q}\left\|u_j^{2p+1} - \omega_{jh}^{2p+1}\right\|_j + \sum_{p=0}^{q}\left\|u_i^{2p} - \omega_{ih}^{2p}\right\|_i \\ &\le k^{2n}\left|u - u^0\right|_{ij} + (n+2)ch^2|\log h|^2.\end{aligned}$$

For $n \in \mathbb{N}$ is an odd number such that $n = 2q + 1$

$$\begin{aligned}\left\|u_i - u_{ih}^{n+1}\right\|_i &\le \left\|u_i - u_i^{n+1}\right\|_i + \left\|u_i^{n+1} - u_{ih}^{n+1}\right\|_i \\ &\le k^{2n}\left|u - u^0\right|_{ij} + \sum_{p=0}^{q+1}\left\|u_i^{2p} - \omega_{ih}^{2p}\right\|_i + \sum_{p=0}^{q}\left\|u_j^{2p+1} - \omega_{jh}^{2p+1}\right\|_j \\ &\le k^{2n}\left|u - u^0\right|_{ij} + (n+3)ch^2|\log h|^2\end{aligned}$$

We suppose that

$$k^{2n} \le h^2$$

and we obtain

$$\left\|u_i - u_{ih}^{n+1}\right\|_i \le ch^2|\log h|^3.$$

For $M = j$ this is similar. □

4 Conclusion

In this work, we have established a error estimate in an L^∞-norm of an overlapping Schwarz algorithm on nonmatching grids for a class of elliptic quasivariational inequalities related to the impulse-control problem. It is important to note that the error estimate obtained in this paper contains an extra power in $|\log h|$ than expected. We will see that this approach may also be extended to other important problems of QVIs.

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