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An intermixed method for solving the combination of mixed variational inequality problems and fixed-point problems

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Abstract

In this paper, we introduce an intermixed algorithm with viscosity technique for finding a common solution of the combination of mixed variational inequality problems and the fixed-point problem of a nonexpansive mapping in a real Hilbert space. Moreover, we propose the mathematical tools related to the combination of mixed variational inequality problems in the second section of this paper. Utilizing our mathematical tools, a strong convergence theorem is established for the proposed algorithm. Furthermore, we establish additional conclusions concerning the split-feasibility problem and the constrained convex-minimization problem utilizing our main result. Finally, we provide numerical experiments to illustrate the convergence behavior of our proposed algorithm.

Keywords: Mixed variational inequality problems; Intermixed algorithm; Strong convergence

1 Introduction

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is the set $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$. A mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

Note that the mapping $I - T$ is demiclosed at zero iff $x \in \mathrm{Fix}(T)$ whenever $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ (see, [1]). It is widely known that if $T : H \to H$ is nonexpansive, then $I - T$ is demiclosed at zero. A mapping $g : C \to C$ is said to be a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$\|g(x) - g(y)\| \le \alpha \|x - y\|, \quad \forall x, y \in C.$$

Let $A : C \to H$ be a mapping and $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous function on H. Now, we consider the mixed variational inequality problem: Find a point $x^* \in C$ such that

$$\langle y - x^*, Ax^* \rangle + f(y) - f(x^*) \geq 0, \tag{1.1}$$

for all $y \in C$. The set of solutions of problem (1.1) is denoted by $VI(C,A,f)$. The problem (1.1) was originally considered by Lescarret [2] and Browder [3] in relation to its various application in mathematical physics. General equilibrium and oligopolistic equilibrium problems, which can be stated as mixed variational inequality problems, were studied by Konnov and Volotskaya [4]. The fixed-point problems and resolvent equations are well known to be equivalent to mixed variational inequality problems. In 1997, Noor, [5] proposed and analyzed a new iterative method for solving mixed variational inequality problems using the resolvent equations technique as follows:

$$\begin{cases} z_n = x_n - \rho A x_n, \\ w_n = z_n - J_{\rho f} z_n + \rho A J_{\rho f} z_n, \\ x_{n+1} = x_n - \gamma w_n, \quad \forall n \geq 1, \end{cases} \tag{1.2}$$

where A is a monotone and Lipschitz continuous operator, $\rho > 0$ is a constant, $J_{\rho f} = (I + \rho \partial f)^{-1}$ is the resolvent operator and I is the identity operator. In 2008, Noor et al. [6] introduced an iterative algorithm to solve the mixed variational inequalities as follows:

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n J_{\rho f}[x_u - \rho A x_n], \quad \forall n \geq 1, \tag{1.3}$$

where $0 \leq \alpha_n \leq 1$ and A is strongly monotone and Lipschitz continuous. In recent years, several researchers have increasingly investigated the problem (1.1) in various directions, for example [5, 7–16] and the references therein.

Note that if C is a closed convex subset of H and $f(x) = \delta_C(x)$, for all $x \in C$, where δ_C is the indicator function of C defined by $\delta_C(x) = 0$ if $x \in C$, and $\delta_C(x) = \infty$ otherwise, then the mixed variational inequality problem (1.1) reduces to the following classical variational inequality problem: find a point $x^* \in C$ such that

$$\langle y - x^*, Ax^* \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of problem (1.4) is denoted by $VI(C,A)$. The variational inequality problem was introduced and studied by Stampacchia in 1966 [17]. The solution of the variational inequality problem is well known to be equivalent to the following fixed-point equation for finding a point $x^* \in C$ such that

$$x^* = P_C(I - \gamma A)x^*,$$

where $\gamma > 0$ is an arbitrary constant and P_C is the metric projection from H onto C (see [18]). This problem is useful in economics, engineering, and mathematics. Many nonlinear analysis problems, such as optimization, optimal control problems, saddle-point problems, and mathematical programming, are included as special cases; see, for example, [19–22]. Furthermore, there have been various methods invented for solving the problem (1.4) and fixed-point problems, for example [23–33] and the references therein.

The intermixed algorithm introduced by Yao et al. [34] is currently one of the most effective methods for solving the fixed-point problem of a nonlinear mapping. This algorithm has the following features: the definition of the sequence $\{x_n\}$ is involved in the sequence $\{y_n\}$ and the definition of the sequence $\{y_n\}$ is also involved in the sequence $\{x_n\}$. They studied the intermixed algorithm for two strict pseudocontractions S and T as follows: For arbitrarily given $x_1 \in C$, $y_1 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = (1-\beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kTx_n], & \forall n \geq 1, \\ y_{n+1} = (1-\beta_n)y_n + \beta_n P_C[\alpha_n g(x_n) + (1-k-\alpha_n)y_n + kSy_n], & \forall n \geq 1, \end{cases} \tag{1.5}$$

where $S, T : C \to C$ are λ-strictly pseudocontraction mappings, $f : C \to H$ is a ρ_1-contraction and $g : C \to H$ is a ρ_2-contraction, $k \in (0, 1-\lambda)$ is a constant and $\{\alpha_n\}$, $\{\beta_n\}$ are two real-number sequences in $(0,1)$. They also proved that the proposed algorithms independently converge strongly to the fixed points of two strict pseudocontractions.

In 2012, Kangtunyakarn [35] modified the set of variational inequality problems as follows:

$$\begin{aligned} &VI\bigl(C, aA + (1-a)B\bigr) = \bigl\{x \in C : \bigl\langle y - x, \bigl(aA + (1-a)B\bigr)x\bigr\rangle \geq 0, \forall y \in C\bigr\}, \\ &\quad \forall a \in (0,1), \end{aligned} \tag{1.6}$$

where A and B are the mappings of C into H. If $A = B$, then the problem (1.6) reduces to the classical variational inequality problem. Moreover, he also gave a new iterative method for solving the proposed problem in Hilbert spaces.

In this article, motivated and inspired by Kangtunyakarn [35], we introduce a problem that is modified by a mixed variational inequality problem as follows: *The combination of mixed variational inequality problems* is to find $x^* \in C$ such that

$$\bigl\langle y - x^*, \bigl(aA + (1-a)B\bigr)x^*\bigr\rangle + f(y) - f\bigl(x^*\bigr) \geq 0, \tag{1.7}$$

for all $y \in C$ and $a \in (0,1)$, where $A, B : C \to H$ are mappings. The set of all solutions to this problem is denoted by $VI(C, aA + (1-a)B, f)$. In particular, if $A = B$, then the problem (1.7) reduces to the mixed variational inequality problem (1.1).

Question. Can we design an intermixed algorithm for solving the combination of mixed variational inequality problems (1.7) above?

In this paper, we give a positive answer to this question. Motivated and inspired by the works in the literature, and by the ongoing research in these directions, we introduce a new intermixed algorithm with viscosity technique for finding a solution of the combination of mixed variational inequality problems and the fixed-point problem of a nonexpansive mapping in a real Hilbert space. Moreover, we propose the mathematical tools related to the combination of mixed variational inequality problems (1.7) in the second section of this paper. Utilizing our mathematical tools, a strong convergence theorem is established for the proposed algorithm. Furthermore, we establish additional conclusions concerning the split-feasibility problem and the constrained convex-minimization problem utilizing our main result. Finally, we provide numerical experiments to illustrate the convergence behavior of our proposed algorithm.

This paper is organized as follows. In Sect. 2, we first recall some basic definitions and lemmas. In Sect. 3, we prove and analyze the strong convergence of the proposed algorithm. In Sect. 4, we also consider the relaxation version of the proposed method. In Sect. 5, some numerical experiments are provided.

2 Preliminary

Let C be a nonempty, closed, and convex subset of a Hilbert space H. The notation I stands for the identity operator on a Hilbert space. Let $\{x_n\}$ be a sequence in H. Weak and strong convergence of $\{x_n\}$ to $x \in H$ are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

Definition 2.1 A mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0 \quad \text{for all } x, y \in C;$$

(ii) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\| \quad \text{for all } x, y \in C;$$

(iii) α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2 \quad \text{for all } x, y \in C;$$

(iv) firmly nonexpansive if

$$\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay \rangle \quad \text{for all } x, y \in C.$$

Throughout this paper, the domain of any function $f : H \to \mathbb{R} \cup \{+\infty\}$, denoted by $\operatorname{dom} f$, is defined as $\operatorname{dom} f := \{x \in H : f(x) < +\infty\}$. The domain of continuity of f is $\operatorname{cont} f = \{x \in H : f(x) \in \mathbb{R}$ and f is continuous at $x\}$.

Definition 2.2 ([36]) Let $f : H \to \mathbb{R}$ be a function. Then,

(i) f is proper if $\{x \in H : f(x) < \infty\} \neq \emptyset$;
(ii) f is lower semicontinuous if $\{x \in H : f(x) \leq a\}$ is closed for each $a \in \mathbb{R}$;
(iii) f is convex if $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$ for every $x, y \in H$ and $t \in [0, 1]$;
(iv) f is Gâteaux differentiable at $x \in H$ if there is $\nabla f(x) \in H$ such that

$$\lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} = \langle y, \nabla f(x) \rangle$$

for each $y \in H$;
(v) f is Fréchet differentiable at $x \in H$ if there is $\nabla f(x)$ such that

$$\lim_{y \to 0} \frac{f(x + y) - f(x) - \langle \nabla f(x), y \rangle}{\|y\|} = 0.$$

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous function on H. The subset

$$\partial f(x) = \{z \in H : \langle z, y - x \rangle + f(x) \leq f(y), \forall y \in H\}$$

is called a *subdifferential* of f at $x \in H$. The function f is said to be *subdifferentiable* at x if $\partial f(x) \neq \emptyset$. The element of $\partial f(x)$ is called the *subgradient* of f at x. It is well known that the subdifferential ∂f is a maximal monotone operator.

Proposition 2.1 ([37] Proposition 17.31) *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper and convex function, and let $x \in \operatorname{dom} f$. Then, the following hold*:

(i) *Suppose that f is Gâteaux differentiable at x. Then $\partial f(x) = \{\nabla f(x)\}$.*

(ii) *Suppose that $x \in \operatorname{cont} f$ and that $\partial f(x)$ consists of a single element u. Then, f is Gâteaux differentiable at x and $u = \nabla f(x)$.*

Definition 2.3 ([38]) For any maximal operator A, the resolvent operator associated with A, for any $\gamma > 0$, is defined as

$$J_{\gamma A}(x) = (I + \gamma A)^{-1}(x), \quad \forall x \in H,$$

where I is the identity operator.

It is well known that an operator A is maximal monotone if and only if its resolvent operator $J_{\gamma A}$ is defined everywhere. It is single valued and nonexpansive. If f is a proper, convex, and lower-semicontinuous function, then its subdifferential ∂f is a maximal monotone operator. In this case, we can define the resolvent operator

$$J_{\gamma f}(x) = (I + \gamma \partial f)^{-1}(x), \quad \forall x \in H,$$

associated with the subdifferential ∂f and $\gamma > 0$ is constant.

Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.2 ([39]) *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \quad \Leftrightarrow \quad \langle u - z, v - u \rangle \geq 0, \quad \forall v \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C.

Lemma 2.3 ([40]) *For given $x \in H$ let $P_C : H \to C$ be a metric projection. Then,*

(a) *$z = P_C x$ if and only if $\langle x - z, y - z \rangle \leq 0, \forall y \in C$;*

(b) *$z = P_C x$ if and only if $\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \forall y \in C$;*

(c) *$\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \forall x, y \in H$.*

Lemma 2.4 ([41]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \delta_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5 *Let C be a nonempty closed convex subset of H and let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous function and let $A, B : C \to H$ be α- and β-inverse strongly monotone operators with $\varepsilon = \min\{\alpha, \beta\}$ and $VI(C,A,f) \cap VI(C,B,f) \neq \emptyset$. Then,*

$$VI(C,A,f) \cap VI(C,B,f) = VI\big(C, aA + (1-a)B, f\big) \tag{2.1}$$

for all $a \in (0,1)$.

Proof Clearly,

$$VI(C,A,f) \cap VI(C,B,f) \subseteq VI\big(C, aA + (1-a)B, f\big). \tag{2.2}$$

Let $x_0 \in VI(C, aA + (1-a)B, f)$ and $x^* \in VI(C,A,f) \cap VI(C,B,f)$. Hence, we have

$$\langle y - x_0, \big(aA + (1-a)B\big)x_0\rangle + f(y) - f(x_0) \ge 0, \quad \forall y \in C. \tag{2.3}$$

It follows from $x^* \in VI(C, aA + (1-a)B, f)$ that

$$\langle y - x^*, \big(aA + (1-a)B\big)x^*\rangle + f(y) - f\big(x^*\big) \ge 0, \quad \forall y \in C. \tag{2.4}$$

From (2.3), (2.4), and the definition of x^*, x_0, we have

$$\langle x^* - x_0, \big(aA + (1-a)B\big)x_0\rangle + f\big(x^*\big) - f(x_0) \ge 0 \tag{2.5}$$

and

$$\langle x_0 - x^*, \big(aA + (1-a)B\big)x^*\rangle + f(x_0) - f\big(x^*\big) \ge 0, \quad \forall y \in C. \tag{2.6}$$

By combining (2.5), (2.6), and the definition of A, B, we obtain

$$\begin{aligned} 0 &\ge \langle x_0 - x^*, a\big(Ax_0 - Ax^*\big) + (1-a)\big(Bx_0 - Bx^*\big)\rangle \\ &= a\langle x_0 - x^*, Ax_0 - Ax^*\rangle + (1-a)\langle x_0 - x^*, Bx_0 - Bx^*\rangle \\ &\ge a\alpha\|Ax_0 - Ax^*\|^2 + (1-a)\beta\|Bx_0 - Bx^*\|^2, \end{aligned}$$

which implies that

$$Ax_0 = Ax^*, \qquad Bx_0 = Bx^*.$$

Let $y \in C$. From $x^* \in VI(C,A,f)$ and $Ax_0 = Ax^*$, we have

$$\langle y - x_0, Ax_0\rangle + f(y) - f(x_0) = \langle y - x^*, Ax^*\rangle + \langle x^* - x_0, Ax_0\rangle$$

$$\begin{aligned}
&+ f(y) - f\left(x^*\right) + f\left(x^*\right) - f(x_0) \\
&\geq \left\langle x^* - x_0, Ax_0 \right\rangle + f\left(x^*\right) - f(x_0).
\end{aligned} \tag{2.7}$$

From $Bx_0 = Bx^*$, $x_0 \in VI(C, aA + (1-a)B, f)$, $x^* \in VI(C, B, f)$, we obtain

$$\begin{aligned}
\left\langle x^* - x_0, aAx_0 \right\rangle + af\left(x^*\right) - af(x_0) &= \left\langle x^* - x_0, aAx_0 + (1-a)Bx_0 \right\rangle \\
&\quad - \left\langle x^* - x_0, (1-a)Bx_0 \right\rangle + af\left(x^*\right) - af(x_0) \\
&= \left\langle x^* - x_0, aAx_0 + (1-a)Bx_0 \right\rangle + f\left(x^*\right) - f(x_0) \\
&\quad - f\left(x^*\right) + f(x_0) - \left\langle x^* - x_0, (1-a)Bx_0 \right\rangle \\
&\quad + af\left(x^*\right) - af(x_0) \\
&\geq \left\langle x_0 - x^*, (1-a)Bx^* \right\rangle + (1-a)f(x_0) \\
&\quad - (1-a)f\left(x^*\right) \\
&= (1-a)\left(\left\langle x_0 - x^*, Bx^* \right\rangle + f(x_0) - f\left(x^*\right)\right) \\
&\geq 0.
\end{aligned}$$

Since $a \in (0,1)$, we have

$$\left\langle x^* - x_0, Ax_0 \right\rangle + f\left(x^*\right) - f(x_0) \geq 0. \tag{2.8}$$

From (2.7) and (2.8), we have

$$\langle y - x_0, Ax_0 \rangle + f(y) - f(x_0) \geq 0. \tag{2.9}$$

This implies that

$$x_0 \in VI(C, A, f). \tag{2.10}$$

Using the same method as (2.10), we have

$$x_0 \in VI(C, B, f). \tag{2.11}$$

From (2.10) and (2.11), we obtain $x_0 \in VI(C, A, f) \cap VI(C, B, f)$. Hence, we can conclude that

$$VI\big(C, aA + (1-a)B, f\big) \subseteq VI(C, A, f) \cap VI(C, B, f). \tag{2.12}$$

From (2.2) and (2.12), we obtain

$$VI(C, A, f) \cap VI(C, B, f) = VI\big(C, aA + (1-a)B, f\big). \tag{2.13}$$

□

Lemma 2.6 *Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous function on H. Let $A : C \to H$ be a mapping. Then,* $\mathrm{Fix}(J_{\gamma f}(I - \gamma A)) = VI(C, A, f)$, *where $J_{\gamma f} : H \to H$ defined as $J_{\gamma f} = (I + \gamma \partial f)^{-1}$ is the resolvent operator, I is the identity operator and $\gamma > 0$ is a constant.*

Proof Let $z \in H$, then

$$\begin{aligned} z \in \mathrm{Fix}\big(J_{\gamma f}(I-\gamma A)\big) \quad &\Leftrightarrow \quad z = J_{\gamma f}(I-\gamma A)z \\ &\Leftrightarrow \quad z = (I+\gamma\partial f)^{-1}(I-\gamma A)z \\ &\Leftrightarrow \quad (I-\gamma A)z \in (I+\gamma\partial f)z \\ &\Leftrightarrow \quad -Az \in \partial f(z) \\ &\Leftrightarrow \quad \langle -Az, y-z\rangle \leq f(y)-f(z), \quad \forall y \in C \\ &\Leftrightarrow \quad z \in VI(C,A,f). \end{aligned} \tag{2.14}$$

Next, we will show that $J_{\gamma f}$ is a firmly nonexpansive mapping.

Let $p = J_{\gamma f}(x) = (I+\gamma\partial f)^{-1}x$ and $q = J_{\gamma f}(y) = (I+\gamma\partial f)^{-1}y$. It follows that $x \in (I+\gamma\partial f)p$ and $y \in (I+\gamma\partial f)q$.

From the definition of $\partial f(p)$ and $\partial f(q)$, we have

$$\frac{x-p}{\gamma} \in \partial f(p) \quad \text{and} \quad \frac{y-q}{\gamma} \in \partial f(q).$$

This implies that

$$\left\langle \frac{x-p}{\gamma}, c-p \right\rangle \leq f(c)-f(p) \quad \text{and} \quad \left\langle \frac{y-q}{\gamma}, c-q \right\rangle \leq f(c)-f(q)$$

for all $c \in H$. Then,

$$\left\langle \frac{x-p}{\gamma}, q-p \right\rangle \leq f(q)-f(p) \tag{2.15}$$

and

$$\left\langle \frac{y-q}{\gamma}, p-q \right\rangle \leq f(p)-f(q). \tag{2.16}$$

By combining (2.15) and (2.16), we obtain

$$\left\langle \frac{x-p}{\gamma} - \frac{y-q}{\gamma}, q-p \right\rangle \leq 0, \tag{2.17}$$

which implies that

$$\langle x-y+q-p, q-p\rangle \leq 0. \tag{2.18}$$

Then, we have

$$\|q-p\|^2 \leq \langle y-x, q-p\rangle.$$

From the definition of p,q, we have

$$\big\|J_{\gamma f}(y)-J_{\gamma f}(x)\big\|^2 \leq \big\langle J_{\gamma f}(y)-J_{\gamma f}(x), y-x\big\rangle.$$

Therefore, $J_{\gamma f}$ is a firmly nonexpansive mapping. □

Remark 2.7 From Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned} VI(C,A,f)\cap VI(C,B,f) &= VI\bigl(C,aA+(1-a)B,f\bigr) \\ &= \operatorname{Fix}(J_{\gamma f}\bigl(I-\gamma\bigl(aA+(1-a)B\bigr)\bigr) \end{aligned} \tag{2.19}$$

for all $\gamma > 0$ and $a \in (0,1)$.

3 Main results

In this section, we introduce a new intermixed algorithm with viscosity technique using Lemmas 2.5 and 2.6 as an important tool for finding a solution of the combination of mixed variational inequality problems and the fixed-point problem of a nonexpansive mapping in a real Hilbert space and establish its strong convergence under some mild conditions.

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of H. For every $i = 1,2$, let $f_i : H \to \mathbb{R}\cup\{+\infty\}$ be a proper, convex, and lower semicontinuous function, let $A_i, B_i : C \to H$ be δ_i^A- and δ_i^B-inverse strongly monotone operators, respectively, with $\delta_i = \min\{\delta_i^A, \delta_i^B\}$ and let $T_i : C \to C$ be nonexpansive mappings. Assume that $\Omega_i = \operatorname{Fix}(T_i) \cap VI(C,A_i,f_i) \cap VI(C,B_i,f_i) \neq \emptyset$, for all $i = 1,2$. Let $g_1, g_2 : H \to H$ be σ_1- and σ_2-contraction mappings with $\sigma_1, \sigma_2 \in (0,1)$ and $\sigma = \max\{\sigma_1,\sigma_2\}$. Let the sequences $\{x_n\}$, $\{y_n\}$ be generated by $x_1, y_1 \in C$ and*

$$\begin{cases} w_n = b_2 y_n + (1-b_2)T_2 y_n, \\ y_{n+1} = (1-\beta_n)w_n + \beta_n P_C(\alpha_n g_2(x_n) \\ \qquad\quad + (1-\alpha_n)J_{\gamma f}^2(y_n - \gamma_2(a_2A_2 + (1-a_2)B_2)y_n)), \\ z_n = b_1 x_n + (1-b_1)T_1 x_n, \\ x_{n+1} = (1-\beta_n)z_n + \beta_n P_C(\alpha_n g_1(y_n) \\ \qquad\quad + (1-\alpha_n)J_{\gamma f}^1(x_n - \gamma_1(a_1A_1 + (1-a_1)B_1)x_n)), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$, $\gamma_i \in (0, 2\delta_i)$, $a_i, b_i \in (0,1)$, and $J_{\gamma f}^i : H \to H$ defined as $J_{\gamma f}^i = (I + \gamma_i \nabla f_i)^{-1}$ is the resolvent operator for all $i = 1,2$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \bar{l} \leq \beta_n \leq l$ *for all* $n \in \mathbb{N}$ *and for some* $\bar{l}, l > 0$;

(iii) $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* = P_{\Omega_1}g_1(y^*)$ and $y^* = P_{\Omega_2}g_2(x^*)$, respectively.

Proof First, we show that $\{x_n\}$ and $\{y_n\}$ are bounded.

We claim that $J_{\gamma f}^i(I - \gamma_i(a_iA_i + (1-a_i)B_i))$ is nonexpansive for all $i = 1,2$. To show this let $x, y \in C$, then

$$\begin{aligned} &\bigl\|J_{\gamma f}^i\bigl(I-\gamma_i\bigl(a_iA_i+(1-a_i)B_i\bigr)\bigr)x - J_{\gamma f}^i\bigl(I-\gamma_i\bigl(a_iA_i+(1-a_i)B_i\bigr)\bigr)y\bigr\|^2 \\ &\quad\leq \bigl\|\bigl(I-\gamma_i\bigl(a_iA_i+(1-a_i)B_i\bigr)\bigr)x - \bigl(I-\gamma_i\bigl(a_iA_i+(1-a_i)B_i\bigr)\bigr)y\bigr\|^2 \\ &\quad= \bigl\|x-y-\gamma_i\bigl(\bigl(a_iA_i+(1-a_i)B_i\bigr)x - \bigl(a_iA_i+(1-a_i)B_i\bigr)y\bigr)\bigr\|^2 \\ &\quad= \bigl\|x-y-\gamma_i\bigl(a_i(A_ix-A_iy)+(1-a_i)(B_ix-B_iy)\bigr)\bigr\|^2 \end{aligned}$$

$$\begin{aligned}
&= \|x-y\|^2 - 2\gamma_i\big\langle a_i(A_ix-A_iy)+(1-a_i)(B_ix-B_iy), x-y\big\rangle \\
&\quad + \gamma_i^2\big\|a_i(A_ix-A_iy)+(1-a_i)(B_ix-B_iy)\big\|^2 \\
&\leq \|x-y\|^2 - 2\gamma_i a_i\langle A_ix-A_iy, x-y\rangle - 2\gamma_i(1-a_i)\langle B_ix-B_iy, x-y\rangle \\
&\quad + \gamma_i^2 a_i\|A_ix-A_iy\|^2 + (1-a_i)\gamma_i^2\|B_ix-B_iy\|^2 \\
&\leq \|x-y\|^2 - 2\gamma_i a_i\delta_i^A\|A_ix-A_iy\|^2 - 2\gamma_i(1-a_i)\delta_i^B\|B_ix-B_iy\|^2 \\
&\quad + \gamma_i^2 a_i\|A_ix-A_iy\|^2 + (1-a_i)\gamma_i^2\|B_ix-B_iy\|^2 \\
&\leq \|x-y\|^2 - 2\gamma_i a_i\delta_i\|A_ix-A_iy\|^2 - 2\gamma_i(1-a_i)\delta_i\|B_ix-B_iy\|^2 \\
&\quad + \gamma_i^2 a_i\|A_ix-A_iy\|^2 + (1-a_i)\gamma_i^2\|B_ix-B_iy\|^2 \\
&\leq \|x-y\|^2 + a_i\gamma_i(\gamma_i-2\delta_i)\|A_ix-A_iy\|^2 + (1-a_i)\gamma_i(\gamma_i-2\delta_i)\|B_ix-B_iy\|^2 \\
&\leq \|x-y\|^2.
\end{aligned} \tag{3.2}$$

Assume that $x^* \in \Omega_1$ and $y^* \in \Omega_2$.

From the definition of z_n and the nonexpansiveness of T_1, we have

$$\begin{aligned}
\|z_n-x^*\| &= \|b_1x_n+(1-b_1)T_1x_n-x^*\| \\
&\leq b_1\|x_n-x^*\| + (1-b_1)\|T_1x_n-x^*\| \\
&\leq b_1\|x_n-x^*\| + (1-b_1)\|x_n-x^*\| \\
&= \|x_n-x^*\|.
\end{aligned} \tag{3.3}$$

Similarly, we have $\|w_n-x^*\| \leq \|y_n-x^*\|$.

Putting $K_i = J_{\gamma f}^i(I-\gamma_i(a_iA_i+(1-a_i)B_i)))$ for all $i=1,2$, from the definition of x_n, the nonexpansiveness of K_i for all $i=1,2$, and (3.3), we have

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \big\|(1-\beta_n)z_n+\beta_nP_C\big(\alpha_ng_1(y_n)+(1-\alpha_n)K_1x_n\big)-x^*\big\| \\
&\leq (1-\beta_n)\|z_n-x^*\| + \beta_n\|\alpha_ng_1(y_n)+(1-\alpha_n)K_1x_n-x^*\| \\
&\leq (1-\beta_n)\|x_n-x^*\| + \beta_n\big(\alpha_n\|g_1(y_n)-x^*\| + (1-\alpha_n)\|K_1x_n-x^*\|\big) \\
&\leq (1-\beta_n)\|x_n-x^*\| + \beta_n\big(\alpha_n\|g_1(y_n)-x^*\| + (1-\alpha_n)\|x_n-x^*\|\big) \\
&= (1-\alpha_n\beta_n)\|x_n-x^*\| + \alpha_n\beta_n\|g_1(y_n)-x^*\| \\
&\leq (1-\alpha_n\beta_n)\|x_n-x^*\| + \alpha_n\beta_n\big(\|g_1(y_n)-g_1(y^*)\| + \|g_1(y^*)-x^*\|\big) \\
&\leq (1-\alpha_n\beta_n)\|x_n-x^*\| + \alpha_n\beta_n\sigma_1\|y_n-y^*\| + \alpha_n\beta_n\|g_1(y^*)-x^*\| \\
&\leq (1-\alpha_n\beta_n)\|x_n-x^*\| + \alpha_n\beta_n\sigma\|y_n-y^*\| + \alpha_n\beta_n\|g_1(y^*)-x^*\|.
\end{aligned} \tag{3.4}$$

Similarly, we obtain

$$\|y_{n+1}-y^*\| \leq (1-\alpha_n\beta_n)\|y_n-y^*\| + \alpha_n\beta_n\sigma\|x_n-x^*\| + \alpha_n\beta_n\|g_2(x^*)-y^*\|. \tag{3.5}$$

Combining (3.4) and (3.5), we have

$$\|x_{n+1}-x^*\| + \|y_{n+1}-y^*\| \leq (1-\alpha_n\beta_n)\|x_n-x^*\| + \alpha_n\beta_n\sigma\|y_n-y^*\|$$

$$
\begin{aligned}
&+ \alpha_n\beta_n \left\| g_1\left(y^*\right) - x^* \right\| \\
&+ (1-\alpha_n\beta_n)\left\| y_n - y^* \right\| + \alpha_n\beta_n\sigma \left\| x_n - x^* \right\| \\
&+ \alpha_n\beta_n \left\| g_2\left(x^*\right) - y^* \right\| \\
={}& (1-\alpha_n\beta_n)\left(\left\| x_n - x^* \right\| + \left\| y_n - y^* \right\|\right) \\
&+ \alpha_n\beta_n\sigma \left(\left\| x_n - x^* \right\| + \left\| y_n - y^* \right\|\right) \\
&+ \alpha_n\beta_n \left(\left\| g_1\left(y^*\right) - x^* \right\| + \left\| g_2\left(x^*\right) - y^* \right\|\right) \\
={}& \left(1-\alpha_n\beta_n(1-\sigma)\right)\left(\left\| x_n - x^* \right\| + \left\| y_n - y^* \right\|\right) \\
&+ \alpha_n\beta_n \left(\left\| g_1\left(y^*\right) - x^* \right\| + \left\| g_2\left(x^*\right) - y^* \right\|\right).
\end{aligned}
$$

We can deduce from induction that

$$
\left\| x_n - x^* \right\| + \left\| y_n - y^* \right\| \le \max\left\{ \left\| x_1 - x^* \right\| + \left\| y_1 - y^* \right\|, \frac{\|g_1(y^*) - x^*\| + \|g_2(x^*) - y^*\|}{1-\sigma} \right\},
$$

for every $n \in \mathbb{N}$. This implies that $\{x_n\}$ and $\{y_n\}$ are bounded. This implies that $\{z_n\}, \{w_n\}$ are also bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$ and $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.

Setting $Q_n = P_C(\alpha_n g_1(y_n) + (1-\alpha_n)K_1 x_n)$ and $Q_n^* = P_C(\alpha_n g_2(x_n) + (1-\alpha_n)K_2 y_n)$. By the nonexpansiveness of K_i for $i = 1, 2$, we have

$$
\begin{aligned}
\|Q_n - Q_{n-1}\| ={}& \left\| P_C\left(\alpha_n g_1(y_n) + (1-\alpha_n)K_1 x_n\right) - P_C\left(\alpha_{n-1} g_1(y_{n-1}) + (1-\alpha_{n-1})K_1 x_{n-1}\right) \right\| \\
\le{}& \left\| \left(\alpha_n g_1(y_n) + (1-\alpha_n)K_1 x_n\right) - \left(\alpha_{n-1} g_1(y_{n-1}) + (1-\alpha_n)K_1 x_{n-1}\right) \right\| \\
={}& \left\| \alpha_n g_1(y_n) - \alpha_n g_1(y_{n-1}) + \alpha_n g_1(y_{n-1}) + (1-\alpha_n)K_1 x_n - (1-\alpha_n)K_1 x_{n-1} \right. \\
&\left. + (1-\alpha_n)K_1 x_{n-1} - \alpha_{n-1} g_1(y_{n-1}) - (1-\alpha_{n-1})K_1 x_{n-1} \right\| \\
={}& \left\| \alpha_n\left(g_1(y_n) - g_1(y_{n-1})\right) + (\alpha_n - \alpha_{n-1}) g_1(y_{n-1}) + (1-\alpha_n)(K_1 x_n - K_1 x_{n-1}) \right. \\
&\left. + (\alpha_{n-1} - \alpha_n)K_1 x_{n-1} \right\| \\
\le{}& \alpha_n \left\| g_1(y_n) - g_1(y_{n-1}) \right\| + |\alpha_n - \alpha_{n-1}| \left\| g_1(y_{n-1}) \right\| \\
&+ (1-\alpha_n)\|K_1 x_n - K_1 x_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}| \|K_1 x_{n-1}\| \\
\le{}& \alpha_n \sigma_1 \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \left\| g_1(y_{n-1}) \right\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}| \|K_1 x_{n-1}\| \\
\le{}& \alpha_n \sigma \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \left\| g_1(y_{n-1}) \right\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}| \|K_1 x_{n-1}\|.
\end{aligned} \tag{3.6}
$$

From the definition of z_n and the nonexpansiveness of T_1, we have

$$
\begin{aligned}
\|z_n - z_{n-1}\| ={}& \left\| b_1 x_n + (1-b_1)T_1 x_n - b_1 x_{n-1} - (1-b_1)T_1 x_{n-1} \right\| \\
\le{}& \left\| b_1(x_n - x_{n-1}) + (1-b_1)(T_1 x_n - T_1 x_{n-1}) \right\|
\end{aligned}
$$

$$\begin{aligned}
&\leq b_1\|x_n - x_{n-1}\| + (1-b_1)\|T_1x_n - T_1x_{n-1}\| \\
&\leq b_1\|x_n - x_{n-1}\| + (1-b_1)\|x_n - x_{n-1}\| \\
&= \|x_n - x_{n-1}\|.
\end{aligned} \tag{3.7}$$

Similarly, we obtain

$$\|w_n - w_{n-1}\| \leq \|x_n - x_{n-1}\|. \tag{3.8}$$

From the definition of x_n, (3.6), and (3.7), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \big\|(1-\beta_n)z_n + \beta_n Q_n - \big((1-\beta_{n-1})z_{n-1} + \beta_{n-1}Q_{n-1}\big)\big\| \\
&\leq (1-\beta_n)\|z_n - z_{n-1}\| + |\beta_{n-1} - \beta_n|\|z_{n-1}\| \\
&\quad + \beta_n\|Q_n - Q_{n-1}\| + |\beta_n - \beta_{n-1}|\|Q_{n-1}\| \\
&\leq (1-\beta_n)\|x_n - x_{n-1}\| + |\beta_{n-1} - \beta_n|\|z_{n-1}\| \\
&\quad + \beta_n\big(\alpha_n\sigma\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big\|g_1(y_{n-1})\big\| \\
&\quad + (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|K_1x_{n-1}\|\big) \\
&\quad + |\beta_n - \beta_{n-1}|\|Q_{n-1}\| \\
&= (1-\beta_n)\|x_n - x_{n-1}\| + |\beta_{n-1} - \beta_n|\|z_{n-1}\| \\
&\quad + \beta_n\alpha_n\sigma\|y_n - y_{n-1}\| + \beta_n|\alpha_n - \alpha_{n-1}|\big\|g_1(y_{n-1})\big\| \\
&\quad + \beta_n(1-\alpha_n)\|x_n - x_{n-1}\| + \beta_n|\alpha_n - \alpha_{n-1}|\|K_1x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\|Q_{n-1}\| \\
&\leq (1-\alpha_n\beta_n)\|x_n - x_{n-1}\| + |\beta_{n-1} - \beta_n|\big(\|z_{n-1}\| + \|Q_{n-1}\|\big) \\
&\quad + \alpha_n\beta_n\sigma\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big(\big\|g_1(y_{n-1})\big\| + \|K_1x_{n-1}\|\big).
\end{aligned} \tag{3.9}$$

Using the same method as derived in (3.9), we have

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq (1-\alpha_n\beta_n)\|y_n - y_{n-1}\| + |\beta_{n-1} - \beta_n|\big(\|w_{n-1}\| + \big\|Q^*_{n-1}\big\|\big) \\
&\quad + \alpha_n\beta_n\sigma\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big(\big\|g_2(x_{n-1})\big\| + \|K_2y_{n-1}\|\big).
\end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| &\leq \big(1 - (1-\sigma)\beta_n\alpha_n\big)(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\| \\
&\quad + |\beta_{n-1} - \beta_n|\big(\|z_{n-1}\| + \|w_{n-1}\| + \|Q_n\| + \big\|Q^*_n\big\|\big) \\
&\quad + |\alpha_n - \alpha_{n-1}|\big(\big\|g_1(y_{n-1})\big\| + \|K_1x_{n-1}\| \\
&\quad + \big\|g_2(x_{n-1})\big\| + \|K_2x_{n-1}\|\big).
\end{aligned}$$

Applying Lemma 2.4 and the condition (iii), we can conclude that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|y_{n+1} - y_n\| = 0. \tag{3.11}$$

Next, we show that $\|x_n - U_n\| \to 0$ as $n \to \infty$, where $U_n = \alpha_n g_1(y_n) + (1-\alpha_n)K_1 x_n$, $\|y_n - V_n\| \to 0$, where $V_n = \alpha_n g_2(x_n) + (1-\alpha_n)K_2 y_n$ as $n \to \infty$, $\|x_n - T_1 x_n\| \to 0$ as $n \to \infty$ and $\|y_n - T_2 y_n\| \to 0$ as $n \to \infty$.

Let $x^* \in \Omega_1$ and $y^* \in \Omega_2$. From the definition of z_n, we obtain

$$\begin{aligned}\|z_n - x^*\|^2 &\le b_1\|x_n - x^*\|^2 + (1-b_1)\|T_1 x_n - x^*\|^2 - b_1(1-b_1)\|x_n - T_1 x_n\|^2 \\ &\le b_1\|x_n - x^*\|^2 + (1-b_1)\|x_n - x^*\|^2 - b_1(1-b_1)\|x_n - T_1 x_n\|^2 \\ &\le \|x_n - x^*\|^2 - b_1(1-b_1)\|x_n - T_1 x_n\|^2. \end{aligned} \tag{3.12}$$

In a similar way, we have

$$\|w_n - x^*\|^2 \le \|y_n - x^*\|^2 - b_2(1-b_2)\|y_n - T_2 y_n\|^2. \tag{3.13}$$

From the definition of x_n, (3.3), and (3.12), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\beta_n)z_n + \beta_n P_C U_n - x^*\|^2 \\ &= (1-\beta_n)\|z_n - x^*\|^2 + \beta_n\|P_C U_n - x^*\|^2 \\ &\quad - (1-\beta_n)\beta_n\|z_n - P_C U_n\|^2 \\ &\le (1-\beta_n)\big(\|x_n - x^*\|^2 - b_1(1-b_1)\|x_n - T_1 x_n\|^2\big) \\ &\quad + \beta_n\|\alpha_n g_1(y_n) + (1-\alpha_n)K_1 x_n - x^*\|^2 \\ &\quad - (1-\beta_n)\beta_n\|z_n - P_C U_n\|^2 \\ &= (1-\beta_n)\|x_n - x^*\|^2 - b_1(1-b_1)(1-\beta_n)\|x_n - T_1 x_n\|^2 \\ &\quad + \beta_n\|\alpha_n\big(g_1(y_n) - K_1 x_n\big) + K_1 x_n - x^*\|^2 \\ &\quad - (1-\beta_n)\beta_n\|z_n - P_C U_n\|^2 \\ &\le (1-\beta_n)\|x_n - x^*\|^2 - b_1(1-b_1)(1-\beta_n)\|x_n - T_1 x_n\|^2 + \beta_n\big(\|K_1 x_n - x^*\|^2 \\ &\quad + 2\alpha_n\big\langle g_1(y_n) - K_1 x_n, \alpha_n g_1(y_n) + (1-\alpha_n)K_1 x_n - x^*\big\rangle\big) \\ &\quad - (1-\beta_n)\beta_n\|z_n - P_C U_n\|^2 \\ &\le (1-\beta_n)\|x_n - x^*\|^2 - b_1(1-b_1)(1-\beta_n)\|x_n - T_1 x_n\|^2 + \beta_n\big(\|K_1 x_n - x^*\|^2 \\ &\quad + 2\alpha_n\|g_1(y_n) - K_1 x_n\|\|\alpha_n g_1(y_n) + (1-\alpha_n)K_1 x_n - x^*\|\big) \\ &\quad - (1-\beta_n)\beta_n\|z_n - P_C U_n\|^2 \\ &\le (1-\beta_n)\|x_n - x^*\|^2 - b_1(1-b_1)(1-\beta_n)\|x_n - T_1 x_n\|^2 + \beta_n\|x_n - x^*\|^2 \\ &\quad + 2\alpha_n\beta_n\|g_1(y_n) - K_1 x_n\|\|\alpha_n g_1(y_n) + (1-\alpha_n)K_1 x_n - x^*\| \\ &\quad - (1-\beta_n)\beta_n\|z_n - P_C U_n\|^2 \\ &= \|x_n - x^*\|^2 + 2\alpha_n\beta_n\|g_1(y_n) - K_1 x_n\|\|\alpha_n g_1(y_n) + (1-\alpha_n)K_1 x_n - x^*\| \\ &\quad - b_1(1-b_1)(1-\beta_n)\|x_n - T_1 x_n\|^2 - (1-\beta_n)\beta_n\|z_n - P_C U_n\|^2. \end{aligned} \tag{3.14}$$

It follows from (3.14) that

$$\begin{aligned}
&b_1(1-b_1)(1-\beta_n)\|x_n - T_1x_n\|^2 + (1-\beta_n)\beta_n\|z_n - P_CU_n\|^2 \\
&\quad\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\qquad + 2\alpha_n\beta_n\|g_1(y_n) - K_1x_n\|\|\alpha_n g_1(y_n) + (1-\alpha_n)K_1x_n - x^*\| \\
&\quad\leq \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\qquad + 2\alpha_n\beta_n\|g_1(y_n) - K_1x_n\|\|\alpha_n g_1(y_n) + (1-\alpha_n)K_1x_n - x^*\|.
\end{aligned}$$

By (3.11) and the conditions i) and ii), we obtain

$$\lim_{n\to\infty}\|P_CU_n - z_n\| = \lim_{n\to\infty}\|x_n - T_1x_n\| = 0. \tag{3.15}$$

From the definition of y_n and applying the same method as (3.15), we have

$$\lim_{n\to\infty}\|P_CV_n - w_n\| = \lim_{n\to\infty}\|y_n - T_2y_n\| = 0. \tag{3.16}$$

From Lemma 2.3, we obtain

$$\|P_CU_n - x^*\|^2 \leq \|U_n - x^*\|^2 - \|U_n - P_CU_n\|^2. \tag{3.17}$$

From the definition of U_n, we obtain

$$\begin{aligned}
\|U_n - x^*\|^2 &= \|\alpha_n(g_1(y_n) - x^*) + (1-\alpha_n)(K_1x_n - x^*)\|^2 \\
&\leq \alpha_n\|g_1(y_n) - x^*\|^2 + (1-\alpha_n)\|K_1x_n - x^*\|^2 \\
&\leq \alpha_n\|g_1(y_n) - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2.
\end{aligned} \tag{3.18}$$

From (3.3), (3.17), and (3.18), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\beta_n)(z_n - x^*) + \beta_n(P_CU_n - x^*)\|^2 \\
&\leq (1-\beta_n)\|z_n - x^*\|^2 + \beta_n\|P_CU_n - x^*\|^2 \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n(\|U_n - x^*\|^2 - \|U_n - P_CU_n\|^2) \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 \\
&\qquad + \beta_n(\alpha_n\|g_1(y_n) - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - \|U_n - P_CU_n\|^2) \\
&\leq (1-\alpha_n\beta_n)\|x_n - x^*\|^2 + \beta_n\alpha_n\|g_1(y_n) - x^*\|^2 - \beta_n\|U_n - P_CU_n\|^2,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\beta_n\|U_n - P_CU_n\|^2 &\leq (1-\alpha_n\beta_n)\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\beta_n\|g_1(y_n) - x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\beta_n\|g_1(y_n) - x^*\|^2 \\
&\leq \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + \alpha_n\beta_n\|g_1(y_n) - x^*\|^2.
\end{aligned}$$

From $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and the conditions (i) and (ii), we have

$$\lim_{n\to\infty} \|U_n - P_C U_n\| = 0. \tag{3.19}$$

From the definition of V_n and applying the same argument as (3.19), we also obtain

$$\lim_{n\to\infty} \|V_n - P_C V_n\| = 0. \tag{3.20}$$

Observe that

$$z_n - x_n = (1 - b_1)(T_1 x_n - x_n). \tag{3.21}$$

From (3.15) and (3.21), we obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.22}$$

Similarly, we also have

$$\lim_{n\to\infty} \|w_n - y_n\| = 0. \tag{3.23}$$

Consider

$$\begin{aligned}\|x_n - U_n\| &= \|x_n - z_n + z_n - P_C U_n + P_C U_n - U_n\| \\ &\leq \|x_n - z_n\| + \|z_n - P_C U_n\| + \|P_C U_n - U_n\|.\end{aligned}$$

From (3.15) and (3.19), we have

$$\lim_{n\to\infty} \|x_n - U_n\| = 0. \tag{3.24}$$

From the definition of y_n and applying the same method as (3.24), we also have

$$\lim_{n\to\infty} \|y_n - V_n\| = 0. \tag{3.25}$$

Next, we show that $\|x_n - K_1 x_n\| \to 0$ as $n \to \infty$ and $\|y_n - K_2 y_n\| \to 0$ as $n \to \infty$, where $K_i = J^i_{\gamma f}(I - \gamma_i(a_i A_i + (1 - a_i)B_i)))$ for all $i = 1, 2$.

Observe that

$$U_n - x_n = \alpha_n\bigl(g_1(y_n) - x_n\bigr) + (1 - \alpha_n)(K_1 x_n - x_n),$$

from which it follows that

$$(1 - \alpha_n)\|K_1 x_n - x_n\| \leq \|U_n - x_n\| + \alpha_n \bigl\|g_1(y_n) - x_n\bigr\|.$$

From (3.24) and the condition (i), we have

$$\lim_{n\to\infty} \|K_1 x_n - x_n\| = \lim_{n\to\infty} \|J^1_{\gamma f}\bigl(I - \gamma_1\bigl(a_1 A_1 + (1 - a_1)B_1\bigr)\bigr)x_n - x_n\| = 0. \tag{3.26}$$

Applying the same argument as (3.26), we also obtain

$$\lim_{n\to\infty}\|K_2y_n-y_n\|=\lim_{n\to\infty}\big\|J_{\gamma f}^2\big(I-\gamma_1\big(a_2A_2+(1-a_2)B_2\big)\big)y_n-y_n\big\|=0. \tag{3.27}$$

Next, we show that $\limsup_{n\to\infty}\langle g_1(y^*)-x^*,U_n-x^*\rangle\le 0$, where $x^*=P_{\Omega_1}g_1(y^*)$ and $\limsup_{n\to\infty}\langle g_2(x^*)-y^*,V_n-y^*\rangle\le 0$, where $y^*=P_{\Omega_2}g_2(x^*)$.

Indeed, take a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ such that

$$\limsup_{n\to\infty}\big\langle g_1\big(y^*\big)-x^*,U_n-x^*\big\rangle=\limsup_{k\to\infty}\big\langle g_1\big(y^*\big)-x^*,U_{n_k}-x^*\big\rangle.$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_k}\rightharpoonup p$ as $k\to\infty$. From (3.24), we obtain $U_{n_k}\rightharpoonup p$ as $k\to\infty$.

Next, we show that $p\in\Omega_1=\mathrm{Fix}(T_1)\cap VI(C,A_1,f_1)\cap VI(C,B_1,f_1)$.

Since K_1 is nonexpansive, then $I-K_1$ is demiclosed at zero. From (3.26) and by the demiclosedness of $I-K_1$ at zero, we obtain that $p\in\mathrm{Fix}(K_1)=\mathrm{Fix}(J_{\gamma f}^1(I-\gamma_1(a_1A_1+(1-a_1)B_1))))$. By Remark 2.7, we have $p\in VI(C,A_1,f_1)\cap VI(C,B_1,f_1)$.

Since T_1 is nonexpansive, then $I-T_1$ is demiclosed at zero. From (3.15) and by the demiclosedness of $I-T_1$ at zero, we obtain that $p\in\mathrm{Fix}(T_1)$. Therefore, $p\in\Omega_1=\mathrm{Fix}(T_1)\cap VI(C,A_1,f_1)\cap VI(C,B_1,f_1)$.

Since $U_{n_k}\rightharpoonup p$ as $k\to\infty$, $p\in\Omega_1$ and Lemma 2.2, we can derive that

$$\begin{aligned}\limsup_{n\to\infty}\big\langle g_1\big(y^*\big)-x^*,U_n-x^*\big\rangle&=\limsup_{k\to\infty}\big\langle g_1\big(y^*\big)-x^*,U_{n_k}-x^*\big\rangle\\&=\big\langle g_1\big(y^*\big)-x^*,p-x^*\big\rangle\\&\le 0.\end{aligned}\tag{3.28}$$

Similarly, take a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ such that

$$\limsup_{n\to\infty}\big\langle g_2\big(x^*\big)-y^*,V_n-y^*\big\rangle=\lim_{k\to\infty}\big\langle g_2\big(x^*\big)-y^*,V_{n_k}-y^*\big\rangle.$$

Since $\{y_n\}$ is bounded, without loss of generality, we may assume that $y_{n_k}\rightharpoonup q$ as $k\to\infty$. From (3.25), we obtain $V_{n_k}\rightharpoonup q$ as $k\to\infty$.

Following the same method as (3.28), we easily obtain that

$$\limsup_{n\to\infty}\big\langle g_2\big(x^*\big)-y^*,V_n-y^*\big\rangle\le 0. \tag{3.29}$$

Finally, we show that $\{x_n\}$ converges strongly to x^*, where $x^*=P_{\Omega_1}g_1(y^*)$ and $\{y_n\}$ converges strongly to y^*, where $y^*=P_{\Omega_2}g_2(x^*)$.

Let $U_n=\alpha_ng_1(y_n)+(1-\alpha_n)K_1x_n$ and $V_n=\alpha_ng_2(x_n)+(1-\alpha_n)K_2y_n$.

From the definition of x_n, we obtain

$$\begin{aligned}\big\|x_{n+1}-x^*\big\|^2&=\big\|(1-\beta_n)z_n+\beta_nP_C\big(\alpha_ng_1(y_n)+(1-\alpha_n)K_1x_n\big)-x^*\big\|^2\\&\le(1-\beta_n)\big\|z_n-x^*\big\|^2+\beta_n\big\|P_C\big(\alpha_ng_1(y_n)+(1-\alpha_n)K_1x_n\big)-x^*\big\|^2\\&\le(1-\beta_n)\big\|x_n-x^*\big\|^2+\beta_n\big\|\alpha_ng_1(y_n)+(1-\alpha_n)K_1x_n-x^*\big\|^2\end{aligned}$$

$$
\begin{aligned}
&= (1-\beta_n)\|x_n - x^*\|^2 \\
&\quad + \beta_n\|\alpha_n(g_1(y_n) - x^*) + (1-\alpha_n)(K_1x_n - x^*)\|^2 \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 \\
&\quad + \beta_n\big((1-\alpha_n)\|K_1x_n - x^*\|^2 + 2\alpha_n\langle g_1(y_n) - x^*, U_n - x^*\rangle\big) \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 \\
&\quad + \beta_n(1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\beta_n\langle g_1(y_n) - x^*, U_n - x^*\rangle \\
&= (1-\alpha_n\beta_n)\|x_n - x^*\|^2 + 2\alpha_n\beta_n\langle g_1(y_n) - x^*, U_n - x^*\rangle \\
&= (1-\alpha_n\beta_n)\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\beta_n\big(\langle g_1(y_n) - g_1(y^*), U_n - x^*\rangle + \langle g_1(y^*) - x^*, U_n - x^*\rangle\big) \\
&\leq (1-\alpha_n\beta_n)\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\beta_n\big(\|g_1(y_n) - g_1(y^*)\|\|U_n - x^*\| + \langle g_1(y^*) - x^*, U_n - x^*\rangle\big) \\
&\leq (1-\alpha_n\beta_n)\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\beta_n\|g_1(y_n) - g_1(y^*)\|\big(\|U_n - x_{n+1}\| + \|x_{n+1} - x^*\|\big) \\
&\quad + 2\alpha_n\beta_n\langle g_1(y^*) - x^*, U_n - x^*\rangle \\
&\leq (1-\alpha_n\beta_n)\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\beta_n\sigma\|y_n - y^*\|\|U_n - x_{n+1}\| + 2\alpha_n\beta_n\sigma\|y_n - y^*\|\|x_{n+1} - x^*\| \\
&\quad + 2\alpha_n\beta_n\langle g_1(y^*) - x^*, U_n - x^*\rangle \\
&\leq (1-\alpha_n\beta_n)\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n\beta_n\sigma\|y_n - y^*\|\|U_n - x_{n+1}\| + \alpha_n\beta_n\sigma\big(\|y_n - y^*\|^2 + \|x_{n+1} - x^*\|^2\big) \\
&\quad + 2\alpha_n\beta_n\langle g_1(y^*) - x^*, U_n - x^*\rangle,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \frac{1-\alpha_n\beta_n}{1-\alpha_n\beta_n\sigma}\|x_n - x^*\|^2 + \frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|\|U_n - x_{n+1}\| \\
&\quad + \frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|^2 + \frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\sigma}\langle g_1(y^*) - x^*, U_n - x^*\rangle \\
&= \left(1 - \frac{\alpha_n\beta_n - \alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\right)\|x_n - x^*\|^2 + \frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|\|U_n - x_{n+1}\| \\
&\quad + \frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|^2 + \frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\sigma}\langle g_1(y^*) - x^*, U_n - x^*\rangle \\
&= \left(1 - \frac{\alpha_n\beta_n(1-\sigma)}{1-\alpha_n\beta_n\sigma}\right)\|x_n - x^*\|^2 + \frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|\|U_n - x_{n+1}\| \\
&\quad + \frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|y_n - y^*\|^2 + \frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\sigma}\langle g_1(y^*) - x^*, U_n - x^*\rangle. \qquad (3.30)
\end{aligned}
$$

Similarly, as previously stated, we have

$$\begin{aligned}\|y_{n+1}-y^*\|^2 &\le \left(1-\frac{\alpha_n\beta_n(1-\sigma)}{1-\alpha_n\beta_n\sigma}\right)\|y_n-y^*\|^2+\frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|x_n-x^*\|\|V_n-y_{n+1}\| \\ &\quad+\frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\|x_n-x^*\|^2+\frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\sigma}\langle g_2(x^*)-y^*,V_n-y^*\rangle. \end{aligned}\tag{3.31}$$

From (3.30) and (3.31), we deduce that

$$\begin{aligned}&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 \\ &\quad\le\left(1-\frac{\alpha_n\beta_n(1-\sigma)}{1-\alpha_n\beta_n\sigma}\right)\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2\right) \\ &\qquad+\frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\left(\|y_n-y^*\|\|U_n-x_{n+1}\|+\|x_n-x^*\|\|V_n-y_{n+1}\|\right) \\ &\qquad+\frac{\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n a}\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2\right) \\ &\qquad+\frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\sigma}\left(\langle g_1(y^*)-x^*,U_n-x^*\rangle+\langle g_2(x^*)-y^*,V_n-y^*\rangle\right) \\ &\quad=\left(1-\frac{\alpha_n\beta_n(1-2\sigma)}{1-\alpha_n\beta_n\sigma}\right)\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2\right) \\ &\qquad+\frac{2\alpha_n\beta_n\sigma}{1-\alpha_n\beta_n\sigma}\left(\|y_n-y^*\|\|U_n-x_{n+1}\|+\|x_n-x^*\|\|V_n-y_{n+1}\|\right) \\ &\qquad+\frac{2\alpha_n\beta_n}{1-\alpha_n\beta_n\sigma}\left(\langle g_1(y^*)-x^*,U_n-x^*\rangle+\langle g_2(x^*)-y^*,V_n-y^*\rangle\right). \end{aligned}\tag{3.32}$$

By (3.11), (3.24), (3.25), (3.28), (3.29), the condition (i), and Lemma 2.4, we have $\lim_{n\to\infty}(\|x_n-x^*\|+\|y_n-y^*\|)=0$. This implies that the sequence $\{x_n\}$, $\{y_n\}$ converges to $x^*=P_{\Omega_1}g_1(y^*)$, $y^*=P_{\Omega_2}g_2(x^*)$, respectively.

This completes the proof. □

As a direct proof of Theorem 3.1, we obtain the following results.

Corollary 3.2 *Let C be a nonempty, closed, and convex subset of H. For every $i=1,2$, let $f_i:H\to\mathbb{R}\cup\{+\infty\}$ be a proper, convex, and lower semicontinuous function, let $A_i,B_i:C\to H$ be δ_i^A- and δ_i^B-inverse strongly monotone operators, respectively, with $\delta_i=\min\{\delta_i^A,\delta_i^B\}$. Assume that $VI(C,A_i,f_i)\cap VI(C,B_i,f_i)\neq\emptyset$, for all $i=1,2$. Let $g_1,g_2:H\to H$ be σ_1- and σ_2-contraction mappings with $\sigma_1,\sigma_2\in(0,1)$ and $\sigma=\max\{\sigma_1,\sigma_2\}$. Let the sequences $\{x_n\}$, $\{y_n\}$ be generated by $x_1,y_1\in C$ and*

$$\begin{cases} y_{n+1}=(1-\beta_n)y_n+\beta_nP_C(\alpha_ng_2(x_n) \\ \qquad\quad+(1-\alpha_n)J^2_{\gamma f}(y_n-\gamma_2(a_2A_2+(1-a_2)B_2)y_n)) \\ x_{n+1}=(1-\beta_n)x_n+\beta_nP_C(\alpha_ng_1(y_n) \\ \qquad\quad+(1-\alpha_n)J^1_{\gamma f}(x_n-\gamma_1(a_1A_1+(1-a_1)B_1)x_n)),\quad \forall n\ge1, \end{cases}\tag{3.33}$$

where $\{\beta_n\},\{\alpha_n\}\subseteq[0,1]$, $\gamma_i\in(0,2\delta_i)$, $a_i\in(0,1)$ and $J^i_{\gamma f}=(I+\gamma_i\nabla f_i)^{-1}$ is the resolvent operator for all $i=1,2$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \bar{l} \le \beta_n \le l$ *for all* $n \in \mathbb{N}$ *and for some* $\bar{l}, l > 0$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $x^* = P_{VI(C,A_1,f_1)\cap VI(C,B_1,f_1)} g_1(y^*)$ *and* $y^* = P_{VI(C,A_2,f_2)\cap VI(C,B_2,f_2)} g_2(x^*)$, *respectively.*

Proof If $T_1 \equiv T_2 \equiv I$ in Theorem 3.1, Hence, from Theorem 3.1, we obtain the desired result. □

Corollary 3.3 *Let C be a nonempty, closed, and convex subset of H. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous function. Let $A, B : C \to H$ be δ^A- and δ^B-inverse strongly monotone operators, respectively, with* $\delta = \min\{\delta^A, \delta^B\}$ *and let $T : C \to C$ be nonexpansive mapping. Assume that* $\Omega = \mathrm{Fix}(T) \cap VI(C,A,f) \cap VI(C,B,f) \neq \emptyset$. *Let $g : H \to H$ be a σ-contraction mapping with $\sigma \in (0,1)$. Let the sequence $\{x_n\}$ be generated by $x \in C$ and*

$$\begin{cases} z_n = bx_n + (1-b)Tx_n \\ x_{n+1} = (1-\beta_n)z_n + \beta_n P_C(\alpha_n g(x_n) \\ \qquad + (1-\alpha_n)J_{\gamma f}(x_n - \gamma(aA + (1-a)B)x_n)), \quad \forall n \ge 1, \end{cases} \tag{3.34}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$, $\gamma \in (0, 2\delta)$, $a, b \in (0,1)$ *and* $J_{\gamma f} = (I + \gamma \nabla f)^{-1}$ *is the resolvent operator. Assume that the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \bar{l} \le \beta_n \le l$ *for all* $n \in \mathbb{N}$ *and for some* $\bar{l}, l > 0$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ *converges strongly to* $x^* = P_\Omega g(x^*)$.

Proof If $g \equiv g_1 \equiv g_2, f \equiv f_1 \equiv f_2, T \equiv T_1 \equiv T_2, A \equiv A_1 \equiv A_2, B \equiv B_1 \equiv B_2$, $w_n = z_n$, and $x_n = y_n$ in Theorem 3.1. Hence, from Theorem 3.1, we obtain the desired result. □

Remark 3.4 We remark here that Corollary 3.3 is modified from Algorithm 3.2 in [6] in the following aspects:

1. From a strongly monotone and Lipschitz continuous operator to two inverse strongly monotone operators.
2. We add a nonexpansive mapping and a contraction mapping in our iterative algorithm.

4 Applications

In this section, we reduce our main problem to the following split-feasibility problem and constrained convex-minimization problem:

4.1 The split-feasibility problem

Let C and Q be nonempty, closed, and convex subsets of Hilbert spaces H_1 and H_2, respectively. The *split-feasibility problem (SFP)* is to find a point

$$x \in C \quad \text{such that } Ax \in Q, \tag{4.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The set of all solutions (SFP) is denoted by $\Gamma = \{x \in C; Ax \in Q\}$. The split-feasibility problem is the first example of the split-inverse problem, which was first introduced by Censor and Elfving [42] in Euclidean spaces. Many mathematical problems, such as the constrained least-squares problem, the linear split-feasibility problem, and the linear programming problem, can be solved using the split-feasibility problem paradigm, and it can be used in real-world applications, for example, in signal processing, in image recovery, in intensity-modulated therapy, in pattern recognition, etc., see [43–46]. Consequently, the split-feasibility problem has been widely studied by many authors, see [47–52] and the references therein.

Proposition 4.1 ([48]) *Given $x^* \in \mathcal{H}_1$, the following statements are equivalent.*
(i) *x^* solves the Γ;*
(ii) *$P_C(I - \lambda A^*(I - P_Q)A)x^* = x^*$, where A^* is the adjoint of A;*
(iii) *x^* solves the variational inequality problem of finding $x^* \in C$ such that*

$$\left\langle \nabla \mathcal{G}\left(x^*\right), x - x^*\right\rangle \geq 0, \quad \forall x \in C, \tag{4.2}$$

where $\nabla \mathcal{G} = A^*(I - P_Q)A$.

If C is a closed and convex subset of H and the function f is the indicator function of C then it is well known that $J_{\gamma f} = P_C$, the projection operator of H, onto the closed convex set C and putting $A_i = B_i$ for all $i = 1, 2$ in Theorem 3.1. Consequently, the following result can be obtained from Theorem 3.1.

Theorem 4.2 *Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty, closed, and convex subsets of real Hilbert space s H_1 and H_2, respectively. Let $A_1, A_2 : H_1 \to H_2$ be bounded linear operators, where A_1^*, A_2^* are adjoints of A_1 and A_2, respectively, where L_1 and L_2 are special radii of $A_1^*A_1$ and $A_2^*A_2$. Let $T_i : C \to C$ be nonexpansive mappings. Assume that $\Xi_i = \mathrm{Fix}(T_i) \cap \Gamma_i \neq \emptyset$, for all $i = 1, 2$. Let $g_1, g_2 : H \to H$ be σ_1- and σ_2-contraction mappings with $\sigma_1, \sigma_2 \in (0, 1)$ and $\sigma = \max\{\sigma_1, \sigma_2\}$. Let the sequences $\{x_n\}$, $\{y_n\}$ be generated by $x_1, y_1 \in C$ and*

$$\begin{cases} w_n = b_2 y_n + (1 - b_2) T_2 y_n \\ y_{n+1} = (1 - \beta_n) w_n + \beta_n P_C(\alpha_n g_2(x_n) + (1 - \alpha_n) P_C(I - \gamma_2 \nabla \mathcal{G}_2) y_n) \\ z_n = b_1 x_n + (1 - b_1) T_1 x_n \\ x_{n+1} = (1 - \beta_n) z_n + \beta_n P_C(\alpha_n g_1(y_n) + (1 - \alpha_n) P_C(I - \gamma_1 \nabla \mathcal{G}_1) x_n), \quad \forall n \geq 1, \end{cases} \tag{4.3}$$

where $\nabla \mathcal{G}_i = A_i^*(I - P_Q)A_i$, $\gamma_i \in (0, \frac{2}{L_i})$, $\{\beta_n\}, \{\alpha_n\} \subseteq [0, 1]$, $b_i \in (0, 1)$ for all $i = 1, 2$. Assume that the following conditions hold:
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \bar{l} \leq \beta_n \leq l$ *for all* $n \in \mathbb{N}$ *and for some* $\bar{l}, l > 0$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.
Then, $\{x_n\}$ and $\{y_n\}$ converges strongly to $x^* = P_{\Xi_1} g_1(y^*)$ and $y^* = P_{\Xi_2} g_2(x^*)$, respectively.

Proof Let $x, y \in C$ and $\nabla \mathcal{G}_i = A_i^*(I - P_Q)A_i$, for all $i = 1, 2$. First, we show that $\nabla \mathcal{G}_i$ is $\frac{1}{L_i}$-inverse strongly monotone for all $i = 1, 2$.

Consider,

$$\begin{aligned} \left\|\nabla\mathcal{G}_i(x)-\nabla\mathcal{G}_i(y)\right\|^2 &= \left\|A_i^*(I-P_Q)A_ix-A_i^*(I-P_Q)A_iy\right\|^2 \\ &\le L_i\left\|(I-P_Q)A_ix-(I-P_Q)A_iy\right\|^2. \end{aligned} \tag{4.4}$$

From the property of P_C, we have

$$\begin{aligned} &\left\|(I-P_Q)A_ix-(I-P_Q)A_iy\right\|^2 \\ &\quad = \left\langle(I-P_Q)A_ix-(I-P_Q)A_iy,(I-P_Q)A_ix-(I-P_Q)A_iy\right\rangle \\ &\quad = \left\langle(I-P_Q)A_ix-(I-P_Q)A_iy,A_ix-A_iy\right\rangle \\ &\qquad - \left\langle(I-P_Q)A_ix-(I-P_Q)A_iy,P_QA_ix-P_QA_iy\right\rangle \\ &\quad = \left\langle A_i^*(I-P_Q)A_ix-A_i^*(I-P_Q)A_iy,x-y\right\rangle \\ &\qquad - \left\langle(I-P_Q)A_ix-(I-P_Q)A_iy,P_QA_ix-P_QA_iy\right\rangle \\ &\quad = \left\langle A_i^*(I-P_Q)A_ix-A_i^*(I-P_Q)A_iy,x-y\right\rangle \\ &\qquad - \left\langle(I-P_Q)A_ix,P_QA_ix-P_QA_iy\right\rangle \\ &\qquad + \left\langle(I-P_Q)A_iy,P_QA_ix-P_QA_iy\right\rangle \\ &\quad \le \left\langle A_i^*(I-P_Q)A_ix-A_i^*(I-P_Q)A_iy,x-y\right\rangle. \end{aligned} \tag{4.5}$$

Substituting (4.5) into (4.4), we have

$$\begin{aligned} \left\|\nabla\mathcal{G}_i(x)-\nabla\mathcal{G}_i(y)\right\|^2 &\le L_i\left\langle A_i^*(I-P_Q)A_ix-A_i^*(I-P_Q)A_iy,x-y\right\rangle \\ &= L_i\left\langle\nabla\mathcal{G}_i(x)-\nabla\mathcal{G}_i(y),x-y\right\rangle. \end{aligned}$$

It follows that

$$\left\langle\nabla\mathcal{G}_i(x)-\nabla\mathcal{G}_i(y),x-y\right\rangle \ge \frac{1}{L_i}\left\|\nabla\mathcal{G}_i(x)-\nabla\mathcal{G}_i(y)\right\|^2.$$

Then, $\nabla\mathcal{G}_i$ is $\frac{1}{L_{A_i}}$-inverse strongly monotone, for all $i=1,2$. Hence, we can conclude Theorem 4.2 from Proposition 4.1 and Theorem 3.1. □

4.2 The constrained convex-minimization problem

Let C be a nonempty, closed, and convex subset of H. Consider that the constrained convex-minimization problem is to find $x^*\in C$ such that

$$\mathcal{Q}\left(x^*\right)=\min_{x\in C}\mathcal{Q}(x), \tag{4.6}$$

where $\mathcal{Q}:H\to\mathbb{R}$ is a continuously differentiable function. Assume that (4.6) is consistent (i.e., it has a solution) and we use Ψ to denote its solution set. It is known that the gradient projection algorithm (GPA) plays an important role in solving constrained convex-minimization problems. It is well known that a necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem (4.6) is that x^* solves the variational inequality:

$$x^* \in C, \left\langle \nabla \mathcal{Q}\left(x^*\right), x - x^*\right\rangle \geq 0, \quad \forall x \in C. \tag{4.7}$$

That is, $\Psi = VI(C, \nabla \mathcal{Q})$, where $\Psi \neq \emptyset$. The following theorem is derived from these results.

Theorem 4.3 *Let C be a nonempty, closed, and convex subset of H. For every $i = 1,2$, let $\mathcal{Q}_i : H \to \mathbb{R}$ be a continuous differentiable function with $\nabla \mathcal{Q}_i$, that is $\frac{1}{L_{\mathcal{Q}_i}}$-inverse strongly monotone. Let $T_i : C \to C$ be nonexpansive mappings. Assume that $\Theta_i = \mathrm{Fix}(T_i) \cap \Psi_i \neq \emptyset$, for all $i = 1,2$. Let $g_1, g_2 : H \to H$ be σ_1- and σ_2-contraction mappings with $\sigma_1, \sigma_2 \in (0,1)$ and $\sigma = \max\{\sigma_1, \sigma_2\}$. Let the sequences $\{x_n\}$, $\{y_n\}$ be generated by $x_1, y_1 \in C$ and*

$$\begin{cases} w_n = b_2 y_n + (1 - b_2) T_2 y_n \\ y_{n+1} = (1 - \beta_n) w_n + \beta_n P_C(\alpha_n g_2(x_n) + (1 - \alpha_n) P_C(I - \gamma_2 \nabla \mathcal{Q}_2) y_n)) \\ z_n = b_1 x_n + (1 - b_1) T_1 x_n \\ x_{n+1} = (1 - \beta_n) z_n + \beta_n P_C(\alpha_n g_1(y_n) + (1 - \alpha_n) P_C(I - \gamma_1 \nabla \mathcal{Q}_1) x_n), \quad \forall n \geq 1, \end{cases} \tag{4.8}$$

where $\{\beta_n\}, \{\alpha_n\} \subseteq [0,1]$, $\gamma_i \in (0, \frac{2}{L_{\mathcal{Q}_i}})$, $b_i \in (0,1)$ for all $i = 1,2$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \bar{l} \leq \beta_n \leq l$ *for all* $n \in \mathbb{N}$ *and for some* $\bar{l}, l > 0$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ and $\{y_n\}$ converges strongly to $x^* = P_{\Theta_1} g_1(y^*)$ and $y^* = P_{\Theta_2} g_2(x^*)$, respectively.

Proof By using Theorem 4.2, we obtain the conclusion. □

5 Numerical experiments

In this section, we give examples to support our main theorem. In the following examples, we choose $\alpha_n = \frac{1}{3n}$, $\beta_n = \frac{n+1}{6n}$, $a_1 = 0.50$, $a_2 = 0.25$, $b_1 = 0.40$, and $b_2 = 0.45$. The stopping criterion used for our computation is $\|x_{n+1} - x_n\| < 10^{-5}$ and $\|y_{n+1} - y_n\| < 10^{-5}$.

Example 5.1 Let $\mathbb{R}$ be the set of real numbers and let $C = [1,10]$. Then, we obtain $P_C x = \max\{\min\{x, 10\}, 1\}$, for all $x \in C$. For every $i = 1,2$, let $A_i, B_i : C \to \mathbb{R}$ defined by $A_1(x) = \frac{3x}{5} - \frac{3}{5}$, $A_2(x) = \frac{2x}{5} - \frac{2}{5}$, $B_1(x) = \frac{2x}{3} - \frac{2}{3}$, and $B_2(x) = \frac{x}{6} - \frac{1}{6}$, for all $x \in C$. For every $i = 1,2$, let $f_i : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = x^2$, $f_2(x) = 2x^2$ for all $x \in \mathbb{R}$. Then, we have $J^1_{\gamma f}, J^2_{\gamma f} : \mathbb{R} \to \mathbb{R}$ defined by $J^1_{\gamma f}(x) = \frac{x}{2}$ and $J^2_{\gamma f}(x) = \frac{5x}{9}$, respectively. For every $i = 1,2$, let $T_i : C \to C$ defined by $T_1(x) = \frac{x}{2} + \frac{1}{2}$ and $T_2(x) = \frac{x}{3} + \frac{2}{3}$, for all $x \in C$. For every $i = 1,2$, let $g_i : \mathbb{R} \to \mathbb{R}$ be defined by $g_1(x) = \frac{x}{5}$ and $g_2(x) = \frac{x}{4}$, for all $x \in \mathbb{R}$. Let the sequences $\{x_n\}$, $\{y_n\}$ be generated by x_1, $y_1 \in C$ and

$$\begin{cases} w_n = 0.45 y_n + 0.55 T_2 y_n \\ y_{n+1} = (1 - \frac{n+1}{6n}) w_n + \frac{n+1}{6n} P_C(\frac{1}{3n} g_2(x_n) + (1 - \frac{1}{3n}) J^2_{\gamma f}(y_n - 0.2(0.25 A_2 + 0.75 B_2) y_n)) \\ z_n = 0.4 x_n + 0.6 T_1 x_n \\ x_{n+1} = (1 - \frac{n+1}{6n}) z_n + \frac{n+1}{6n} P_C(\frac{1}{3n} g_1(y_n) + (1 - \frac{1}{3n}) J^1_{\gamma f}(x_n - 0.5(0.5 A_1 + 0.5 B_1) x_n)). \end{cases}$$

Table 1 The values of $\{x_n\}$ and $\{y_n\}$ with initial values $x_1 = 10, y_1 = 8$ in Example 5.1

n	x_n	y_n	$\|x_{n+1} - x_n\|$	$\|y_{n+1} - y_n\|$
1	10	8	–	–
2	5.83889	4.84877	4.16111E+00	3.15123E+00
3	3.77942	3.18014	2.05946E+00	1.66863E+00
4	2.59307	2.21325	1.18635E+00	9.66891E-01
5	1.88283	1.64025	7.10245E-01	5.72994E-01
⋮	⋮	⋮	⋮	⋮
21	1.00012	1.00002	8.54581E-05	1.56096E-05
22	1.00007	1.00001	4.93196E-05	8.15171E-06
23	1.00004	1.00000	2.84787E-05	4.25928E-06
24	1.00002	1.00000	1.64526E-05	2.22654E-06
25	1.00001	1.00000	9.50912E-06	1.16443E-06

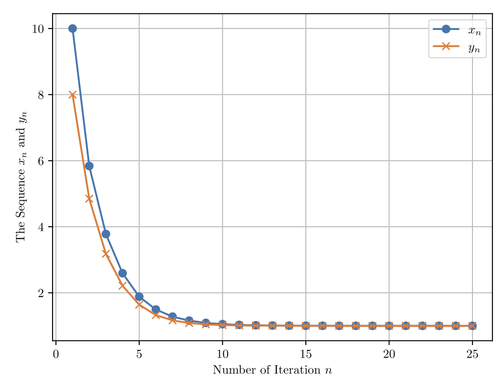


Figure 1 The convergence behavior of $\{x_n\}$ and $\{y_n\}$ with $x_1 = 10, y_1 = 8$ in Example 5.1

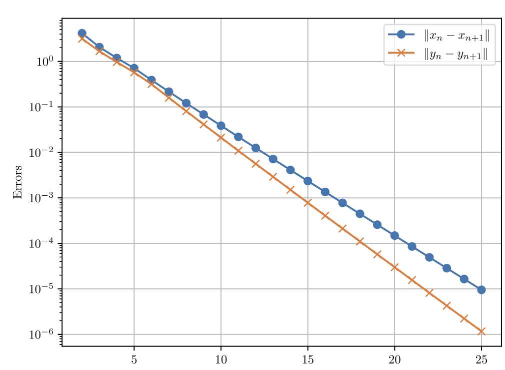


Figure 2 Error plotting of $\|x_{n+1} - x_n\|$ and $\|y_{n+1} - y_n\|$ in Example 5.1, the y-axis is illustrated on a logscale

According to the definition of A_i, B_i, T_i, f_i, for all $i = 1, 2$, we obtain $1 \in \mathrm{Fix}(T_i) \cap VI(C, A_i, f_i), \cap VI(C, B_i, f_i)$. From Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to 1.

The numerical and graphical results of Example 5.1 are shown in Table 1 and Figs. 1 and 2.

Next, we consider the problem in the infinite-dimensional Hilbert space.

Example 5.2 Let $H = L_2([0,1])$ with the inner product defined by

$$\langle x,y\rangle := \int_0^1 x(t)y(t)\,dt, \quad \forall x,y \in H$$

and the induced norm by

$$\|x\| := \left(\int_0^1 \left|x(t)\right|^2 dt\right)^{\frac{1}{2}}, \quad \forall x \in H.$$

Let $C := \{x \in L_2([0,1]) : \|x\| \leq 1\}$ be the unit ball. Then, we have

$$P_C\big(x(t)\big) = \begin{cases} x(t) & \text{if } \|x(t)\| \leq 1, \\ \frac{x(t)}{\|x(t)\|} & \text{if } \|x(t)\| > 1. \end{cases} \tag{5.1}$$

For every $i = 1,2$, let $A_i, B_i : C \to H$ be defined by $A_1(x(t)) = x(t)$, $A_2(x(t)) = \frac{3x(t)}{2}$, $B_1(x(t)) = 2x(t)$, and $B_2(x(t)) = \frac{5x(t)}{3}$, for all $t \in [0,1]$, $x \in C$. For every $i = 1,2$, let $f_i : H \to \mathbb{R}$ be defined by $f_1(x(t)) = \frac{3x(t)^2}{2}$, $f_2(x(t)) = \frac{x(t)^2}{2}$ for all $t \in [0,1]$, $x \in H$. Then, we have $J^1_{\gamma f}, J^2_{\gamma f} : H \to H$ defined by $J^1_{\gamma f}(x(t)) = \frac{4x(t)}{7}$ and $J^2_{\gamma f}(x(t)) = \frac{5x(t)}{6}$, for all $t \in [0,1]$, respectively. For every $i = 1,2$, let $T_i : C \to C$ be defined by $T_1(x(t)) = \frac{x(t)}{2}$ and $T_2(x(t)) = \frac{x(t)}{3}$, for all $t \in [0,1]$, $x \in C$. For every $i = 1,2$, let $g_i : H \to H$ be defined by $g_1(x(t)) = \frac{x(t)}{9}$ and $g_2(x(t)) = \frac{x(t)}{16}$, for all $t \in [0,1]$, $x \in H$. Let the sequences $\{x_n\}$, $\{y_n\}$ be generated by $x_1, y_1 \in C$ and

$$\begin{cases} w_n = 0.45y_n + 0.55T_2y_n, \\ y_{n+1} = (1 - \frac{n+1}{6n})w_n \\ \qquad + \frac{n+1}{6n}P_C(\frac{1}{3n}g_2(x_n) + (1 - \frac{1}{3n})J^2_{\gamma f}(y_n - 0.2(0.25A_2 + 0.75B_2)y_n)), \\ z_n = 0.4x_n + 0.6T_1x_n, \\ x_{n+1} = (1 - \frac{n+1}{6n})z_n \\ \qquad + \frac{n+1}{6n}P_C(\frac{1}{3n}g_1(y_n) + (1 - \frac{1}{3n})J^1_{\gamma f}(x_n - 0.25(0.5A_1 + 0.5)B_1)x_n)). \end{cases} \tag{5.2}$$

According to the definition of A_i, B_i, T_i, f_i, for all $i = 1,2$, then the solution of this problem is $x(t) = \mathbf{0}$, where $\mathbf{0} \in \operatorname{Fix}(T_i) \cap VI(C, A_i, f_i), \cap VI(C, B_i, f_i)$. From Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $x(t) = \mathbf{0}$.

We test the algorithms for three different starting points and use $\|x_{n+1} - x_n\| < 10^{-5}$ and $\|y_{n+1} - y_n\| < 10^{-5}$ as the stopping criterion.

Case 1: $x_1 = 0.2t$ and $y_1 = 0.8t$;

Case 2: $x_1 = e^{-2t}$ and $y_1 = t^2$;

Case 3: $x_1 = \sin(t)$ and $y_1 = \cos(t)$.

The computational and graphical results of Example 5.2 are shown in Tables 2, 3, and 4 and Figs. 3, 4, 5, and 6.

We next give a comparison between Algorithm (5.3) in Corollary 3.3 and Algorithm 3.2 in [6].

Example 5.3 In this example, we use the same mappings and parameters as in Example 5.2. Putting the sequence $\{x_n\} = \{y_n\}$ and $\{w_n\} = \{z_n\}$, the mapping $A_1 \equiv A_2 \equiv B_1 \equiv B_2, f_1 \equiv f_2$,

Table 2 Computational results of Case 1 for Example 5.2

n	$x_n(t)$	$y_n(t)$	$\|x_{n+1}-x_n\|$	$\|y_{n+1}-y_n\|$
1	$0.2t$	$0.8t$	–	–
2	$0.11908t$	$0.43917t$	0.046718	0.20833
3	$0.073412t$	$0.26038t$	0.026368	0.10322
4	$0.045862t$	$0.15731t$	0.015906	0.05951
5	$0.028847t$	$0.095819t$	0.0098239	0.035499
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
18	$8.1465\cdot 10^{-5}t$	$0.00017912t$	$2.6556\cdot 10^{-5}$	$6.3657\cdot 10^{-5}$
19	$5.2082\cdot 10^{-5}t$	$0.0001109t$	$1.6964\cdot 10^{-5}$	$3.9387\cdot 10^{-5}$
20	$3.3305\cdot 10^{-5}t$	$6.8675\cdot 10^{-5}t$	$1.0841\cdot 10^{-5}$	$2.4377\cdot 10^{-5}$
21	$2.1303\cdot 10^{-5}t$	$4.2536\cdot 10^{-5}t$	$6.9296\cdot 10^{-6}$	$1.5091\cdot 10^{-5}$
22	$1.3629\cdot 10^{-5}t$	$2.6351\cdot 10^{-5}t$	$4.4307\cdot 10^{-6}$	$9.3446\cdot 10^{-6}$

Table 3 Computational results of Case 2 for Example 5.2

n	$x_n(t)$	$y_n(t)$	$\|x_{n+1}-x_n\|$	$\|y_{n+1}-y_n\|$
1	e^{-2t}	t^2	–	–
2	$0.012346t^2+0.54603e^{-2t}$	$0.54722t^2+0.0069444e^{-2t}$	0.22294	0.20126
3	$0.0099335t^2+0.32733e^{-2t}$	$0.32409t^2+0.0055344e^{-2t}$	0.10874	0.10004
4	$0.0069982t^2+0.20132e^{-2t}$	$0.19567t^2+0.0038463e^{-2t}$	0.062915	0.057742
5	$0.0047329t^2+0.1253e^{-2t}$	$0.11913t^2+0.0025601e^{-2t}$	0.03804	0.034466
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	$7.0577\cdot 10^{-6}t^2+0.0001383e^{-2t}$	$8.5148\cdot 10^{-5}t^2+2.784\cdot 10^{-6}e^{-2t}$	$3.9419\cdot 10^{-5}$	$2.3729\cdot 10^{-5}$
21	$4.5372\cdot 10^{-6}t^2+8.8366\cdot 10^{-5}e^{-2t}$	$5.2733\cdot 10^{-5}t^2+1.7493\cdot 10^{-6}e^{-2t}$	$2.5168\cdot 10^{-5}$	$1.4691\cdot 10^{-5}$
22	$2.9161\cdot 10^{-6}t^2+5.6479\cdot 10^{-5}e^{-2t}$	$3.2664\cdot 10^{-5}t^2+1.0988\cdot 10^{-6}e^{-2t}$	$1.6075\cdot 10^{-5}$	$9.0977\cdot 10^{-6}$
23	$1.8739\cdot 10^{-6}t^2+3.6109\cdot 10^{-5}e^{-2t}$	$2.0236\cdot 10^{-5}t^2+6.9006\cdot 10^{-7}e^{-2t}$	$1.0271\cdot 10^{-5}$	$5.635\cdot 10^{-6}$
24	$1.204\cdot 10^{-6}t^2+2.3091\cdot 10^{-5}e^{-2t}$	$1.2538\cdot 10^{-5}t^2+4.3325\cdot 10^{-7}e^{-2t}$	$6.5643\cdot 10^{-6}$	$3.4909\cdot 10^{-6}$

Table 4 Computational results of Case 3 for Example 5.2

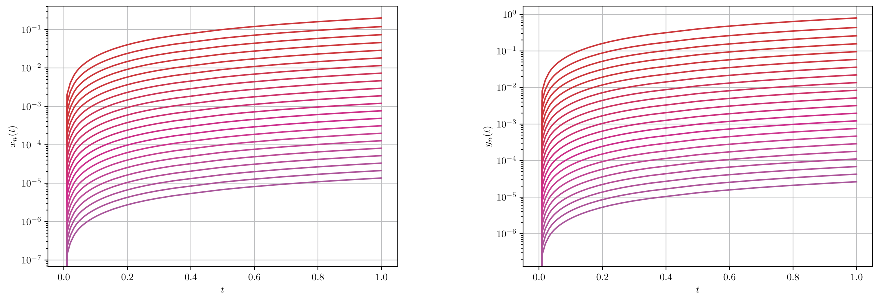
n	$x_n(t)$	$y_n(t)$	$\|x_{n+1}-x_n\|$	$\|y_{n+1}-y_n\|$
1	$\sin(t)$	$\cos(t)$	–	–
2	$0.54603\sin(t)+0.012346\cos(t)$	$0.0069444\sin(t)+0.54722\cos(t)$	0.22877	0.38327
3	$0.32733\sin(t)+0.0099335\cos(t)$	$0.0055344\sin(t)+0.32409\cos(t)$	0.11585	0.19088
4	$0.20132\sin(t)+0.0069982\cos(t)$	$0.0038463\sin(t)+0.19567\cos(t)$	0.067807	0.11022
5	$0.1253\sin(t)+0.0047329\cos(t)$	$0.0025601\sin(t)+0.11913\cos(t)$	0.041247	0.065808
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	$0.0001383\sin(t)+7.0577\cdot 10^{-6}\cos(t)$	$2.784\cdot 10^{-6}\sin(t)+8.5148\cdot 10^{-5}\cos(t)$	$4.3544\cdot 10^{-5}$	$4.5346\cdot 10^{-5}$
21	$8.8366\cdot 10^{-5}\sin(t)+4.5372\cdot 10^{-6}\cos(t)$	$1.7493\cdot 10^{-6}\sin(t)+5.2733\cdot 10^{-5}\cos(t)$	$2.7812\cdot 10^{-5}$	$2.8076\cdot 10^{-5}$
22	$5.6479\cdot 10^{-5}\sin(t)+2.9161\cdot 10^{-6}\cos(t)$	$1.0988\cdot 10^{-6}\sin(t)+3.2664\cdot 10^{-5}\cos(t)$	$1.777\cdot 10^{-5}$	$1.7387\cdot 10^{-5}$
23	$3.6109\cdot 10^{-5}\sin(t)+1.8739\cdot 10^{-6}\cos(t)$	$6.9006\cdot 10^{-7}\sin(t)+2.0236\cdot 10^{-5}\cos(t)$	$1.1357\cdot 10^{-5}$	$1.0769\cdot 10^{-5}$
24	$2.3091\cdot 10^{-5}\sin(t)+1.204\cdot 10^{-6}\cos(t)$	$4.3325\cdot 10^{-7}\sin(t)+1.2538\cdot 10^{-5}\cos(t)$	$7.2602\cdot 10^{-6}$	$6.6718\cdot 10^{-6}$

$g_1\equiv g_2$, and $T_1\equiv T_2\equiv I$, we can rewrite (3.34) as follows:

$$x_{n+1}=\left(1-\frac{n+1}{6n}\right)x_n+\frac{n+1}{6n}P_C\left(\frac{1}{3n}g_1(x_n)+\left(1-\frac{1}{3n}\right)J^1_{\gamma f}(x_n-0.25A_1x_n)\right). \tag{5.3}$$

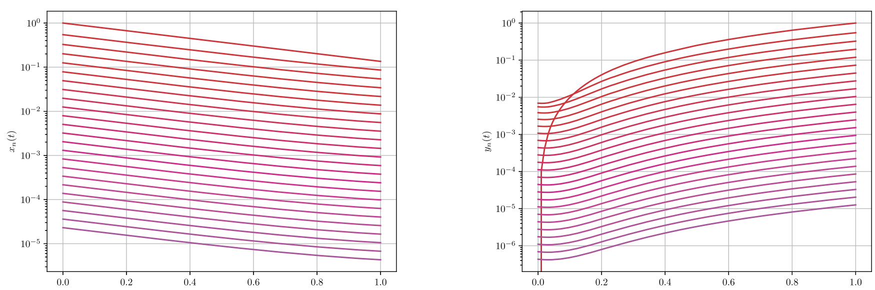
Also, we modify Algorithm 3.2 in [6] by putting $A\equiv A_1$ that is an inverse strongly monotone operator and choose the same mappings and parameters as in Example 5.2. Hence, we can rewrite as follows:

$$x_{n+1}=\frac{1}{3n}x_n+\left(1-\frac{1}{3n}\right)J^1_{\gamma f}(x_n-0.25A_1x_n). \tag{5.4}$$



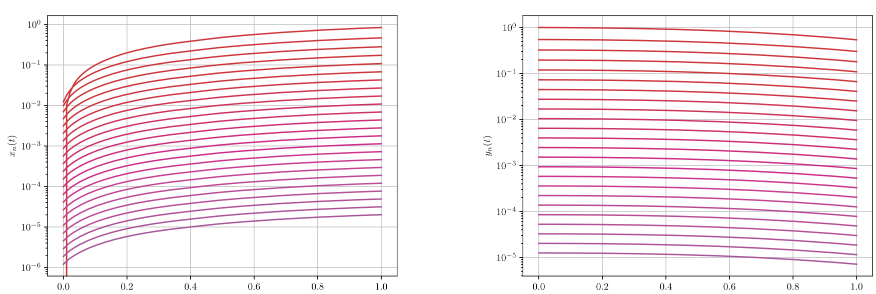
(a) Case $1 : x_1 = 0.2t$ for $n = 1, 2, 3, ..., 22$. (b) Case $1 : y_1 = 0.8t$ for $n = 1, 2, 3, ..., 22$.

Figure 3 The convergence behavior of $\{x_n(t)\}$ and $\{y_n(t)\}$ with $x_1 = 0.2t$ and $y_1 = 0.8t$ (Case 1) in Example 5.2, the y-axis is illustrated on a logscale



(a) Case $2 : x_1 = e^{-2t}$ for $n = 1, 2, 3, ..., 24$. (b) Case $2 : y_1 = t^2$ for $n = 1, 2, 3, ..., 24$.

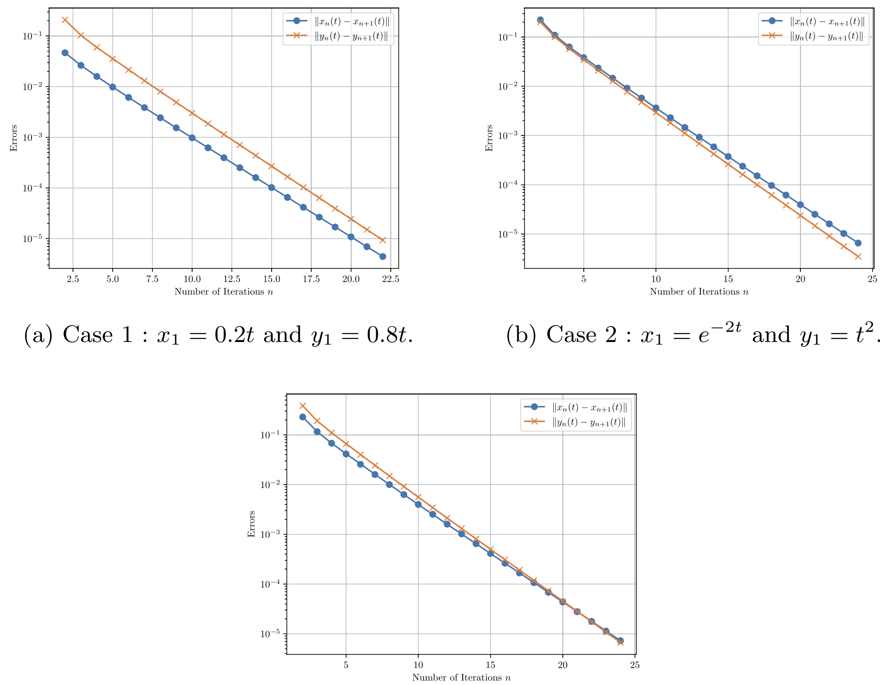
Figure 4 The convergence behavior of $\{x_n(t)\}$ and $\{y_n(t)\}$ with $x_1 = e^{-2t}$ and $y_1 = t^2$ (Case 2) in Example 5.2, the y-axis is illustrated on a logscale



(a) Case $3 : x_1 = \sin(t)$ for $n = 1, 2, 3, ..., 24$. (b) Case $3 : y_1 = \cos(t)$ for $n = 1, 2, 3, ..., 24$.

Figure 5 The convergence behavior of $\{x_n(t)\}$ and $\{y_n(t)\}$ with $x_1 = \sin(t)$ and $y_1 = \cos(t)$ (Case 3) in Example 5.2, the y-axis is illustrated on a logscale

The comparison of Algorithm (5.3) and Algorithm (5.4), which is modified from Algorithm 3.2 in [6], in terms of the CPU time and the number of iterations with different starting points, is reported in Table 5.



(a) Case 1 : $x_1 = 0.2t$ and $y_1 = 0.8t$.

(b) Case 2 : $x_1 = e^{-2t}$ and $y_1 = t^2$.

(c) Case 3 : $x_1 = \sin(t)$ and $y_1 = \cos(t)$

Figure 6 Error plotting of $\|x_{n+1} - x_n\|$ and $\|y_{n+1} - y_n\|$ in Example 5.2, the y-axis is illustrated on a logscale

Table 5 Numerical values of Algorithm (5.3) and Algorithm (5.4)

Starting point		Algorithm (5.3)	Algorithm (5.4)
$x_1 = 0.2t$	No. of Iter.	93	580
$y_1 = 0.8t$	CPU Time (s)	1.709670	8.759806
$x_1 = e^{-2t}$	No. of Iter.	80	1821
$y_1 = t^2$	CPU Time (s)	10.569948	27.241337
$x_1 = \sin(t)$	No. of Iter.	80	3145
$y_1 = \cos(t)$	CPU Time (s)	6.652260	134.727010

Remark 5.4 From our numerical experiments in Examples 5.1, 5.2, and 5.3, we make the following observations.

1. Table 1 and Figs. 1 and 2 show that $\{x_n\}$ and $\{y_n\}$ converge to 1, where $1 \in \mathrm{Fix}(T_i) \cap VI(C, A_i, f_i), \cap VI(C, B_i, f_i)$, for all $i = 1, 2$. The convergence of $\{x_n\}$ and $\{y_n\}$ of Example 5.1 can be guaranteed by Theorem 3.1.
2. Tables 2, 3, and 4 and Figs. 3, 4, 5, and 6 show that $\{x_n\}$ and $\{y_n\}$ converge to $x(t) = \mathbf{0}$, where $\mathbf{0} \in \mathrm{Fix}(T_i) \cap VI(C, A_i, f_i), \cap VI(C, B_i, f_i)$, for all $i = 1, 2$. The convergence of $\{x_n\}$ and $\{y_n\}$ of Example 5.2 can be guaranteed by Theorem 3.1.
3. From Table 5, we see that the sequence generated by our Algorithm (5.3) has a better convergence than Algorithm (5.4), which is modified from Algorithm 3.2 in [6], in terms of the number of iterations and the CPU time.

6 Conclusion

In this paper, we have proposed a new problem, called the combination of mixed variational inequality problems (1.7). This problem can be reduced to a classical variational inequalities problem (1.4). Using the intermixed method with viscosity technique, we introduce a new intermixed algorithm with viscosity technique for finding a solution of the combination of mixed variational inequality problems and the fixed-point problem of a nonexpansive mapping in a real Hilbert space. Moreover, we propose Lemmas 2.5 and 2.6 related to the combination of mixed variational inequality problems (1.7) in Sect. 2. Under some suitable conditions, a strong convergence theorem (Theorem 3.1) is established for the proposed Algorithm (3.1). We apply our theorem to solve the split-feasibility problem and the constrained convex-minimization problem. The effectiveness and numerical results of the proposed method for solving some examples in Hilbert space are illustrated (see Tables 1, 2, 3, 4, and 5 and Figs. 1, 2, 3, 4, 5, and 6). The obtained results improve and extend several previously published results in this field.

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
AK dealt with the conceptualization, formal analysis, supervision, writing—review and editing. WK writing—original draft, formal analysis, computation. Both authors read and approved the final manuscript.

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