Hu et al. *Journal of Inequalities and Applications* (2023) 2023:2
https://doi.org/10.1186/s13660-022-02906-w

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

# Quantitative versions of the two-dimensional Gaussian product inequalities

Ze-Chun Hu[1], Han Zhao[1] and Qian-Qian Zhou[2*]

*Correspondence:
qianqzhou@yeah.net
[2]School of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, China
Full list of author information is available at the end of the article

**Abstract**

The Gaussian product-inequality (GPI) conjecture is one of the most famous inequalities associated with Gaussian distributions and has attracted much attention. In this note, we investigate the quantitative versions of the two-dimensional Gaussian product inequalities. For any centered, nondegenerate, and two-dimensional Gaussian random vector $(X_1, X_2)$ with $E[X_1^2] = E[X_2^2] = 1$ and the correlation coefficient $\rho$, we prove that for any real numbers $\alpha_1, \alpha_2 \in (-1, 0)$ or $\alpha_1, \alpha_2 \in (0, \infty)$, it holds that

$$\mathbf{E}\big[|X_1|^{\alpha_1}|X_2|^{\alpha_2}\big] - \mathbf{E}\big[|X_1|^{\alpha_1}\big]\mathbf{E}\big[|X_2|^{\alpha_2}\big] \geq f(\alpha_1, \alpha_2, \rho) \geq 0,$$

where the function $f(\alpha_1, \alpha_2, \rho)$ will be given explicitly by the Gamma function and is positive when $\rho \neq 0$. When $-1 < \alpha_1 < 0$ and $\alpha_2 > 0$, Russell and Sun (Statist. Probab. Lett. 191:109656, 2022) proved the "opposite Gaussian product inequality", of which we will also give a quantitative version. These quantitative inequalities are derived by employing the hypergeometric functions and the generalized hypergeometric functions.

**MSC:** Primary 60E15; secondary 62H12

**Keywords:** Gaussian product-inequality conjecture; Quantitative inequality; Hypergeometric function

## 1 Introduction

The Gaussian product-inequality (GPI) conjecture is one of the most famous inequalities associated with Gaussian distributions, which states that for any $d$-dimensional, real-valued, and centered Gaussian random vector $(X_1, \ldots, X_d)$,

$$\mathbf{E}\left[\prod_{j=1}^{d} X_j^{2m}\right] \geq \prod_{j=1}^{d} \mathbf{E}\big[X_j^{2m}\big], \quad m \in \mathbb{N}. \tag{1.1}$$

In [12], Li and Wei proposed the following improved version of the GPI conjecture:

$$\mathbf{E}\left[\prod_{j=1}^{d}|X_j|^{\alpha_j}\right] \geq \prod_{j=1}^{d}\mathbf{E}\big[|X_j|^{\alpha_j}\big], \tag{1.2}$$

where $\alpha_j, j = 1, 2, \dots, d$, are nonnegative real numbers.

Up to now, the GPI (1.1) and the GPI (1.2) are still open, however, some special cases have been proved by using various tools. Frenkel [6] proved (1.1) with $m = 1$ (or (1.2) for the case $\alpha_j = 2$) by using algebraic methods. Malicet et al. [13] gave an analytic proof to Frenkel's result among other issues, such as giving new lower bounds on homogeneous polynomials (see [13, Theorem 1.5]), and presenting a new result supporting the U-conjecture (see [13, Theorem 1.7], the U-conjecture is still open). Wei [21] proved a stronger version of (1.2) for $\alpha_j \in (-1, 0)$ as follows:

$$\mathbf{E}\left[\prod_{j=1}^{d}|X_j|^{\alpha_j}\right] \geq \mathbf{E}\left[\prod_{j=1}^{k}|X_j|^{\alpha_j}\right]\mathbf{E}\left[\prod_{j=k+1}^{d}|X_j|^{\alpha_j}\right], \quad \forall 1 \leq k \leq d-1. \tag{1.3}$$

From Karlin and Rinott [10], we know that (1.1) and (1.2) hold for $\mathbf{X} = (X_1, \dots, X_d)$ when the density of $|\mathbf{X}| = (|X_1|, |X_2|, \dots, |X_d|)$ satisfies the so-called condition $\mathbf{MTP}_2$, and for any nondegenerate, 2-dimensional, and centered Gaussian random vector $(X_1, X_2)$, $(|X_1|, |X_2|)$ has an $\mathbf{MTP}_2$ density. Thus, (1.1) and (1.2) hold for $d = 2$.

Lan et al. [11] used the hypergeometric functions to prove the following 3-dimensional GPI: for any $m_1, m_2 \in \mathbb{N}$ and any centered Gaussian random vector $(X_1, X_2, X_3)$,

$$\mathbf{E}\big[X_1^{2m_1}X_2^{2m_2}X_3^{2m_2}\big] \geq \mathbf{E}\big[X_1^{2m_1}\big]\mathbf{E}\big[X_2^{2m_2}\big]\mathbf{E}\big[X_3^{2m_2}\big]. \tag{1.4}$$

Russell and Sun [20] proved the following 3-dimensional GPI: for any $m_2, m_3 \in \mathbb{N}$ and any centered Gaussian random vector $(X_1, X_2, X_3)$,

$$\mathbf{E}\big[X_1^{2}X_2^{2m_2}X_3^{2m_3}\big] \geq \mathbf{E}\big[X_1^{2}\big]\mathbf{E}\big[X_2^{2m_2}\big]\mathbf{E}\big[X_3^{2m_3}\big]. \tag{1.5}$$

Herry et al. [9] proved the following 3-dimensional GPI: for any $m_1, m_2, m_3 \in \mathbb{N}$ and any centered Gaussian random vector $(X_1, X_2, X_3)$,

$$\mathbf{E}\big[X_1^{2m_1}X_2^{2m_2}X_3^{2m_3}\big] \geq \mathbf{E}\big[X_1^{2m_1}\big]\mathbf{E}\big[X_2^{2m_2}\big]\mathbf{E}\big[X_3^{2m_3}\big]. \tag{1.6}$$

Genest and Ouimet [7] proved that if there exists a matrix $C \in [0, +\infty)^{d\times d}$ such that$(X_1, X_2, \dots, X_d) = (Z_1, Z_2, \dots, Z_d)C$ in law, where $(Z_1, Z_2, \dots, Z_d)$ is a $d$-dimensional, standard Gaussian random vector, the following stronger version of (1.1) holds:

$$\mathbf{E}\left[\prod_{j=1}^{d}X_j^{2m_j}\right] \geq \mathbf{E}\left[\prod_{j=1}^{k}X_j^{2m_j}\right]\mathbf{E}\left[\prod_{j=k+1}^{n}X_j^{2m_j}\right], \quad m_j \in \mathbb{N}, j = 1, \dots, n, \forall 1 \leq k \leq n-1. \tag{1.7}$$

Russell and Sun [17] proved among other things that (1.7) holds if all the correlation coefficients are nonnegative. Edelmann et al. [5] extended (1.7) to the multivarite gamma distributions. As to other related work, we refer to Genest and Ouiment [8], Russell and Sun [18], and Russell and Sun [19].

Recently, De et al. [3] obtained a range of quantitative correlation inequalities, including the quantitative version of the Gaussian correlation inequality proved by Royen [16], and the quantitative version of the well-known Fortuin–Kasteleyn–Ginibre (FKG) inequality for monotone functions over any finite product probability space. A further related work is De et al. [4].

To our understanding, quantitative correlation inequalities contain more information than qualitative correlation inequalities, and can imply the latter. A quantitative inequality with respect to (w.r.t. for short) the corresponding qualitative inequality is like the law of iterated logarithm w.r.t. the strong law of large numbers. In this note, we consider the quantitative versions of the two-dimensional Gaussian product inequalities.

For any centered, nondegenerate, and two-dimensional Gaussian random vector $(X_1, X_2)$ with the correlation coefficient $\rho$, without loss of generality, we assume that $\mathbf{E}[X_1^2] = \mathbf{E}[X_2^2] = 1$. We will prove that for any real numbers $\alpha_1, \alpha_2 \in (-1, 0)$ or $\alpha_1, \alpha_2 \in (0, \infty)$, it holds that

$$\mathbf{E}\big[|X_1|^{\alpha_1}|X_2|^{\alpha_2}\big] - \mathbf{E}\big[|X_1|^{\alpha_1}\big]\mathbf{E}\big[|X_2|^{\alpha_2}\big] \geq f(\alpha_1, \alpha_2, \rho) \geq 0,$$

where the function $f(\alpha_1, \alpha_2, \rho)$ will be given explicitly by the Gamma function and is positive when $\rho \neq 0$. When $-1 < \alpha_1 < 0$ and $\alpha_2 > 0$, Russell and Sun [18] proved the "opposite Gaussian product inequality", of which we will also give a quantitative version. These quantitative inequalities are derived by employing the hypergeometric functions and the generalized hypergeometric functions.

The rest of this note is organized as follows. In Sect. 2, we present the main results and some corollaries. In Sect. 3, we give the proofs. In Sect. 4, we give some remarks. In Sect. 5, we give the conclusion and introduce some related questions.

## 2 Main results

Throughout this note, any Gaussian random variable is assumed to be real-valued, nondegenerate, and standard. Our main results are as follows.

**Theorem 2.1** *Let $(X_1, X_2)$ be centered, bivariate Gaussian random variables with the correlation coefficient $\rho$ satisfying $|\rho| < 1$. Then, for any real numbers $\alpha_1, \alpha_2 \in (-1, 0)$ or $\alpha_1, \alpha_2 \in (0, \infty)$,*

$$\boldsymbol{E}\big[|X_1|^{\alpha_1}|X_2|^{\alpha_2}\big] - \boldsymbol{E}\big[|X_1|^{\alpha_1}\big]\boldsymbol{E}\big[|X_2|^{\alpha_2}\big] \geq f(\alpha_1, \alpha_2, \rho), \tag{2.1}$$

*where the nonnegative function $f(\cdot)$ is defined by*

$$f(\alpha_1, \alpha_2, \rho) := \begin{cases} \frac{2^{\frac{\alpha_1+\alpha_2}{2}}\alpha_1\alpha_2\rho^2}{2\pi}\Gamma(\frac{\alpha_1+1}{2})\Gamma(\frac{\alpha_2+1}{2}), & \textit{if } -1 < \alpha_1, \alpha_2 < 0 \textit{ or } 0 < \alpha_1, \alpha_2 \leq 2 \textit{ or } \alpha_1, \alpha_2 > 2, \\ \frac{2^{\frac{\alpha_1+\alpha_2}{2}}\alpha_1\alpha_2\rho^2}{4\sqrt{\pi}}\Gamma(\frac{\alpha_1+\alpha_2-1}{2}), & \textit{if } \alpha_1 > 2, 0 < \alpha_2 \leq 2 \textit{ or } 0 < \alpha_1 \leq 2, \alpha_2 > 2. \end{cases}$$

When the real numbers $\alpha_1$ and $\alpha_2$ in Theorem 2.1 have opposite signs, Russell and Sun [18] proved the following opposite GPI:

$$\mathbf{E}\big[|X_1|^{\alpha_1}|X_2|^{\alpha_2}\big] \le \mathbf{E}\big[|X_1|^{\alpha_1}\big]\mathbf{E}\big[|X_2|^{\alpha_2}\big]. \tag{2.2}$$

We have the quantitative version of the GPI (2.2) as follows.

**Theorem 2.2** *Let $(X_1, X_2)$ be centered, bivariate Gaussian random variables with the correlation coefficient $\rho$ satisfying $|\rho| < 1$. Let $\alpha_1 \in (-1, 0)$ and $\alpha_2 \in (0, \infty)$.*

(i) *When $0 < \alpha_2 \le 2$,*

$$\begin{aligned}
&\frac{2^{\frac{\alpha_1+\alpha_2}{2}}\alpha_1\alpha_2\rho^2}{2\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right)F\left(1-\frac{\alpha_1}{2}, 1-\frac{\alpha_2}{2}; \frac{3}{2}; 1\right)\\
&\quad\le \boldsymbol{E}\big[|X_1|^{\alpha_1}|X_2|^{\alpha_2}\big] - \boldsymbol{E}\big[|X_1|^{\alpha_1}\big]\boldsymbol{E}\big[|X_2|^{\alpha_2}\big]\\
&\quad\le \frac{2^{\frac{\alpha_1+\alpha_2}{2}}\alpha_1\alpha_2\rho^2}{2\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right) \le 0,
\end{aligned} \tag{2.3}$$

*where $F(\cdot)$ is the hypergeometric function defined by*

$$F(a, b; c; z) := \sum_{n=0}^{+\infty} \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{z^n}{n!}, \ |z| \le 1, \tag{2.4}$$

*and, for $\alpha \in \mathbb{R}$,*

$$(\alpha)_n := \begin{cases} \alpha(\alpha+1)\ldots(\alpha+n-1), & n \ge 1,\\ 1, & n = 0, \alpha \neq 0. \end{cases}$$

(ii) *When $\alpha_2 > 2$,*

$$\begin{aligned}
&\frac{2^{\frac{\alpha_1+\alpha_2}{2}}\alpha_1\alpha_2\rho^2}{2\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right)\\
&\quad\le \boldsymbol{E}\big[|X_1|^{\alpha_1}|X_2|^{\alpha_2}\big] - \boldsymbol{E}\big[|X_1|^{\alpha_1}\big]\boldsymbol{E}\big[|X_2|^{\alpha_2}\big]\\
&\quad\le \min\left\{\frac{2^{\frac{\alpha_1+\alpha_2}{2}}\alpha_1\alpha_2\rho^2}{2\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right)F\left(1-\frac{\alpha_1}{2}, 1-\frac{\alpha_2}{2}; \frac{3}{2}; 1\right), 0\right\}.
\end{aligned} \tag{2.5}$$

By Theorem 2.1, we have the following corollaries.

**Corollary 2.3** *Let $\alpha_1, \alpha_2 \in \{1, 2\}$ in Theorem 2.1, then we have*

$$\begin{aligned}
&\boldsymbol{E}\big[|X_1||X_2|\big] - \boldsymbol{E}\big[|X_1|\big]\boldsymbol{E}\big[|X_2|\big] \ge \frac{\rho^2}{\pi},\\
&\boldsymbol{E}\big[|X_1|X_2{}^2\big] - \boldsymbol{E}\big[|X_1|\big]\boldsymbol{E}\big[X_2{}^2\big] \ge \frac{\sqrt{2}\rho^2}{\sqrt{\pi}},\\
&\boldsymbol{E}\big[X_1{}^2X_2{}^2\big] - \boldsymbol{E}\big[X_1{}^2\big]\boldsymbol{E}\big[X_2{}^2\big] \ge 2\rho^2.
\end{aligned}$$

**Corollary 2.4** *Under the conditions of Theorem 2.1, suppose that $\alpha_2 = 1$, and $\alpha_1$ is an integer satisfying $\alpha_1 = m > 2$. Then,*

$$\boldsymbol{E}\big[|X_1|^m|X_2|\big] - \boldsymbol{E}\big[|X_1|^m\big]\boldsymbol{E}\big[|X_2|\big] \geq \begin{cases} \frac{(m-2)!!m\rho^2}{\sqrt{2\pi}}, & \text{if } m \text{ is even},\\ \frac{(m-2)!!m\rho^2}{2}, & \text{if } m \text{ is odd}.\end{cases}$$

**Corollary 2.5** *Under the conditions of Theorem 2.1, suppose that $\alpha_1$ and $\alpha_2$ are integers and $\alpha_1 = m$, $\alpha_2 = n$ with $m, n \in (2,\infty)$,*

(i) *if $m$, $n$ are both even integers, then*

$$\boldsymbol{E}\big[X_1{}^m X_2{}^n\big] - \boldsymbol{E}\big[X_1{}^m\big]\boldsymbol{E}\big[X_2{}^n\big] \geq \frac{(m-1)!!(n-1)!!mn\rho^2}{2};$$

(ii) *if $m$, $n$ are both odd integers, then*

$$\boldsymbol{E}\big[|X_1|^m|X_2|^n\big] - \boldsymbol{E}\big[|X_1|^m\big]\boldsymbol{E}\big[|X_2|^n\big] \geq \frac{(m-1)!!(n-1)!!mn\rho^2}{\pi};$$

(iii) *if one of $m$ and $n$ is odd and the other is even, then*

$$\boldsymbol{E}\big[|X_1|^m|X_2|^n\big] - \boldsymbol{E}\big[|X_1|^m\big]\boldsymbol{E}\big[|X_2|^n\big] \geq \frac{(m-1)!!(n-1)!!mn\rho^2}{\sqrt{2\pi}}.$$

## 3 Proofs

In this section, we will give the proofs of Theorems 2.1 and 2.2. First, we recall some properties of hypergeometric functions and generalized hypergeometric functions in Sect. 3.1. The proofs of Theorems 2.1 and 2.2 will be presented in Sects. 3.2 and 3.3, respectively.

### 3.1 Preliminaries

**Definition 3.1** (see Andrews et al. [1, Chap. 2] or Rainville [15, Chap. 5]) The generalized hypergeometric function ${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)$ is defined by

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) := \sum_{k=0}^{+\infty} \frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\cdot\frac{z^n}{n!}, \tag{3.1}$$

where

$$(\alpha)_n := \begin{cases} \alpha(\alpha+1)\dots(\alpha+n-1), & n \geq 1,\\ 1, & n = 0, \alpha \neq 0.\end{cases}$$

In particular, when $p = 2$ and $q = 1$, ${}_2F_1(a_1,a_2;b_1;z)$ is called the hypergeometric function that is written as $F(a,b;c;z)$ in short (see (2.4)).

**Theorem 3.2** (see Andrews et al. [1, Theorem 2.2.5] or Rainville [15, Theorem 21]) *If $|z| < 1$,*

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z). \tag{3.2}$$

**Theorem 3.3** (see Rainville [15, Theorem 28]) *If $p \le q+1$, $Re(b_1)> Re(a_1)> 0$, no one of $b_1, b_2, \dots, b_q$ is zero or a negative integer, and $|z| < 1$,*

$$\begin{aligned} &{}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) \\ &\quad = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1 - a_1)} \int_0^1 t^{(a_1-1)}(1-t)^{(b_1-a_1-1)}_{p-1}F_{q-1}(a_2, \dots, a_p; b_2, \dots, b_q; z). \end{aligned} \tag{3.3}$$

*If $p \le q$, the condition $|z| < 1$ may be omitted.*

**Theorem 3.4** (see Rainville [15, Theorem 18]) *If* $\mathrm{Re}(c-a-b) > 0$ *and if $c$ is neither zero nor a negative integer,*

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{3.4}$$

In addition, from Andrews et al. [1, (2.5.1)], we know that the hypergeometric function satisfies the following differential equation:

$$\frac{d}{dz}F(a, b; c; z) = \frac{ab}{c}F(a+1, b+1; c+1; z). \tag{3.5}$$

### 3.2 Proof of Theorem 2.1

Obviously, we can assume that $\rho \neq 0$. From the density function of a centered Gaussian random variable and the definition of the Gamma function, we have that for $i = 1, 2$,

$$\begin{aligned} \mathbf{E}\bigl[|X_i|^{\alpha_i}\bigr] &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} |x|^{\alpha_i} e^{-\frac{x^2}{2}}\,dx = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{+\infty} x^{\alpha_i} e^{-\frac{x^2}{2}}\,dx \\ &= \frac{2^{\frac{\alpha_i}{2}}}{\sqrt{\pi}} \int_0^{+\infty} y^{\frac{\alpha_i - 1}{2}} e^{-y}\,dy = \frac{2^{\frac{\alpha_i}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{\alpha_i + 1}{2}\right). \end{aligned} \tag{3.6}$$

Then,

$$\mathbf{E}\bigl[|X_1|^{\alpha_1}\bigr]\mathbf{E}\bigl[|X_2|^{\alpha_2}\bigr] = \frac{2^{\frac{\alpha_1+\alpha_2}{2}}}{\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right). \tag{3.7}$$

Since $|\rho| < 1$, by Nabeya [14], we know that

$$\mathbf{E}\bigl[|X_1|^{\alpha_1}|X_2|^{\alpha_2}\bigr] = \frac{2^{\frac{\alpha_1+\alpha_2}{2}}}{\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right)F\left(-\frac{\alpha_1}{2}, -\frac{\alpha_2}{2}; \frac{1}{2}; \rho^2\right), \tag{3.8}$$

where $F(\cdot)$ is the hypergeometric function defined by (2.4).

It follows from (3.7) and (3.8) that

$$\begin{aligned} &\mathbf{E}\bigl[|X_1|^{\alpha_1}|X_2|^{\alpha_2}\bigr] - \mathbf{E}\bigl[|X_1|^{\alpha_1}\bigr]\mathbf{E}\bigl[|X_2|^{\alpha_2}\bigr] \\ &\quad = \frac{2^{\frac{\alpha_1+\alpha_2}{2}}}{\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right)\left[F\left(-\frac{\alpha_1}{2}, -\frac{\alpha_2}{2}; \frac{1}{2}; \rho^2\right) - 1\right]. \end{aligned} \tag{3.9}$$

Since $\frac{2^{\frac{\alpha_1+\alpha_2}{2}}}{\pi}\Gamma(\frac{\alpha_1+1}{2})\Gamma(\frac{\alpha_2+1}{2}) > 0$ for any $-1 < \alpha_1, \alpha_2 < 0$ or $\alpha_1, \alpha_2 > 0$, it is enough to find the lower bound of $F(-\frac{\alpha_1}{2}, -\frac{\alpha_2}{2}; \frac{1}{2}; \rho^2) - 1$. First, we have that

$$\begin{aligned} F\left(-\frac{\alpha_1}{2}, -\frac{\alpha_2}{2}; \frac{1}{2}; \rho^2\right) - 1 &= \sum_{k=1}^{+\infty} \frac{(-\frac{\alpha_1}{2})_k(-\frac{\alpha_2}{2})_k}{(\frac{1}{2})_k} \cdot \frac{(\rho^2)^k}{k!} \\ &= \sum_{k=0}^{+\infty} \frac{(-\frac{\alpha_1}{2})_{k+1}(-\frac{\alpha_2}{2})_{k+1}}{(\frac{1}{2})_{k+1}} \cdot \frac{(\rho^2)^{k+1}}{(k+1)!} \\ &= \frac{\rho^2\alpha_1\alpha_2}{2} \sum_{k=0}^{+\infty} \frac{(1-\frac{\alpha_1}{2})_k(1-\frac{\alpha_2}{2})_k(1)_k}{(\frac{3}{2})_k(2)_k} \cdot \frac{\rho^{2k}}{(k)!} \\ &= \frac{\rho^2\alpha_1\alpha_2}{2}\, {}_3F_2\left(1-\frac{\alpha_1}{2}, 1-\frac{\alpha_2}{2}, 1; \frac{3}{2}, 2; \rho^2\right), \end{aligned} \tag{3.10}$$

where ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ is the generalized hypergeometric function defined by (3.1). By the property of the generalized hypergeometric function in (3.3) and (3.10), we obtain that

$$\begin{aligned} F\left(-\frac{\alpha_1}{2}, -\frac{\alpha_2}{2}; \frac{1}{2}; \rho^2\right) - 1 &= \frac{\rho^2\alpha_1\alpha_2}{2} \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \int_0^1 F\left(1-\frac{\alpha_1}{2}, 1-\frac{\alpha_2}{2}; \frac{3}{2}; \rho^2 t\right) dt \\ &= \frac{\alpha_1\alpha_2}{2} \int_0^{\rho^2} F\left(1-\frac{\alpha_1}{2}, 1-\frac{\alpha_2}{2}; \frac{3}{2}; h\right) dh. \end{aligned} \tag{3.11}$$

Secondly, by (3.5) and the Euler transformation (3.2), we have that

$$\begin{aligned} &\frac{d}{dh} F\left(1-\frac{\alpha_1}{2}, 1-\frac{\alpha_2}{2}; \frac{3}{2}; h\right) \\ &= \frac{2}{3}\left(1-\frac{\alpha_1}{2}\right)\left(1-\frac{\alpha_2}{2}\right) F\left(2-\frac{\alpha_1}{2}, 2-\frac{\alpha_2}{2}; \frac{5}{2}; h\right) \\ &= \frac{2}{3}\left(1-\frac{\alpha_1}{2}\right)\left(1-\frac{\alpha_2}{2}\right)(1-h)^{\frac{1+\alpha_1+\alpha_2}{2}-2} F\left(\frac{1}{2}+\frac{\alpha_1}{2}, \frac{1}{2}+\frac{\alpha_2}{2}; \frac{5}{2}; h\right). \end{aligned} \tag{3.12}$$

Therefore, in order to estimate $F(-\frac{\alpha_1}{2}, -\frac{\alpha_2}{2}; \frac{1}{2}; \rho^2) - 1$, we divide it into the following two cases.

**Case I**: When $-1 < \alpha_1, \alpha_2 < 0$ or $0 < \alpha_1, \alpha_2 \le 2$, or $\alpha_1, \alpha_2 > 2$.

In this case, $(1-\frac{\alpha_1}{2})(1-\frac{\alpha_2}{2}) \ge 0$ and $\alpha_1\alpha_2 > 0$. Thus, the derivative of $\frac{d}{dh}F(1-\frac{\alpha_1}{2}, 1-\frac{\alpha_2}{2}; \frac{3}{2}; h)$ in (3.12) is nonnegative. Therefore, by (3.11) we obtain that

$$F\left(-\frac{\alpha_1}{2}, -\frac{\alpha_2}{2}; \frac{1}{2}; \rho^2\right) - 1 \ge \frac{\alpha_1\alpha_2}{2} \int_0^{\rho^2} F\left(1-\frac{\alpha_1}{2}, 1-\frac{\alpha_2}{2}; \frac{3}{2}; 0\right) dh = \frac{\alpha_1\alpha_2}{2} \cdot \rho^2.$$

Thus, by the above statements, we obtain that

$$\begin{aligned} \mathbf{E}\left[|X_1|^{\alpha_1}|X_2|^{\alpha_2}\right] - \mathbf{E}\left[|X_1|^{\alpha_1}\right]\mathbf{E}\left[|X_2|^{\alpha_2}\right] &\ge \frac{2^{\frac{\alpha_1+\alpha_2}{2}}}{\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right)\frac{\alpha_1\alpha_2}{2} \cdot \rho^2 \\ &= \frac{2^{\frac{\alpha_1+\alpha_2}{2}}\alpha_1\alpha_2\rho^2}{2\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right). \end{aligned}$$

**Case II**: When $\alpha_1 > 2$ and $0 < \alpha_2 \leq 2$ or $0 < \alpha_1 \leq 2$ and $\alpha_2 > 2$.

Without loss of generality, it is assumed that $\alpha_1 > 2$ and $0 < \alpha_2 \leq 2$. Then, $(1 - \frac{\alpha_1}{2})(1 - \frac{\alpha_2}{2}) \leq 0$ and $\alpha_1\alpha_2 > 0$. Thus, in this case, by (3.12), we know that

$$\frac{d}{dh}F\left(1-\frac{\alpha_1}{2},1-\frac{\alpha_2}{2};\frac{3}{2};h\right) \leq 0.$$

Then, $F(1-\frac{\alpha_1}{2},1-\frac{\alpha_2}{2};\frac{3}{2};h)$ reaches its minimum at $h = 1$, and the minimum value is

$$F\left(1-\frac{\alpha_1}{2},1-\frac{\alpha_2}{2};\frac{3}{2};1\right) = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{\alpha_1+\alpha_2}{2}-\frac{1}{2})}{\Gamma(\frac{\alpha_1}{2}+\frac{1}{2})\Gamma(\frac{\alpha_2}{2}+\frac{1}{2})},$$

where the equality holds by (3.4). Then, from (3.11), we obtain

$$F\left(-\frac{\alpha_1}{2},-\frac{\alpha_2}{2};\frac{1}{2};\rho^2\right) - 1 \geq \frac{\alpha_1\alpha_2}{2}\cdot\rho^2\frac{\Gamma(\frac{3}{2})\Gamma(\frac{\alpha_1+\alpha_2}{2}-\frac{1}{2})}{\Gamma(\frac{\alpha_1}{2}+\frac{1}{2})\Gamma(\frac{\alpha_2}{2}+\frac{1}{2})}.$$

From (3.9), we obtain

$$\begin{aligned}
&\mathbf{E}\big[|X_1|^{\alpha_1}|X_2|^{\alpha_2}\big] - \mathbf{E}\big[|X_1|^{\alpha_1}\big]\mathbf{E}\big[|X_2|^{\alpha_2}\big] \\
&\quad\geq \frac{2^{\frac{\alpha_1+\alpha_2}{2}}}{\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right)\frac{\alpha_1\alpha_2}{2}\cdot\rho^2\frac{\Gamma(\frac{3}{2})\Gamma(\frac{\alpha_1+\alpha_2}{2}-\frac{1}{2})}{\Gamma(\frac{\alpha_1}{2}+\frac{1}{2})\Gamma(\frac{\alpha_2}{2}+\frac{1}{2})} \\
&\quad= \frac{2^{\frac{\alpha_1+\alpha_2}{2}}\alpha_1\alpha_2\rho^2}{4\sqrt{\pi}}\Gamma\left(\frac{\alpha_1+\alpha_2-1}{2}\right).
\end{aligned}$$

The proof is complete.

### 3.3 Proof of Theorem 2.2

From the proof of Theorem 2.1, we know that

$$\begin{aligned}
&\mathbf{E}\big[|X_1|^{\alpha_1}|X_2|^{\alpha_2}\big] - \mathbf{E}\big[|X_1|^{\alpha_1}\big]\mathbf{E}\big[|X_2|^{\alpha_2}\big] \\
&\quad= \frac{2^{\frac{\alpha_1+\alpha_2}{2}}}{\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right)\left[F\left(-\frac{\alpha_1}{2},-\frac{\alpha_2}{2};\frac{1}{2};\rho^2\right)-1\right], \qquad (3.13)\\
&F\left(-\frac{\alpha_1}{2},-\frac{\alpha_2}{2};\frac{1}{2};\rho^2\right) - 1 = \frac{\alpha_1\alpha_2}{2}\int_0^{\rho^2}F\left(1-\frac{\alpha_1}{2},1-\frac{\alpha_2}{2};\frac{3}{2};h\right)dh, \qquad (3.14)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dh}F\left(1-\frac{\alpha_1}{2},1-\frac{\alpha_2}{2};\frac{3}{2};h\right) \\
&\quad= \frac{2}{3}\left(1-\frac{\alpha_1}{2}\right)\left(1-\frac{\alpha_2}{2}\right)(1-h)^{\frac{1+\alpha_1+\alpha_2}{2}-2}F\left(\frac{1}{2}+\frac{\alpha_1}{2},\frac{1}{2}+\frac{\alpha_2}{2};\frac{5}{2};h\right).
\end{aligned}$$

**Case I**: $0 < \alpha_2 \leq 2$. In this case $1 - \frac{\alpha_2}{2} \geq 0$ and thus

$$\frac{d}{dh}F\left(1-\frac{\alpha_1}{2},1-\frac{\alpha_2}{2};\frac{3}{2};h\right) \geq 0.$$

Then, from (3.14), we obtain

$$\frac{\alpha_1\alpha_2\rho^2}{2}F\left(1-\frac{\alpha_1}{2},1-\frac{\alpha_2}{2};\frac{3}{2};1\right)\le F\left(-\frac{\alpha_1}{2},-\frac{\alpha_2}{2};\frac{1}{2};\rho^2\right)-1\le\frac{\alpha_1\alpha_2\rho^2}{2}\le 0,$$

which together with (3.13) implies (2.3).

**Case II**: $\alpha_2 > 2$. In this case, $1-\frac{\alpha_2}{2}<0$, and thus

$$\frac{d}{dh}F\left(1-\frac{\alpha_1}{2},1-\frac{\alpha_2}{2};\frac{3}{2};h\right)\le 0.$$

Then, from (3.14), it can be derived that

$$\frac{\alpha_1\alpha_2\rho^2}{2}\le F\left(-\frac{\alpha_1}{2},-\frac{\alpha_2}{2};\frac{1}{2};\rho^2\right)-1\le\frac{\alpha_1\alpha_2\rho^2}{2}F\left(1-\frac{\alpha_1}{2},1-\frac{\alpha_2}{2};\frac{3}{2};1\right),$$

which together with (3.13) implies (2.5).

The proof is complete. □

## 4 Remarks

*Remark* 4.1 If $\rho$ in Theorem 2.1 satisfies that $|\rho|=1$, then $X_2=\pm X_1$ in law and thus $\mathbf{E}[|X_1|^{\alpha_1}|X_2|^{\alpha_2}]=\mathbf{E}[|X_1|^{\alpha_1+\alpha_2}]$. From (3.6) and $\Gamma(\frac{1}{2})=\sqrt{\pi}$, we know that

$$\begin{aligned}
&\mathbf{E}\big[|X_1|^{\alpha_1}|X_2|^{\alpha_2}\big]-\mathbf{E}\big[|X_1|^{\alpha_1}|\big]\mathbf{E}\big[|X_2|^{\alpha_2}\big]\\
&=\mathbf{E}\big[|X_1|^{\alpha_1+\alpha_2}\big]-\mathbf{E}\big[|X_1|^{\alpha_1}|\big]\mathbf{E}\big[|X_2|^{\alpha_2}\big]\\
&=\frac{2^{\frac{\alpha_1+\alpha_2}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha_1+\alpha_2+1}{2}\right)-\frac{2^{\frac{\alpha_1+\alpha_2}{2}}}{\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right)\\
&=\frac{2^{\frac{\alpha_1+\alpha_2}{2}}}{\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right)\left[\frac{\Gamma(\frac{\alpha_1+\alpha_2+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{\alpha_1+1}{2})\Gamma(\frac{\alpha_2+1}{2})}-1\right]\\
&=\frac{2^{\frac{\alpha_1+\alpha_2}{2}}}{\pi}\Gamma\left(\frac{\alpha_1+1}{2}\right)\Gamma\left(\frac{\alpha_2+1}{2}\right)\left[F\left(-\frac{\alpha_1}{2},-\frac{\alpha_2}{2};\frac{1}{2};1\right)-1\right],
\end{aligned}$$

the last equality follows from (3.4). That is to say, (3.9) still holds in this case. Hence, the result of Theorem 2.1 still holds if $|\rho|=1$ and one of the following two conditions holds:

(i) $\alpha_1,\alpha_2\in(0,\infty)$;

(ii) $\alpha_1,\alpha_2\in(-1,0)$ and $\alpha_1+\alpha_2\in(-1,0)$.

*Remark* 4.2 If $\rho$ in Theorem 2.2 satisfies that $|\rho|=1$, then by Remark 4.1 and the proof of Theorem 2.2 we know that the result of Theorem 2.2 still holds in this case.

## 5 Conclusions

In this note, we gave several quantitative versions of the two-dimensional Gaussian product inequalities, which can imply the corresponding qualitative inequalities and contain more information. The main results are Theorems 2.1 and 2.2.

We hope that this note is a good starting point in this aspect and can stimulate more related work. In fact, we can ask the following questions:

**Question 1** *How about the quantitative versions of the* 3*-dimensional Gaussian product inequalities* (1.4), (1.5), *and* (1.6)*?*

**Question 2** *How about the quantitative version of the product inequality* (1.7)*?*

**Question 3** *How about the quantitative version of the product inequality* (1.3)*?*

**Acknowledgements**
We thank the referee for helpful suggestions that helped improve the presentation of the paper.

**Funding**
This work was supported by the National Natural Science Foundation of China (12171335), the Science Development Project of Sichuan University (2020SCUNL201), and the Scientific Foundation of Nanjing University of Posts and Telecommunications (NY221026).

**Availability of data and materials**
Not applicable.

## Declarations

**Competing interests**
The authors declare no competing interests.

**Author contributions**
Zechun Hu proposed this subject and presented the main ideas of the results. Han Zhao contributed to the proofs of the theorems. Qianqian Zhou checked all the proofs of the main results and was a major contributor in writing the manuscript. All authors read and approved the final manuscript.

**Author details**
[1]College of Mathematics, Sichuan University, Chengdu 610065, China. [2]School of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, China.

## Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 14 November 2022 Accepted: 19 December 2022 Published online: 04 January 2023

**References**

1. Andrews, G.E., Askey, R., Roy, R. (eds.): Special Functions Cambridge University Press, Cambridge (1999)
2. Benitez, C., Sarantopoulos, Y., Tonge, A.: Lower bounds for norms of products of polynomials. Math. Proc. Camb. Philos. Soc. **124**, 395–408 (1998)
3. De, A., Nadimpalli, S., Servedio, R.A.: Quantitative correlation inequalities via semigroup interpolation. In: 12th Innovations in Theoretical Computer Science Conference (2021). Art. No. 69, 20 pp. LIPIcs. Leibniz Int. Proc. Inform., 185, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern
4. De, A., Nadimpalli, S., Servedio, R.A.: Quantitative correlation inequalities via extremal power series. Probab. Theory Relat. Fields **183**, 649–675 (2022)
5. Edelmann, D., Richards, D., Royen, T.: Product Inequalities for multivariate Gaussian, Gamma, and positively upper orthant dependent distributions (2022). arXiv:2204.06220v1
6. Frenkel, P.E.: Pfaffians, Hafnians and products of real linear functionals. Math. Res. Lett. **15**, 351–358 (2008)
7. Genest, C., Ouimet, F.: A combinatorial proof of the Gaussian product inequality conjecture beyond the $\mathrm{MTP}_2$ case. Depend. Model. **10**, 236–244 (2022)
8. Genest, C., Ouimet, F.: Miscellaneous results related to the Gaussian product inequality conjecture for the joint distribution of traces of Wishart matrices (2022). arXiv:2206.01976v1
9. Herry, R., Malicet, D., Poly, G.: A short proof of the strong three dimensional Gaussian product inequality (2022). arXiv:2211.07314v1
10. Karlin, S., Rinott, Y.: Total positivity properties of absolute value multinormal variables with applications to confidence interval estimates and related probabilistic inequalities. Ann. Stat. **9**, 1035–1049 (1981)
11. Lan, G.L., Hu, Z.C., Sun, W.: The three-dimensional Gaussian product inequality. J. Math. Anal. Appl. **485**, 123858 (2020)
12. Li, W.V., Wei, A.: A Gaussian inequality for expected absolute products. J. Theor. Probab. **25**, 92–99 (2012)
13. Malicet, D., Nourdin, I., Peccati, G., Poly, G.: Squared chaotic random variables: new moment inequalities with applications. J. Funct. Anal. **270**, 649–670 (2016)
14. Nabeya, S.: Absolute moments in 2-dimensional normal distribution. Ann. Inst. Stat. Math. **3**, 1–6 (1951)
15. Rainville, E.D.: Special Functions. The Macmillan Company, New York (1960)
16. Royen, T.: A simple proof of the Gaussian correlation conjecture extended to multivariate gamma distributions. Far East J. Their. Stat. **48**, 139–145 (2014)
17. Russell, O., Sun, W.: Some new Gaussian product inequalities. J. Math. Anal. Appl. **551**, 126439 (2022)

18. Russell, O., Sun, W.: An opposite Gaussian product inequality. Stat. Probab. Lett. **191**, 109656 (2022)
19. Russell, O., Sun, W.: Using sums-of-squares to prove Gaussian product inequalities (2022). arXiv:2205.02127v1
20. Russell, O., Sun, W.: Moment ratio inequality of bivariate Gaussian distribution and three-dimensional Gaussian product inequality (2022). arXiv:2208.13957v1
21. Wei, A.: Representations of the absolute value function and applications in Gaussian estimates. J. Theor. Probab. **27**, 1059–1070 (2014)