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New Minkowski and related inequalities via general kernels and measures

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Abstract

In this article, we introduce a class of functions $\mathfrak{U}(\mathfrak{p})$ with integral representation defined over a measure space with σ-finite measure. The main purpose of this paper is to extend the Minkowski and related inequalities by considering general kernels. As a consequence of our general results, we connect our results with various variants for the fractional integrals operators. Such applications have wide use and importance in the field of applied sciences.

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1 Introduction

Fractional calculus is generally referred to as the calculus of noninteger order. In the last few decades, the concept of fractional calculus has been comprehensively studied by various mathematicians [1–6]. Studying different aspects of the subject has stimulated many mathematicians to put their continued efforts into different time scales. Continuously, researchers have given the generalizations of fractional integrals by using different techniques. It is always interesting and motivating for us to provide the generalization of inequalities that cover all possible results, which are proven till now for different fractional integrals.

Recently, various inequalities have been given in the sense of generalizations and improvements for different fractional integrals. We state some of them here; the variants of Minkowski, Wirtinger, Hardy, Opial, Ostrowski, Hermite–Hadamard, Lyenger, Grüss, Cebyšev, and Pólya–Szegö [7–15]. Such applications of fractional integral operators compelled us to show the generalization of the reverse Minkowski inequality [7–9] involving general kernels.

Let (Δ, Σ, π) be a measure space with a positive σ-finite measure, $\mathfrak{p} : \Delta \times \Delta \to \mathbb{R}$ be a nonnegative function, and

$$\Theta(\varrho) = \int_{\Delta} \mathfrak{p}(\varrho, \chi)\, d\pi(\chi), \quad \varrho \in \Delta. \tag{1.1}$$

Throughout this paper, we suppose $\Theta(\varrho) > 0$ a.e. on Δ.

Let $\mathfrak{U}(\mathfrak{p})$ denote the class of functions $L : \Delta \to \mathbb{R}$ with the representation

$$\mathfrak{L}(\varrho) = \int_{\Delta} \mathfrak{p}(\varrho, \chi) L(\chi)\, d\pi(\chi),$$

where $\mathfrak{L} : \Delta \to \mathbb{R}$ is a measurable function.

Definition 1.1 ([15]) Let $f \in L_1([a,b])$ (the Lebesgue measure). The left-sided and right-sided Riemann–Liouville fractional integrals $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b^-}f$ of order $\alpha > 0$ are defined by

$$I^{\alpha}_{a^+}f(\varrho) = \frac{1}{\Gamma(\alpha)} \int_a^{\varrho} f(\chi)(\varrho - \chi)^{\alpha-1}\, d\chi, \quad (\varrho > a)$$

and

$$I^{\alpha}_{b^-}f(\varrho) = \frac{1}{\Gamma(\alpha)} \int_{\varrho}^{b} f(\chi)(\chi - \varrho)^{\alpha-1}\, d\chi, \quad (\varrho < b),$$

where $\Gamma(\alpha)$ is the usual gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{x} x^{\alpha-1}\, dx, \quad \operatorname{Re}(\alpha) > 0.$$

Definition 1.2 ([16]) Let $f \in L_1([a,b])$ (the Lebesgue measure). The left-sided and right-sided Riemann–Liouville k-fractional integrals $I^{\alpha,k}_{a^+}f$ and $I^{\alpha,k}_{b^-}f$ of order $\alpha > k$ are defined by

$$I^{\alpha,k}_{a^+}f(\varrho) = \frac{1}{k\Gamma_k(\alpha)} \int_a^{\varrho} f(\chi)(\varrho - \chi)^{\frac{\alpha}{k}-1}\, d\chi, \quad (\varrho > a)$$

and

$$I^{\alpha,k}_{b^-}f(\varrho) = \frac{1}{k\Gamma_k(\alpha)} \int_{\varrho}^{b} f(\chi)(\chi - \varrho)^{\frac{\alpha}{k}-1}\, d\chi, \quad (\varrho < b),$$

where $\Gamma_k(\alpha)$ is the k-gamma function defined by

$$\Gamma_k(\alpha) = \int_0^{\infty} e^{-\frac{x^k}{k}} x^{\alpha-1}\, dx, \quad \operatorname{Re}(\alpha) > 0.$$

A more general form of Definition 1.2 is given in the next definition.

Definition 1.3 Let $k > 0$, $(a,b)(-\infty \le a < b \le \infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also, let $\mathfrak{g}$ be an increasing and positive monotone on $(a,b]$. The left- and right-sided fractional integrals of a function f with respect to another function $\mathfrak{g}$ of order $\alpha, k > 0$ in $[a,b]$ are given by

$$I^{\alpha,k}_{a+;\mathfrak{g}}f(\varrho) = \frac{1}{k\Gamma_k(\alpha)} \int_a^{\varrho} \frac{\mathfrak{g}'(\chi)f(\chi)\, d\chi}{[\mathfrak{g}(\varrho) - \mathfrak{g}(\chi)]^{1-\frac{\alpha}{k}}}, \quad \varrho > a$$

and

$$I^{\alpha,k}_{b-;\mathfrak{g}}f(\varrho) = \frac{1}{k\Gamma_k(\alpha)} \int_{\varrho}^{b} \frac{\mathfrak{g}'(\chi)f(\chi)\, d\chi}{[\mathfrak{g}(\chi) - \mathfrak{g}(\varrho)]^{1-\frac{\alpha}{k}}}, \quad \varrho < b.$$

Definition 1.4 ([4]) Let $(a,b)(0 \leq a < b \leq \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$. Also, let $\alpha > 0, \sigma > 0$, and $\eta \in \mathbb{R}$. We consider the left- and right-sided integrals of order $\alpha \in \mathbb{R}$ defined by

$$I^{\alpha}_{a+;\sigma;\eta} f(\varrho) = \frac{\sigma \varrho^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^{\varrho} \frac{\chi^{\sigma\eta+\sigma-1} f(\chi)\,d\chi}{(\varrho^{\sigma} - \chi^{\sigma})^{1-\alpha}} \tag{1.2}$$

and

$$I^{\alpha}_{b-;\sigma;\eta} f(\varrho) = \frac{\sigma \varrho^{\sigma\eta}}{\Gamma(\alpha)} \int_{\varrho}^{b} \frac{t^{\sigma(1-\eta-\alpha)-1} f(\chi)\,d\chi}{(\chi^{\sigma} - x^{\sigma})^{1-\alpha}}, \tag{1.3}$$

respectively. Integrals (1.2) and (1.3) are called Erdélyi–Kober-type fractional integrals.

Consider the space $X^p_c(a,b)(c \in \mathbb{R}, 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{X^p_c(a,b)} < \infty$, where the norm is defined by

$$\|f\|_{X^p_c(a,b)} = \int_a^b \left|\chi^c f(\chi)\right|^p \frac{d\chi}{\chi} < \infty.$$

Definition 1.5 ([17]) Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-sided Katugampola fractional integrals of order $\alpha > 0$ of $f \in X^p_c(a,b)$ are defined by

$${}^{\rho}I^{\alpha}_{a+} f(\varrho) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^{\varrho} \frac{t^{\rho-1} f(\chi)\,d\chi}{(\varrho^{\rho} - \chi^{\rho})^{1-\alpha}}$$

and

$${}^{\rho}I^{\alpha}_{b-} f(\varrho) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^{\varrho} \frac{\chi^{\rho-1} f(\chi)\,d\chi}{(\chi^{\rho} - \varrho^{\rho})^{1-\alpha}},$$

with $a < \varrho < b$ and $\rho > 0$, if the integrals exist.

Definition 1.6 ([1]) Let $\beta \in \mathbb{C}$ and $\mathbb{R}(\beta) > 0$. We define the left-fractional conformable integral operator and right-fractional conformable integral operator by

$${}^{\rho}_{\alpha}\mathfrak{J}^{\alpha} f(\varrho) = \frac{1}{\Gamma(\beta)} \int_a^{\varrho} \left(\frac{(\varrho - a)^{\alpha} - (\chi - a)^{\alpha}}{\alpha} \right)^{\beta-1} f(\chi) \frac{d\chi}{(\chi - a)^{1-\alpha}}$$

and

$${}^{\rho}\mathfrak{J}^{\alpha}_{\beta} f(\varrho) = \frac{1}{\Gamma(\beta)} \int_{\varrho}^{b} \left(\frac{(b - \varrho)^{\alpha} - (b - \chi)^{\alpha}}{\alpha} \right)^{\beta-1} f(\chi) \frac{d\chi}{(b - \chi)^{1-\alpha}},$$

respectively.

Definition 1.7 ([3]) Let ϕ be a conformable fractional integral on the interval $[p,q] \subseteq (0,\infty)$. The right-sided and left-sided generalized conformable fractional integrals ${}^{\tau}_{\alpha}K^{\beta}_{p^+}$ and ${}^{\tau}_{\alpha}K^{\beta}_{q^-}$ of order $\beta > 0, \tau \in \mathbb{R}, \alpha + \tau \neq 0$, are defined by

$${}^{\tau}_{\alpha}K^{\beta}_{p^+} \phi(r) = \frac{1}{\Gamma(\beta)} \int_p^r \left(\frac{r^{\alpha+\tau} - w^{\alpha+\tau}}{\alpha + \tau} \right)^{\beta-1} \frac{\phi(w)}{w^{1-\tau-\alpha}}\,dw$$

and

$$ {}_{\alpha}^{\tau}K_{q_-}^{\beta}\phi(r) = \frac{1}{\Gamma(\beta)}\int_r^q \left(\frac{w^{\alpha+\tau}-r^{\alpha+\tau}}{\alpha+\tau}\right)^{\beta-1}\frac{\phi(w)}{w^{1-\tau-\alpha}}\,dw, $$

respectively, with ${}_{\alpha}^{\tau}K_{p^+}^{0}\phi(r) = {}_{\alpha}^{\tau}K_{q_-}^{0}\phi(r) = \phi(r)$.

2 Preliminaries

This section is dedicated to some known results.

Theorem 2.1 ([18]) *For $p \geq 1$, let there be two positive functions q_1 and q_2 on $[0,\infty)$. If $0 < \nu_1 \leq \frac{q_1(\zeta)}{q_2(\zeta)} \leq \nu_2$ and $\chi \in [\kappa_1,\kappa_2]$, then*

$$ \left(\int_{\kappa_1}^{\kappa_2} q_1^p(\chi)\,d\chi\right)^{\frac{1}{p}} + \left(\int_{\kappa_1}^{\kappa_2} q_2^p(\chi)\,d\chi\right)^{\frac{1}{p}} \leq \frac{1+\nu_2(\nu_1+2)}{(\nu_1+1)(\nu_2+2)}\left(\int_{\kappa_1}^{\kappa_2} (q_1+q_2)^p(\chi)\,d\chi\right)^{\frac{1}{p}}. $$

Theorem 2.2 ([18]) *For $p \geq 1$, let there be two positive functions q_1 and q_2 on $[0,\infty)$. If $0 < \nu_1 \leq \frac{q_1(\zeta)}{q_2(\zeta)} \leq \nu_2$ and $\chi \in [\kappa_1,\kappa_2]$, then*

$$ \begin{aligned} &\left(\int_{\kappa_1}^{\kappa_2} q_1^p(\chi)\,d\chi\right)^{\frac{2}{p}} + \left(\int_{\kappa_1}^{\kappa_2} q_2^p(\chi)\,d\chi\right)^{\frac{2}{p}} \\ &\quad\geq \left(\frac{(1+\nu_2)(\nu_1+1)}{\nu_2}-2\right)\left(\int_{\kappa_1}^{\kappa_2} q_1^p(\chi)\,d\chi\right)^{\frac{1}{p}}\left(\int_{\kappa_1}^{\kappa_2} q_2^p(\chi)\,d\chi\right)^{\frac{1}{p}}. \end{aligned} $$

Dahmani [8] used the Riemann–Liouville fractional integral to prove the new variant of the previous theorems.

Theorem 2.3 *For $p \geq 1$, let there be two positive functions q_1 and q_2 on $[0,\infty)$. If $0 < \nu_1 \leq \frac{q_1(\zeta)}{q_2(\zeta)} \leq \nu_2$ and $\chi \in [\kappa_1,\kappa_2]$, then*

$$ \begin{aligned} &\left(\int_{\kappa_1}^{\kappa_2} {}_{\varrho}^{y}K_{\kappa_1^+}^{\zeta} q_1^p(\chi)\,d\chi\right)^{\frac{1}{p}} + \left(\int_{\kappa_1}^{\kappa_2} {}_{\varrho}^{y}K_{\kappa_1^+}^{\zeta} q_2^p(\chi)\,d\chi\right)^{\frac{1}{p}} \\ &\quad\leq \frac{1+\nu_2(\nu_1+2)}{(\nu_1+1)(\nu_2+2)}\left(\int_{\kappa_1}^{\kappa_2} {}_{\varrho}^{y}K_{\kappa_1^+}^{\zeta}(q_1+q_2)^p(\chi)\,d\chi\right)^{\frac{1}{p}}. \end{aligned} $$

Theorem 2.4 *For $p \geq 1$, let there be two positive functions q_1 and q_2 on $[0,\infty)$. If $0 < \nu_1 \leq \frac{q_1(\zeta)}{q_2(\zeta)} \leq \nu_2$ and $\chi \in [\kappa_1,\kappa_2]$, then*

$$ \begin{aligned} &\left(\int_{\kappa_1}^{\kappa_2} {}_{\varrho}^{y}K_{\kappa_1^+}^{\zeta} q_1^p(\chi)\,d\chi\right)^{\frac{2}{p}} + \left(\int_{\kappa_1}^{\kappa_2} {}_{\varrho}^{y}K_{\kappa_1^+}^{\zeta} q_2^p(\chi)\,d\chi\right)^{\frac{2}{p}} \\ &\quad\geq \left(\frac{(1+\nu_2)(\nu_1+1)}{\nu_2}-2\right)\left(\int_{\kappa_1}^{\kappa_2} {}_{\varrho}^{y}K_{\kappa_1^+}^{\zeta} q_1^p(\chi)\,d\chi\right)^{\frac{1}{p}}\left(\int_{\kappa_1}^{\kappa_2} {}_{\varrho}^{y}K_{\kappa_1^+}^{\zeta} q_2^p(\chi)\,d\chi\right)^{\frac{1}{p}}. \end{aligned} $$

Recently, Rashid et al. [19] used the generalized fractional conformable integrals to prove the new inequalities that generalize the previous results of [8] and [18]. It is motivating for us to give the generalization of the results presented in [19] for general kernels with a measure space.

3 Reverse Minkowski inequalities involving general kernels

Theorem 3.1 *Let (Δ,Σ,π) be a measure space with positive σ-finite measure. For $p\geq 1$, let there be two positive functions q_1 and q_2 on $[0,\infty)$ such that $q_1,q_2\in\mathfrak{U}(\mathfrak{p})$. If $0<\nu_1\leq\frac{q_1(\zeta)}{q_2(\zeta)}\leq\nu_2$ for $\nu_1,\nu_2\in\mathbb{R}^+$ and for all $\varrho\in[\kappa_1,\chi]$, $(\mathfrak{L}q_1^p(\chi))<\infty$ and $(\mathfrak{L}q_2^p(\chi))<\infty$, then*

$$\bigl(\mathfrak{L}q_1^p(\chi)\bigr)^{\frac{1}{p}}+\bigl(\mathfrak{L}q_2^p(\chi)\bigr)^{\frac{1}{p}}\leq\left(\frac{1+\nu_2(\nu_1+1)}{\nu_2}\right)\bigl(\mathfrak{L}q_1^p(\chi)\bigr)^{\frac{1}{p}}\bigl(\mathfrak{L}q_2^p(\chi)\bigr)^{\frac{1}{p}}. \tag{3.1}$$

Proof By using the assumption $\frac{q_1(\zeta)}{q_2(\zeta)}\leq\nu_2$ and $\kappa_1\leq\zeta\leq\chi$, we obtain

$$(\nu_2+1)^p q_1^p(\zeta)\leq\nu_2^p\bigl(q_1(\zeta)+q_2(\zeta)\bigr)^p. \tag{3.2}$$

Multiplying both sides of the inequality (3.2) by $\mathfrak{p}(\chi,\zeta)$ and integrating with respect to ζ over measure space Δ, we obtain

$$(\nu_2+1)^p\int_\Delta\mathfrak{p}(\chi,\zeta)q_1^p(\zeta)\,d\pi(\zeta)\leq\nu_2^p\int_\Delta\mathfrak{p}(\chi,\zeta)\bigl(q_1(\zeta)+q_2(\zeta)\bigr)^p\,d\pi(\zeta),$$

which can be written as

$$\bigl(\mathfrak{L}q_1^p(\chi)\bigr)^{\frac{1}{p}}\leq\frac{\nu_2}{\nu_2+1}\bigl(\mathfrak{L}\bigl(q_1(\chi)+q_2(\chi)\bigr)^p\bigr)^{\frac{1}{p}}. \tag{3.3}$$

On the other hand, we have $0<\nu_1\leq\frac{q_1(\zeta)}{q_2(\zeta)}$, it follows that

$$\left(1+\frac{1}{\nu_1}\right)^p q_2^p(\zeta)\leq\left(\frac{1}{\nu_1}\right)^p\bigl(q_1(\zeta)+q_2(\zeta)\bigr)^p. \tag{3.4}$$

One can readily see that

$$\left(1+\frac{1}{\nu_1}\right)^p\int_\Delta\mathfrak{p}(\chi,\zeta)q_2^p(\zeta)\,d\pi(\zeta)\leq\left(\frac{1}{\nu_1}\right)^p\int_\Delta\mathfrak{p}(\chi,\zeta)\bigl(q_1(\zeta)+q_2(\zeta)\bigr)^p\,d\pi(\zeta),$$

or

$$\bigl(\mathfrak{L}q_2^p(\chi)\bigr)^{\frac{1}{p}}\leq\left(\frac{1}{\nu_1+1}\right)\bigl(\mathfrak{L}\bigl(q_1(\chi)+q_2(\chi)\bigr)^p\bigr)^{\frac{1}{p}}. \tag{3.5}$$

Adding (3.3) and (3.5) produces the desired inequality (3.1). □

Corollary 3.2 *Applying Theorem 3.1 with $\Delta=(a,b)$, $d\pi(\zeta)=d\zeta$ and*

$$\mathfrak{p}(\chi,\zeta)=\begin{cases}\frac{\mathfrak{g}'(\zeta)}{k\Gamma_k(\alpha)(\mathfrak{g}(\chi)-\mathfrak{g}(\zeta))^{1-\frac{\alpha}{k}}} & \text{for } a\leq\zeta\leq\chi;\\ 0 & \text{for } \chi<\zeta\leq b.\end{cases} \tag{3.6}$$

Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}}=(I^{\alpha,k}_{a+;\mathfrak{g}}q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}}=(I^{\alpha,k}_{a+;\mathfrak{g}}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the following inequality

$$\bigl(I^{\alpha,k}_{a+;\mathfrak{g}}q_1^p(\chi)\bigr)^{\frac{1}{p}}+\bigl(I^{\alpha,k}_{a+;\mathfrak{g}}q_2^p(\chi)\bigr)^{\frac{1}{p}}\leq\left(\frac{1+\nu_2(\nu_1+1)}{\nu_2}\right)\bigl(I^{\alpha,k}_{a+;\mathfrak{g}}q_1^p(\chi)\bigr)^{\frac{1}{p}}\bigl(I^{\alpha,k}_{a+;\mathfrak{g}}q_2^p(\chi)\bigr)^{\frac{1}{p}}. \tag{3.7}$$

Example 3.3 Taking $\mathfrak{g}(\chi) = \chi$ in Corollary 3.2, the corresponding $\mathfrak{p}(\chi,\zeta)$ defined by (3.6) takes the form

$$\mathfrak{p}(\chi,\zeta) = \begin{cases} \frac{1}{k\Gamma_k(\alpha)(\chi-\zeta)^{1-\frac{\alpha}{k}}} & \text{for } a \le \zeta \le \chi; \\ 0 & \text{for } \chi < \zeta \le b \end{cases} \tag{3.8}$$

and (3.1) becomes

$$\left(I^{\alpha}_{a+}q_1^p(\chi)\right)^{\frac{1}{p}} + \left(I^{\alpha}_{a+}q_2^p(\chi)\right)^{\frac{1}{p}} \le \left(\frac{1+\nu_2(\nu_1+1)}{\nu_2}\right)\left(I^{\alpha}_{a+}q_1^p(\chi)\right)^{\frac{1}{p}}\left(I^{\alpha}_{a+}q_2^p(\chi)\right)^{\frac{1}{p}}.$$

Example 3.4 Taking $\mathfrak{g}(\chi) = \log(\chi)$ and $k = 1$ in Corollary 3.2, the corresponding $\mathfrak{p}(\chi,\zeta)$ defined by (3.6) takes the form

$$\mathfrak{p}(\chi,\zeta) = \begin{cases} \frac{1}{\zeta k\Gamma_k(\alpha)(\log\chi-\log\zeta)^{1-\frac{\alpha}{k}}} & \text{for } a \le \zeta \le \chi; \\ 0 & \text{for } \chi < \nu \le b \end{cases} \tag{3.9}$$

and (3.1) becomes the well-known Hadamard fractional integrals, i.e.,

$$\left(J^{\alpha}_{a_+}q_1^p(\chi)\right)^{\frac{1}{p}} + \left(J^{\alpha}_{a_+}q_2^p(\chi)\right)^{\frac{1}{p}} \le \left(\frac{1+\nu_2(\nu_1+1)}{\nu_2}\right)\left(J^{\alpha}_{a_+}q_1^p(\chi)\right)^{\frac{1}{p}}\left(J^{\alpha}_{a_+}q_2^p(\chi)\right)^{\frac{1}{p}}.$$

Corollary 3.5 *Applying Theorem 3.1 with $\Delta = (a,b), d\pi(\zeta) = d\zeta$ and*

$$\mathfrak{p}(\chi,\zeta) = \begin{cases} \frac{1}{\Gamma(\alpha)}\frac{\sigma\chi^{-\sigma(\alpha+\eta)}}{(\chi^\sigma-\zeta^\sigma)^{1-\alpha}}\zeta^{\sigma\eta+\sigma-1} & \text{for } a \le \zeta \le \chi; \\ 0 & \text{for } \chi < \zeta \le b. \end{cases} \tag{3.10}$$

Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}} = (I^{\alpha}_{a_+;\sigma;\eta}q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}} = (I^{\alpha}_{a_+;\sigma;\eta}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the inequality for the Erdélyi–Kober-type fractional integral, i.e.,

$$\begin{aligned}&\left(I^{\alpha}_{a_+;\sigma;\eta}q_1^p(\chi)\right)^{\frac{1}{p}} + \left(I^{\alpha}_{a_+;\sigma;\eta}q_2^p(\chi)\right)^{\frac{1}{p}} \\ &\quad\le \left(\frac{1+\nu_2(\nu_1+1)}{\nu_2}\right)\left(I^{\alpha}_{a_+;\sigma;\eta}q_1^p(\chi)\right)^{\frac{1}{p}}\left(I^{\alpha}_{a_+;\sigma;\eta}q_2^p(\chi)\right)^{\frac{1}{p}}.\end{aligned} \tag{3.11}$$

Remark 3.6 Taking $\beta > 0, \mathfrak{g}(\lambda) = \frac{\lambda^\beta}{\beta}$ and $k = 1$ in Corollary 3.2, we obtain the inequality for the Katugampola fractional integrals in the literature [17], i.e.,

$$\left({}^\rho I^{\alpha}_{a_+}q_1^p(\chi)\right)^{\frac{1}{p}} + \left({}^\rho I^{\alpha}_{a_+}q_2^p(\chi)\right)^{\frac{1}{p}} \le \left(\frac{1+\nu_2(\nu_1+1)}{\nu_2}\right)\left({}^\rho I^{\alpha}_{a_+}q_1^p(\chi)\right)^{\frac{1}{p}}\left({}^\rho I^{\alpha}_{a_+}q_2^p(\chi)\right)^{\frac{1}{p}}.$$

Remark 3.7 Taking $\beta > 0, \mathfrak{g}(\lambda) = \frac{(\lambda-a)^\beta}{\beta}$ and $k = 1$ in Corollary 3.2, we obtain the inequality for the conformable fractional integral operators defined by Jarad et al. [1] and the inequality takes the form

$$\left({}^\beta_\alpha\mathfrak{J}^\alpha q_1^p(\chi)\right)^{\frac{1}{p}} + \left({}^\beta_\alpha\mathfrak{J}^\alpha q_2^p(\chi)\right)^{\frac{1}{p}} \le \left(\frac{1+\nu_2(\nu_1+1)}{\nu_2}\right)\left({}^\beta_\alpha\mathfrak{J}^\alpha q_1^p(\chi)\right)^{\frac{1}{p}}\left({}^\beta_\alpha\mathfrak{J}^\alpha q_2^p(\chi)\right)^{\frac{1}{p}}.$$

Remark 3.8 Taking $\beta > 0, \mathfrak{g}(\lambda) = \frac{\lambda^{\mu+\nu}}{\mu+\nu}$ and $k = 1$ in Corollary 3.2, we obtain the inequality for the conformable fractional integral operators defined by Khan [3] and the inequality takes the form

$$\left({}^{\tau}_{\alpha}K^{\beta}_{p^+}q_1^p(\chi)\right)^{\frac{1}{p}} + \left({}^{\tau}_{\alpha}K^{\beta}_{p^+}q_2^p(\chi)\right)^{\frac{1}{p}} \leq \left(\frac{1+\nu_2(\nu_1+1)}{\nu_2}\right)\left({}^{\tau}_{\alpha}K^{\beta}_{p^+}q_1^p(\chi)\right)^{\frac{1}{p}}\left({}^{\tau}_{\alpha}K^{\beta}_{p^+}q_2^p(\chi)\right)^{\frac{1}{p}}.$$

Theorem 3.9 *Let (Δ, Σ, π) be a measure space with positive σ-finite measure. For $p \geq 1$, let there be two positive functions q_1 and q_2 on $[0,\infty)$ such that $q_1, q_2 \in \mathfrak{U}(\mathfrak{p})$. If $0 < \nu_1 \leq \frac{q_1(\zeta)}{q_2(\zeta)} \leq \nu_2$ for $\nu_1, \nu_2 \in \mathbb{R}^+$ and for all $\varrho \in [\kappa_1, \chi]$, $\mathfrak{L}q_1^p(\chi), \mathfrak{L}q_2^p(\chi) < \infty$, then*

$$\left(\mathfrak{L}q_1^p(\chi)\right)^{\frac{1}{p}} + \left(\mathfrak{L}q_2^p(\chi)\right)^{\frac{1}{p}} \leq \left(\frac{(1+\nu_2)(\nu_1+1)}{\nu_2} - 2\right)\left(\mathfrak{L}q_1^p(\chi)\right)^{\frac{1}{p}}\left(\mathfrak{L}q_2^p(\chi)\right)^{\frac{1}{p}}. \tag{3.12}$$

Proof Taking the product of (3.3) and (3.5) yields that

$$\left(\frac{(\nu_1+1)(\nu_2+1)}{\nu_2} - 2\right)\left(\mathfrak{L}q_1^p(\chi)\right)^{\frac{1}{p}}\left(\mathfrak{L}q_2^p(\chi)\right)^{\frac{1}{p}} \leq \left[\left(\mathfrak{L}(q_1+q_2)^p(\chi)\right)^{\frac{1}{p}}\right]^2. \tag{3.13}$$

Using Minkowski's inequality on the right-hand side of (3.13), we obtain

$$\begin{aligned}\left[\left(\mathfrak{L}(q_1+q_2)^p(\chi)\right)^{\frac{1}{p}}\right]^2 &\leq \left[\left(\mathfrak{L}q_1^p(\chi)\right)^{\frac{1}{p}} + \left(\mathfrak{L}q_2^p(\chi)\right)^{\frac{1}{p}}\right]^2 \\ &\leq \left(\mathfrak{L}q_1^p(\chi)\right)^{\frac{2}{p}} + \left(\mathfrak{L}q_2^p(\chi)\right)^{\frac{2}{p}} + 2\left(\mathfrak{L}q_1^p(\chi)\right)^{\frac{1}{p}}\left(\mathfrak{L}q_2^p(\chi)\right)^{\frac{1}{p}}.\end{aligned} \tag{3.14}$$

Thus, from (3.13) and (3.14), we obtain (3.12) as desired. □

Corollary 3.10 *Applying Theorem 3.9 with $\Delta = (a,b), d\pi(\zeta) = d\zeta$ and $\mathfrak{p}(\chi,\zeta)$ defined by (3.6). Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}} = (I^{\alpha,k}_{a+;\mathfrak{g}}q_1^p(\chi))^{\frac{1}{p}}$, and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}} = (I^{\alpha,k}_{a+;\mathfrak{g}}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the following inequality*

$$\begin{aligned}&\left(I^{\alpha,k}_{a+;\mathfrak{g}}q_1^p(\chi)\right)^{\frac{1}{p}} + \left(I^{\alpha,k}_{a+;\mathfrak{g}}q_2^p(\chi)\right)^{\frac{1}{p}} \\ &\quad\leq \left(\frac{(1+\nu_2)(\nu_1+1)}{\nu_2} - 2\right)\left(I^{\alpha,k}_{a+;\mathfrak{g}}q_1^p(\chi)\right)^{\frac{1}{p}}\left(I^{\alpha,k}_{a+;\mathfrak{g}}q_2^p(\chi)\right)^{\frac{1}{p}}.\end{aligned} \tag{3.15}$$

Example 3.11 Taking $\mathfrak{g}(\chi) = \chi$ in Corollary 3.10, $\mathfrak{p}(\chi,\zeta)$ defined by (3.8) and (3.12) becomes

$$\left(I^{\alpha,k}_{a+}q_1^p(\chi)\right)^{\frac{1}{p}} + \left(I^{\alpha,k}_{a+}q_2^p(\chi)\right)^{\frac{1}{p}} \leq \left(\frac{(1+\nu_2)(\nu_1+1)}{\nu_2} - 2\right)\left(I^{\alpha,k}_{a+}q_1^p(\chi)\right)^{\frac{1}{p}}\left(I^{\alpha,k}_{a+}q_2^p(\chi)\right)^{\frac{1}{p}}.$$

Example 3.12 Taking $\mathfrak{g}(\chi) = \log(\chi)$ and $k = 1$ in Corollary 3.10 and $\mathfrak{p}(\chi,\zeta)$ defined by (3.9), (3.12) becomes

$$\left(J^{\alpha}_{a_+}q_1^p(\chi)\right)^{\frac{1}{p}} + \left(J^{\alpha}_{a_+}q_2^p(\chi)\right)^{\frac{1}{p}} \leq \left(\frac{(1+\nu_2)(\nu_1+1)}{\nu_2} - 2\right)\left(J^{\alpha}_{a_+}q_1^p(\chi)\right)^{\frac{1}{p}}\left(J^{\alpha}_{a_+}q_2^p(\chi)\right)^{\frac{1}{p}}.$$

Remark 3.13 Applying Theorem 3.9 with $\Delta = (a,b), d\pi(\zeta) = d\zeta$ and $\mathfrak{p}(\chi,\zeta)$ defined by (3.10). Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}} = (I^{\alpha}_{a_+;\sigma;\eta}q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}} = (I^{\alpha}_{a_+;\sigma;\eta}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the inequality for the Erdélyi–Kober-type fractional integral, i.e.,

$$\begin{aligned}&\left(I^{\alpha}_{a_+;\sigma;\eta}q_1^p(\chi)\right)^{\frac{1}{p}}+\left(I^{\alpha}_{a_+;\sigma;\eta}q_2^p(\chi)\right)^{\frac{1}{p}}\\&\quad\le\left(\frac{(1+\nu_2)(\nu_1+1)}{\nu_2}-2\right)\left(I^{\alpha}_{a_+;\sigma;\eta}q_1^p(\chi)\right)^{\frac{1}{p}}\left(I^{\alpha}_{a_+;\sigma;\eta}q_2^p(\chi)\right)^{\frac{1}{p}}.\end{aligned}$$

Remark 3.14 Taking $\beta>0$, $\mathfrak{g}(\chi)=\frac{\chi^\beta}{\beta}$ and $k=1$ in Corollary 3.10, we obtain the inequality for the Katugampola fractional integrals, i.e.,

$$\begin{aligned}&\left({}^\rho I^{\alpha}_{a_+}q_1^p(\chi)\right)^{\frac{1}{p}}+\left({}^\rho I^{\alpha}_{a_+}q_2^p(\chi)\right)^{\frac{1}{p}}\\&\quad\le\left(\frac{(1+\nu_2)(\nu_1+1)}{\nu_2}-2\right)\left({}^\rho I^{\alpha}_{a_+}q_1^p(\chi)\right)^{\frac{1}{p}}\left({}^\rho I^{\alpha}_{a_+}q_2^p(\chi)\right)^{\frac{1}{p}}.\end{aligned}$$

Remark 3.15 Taking $\beta>0,\mathfrak{g}(\chi)=\frac{(\chi-a)^\beta}{\beta}$ and $k=1$ in Corollary 3.10, we obtain the inequality for the conformable fractional integral and the inequality takes the form

$$\begin{aligned}&\left({}^\beta_\alpha\mathfrak{J}^{\alpha}q_1^p(\chi)\right)^{\frac{1}{p}}+\left({}^\beta_\alpha\mathfrak{J}^{\alpha}q_2^p(\chi)\right)^{\frac{1}{p}}\\&\quad\le\left(\frac{(1+\nu_2)(\nu_1+1)}{\nu_2}-2\right)\left({}^\beta_\alpha\mathfrak{J}^{\alpha}q_1^p(\chi)\right)^{\frac{1}{p}}\left({}^\beta_\alpha\mathfrak{J}^{\alpha}q_2^p(\chi)\right)^{\frac{1}{p}}.\end{aligned}$$

Remark 3.16 Taking $\beta>0,\mathfrak{g}(\chi)=\frac{\chi^{\mu+\nu}}{\mu+\nu}$ and $k=1$ in Corollary 3.10, we obtain the inequality for the conformable fractional integral, i.e.,

$$\begin{aligned}&\left({}^\tau_\alpha K^{\beta}_{p^+}q_1^p(\chi)\right)^{\frac{1}{p}}+\left({}^\tau_\alpha K^{\beta}_{p^+}q_2^p(\chi)\right)^{\frac{1}{p}}\\&\quad\le\left(\frac{(1+\nu_2)(\nu_1+1)}{\nu_2}-2\right)\left({}^\tau_\alpha K^{\beta}_{p^+}q_1^p(\chi)\right)^{\frac{1}{p}}\left({}^\tau_\alpha K^{\beta}_{p^+}q_2^p(\chi)\right)^{\frac{1}{p}}.\end{aligned}$$

4 Certain associated inequalities involving a general kernel

This section is dedicated to certain associated inequalities involving a general kernel with application for fractional calculus operators.

Theorem 4.1 *Let (Δ,Σ,π) be a measure space with positive σ-finite measure. For $p,q\ge 1$ with $\frac{1}{p}+\frac{1}{q}=1$. Suppose that there are two positive functions q_1 and q_2 on $[0,\infty)$ and $q_1,q_2\in\mathfrak{U}(\mathfrak{p})$ such that $\chi>\kappa_1$ and $\mathfrak{L}q_1^p(\chi),\mathfrak{L}q_2^p(\chi),\mathfrak{L}q_1^{\frac{1}{p}}(\chi)q_2^{\frac{1}{q}}(\chi)<\infty$. If $0<\nu_1\le\frac{q_1(\zeta)}{q_2(\zeta)}\le\nu_2$ for $\nu_1,\nu_2\in\mathbb{R}^+$ and for all $\varrho\in[\kappa_1,\chi]$, then*

$$\left(\mathfrak{L}q_1^p(\chi)\right)^{\frac{1}{p}}\left(\mathfrak{L}q_2^q(\chi)\right)^{\frac{1}{q}}\le\left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{pq}}\left(\mathfrak{L}q_1^{\frac{1}{p}}(\chi)q_2^{\frac{1}{q}}(\chi)\right).\tag{4.1}$$

Proof By using the assumption $\frac{q_1(\zeta)}{q_2(\zeta)}\le\nu_2$ and $\kappa_1\le\eta\le^{\frac{1}{p}}$, we have

$$q_2^{\frac{1}{q}}(\zeta)\ge\nu_2^{-\frac{1}{q}}(\zeta)q_1^{\frac{1}{q}}(\zeta).\tag{4.2}$$

Taking the products of both sides of (4.2) by $q_1^{\frac{1}{p}}(\zeta)$, it follows that

$$q_1^{\frac{1}{p}}(\zeta)q_2^{\frac{1}{q}}(\zeta) \geq \nu_2^{-\frac{1}{q}}(\zeta)q_1(\zeta).$$

One obtains after some settings

$$\int_{\Delta} \mathfrak{p}(\chi,\zeta)q_1^{\frac{1}{p}}(\zeta)q_2^{\frac{1}{q}}(\zeta)\,d\pi(\zeta) \geq \int_{\Delta} \mathfrak{p}(\chi,\zeta)\nu_2^{\frac{-1}{q}}(\zeta)q_1(\zeta)\,d\pi(\zeta),$$

which implies that

$$\nu_2^{\frac{-1}{pq}}(\zeta)\big(\mathfrak{L}q_1(\chi)\big)^{\frac{1}{p}} \leq \big(\mathfrak{L}q_1^{\frac{1}{p}}(\chi)q_2^{\frac{1}{q}}(\chi)\big)^{\frac{1}{p}}. \tag{4.3}$$

In contrast to the above $\nu_1 q_2(\zeta) \leq q_1(\zeta)$, we have

$$\nu_1^{\frac{1}{p}}(\zeta)q_2^{\frac{1}{p}}(\zeta) \leq q_1^{\frac{1}{p}}(\zeta). \tag{4.4}$$

Taking the products of both sides of (4.4) by $q_2^{\frac{1}{q}}(\zeta)$, it follows that after some necessary settings

$$\nu_1^{\frac{1}{pq}}\big(\mathfrak{L}q_2(\chi)\big)^{\frac{1}{q}} \leq \big(\mathfrak{L}q_1^{\frac{1}{p}}(\chi)\mathfrak{L}q_2^{\frac{1}{q}}(\chi)\big)^{\frac{1}{q}}. \tag{4.5}$$

Multiplying (4.3) and (4.5), we obtain the desired inequality. □

Corollary 4.2 *Applying Theorem 4.1 with $\Delta = (a,b), d\pi(\zeta) = d\zeta$ and $\mathfrak{p}(\chi,\zeta)$ defined by (3.6). Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}} = (I_{a+;\mathfrak{g}}^{\alpha,k}q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}} = (I_{a+;\mathfrak{g}}^{\alpha,k}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the following inequality*

$$\big(I_{a+;g}^{\alpha,k}q_1^p(\chi)\big)^{\frac{1}{p}}\big(I_{a+;g}^{\alpha,k}q_2^q(\chi)\big)^{\frac{1}{q}} \leq \left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{pq}}\big(I_{a+;g}^{\alpha,k}q_1^{\frac{1}{p}}(\chi)q_2^{\frac{1}{q}}(\chi)\big). \tag{4.6}$$

Remark 4.3 Applying Corollary 4.2 with $\mathfrak{g}(\chi) = \chi$ and the corresponding $\mathfrak{p}(\chi,\zeta)$ defined by (3.6), we obtain the inequality for Riemann–Liouville fractional integrals, i.e.,

$$\big(I_{a+}^{\alpha,k}q_1^p(\chi)\big)^{\frac{1}{p}}\big(I_{a+}^{\alpha,k}q_2^q(\chi)\big)^{\frac{1}{q}} \leq \left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{pq}}\big(I_{a+}^{\alpha,k}q_1^{\frac{1}{p}}(\chi)q_2^{\frac{1}{q}}(\chi)\big).$$

Example 4.4 Taking $\mathfrak{g}(\chi) = \log(\chi)$ and $k = 1$ in Corollary 4.2 and $\mathfrak{p}(\chi,\zeta)$ defined by (3.9), (4.1) reduces to

$$\big(J_{a_+}^{\alpha}q_1^p(\chi)\big)^{\frac{1}{p}}\big(J_{a_+}^{\alpha}q_2^q(\chi)\big)^{\frac{1}{q}} \leq \left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{pq}}\big(J_{a_+}^{\alpha}q_1^{\frac{1}{p}}(\chi)q_2^{\frac{1}{q}}(\chi)\big).$$

Remark 4.5 Applying Theorem 4.1 with $\Delta = (a,b), d\pi(\zeta) = d\zeta$ and $\mathfrak{p}(\chi,\zeta)$ defined by (3.10). Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}} = (I_{a_+;\sigma;\eta}^{\alpha}q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}} = (I_{a_+;\sigma;\eta}^{\alpha}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the inequality for the Erdélyi–Kober-type fractional integral, i.e.,

$$\big(I_{a_+;\sigma;\eta}^{\alpha}q_1^p(\chi)\big)^{\frac{1}{p}}\big(I_{a_+;\sigma;\eta}^{\alpha}q_2^q(\chi)\big)^{\frac{1}{q}} \leq \left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{pq}}\big(I_{a_+;\sigma;\eta}^{\alpha}q_1^{\frac{1}{p}}(\chi)q_2^{\frac{1}{q}}(\chi)\big).$$

Example 4.6 Taking $\beta > 0, \mathfrak{g}(\chi) = \frac{\chi^\beta}{\beta}$ and $k = 1$ in Corollary 4.2, we obtain the inequality for the Katugampola fractional integrals [17] and the inequality takes the form

$$\big({}^\rho I^\alpha_{a_+} q_1^p(\chi)\big)^{\frac{1}{p}} \big({}^\rho I^\alpha_{a_+} q_2^q(\chi)\big)^{\frac{1}{q}} \leq \left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{pq}} \big({}^\rho I^\alpha_{a_+} q_1^{\frac{1}{p}}(\chi) q_2^{\frac{1}{q}}(\chi)\big).$$

Remark 4.7 Taking $\beta > 0, \mathfrak{g}(\chi) = \frac{(\chi-a)^\beta}{\beta}$ and $k = 1$ in Corollary 4.2, we obtain the inequality for the conformable fractional integral, i.e.,

$$\big({}^\beta_\alpha \mathfrak{J}^\alpha q_1^p(\chi)\big)^{\frac{1}{p}} \big({}^\beta_\alpha \mathfrak{J}^\alpha q_2^q(\chi)\big)^{\frac{1}{q}} \leq \left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{pq}} \big({}^\beta_\alpha \mathfrak{J}^\alpha q_1^{\frac{1}{p}}(\chi) q_2^{\frac{1}{q}}(\chi)\big).$$

Remark 4.8 Taking $\beta > 0, \mathfrak{g}(\chi) = \frac{\chi^{\mu+\nu}}{\mu+\nu}$ and $k = 1$ in Corollary 4.2, we obtain the inequality for the conformable fractional integral, i.e.,

$$\big({}^\tau_\alpha K^\beta_{p^+} q_1^p(\chi)\big)^{\frac{1}{p}} \big({}^\tau_\alpha K^\beta_{p^+} q_2^q(\chi)\big)^{\frac{1}{q}} \leq \left(\frac{\nu_2}{\nu_1}\right)^{\frac{1}{pq}} \big({}^\tau_\alpha K^\beta_{p^+} q_1^{\frac{1}{p}}(\chi) q_2^{\frac{1}{q}}(\chi)\big).$$

Theorem 4.9 *Let (Δ, Σ, π) be a measure space with positive σ-finite measure. For $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that there are two positive functions q_1 and q_2 on $[0, \infty)$ and $q_1, q_2 \in \mathfrak{U}(\mathfrak{p})$ such that $\chi > \kappa_1, (\mathfrak{L} q_1^p(\chi)) < \infty$ and $(\mathfrak{L} q_2^p(\chi)) < \infty$. If $0 < \nu_1 \leq \frac{q_1(\zeta)}{q_2(\zeta)} \leq \nu_2$ for $\nu_1, \nu_2 \in \mathbb{R}^+$ and for all $\varrho \in [\kappa_1, \chi]$, then*

$$\begin{aligned}\big(\mathfrak{L} q_1(\chi) q_2(\chi)\big) \leq{}& \frac{2^{p-1}\nu_2^p}{p(\nu_2+1)^p(\chi)} \big(\mathfrak{L}\big(q_1^p(\chi) + q_2^p(\chi)\big)\big) \\ &+ \frac{2^{q-1}}{p(\nu_1+1)^p} \big(\mathfrak{L}\big(q_1^q(\chi) + q_2^q(\chi)\big)\big).\end{aligned} \tag{4.7}$$

Proof By using the assumption $\frac{q_1(\zeta)}{q_2(\zeta)} \leq \nu_2, \kappa_1 \leq \eta \leq \chi$ and

$$(\nu_2+1)^p q_1^p(\zeta) \leq \nu_2^p \big(q_1(\zeta) + q_2(\zeta)\big)^p,$$

which implies that

$$(\nu_2+1)^p \int_\Delta \mathfrak{p}(\chi,\zeta) q_1^p(\zeta)\, d\pi(\zeta) \leq \nu_2^p \int_\Delta \mathfrak{p}(\chi,\zeta) \big(q_1(\zeta) + q_2(\zeta)\big)^p\, d\pi(\zeta).$$

This can be written as

$$\big(\mathfrak{L} q_1^p(\chi)\big)^{\frac{1}{p}} \leq \frac{\nu_2}{\nu_2+1} \big(\mathfrak{L}\big(q_1(\chi) + q_2(\chi)\big)^p\big)^{\frac{1}{p}}. \tag{4.8}$$

Now,

$$(\nu_1+1)^q q_2^q(\zeta) \leq \big(q_1(\zeta) + q_2(\zeta)\big)^q.$$

Similarly,

$$(\nu_1+1)^q \int_\Delta \mathfrak{p}(\chi,\zeta) q_1^q(\zeta)\, d\pi(\zeta) \leq \nu_1^q \int_\Delta \mathfrak{p}(\chi,\zeta) \big(q_1(\zeta) + q_2(\zeta)\big)^q\, d\pi(\zeta),$$

which can be written as

$$\big(\mathfrak{L}q_2^q(\chi)\big) \leq \frac{1}{(\nu_1+1)^q}\big(\mathfrak{L}\big(q_1(\chi)+q_2(\chi)\big)^q\big). \tag{4.9}$$

Now, taking into account Young's inequality

$$q_1(\zeta)q_2(\zeta) \leq \frac{q_1^p(\zeta)}{p} + \frac{q_2^q(\zeta)}{q}. \tag{4.10}$$

Multiplying both sides of (4.10) with $\mathfrak{p}(\chi,\zeta)$ and integrating with respect to ζ over measure space Δ, we obtain that

$$\mathfrak{L}q_1(\chi)q_2(\chi) \leq \frac{\mathfrak{L}q_1^p(\chi)}{p} + \frac{\mathfrak{L}q_2^q(\chi)}{q}. \tag{4.11}$$

Putting (4.8) and (4.9) into (4.10), we obtain

$$\begin{aligned}\mathfrak{L}q_1(\chi)q_2(\chi) &\leq \frac{\mathfrak{L}q_1^p(\chi)}{p} + \frac{\mathfrak{L}q_2^q(\chi)}{q}\\ &\leq \frac{\nu_2^p(\mathfrak{L}(q_1(\chi)+q_2(\chi))^p)}{p(\nu_2+1)^p} + \frac{(\mathfrak{L}(q_1(\chi)+q_2(\chi))^q)}{q(\nu_1+1)^q}.\end{aligned} \tag{4.12}$$

Using the inequality

$$(\phi+\psi)^s \leq 2^{s-1}\big(\phi^s+\psi^s\big), \quad s,\phi,\psi > 0,$$

we obtain

$$\mathfrak{L}\big(q_1(\zeta)+q_2(\zeta)\big)^p \leq 2^{s-1}\mathfrak{L}\big(q_1^p(\zeta)+q_2^p(\zeta)\big), \tag{4.13}$$

and

$$\mathfrak{L}\big(q_1(\chi)+q_2(\chi)\big)^q \leq 2^{s-1}\mathfrak{L}\big(q_1^q(\chi)+q_2^q(\chi)\big). \tag{4.14}$$

The required result can be obtained by collective use of (4.12), (4.13), and (4.14). □

Corollary 4.10 *Applying Theorem 4.9 with $\Delta = (a,b), d\pi(\zeta) = d\zeta$ and $\mathfrak{p}(\chi,\zeta)$ defined by (3.6). Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}} = (I_{a+;\mathfrak{g}}^{\alpha,k}q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}} = (I_{a+;\mathfrak{g}}^{\alpha,k}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the following inequality*

$$\begin{aligned}\big(I_{a+;\mathfrak{g}}^{\alpha,k}q_1(\chi)q_2(\chi)\big) &\leq \frac{2^{p-1}\nu_2^p}{p(\nu_2+1)^p(\chi)}\big(I_{a+;\mathfrak{g}}^{\alpha,k}\big(q_1^p(\chi)+q_2^p(\chi)\big)\big)\\ &\quad + \frac{2^{q-1}}{p(\nu_1+1)^p}\big(I_{a+;\mathfrak{g}}^{\alpha,k}\big(q_1^q(\chi)+q_2^q(\chi)\big)\big).\end{aligned} \tag{4.15}$$

Example 4.11 Applying Corollary 4.10 with $\mathfrak{g}(\chi)=\chi, k=1$ and the corresponding $\mathfrak{p}(\chi,\zeta)$ defined by (3.6), we have

$$\big(I_{a+}^{\alpha}q_1(\chi)q_2(\chi)\big) \leq \frac{2^{p-1}\nu_2^p}{p(\nu_2+1)^p(\chi)}\big(I_{a+}^{\alpha}\big(q_1^p(\chi)+q_2^p(\chi)\big)\big)$$

$$+\frac{2^{q-1}}{p(\nu_1+1)^p}\big(I^{\alpha}_{a+}\big(q_1^q(\chi)+q_2^q(\chi)\big)\big).$$

Example 4.12 Taking $\mathfrak{g}(\chi)=\log(\chi)$ and $k=1$ in Corollary 4.10 and $\mathfrak{p}(\chi,\zeta)$ defined by (3.9), we have

$$\begin{aligned}&\big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}q_1(\chi)q_2(\chi)\big)\\&\quad\le\frac{2^{p-1}\nu_2^p}{p(\nu_2+1)^p(\chi)}\big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\big(q_1^p(\chi)+q_2^p(\chi)\big)\big)+\frac{2^{q-1}}{p(\nu_1+1)^p}\big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\big(q_1^q(\chi)+q_2^q(\chi)\big)\big).\end{aligned}$$

Remark 4.13 Applying Theorem 4.9 with $\Delta=(a,b), d\pi(\zeta)=d\zeta$ and $\mathfrak{p}(\chi,\zeta)$ defined by (3.10). Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}}=(I^{\alpha}_{a_+;\sigma;\eta}q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}}=(I^{\alpha}_{a_+;\sigma;\eta}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the inequality for the Erdélyi–Kober-type fractional integral, i.e.,

$$\begin{aligned}&\big(I^{\alpha}_{a_+;\sigma;\eta}q_1(\chi)q_2(\chi)\big)\\&\quad\le\frac{2^{p-1}\nu_2^p}{p(\nu_2+1)^p(\chi)}\big(I^{\alpha}_{a_+;\sigma;\eta}\big(q_1^p(\chi)+q_2^p(\chi)\big)\big)+\frac{2^{q-1}}{p(\nu_1+1)^p}\big(I^{\alpha}_{a_+;\sigma;\eta}\big(q_1^q(\chi)+q_2^q(\chi)\big)\big).\end{aligned}$$

Remark 4.14 Taking $\beta>0, \mathfrak{g}(\chi)=\frac{\chi^{\beta}}{\beta}$ and $k=1$ in Corollary 4.10, we obtain the inequality for the Katugampola fractional integrals, i.e.,

$$\begin{aligned}&\big({}^{\rho}I^{\alpha}_{a_+}q_1(\chi)q_2(\chi)\big)\\&\quad\le\frac{2^{p-1}\nu_2^p}{p(\nu_2+1)^p(\chi)}\big({}^{\rho}I^{\alpha}_{a_+}\big(q_1^p(\chi)+q_2^p(\chi)\big)\big)+\frac{2^{q-1}}{p(\nu_1+1)^p}\big({}^{\rho}I^{\alpha}_{a_+}\big(q_1^q(\chi)+q_2^q(\chi)\big)\big).\end{aligned}$$

Remark 4.15 Taking $\beta>0, \mathfrak{g}(\chi)=\frac{(\chi-a)^{\beta}}{\beta}$ and $k=1$ in Corollary 4.10, we obtain the inequality for the conformable fractional integral, i.e.,

$$\begin{aligned}&\big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}q_1(\chi)q_2(\chi)\big)\\&\quad\le\frac{2^{p-1}\nu_2^p}{p(\nu_2+1)^p(\chi)}\big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\big(q_1^p(\chi)+q_2^p(\chi)\big)\big)+\frac{2^{q-1}}{p(\nu_1+1)^p}\big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\big(q_1^q(\chi)+q_2^q(\chi)\big)\big).\end{aligned}$$

Remark 4.16 Taking $\beta>0, \mathfrak{g}(\chi)=\frac{\chi^{\mu+\nu}}{\mu+\nu}$ and $k=1$ in Corollary 4.10, we obtain the inequality for the generalized conformable fractional, i.e.,

$$\begin{aligned}&\big({}^{\tau}_{\alpha}K^{\beta}_{p^+}q_1(\chi)q_2(\chi)\big)\\&\quad\le\frac{2^{p-1}\nu_2^p}{p(\nu_2+1)^p(\chi)}\big({}^{\tau}_{\alpha}K^{\beta}_{p^+}\big(q_1^p(\chi)+q_2^p(\chi)\big)\big)+\frac{2^{q-1}}{p(\nu_1+1)^p}\big({}^{\tau}_{\alpha}K^{\beta}_{p^+}\big(q_1^q(\chi)+q_2^q(\chi)\big)\big).\end{aligned}$$

Theorem 4.17 *Let (Δ,Σ,π) be a measure space with positive σ-finite measure. For $p\ge1$, suppose that there are two positive functions q_1 and q_2 on $[0,\infty)$ and $q_1,q_2\in\mathfrak{U}(\mathfrak{p})$ such that $\chi>\kappa_1, (\mathfrak{L}q_1^p(\chi))<\infty$ and $(\mathfrak{L}q_2^p(\chi))<\infty$. If $0<\nu_1\le\frac{q_1(\zeta)}{q_2(\zeta)}\le\nu_2$ for $\nu_1,\nu_2\in\mathbb{R}^+$ and for all $\varrho\in[\kappa_1,\chi]$, then*

$$\frac{\nu_2+1}{\nu_2-\lambda}\big(\mathfrak{L}\big(q_1(\chi)-\lambda q_2(\chi)\big)\big)\le\big(\mathfrak{L}\big(q_1^p(\chi)\big)\big)^{\frac{1}{p}}+\big(\mathfrak{L}\big(q_2^p(\chi)\big)\big)^{\frac{1}{p}}$$

$$\leq \frac{\nu_1+1}{\nu_1-\lambda}\big(\mathfrak{L}\big(q_1(\chi)-\lambda q_2(\chi)\big)\big)^{\frac{1}{p}}. \tag{4.16}$$

Proof Under the assumption $0<\lambda<\nu_1\leq\frac{q_1(\zeta)}{q_2(\zeta)}\leq\nu_2$, we have

$$\nu_1\leq\nu_2 \quad\Rightarrow\quad (\nu_2+1)(\nu_1-\lambda)\leq(\nu_1+1)(\nu_2-\lambda).$$

It follows that

$$\frac{\nu_2+1}{\nu_2-\lambda}\leq\frac{\nu_1+1}{\nu_1-\lambda}.$$

Also, we have

$$\nu_1-\lambda\leq\frac{q_1(\zeta)-\lambda q_2(\zeta)}{q_2(\zeta)}\leq\nu_2-\lambda,$$

implying

$$\frac{(q_1(\zeta)-\lambda q_2(\zeta))^p}{(\nu_2-\lambda)^p}\leq q_2^p(\zeta)\leq\frac{(q_1(\zeta)-\lambda q_2(\zeta))^p}{(\nu_1-\lambda)^p}.$$

Furthermore, we have

$$\frac{1}{\nu_2}\leq\frac{q_2(\zeta)}{q_1(\zeta)}\leq\frac{1}{\nu_1} \quad\Rightarrow\quad \frac{\nu_1-\lambda}{\nu_1\lambda}\leq\frac{q_1(\zeta)-\lambda q_2(\zeta)}{\lambda q_1(\zeta)}\leq\frac{\nu_2-\lambda}{\nu_2\lambda}.$$

It follows that

$$\left(\frac{1}{\nu_2-\lambda}\right)^p\big(\big(q_1(\zeta)-\lambda q_2(\zeta)\big)^p\big)^{\frac{1}{p}}\leq q_2^p(\zeta)\leq\left(\frac{1}{\nu_1-\lambda}\right)^p\big(\big(q_1(\zeta)-\lambda q_2(\zeta)\big)^p\big)^{\frac{1}{p}}.$$

One can readily see that

$$\begin{aligned}
&\left(\frac{1}{\nu_2-\lambda}\right)^p\int_\Delta \mathfrak{p}(\chi,\zeta)\big(q_1(\zeta)-\lambda q_2(\zeta)\big)^p\,d\pi(\zeta)\\
&\quad\leq\int_\Delta \mathfrak{p}(\chi,\zeta)q_2^p(\zeta)\,d\pi(\zeta)\\
&\quad\leq\left(\frac{1}{\nu_1-\lambda}\right)^p\int_\Delta \mathfrak{p}(\chi,\zeta)\big(q_1(\zeta)-\lambda q_2(\zeta)\big)^p\,d\pi(\zeta).
\end{aligned}$$

This can be written as

$$\begin{aligned}
&\left(\frac{1}{\nu_2-\lambda}\right)\big(\mathfrak{L}\big(q_1(\chi)-\lambda q_2(\chi)\big)^p\big)^{\frac{1}{p}}\leq\big(\mathfrak{L}q_2^p(\chi)\big)^{\frac{1}{p}}\\
&\qquad\qquad\leq\left(\frac{1}{\nu_1-\lambda}\right)\mathfrak{L}\big(q_1(\chi)-\lambda q_2(\chi)\big)^{\frac{1}{p}}.
\end{aligned}\tag{4.17}$$

Using the same technique, we have

$$\left(\frac{1}{\nu_2-\lambda}\right)\big(\mathfrak{L}\big(q_1(\zeta)-\lambda q_2(\zeta)\big)^p\big)^{\frac{1}{p}}\leq\big(\mathfrak{L}q_1^p(\zeta)\big)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{\nu_1-\lambda}\right)\left(\mathfrak{L}\left(q_1(\zeta)-\lambda q_2(\zeta)\right)^p\right)^{\frac{1}{p}}. \tag{4.18}$$

Adding (4.17) and (4.18), we have the desired inequality. □

Corollary 4.18 *Applying Theorem 4.9 with $\Delta=(a,b), d\pi(\zeta)=d\zeta$ and $\mathfrak{p}(\chi,\zeta)$ defined by (3.6). Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}}=(I^{\alpha,k}_{a+;\mathfrak{g}}q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}}=(I^{\alpha,k}_{a+;\mathfrak{g}}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the following inequality*

$$\begin{aligned}\frac{\nu_2+1}{\nu_2-\lambda}\left(I^{\alpha,k}_{a+;\mathfrak{g}}\left(q_1(\chi)-\lambda q_2(\chi)\right)\right) &\leq \left(I^{\alpha,k}_{a+;\mathfrak{g}}\left(q_1^p(\chi)\right)\right)^{\frac{1}{p}}+\left(I^{\alpha,k}_{a+;\mathfrak{g}}\left(q_2^p(\chi)\right)\right)^{\frac{1}{p}}\\ &\leq \frac{\nu_1+1}{\nu_1-\lambda}\left(I^{\alpha,k}_{a+;\mathfrak{g}}\left(q_1(\chi)-\lambda q_2(\chi)\right)\right)^{\frac{1}{p}}.\end{aligned} \tag{4.19}$$

Remark 4.19 Applying Corollary 4.18 with $\mathfrak{g}(\chi)=\chi, k=1$ and corresponding $\mathfrak{p}(\chi,\zeta)$ defined by (3.6), then we have

$$\begin{aligned}\frac{\nu_2+1}{\nu_2-\lambda}\left(I^{\alpha}_{a+}\left(q_1(\chi)-\lambda q_2(\chi)\right)\right) &\leq \left(I^{\alpha}_{a+}\left(q_1^p(\chi)\right)\right)^{\frac{1}{p}}+\left(I^{\alpha}_{a+}\left(q_2^p(\chi)\right)\right)^{\frac{1}{p}}\\ &\leq \frac{\nu_1+1}{\nu_1-\lambda}\left(I^{\alpha}_{a+}\left(q_1(\chi)-\lambda q_2(\chi)\right)\right)^{\frac{1}{p}}.\end{aligned}$$

Example 4.20 Taking $\mathfrak{g}(\chi)=\log(\chi)$ and $k=1$ in Corollary 4.18 and $\mathfrak{p}(\chi,\zeta)$ defined by (3.9), then

$$\begin{aligned}\frac{\nu_2+1}{\nu_2-\lambda}\left({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\left(q_1(\chi)-\lambda q_2(\chi)\right)\right) &\leq \left({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\left(q_1^p(\chi)\right)\right)^{\frac{1}{p}}+\left({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\left(q_2^p(\chi)\right)\right)^{\frac{1}{p}}\\ &\leq \frac{\nu_1+1}{\nu_1-\lambda}\left({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\left(q_1(\chi)-\lambda q_2(\chi)\right)\right)^{\frac{1}{p}}.\end{aligned}$$

Remark 4.21 Applying Theorem 4.17 with $\Delta=(a,b), d\pi(\zeta)=d\zeta$ and $\mathfrak{p}(\chi,\zeta)$ defined by (3.10). Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}}=(I^{\alpha}_{a+;\sigma;\eta}q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}}=(I^{\alpha}_{a+;\sigma;\eta}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the inequality for the Erdélyi–Kober-type fractional integral, i.e.,

$$\begin{aligned}\frac{\nu_2+1}{\nu_2-\lambda}\left(I^{\alpha}_{a+;\sigma;\eta}\left(q_1(\chi)-\lambda q_2(\chi)\right)\right) &\leq \left(I^{\alpha}_{a+;\sigma;\eta}\left(q_1^p(\chi)\right)\right)^{\frac{1}{p}}+\left(I^{\alpha}_{a+;\sigma;\eta}\left(q_2^p(\chi)\right)\right)^{\frac{1}{p}}\\ &\leq \frac{\nu_1+1}{\nu_1-\lambda}\left(I^{\alpha}_{a+;\sigma;\eta}\left(q_1(\chi)-\lambda q_2(\chi)\right)\right)^{\frac{1}{p}}.\end{aligned}$$

Remark 4.22 Taking $\beta>0, \mathfrak{g}(\chi)=\frac{\chi^\beta}{\beta}$ and $k=1$ in Corollary 4.18, we obtain the inequality for the Katugampola fractional integrals, i.e.,

$$\begin{aligned}\frac{\nu_2+1}{\nu_2-\lambda}\left({}^{\rho}I^{\alpha}_{a_+}\left(q_1(\chi)-\lambda q_2(\chi)\right)\right) &\leq \left({}^{\rho}I^{\alpha}_{a_+}\left(q_1^p(\chi)\right)\right)^{\frac{1}{p}}+\left({}^{\rho}I^{\alpha}_{a_+}\left(q_2^p(\chi)\right)\right)^{\frac{1}{p}}\\ &\leq \frac{\nu_1+1}{\nu_1-\lambda}\left({}^{\rho}I^{\alpha}_{a_+}\left(q_1(\chi)-\lambda q_2(\chi)\right)\right)^{\frac{1}{p}}.\end{aligned}$$

Remark 4.23 Taking $\beta>0, \mathfrak{g}(\chi)=\frac{(\chi-a)^\beta}{\beta}$ and $k=1$ in Corollary 4.18, we obtain the inequality for conformable fractional integral, i.e.,

$$\frac{\nu_2+1}{\nu_2-\lambda}\left({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\left(q_1(\chi)-\lambda q_2(\chi)\right)\right) \leq \left({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\left(q_1^p(\chi)\right)\right)^{\frac{1}{p}}+\left({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\left(q_2^p(\chi)\right)\right)^{\frac{1}{p}}$$

$$\leq \frac{\nu_1 + 1}{\nu_1 - \lambda}\left({}_{\alpha}^{\beta}\mathfrak{J}^{\alpha}\left(q_1(\chi) - \lambda q_2(\chi)\right)\right)^{\frac{1}{p}}.$$

Remark 4.24 Taking $\beta > 0, \mathfrak{g}(\chi) = \frac{\chi^{\mu+\nu}}{\mu+\nu}$ and $k = 1$ in Corollary 4.18, we obtain the inequality for the generalized conformable fractional, i.e.,

$$\begin{aligned}\frac{\nu_2 + 1}{\nu_2 - \lambda}\left({}_{\alpha}^{\tau}K_{p^+}^{\beta}\left(q_1(\chi) - \lambda q_2(\chi)\right)\right) &\leq \left({}_{\alpha}^{\tau}K_{p^+}^{\beta}\left(q_1^p(\chi)\right)\right)^{\frac{1}{p}} + \left({}_{\alpha}^{\tau}K_{p^+}^{\beta}\left(q_2^p(\chi)\right)\right)^{\frac{1}{p}} \\ &\leq \frac{\nu_1 + 1}{\nu_1 - \lambda}\left({}_{\alpha}^{\tau}K_{p^+}^{\beta}\left(q_1(\chi) - \lambda q_2(\chi)\right)\right)^{\frac{1}{p}}.\end{aligned}$$

Theorem 4.25 *For $p \geq 1$, let there be two positive functions q_1 and q_2 on $[0,\infty)$. If $0 < \mathfrak{h} \leq q_1(\zeta) \leq \mathfrak{H}, 0 < \mathfrak{m} \leq q_2(\zeta) \leq \mathfrak{M}$ and $\chi \in [\kappa_1, \kappa_2]$, then*

$$\left(\mathfrak{L}\left(q_1^p(\chi)\right)^{\frac{1}{p}}\right) + \left(\mathfrak{L}\left(q_2^p(\chi)\right)^{\frac{1}{p}}\right) \leq \frac{\mathfrak{H}(\mathfrak{h} + \mathfrak{M}) + \mathfrak{M}(\mathfrak{H} + \mathfrak{m})}{(\mathfrak{m} + \mathfrak{H})(\mathfrak{h} + \mathfrak{M})}\left(\mathfrak{L}\left(q_1(\chi) + q_2(\chi)\right)^p\right)^{\frac{1}{p}}. \tag{4.20}$$

Proof Under the supposition, we observe that

$$\frac{1}{\mathfrak{M}} \leq \frac{1}{q_2(\zeta)} \leq \frac{1}{\mathfrak{m}}$$

and we have

$$\frac{\mathfrak{h}}{\mathfrak{M}} \leq \frac{q_1(\zeta)}{q_2(\zeta)} \leq \frac{\mathfrak{H}}{\mathfrak{m}}. \tag{4.21}$$

From (4.21), we have

$$q_2^p(\zeta) \leq \left(\frac{\mathfrak{M}}{\mathfrak{h} + \mathfrak{M}}\right)^p \left(q_1(\zeta) + q_1(\zeta)\right)^p \tag{4.22}$$

and

$$q_1^p(\zeta) \leq \left(\frac{\mathfrak{H}}{\mathfrak{m} + \mathfrak{H}}\right)^p \left(q_1(\zeta) + q_1(\zeta)\right)^p. \tag{4.23}$$

After some necessary settings, we have

$$\int_{\Delta} \mathfrak{p}(\chi,\zeta) q_1^p(\zeta)\, d\pi(\zeta) \leq \left(\frac{\mathfrak{H}}{\mathfrak{m} + \mathfrak{H}}\right)^p \int_{\Delta} \mathfrak{p}(\chi,\zeta)\left(q_1(\zeta) + q_1(\zeta)\right)^p d\pi(\zeta), \tag{4.24}$$

which can be written as

$$\left(\mathfrak{L}q_1^p(\zeta)\right)^{\frac{1}{p}} \leq \left(\frac{\mathfrak{H}}{\mathfrak{m} + \mathfrak{H}}\right)^p \left(\mathfrak{L}\left(q_1(\zeta) + q_1(\zeta)\right)^p\right)^{\frac{1}{p}}. \tag{4.25}$$

Similarly, we have

$$\left(\mathfrak{L}q_2^p(\zeta)\right)^{\frac{1}{p}} \leq \left(\frac{\mathfrak{H}}{\mathfrak{m} + \mathfrak{H}}\right)^p \left(\mathfrak{L}\left(q_1(\zeta) + q_1(\zeta)\right)^p\right)^{\frac{1}{p}}. \tag{4.26}$$

Adding (4.25) and (4.26), we obtain the required inequality. □

Corollary 4.26 *Applying Theorem 4.25 with* $\Delta = (a,b), d\pi(\zeta) = d\zeta$ *and* $\mathfrak{p}(\chi,\zeta)$ *defined by* (3.6). *Substituting* $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}} = (I_{a+;\mathfrak{g}}^{\alpha,k} q_1^p(\chi))^{\frac{1}{p}}$ *and* $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}} = (I_{a+;\mathfrak{g}}^{\alpha,k} q_2^p(\chi))^{\frac{1}{p}}$, *we obtain the following inequality*

$$\begin{aligned}&\big(I_{a+;\mathfrak{g}}^{\alpha,k}\big(q_1^p(\chi)\big)^{\frac{1}{p}}\big) + \big(I_{a+;\mathfrak{g}}^{\alpha,k}\big(q_2^p(\chi)\big)^{\frac{1}{p}}\big)\\ &\quad\leq \frac{\mathfrak{H}(\mathfrak{h}+\mathfrak{M}) + \mathfrak{M}(\mathfrak{H}+\mathfrak{m})}{(\mathfrak{m}+\mathfrak{H})(\mathfrak{h}+\mathfrak{M})}\big(I_{a+;\mathfrak{g}}^{\alpha,k}\big(q_1(\chi)+q_2(\chi)\big)^p\big)^{\frac{1}{p}}.\end{aligned} \tag{4.27}$$

Remark 4.27 Applying Corollary 4.26 with $\mathfrak{g}(\chi) = \chi, k = 1$ and the corresponding $\mathfrak{p}(\chi,\zeta)$ defined by (3.6), we have

$$\begin{aligned}&\big(I_{a+}^{\alpha}\big(q_1^p(\chi)\big)^{\frac{1}{p}}\big) + \big(I_{a+}^{\alpha}\big(q_2^p(\chi)\big)^{\frac{1}{p}}\big)\\ &\quad\leq \frac{\mathfrak{H}(\mathfrak{h}+\mathfrak{M}) + \mathfrak{M}(\mathfrak{H}+\mathfrak{m})}{(\mathfrak{m}+\mathfrak{H})(\mathfrak{h}+\mathfrak{M})}\big(I_{a+}^{\alpha}\big(q_1(\chi)+q_2(\chi)\big)^p\big)^{\frac{1}{p}}.\end{aligned}$$

Example 4.28 Taking $\mathfrak{g}(\chi) = \log(\chi)$ and $k = 1$ in Corollary 4.26 and $\mathfrak{p}(\chi,\zeta)$ defined by (3.9), we have

$$\begin{aligned}&\big({}_{\alpha}^{\beta}\mathfrak{J}^{\alpha}\big(q_1^p(\chi)\big)^{\frac{1}{p}}\big) + \big({}_{\alpha}^{\beta}\mathfrak{J}^{\alpha}\big(q_2^p(\chi)\big)^{\frac{1}{p}}\big)\\ &\quad\leq \frac{\mathfrak{H}(\mathfrak{h}+\mathfrak{M}) + \mathfrak{M}(\mathfrak{H}+\mathfrak{m})}{(\mathfrak{m}+\mathfrak{H})(\mathfrak{h}+\mathfrak{M})}\big({}_{\alpha}^{\beta}\mathfrak{J}^{\alpha}\big(q_1(\chi)+q_2(\chi)\big)^p\big)^{\frac{1}{p}}.\end{aligned}$$

Remark 4.29 Applying Theorem 4.25 with $\Delta = (a,b), d\pi(\zeta) = d\zeta$ and $\mathfrak{p}(\chi,\zeta)$ defined by (3.10). Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}} = (I_{a_+;\sigma;\eta}^{\alpha} q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}} = (I_{a_+;\sigma;\eta}^{\alpha} q_2^p(\chi))^{\frac{1}{p}}$, we obtain the inequality for the Erdélyi–Kober-type fractional integral, i.e.,

$$\begin{aligned}&\big(I_{a_+;\sigma;\eta}^{\alpha}\big(q_1^p(\chi)\big)^{\frac{1}{p}}\big) + \big(I_{a_+;\sigma;\eta}^{\alpha}\big(q_2^p(\chi)\big)^{\frac{1}{p}}\big)\\ &\quad\leq \frac{\mathfrak{H}(\mathfrak{h}+\mathfrak{M}) + \mathfrak{M}(\mathfrak{H}+\mathfrak{m})}{(\mathfrak{m}+\mathfrak{H})(\mathfrak{h}+\mathfrak{M})}\big(I_{a_+;\sigma;\eta}^{\alpha}\big(q_1(\chi)+q_2(\chi)\big)^p\big)^{\frac{1}{p}}.\end{aligned}$$

Remark 4.30 Taking $\beta > 0, \mathfrak{g}(\chi) = \frac{\chi^{\beta}}{\beta}$ and $k = 1$ in Corollary 4.26, we obtain the inequality for the Katugampola fractional integrals, i.e.,

$$\begin{aligned}&\big({}^{\rho}I_{a_+}^{\alpha}\big(q_1^p(\chi)\big)^{\frac{1}{p}}\big) + \big({}^{\rho}I_{a_+}^{\alpha}\big(q_2^p(\chi)\big)^{\frac{1}{p}}\big)\\ &\quad\leq \frac{\mathfrak{H}(\mathfrak{h}+\mathfrak{M}) + \mathfrak{M}(\mathfrak{H}+\mathfrak{m})}{(\mathfrak{m}+\mathfrak{H})(\mathfrak{h}+\mathfrak{M})}\big({}^{\rho}I_{a_+}^{\alpha}\big(q_1(\chi)+q_2(\chi)\big)^p\big)^{\frac{1}{p}}.\end{aligned}$$

Remark 4.31 Taking $\beta > 0, \mathfrak{g}(\chi) = \frac{(\chi-a)^{\beta}}{\beta}$ and $k = 1$ in Corollary 4.26, we obtain the inequality for the conformable fractional integral, i.e.,

$$\begin{aligned}&\big({}_{\alpha}^{\beta}\mathfrak{J}^{\alpha}\big(q_1^p(\chi)\big)^{\frac{1}{p}}\big) + \big({}_{\alpha}^{\beta}\mathfrak{J}^{\alpha}\big(q_2^p(\chi)\big)^{\frac{1}{p}}\big)\\ &\quad\leq \frac{\mathfrak{H}(\mathfrak{h}+\mathfrak{M}) + \mathfrak{M}(\mathfrak{H}+\mathfrak{m})}{(\mathfrak{m}+\mathfrak{H})(\mathfrak{h}+\mathfrak{M})}\big({}_{\alpha}^{\beta}\mathfrak{J}^{\alpha}\big(q_1(\chi)+q_2(\chi)\big)^p\big)^{\frac{1}{p}}.\end{aligned}$$

Remark 4.32 Taking $\beta > 0, \mathfrak{g}(\chi) = \frac{\chi^{\mu+\nu}}{\mu+\nu}$, and $k = 1$ in Corollary 4.26, we obtain the inequality for the generalized conformable fractional, i.e.,

$$\begin{aligned}&\left({}^{\tau}_{\alpha}K^{\beta}_{p^+}\left(q_1^p(\chi)\right)^{\frac{1}{p}}\right) + \left({}^{\tau}_{\alpha}K^{\beta}_{p^+}\left(q_2^p(\chi)\right)^{\frac{1}{p}}\right)\\ &\quad \leq \frac{\mathfrak{H}(\mathfrak{h}+\mathfrak{M}) + \mathfrak{M}(\mathfrak{H}+\mathfrak{m})}{(\mathfrak{m}+\mathfrak{H})(\mathfrak{h}+\mathfrak{M})}\left({}^{\tau}_{\alpha}K^{\beta}_{p^+}\left(q_1(\chi)+q_2(\chi)\right)^p\right)^{\frac{1}{p}}.\end{aligned}$$

Theorem 4.33 *Let (Δ, Σ, π) be a measure space with positive σ-finite measure. For $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that there are two positive functions q_1 and q_2 on $[0, \infty)$ and $q_1, q_2 \in \mathfrak{U}(\mathfrak{p})$ such that $\chi > \kappa_1, (\mathfrak{L}q_1^p(\chi)) < \infty$ and $(\mathfrak{L}q_2^p(\chi)) < \infty$. If $0 < \mathfrak{h} \leq q_1(\zeta) \leq \mathfrak{H}, 0 < \mathfrak{m} \leq q_2(\zeta) \leq \mathfrak{M}$ and $\chi \in [\kappa_1, \kappa_2]$, then*

$$\begin{aligned}\frac{1}{\nu_2}\left(\mathfrak{L}\left(q_1(\chi)q_2(\chi)\right)\right) &\leq \frac{1}{(\nu_1+1)(\nu_2+1)}\left(\mathfrak{L}\left(q_1(\chi)+q_2(\chi)\right)\right)^2\\ &\leq \frac{1}{\nu_1}\left(\mathfrak{L}\left(q_1(\chi)q_2(\chi)\right)\right).\end{aligned} \tag{4.28}$$

Proof Under the supposition, we observe that

$$\nu_1 \leq \frac{q_1(\zeta)}{q_2(\zeta)} \leq \nu_2,$$

it follows that

$$q_2(\zeta)(\nu_1+1) \leq q_1(\zeta) + q_2(\zeta) \leq q_2(\zeta)(\nu_2+1). \tag{4.29}$$

Additionally, we have

$$\frac{1}{\nu_2} \leq \frac{q_2(\zeta)}{q_1(\zeta)} \leq \frac{1}{\nu_1}, \tag{4.30}$$

which yields that

$$\frac{\nu_2+1}{\nu_2}q_1(\zeta) \leq q_2(\zeta) + q_1(\zeta) \leq \frac{\nu_1+1}{\nu_1}q_1(\zeta). \tag{4.31}$$

From (4.29) and (4.31), we have

$$\frac{q_1(\zeta)q_2(\zeta)}{\nu_2} \leq \frac{(q_2(\zeta)+q_1(\zeta))^2}{(\nu_1+1)(\nu_2+1)} \leq \frac{q_1(\zeta)q_2(\zeta)}{\nu_1}. \tag{4.32}$$

Multiplying both sides of the above inequality with $\mathfrak{p}(\chi, \zeta)$ and integrating with respect to ζ over measure space Δ, we obtain

$$\begin{aligned}\int_{\Delta}\mathfrak{p}(\chi,\zeta)\frac{q_1(\zeta)q_2(\zeta)}{\nu_2} &\leq \int_{\Delta}\mathfrak{p}(\chi,\zeta)\frac{(q_2(\zeta)+q_1(\zeta))^2}{(\nu_1+1)(\nu_2+1)}\\ &\leq \int_{\Delta}\mathfrak{p}(\chi,\zeta)\frac{q_1(\zeta)q_2(\zeta)}{\nu_1}.\end{aligned} \tag{4.33}$$

This can be written as

$$\mathfrak{L}\frac{q_1(\zeta)q_2(\zeta)}{\nu_2} \leq \mathfrak{L}\frac{(q_2(\zeta)+q_1(\zeta))^2}{(\nu_1+1)(\nu_2+1)} \leq \mathfrak{L}\frac{q_1(\zeta)q_2(\zeta)}{\nu_1}, \tag{4.34}$$

which is the required result. □

Corollary 4.34 *Applying Theorem 4.33 with $\Delta = (a,b), d\pi(\zeta) = d\zeta$ and $\mathfrak{p}(\chi,\zeta)$ defined by (3.6). Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}} = (I^{\alpha,k}_{a+;\mathfrak{g}}q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}} = (I^{\alpha,k}_{a+;\mathfrak{g}}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the following inequality*

$$\begin{aligned}\frac{1}{\nu_2}\big(I^{\alpha,k}_{a+;\mathfrak{g}}\big(q_1(\chi)q_2(\chi)\big)\big) &\leq \frac{1}{(\nu_1+1)(\nu_2+1)}\big(I^{\alpha,k}_{a+;\mathfrak{g}}\big(q_1(\chi)+q_2(\chi)\big)\big)^2 \\ &\leq \frac{1}{\nu_1}\big(I^{\alpha,k}_{a+;\mathfrak{g}}\big(q_1(\chi)q_2(\chi)\big)\big).\end{aligned} \tag{4.35}$$

Remark 4.35 Applying Corollary 4.34 with $\mathfrak{g}(\chi) = \chi$ and the corresponding corresponding $\mathfrak{p}(\chi,\zeta)$ defined by (3.6), we have

$$\begin{aligned}\frac{1}{\nu_2}\big(I^{\alpha}_{a+}\big(q_1(\chi)q_2(\chi)\big)\big) &\leq \frac{1}{(\nu_1+1)(\nu_2+1)}\big(I^{\alpha}_{a+}\big(q_1(\chi)+q_2(\chi)\big)\big)^2 \\ &\leq \frac{1}{\nu_1}\big(I^{\alpha}_{a+}\big(q_1(\chi)q_2(\chi)\big)\big).\end{aligned}$$

Example 4.36 Taking $\mathfrak{g}(\chi) = \log(\chi)$ and $k = 1$ in Corollary 4.34 and $\mathfrak{p}(\chi,\zeta)$ defined by (3.9), (3.12) becomes

$$\begin{aligned}\frac{1}{\nu_2}\big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\big(q_1(\chi)q_2(\chi)\big)\big) &\leq \frac{1}{(\nu_1+1)(\nu_2+1)}\big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\big(q_1(\chi)+q_2(\chi)\big)\big)^2 \\ &\leq \frac{1}{\nu_1}\big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\big(q_1(\chi)q_2(\chi)\big)\big).\end{aligned}$$

Remark 4.37 Applying Theorem 4.33 with $\Delta = (a,b), d\pi(\zeta) = d\zeta$ and $\mathfrak{p}(\chi,\zeta)$ defined by (3.10). Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}} = (I^{\alpha}_{a+;\sigma;\eta}q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}} = (I^{\alpha}_{a+;\sigma;\eta}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the inequality for the Erdélyi–Köber fractional integral, i.e.,

$$\begin{aligned}\frac{1}{\nu_2}\big(I^{\alpha}_{a+;\sigma;\eta}\big(q_1(\chi)q_2(\chi)\big)\big) &\leq \frac{1}{(\nu_1+1)(\nu_2+1)}\big(I^{\alpha}_{a+;\sigma;\eta}\big(q_1(\chi)+q_2(\chi)\big)\big)^2 \\ &\leq \frac{1}{\nu_1}\big(I^{\alpha}_{a+;\sigma;\eta}\big(q_1(\chi)q_2(\chi)\big)\big).\end{aligned}$$

Remark 4.38 Taking $\beta > 0, \mathfrak{g}(\chi) = \frac{\chi^\beta}{\beta}$ and $k = 1$ in Corollary 4.34, we obtain the inequality for the Katugampola fractional integral operators in the literature [17] and the inequality takes the form

$$\begin{aligned}\frac{1}{\nu_2}\big({}^{\tau}_{\alpha}K^{\beta}_{p^+}\big(q_1(\chi)q_2(\chi)\big)\big) &\leq \frac{1}{(\nu_1+1)(\nu_2+1)}\big({}^{\tau}_{\alpha}K^{\beta}_{p^+}\big(q_1(\chi)+q_2(\chi)\big)\big)^2 \\ &\leq \frac{1}{\nu_1}\big({}^{\tau}_{\alpha}K^{\beta}_{p^+}\big(q_1(\chi)q_2(\chi)\big)\big).\end{aligned}$$

Remark 4.39 Taking $\beta > 0, \mathfrak{g}(\chi) = \frac{(\chi-a)^\beta}{\beta}$ and $k = 1$ in Corollary 4.34, we obtain the inequality for the conformable fractional integral operators defined by Jarad et al. [1] and the inequality takes the form

$$\begin{aligned}\frac{1}{\nu_2}\left({}_{\alpha}^{\beta}\mathfrak{J}^{\alpha}\left(q_1(\chi)q_2(\chi)\right)\right) &\leq \frac{1}{(\nu_1+1)(\nu_2+1)}\left({}_{\alpha}^{\beta}\mathfrak{J}^{\alpha}\left(q_1(\chi)+q_2(\chi)\right)\right)^2 \\ &\leq \frac{1}{\nu_1}\left({}_{\alpha}^{\beta}\mathfrak{J}^{\alpha}\left(q_1(\chi)q_2(\chi)\right)\right).\end{aligned}$$

Remark 4.40 Taking $\beta > 0, \mathfrak{g}(\chi) = \frac{\chi^{\mu+\nu}}{\mu+\nu}$ and $k = 1$ in Corollary 4.34, we obtain the inequality for the conformable fractional integral operators defined by Khan et al. [3] and the inequality takes the form

$$\begin{aligned}\frac{1}{\nu_2}\left({}_{\alpha}^{\tau}K_{p^+}^{\beta}\left(q_1(\chi)q_2(\chi)\right)\right) &\leq \frac{1}{(\nu_1+1)(\nu_2+1)}\left({}_{\alpha}^{\tau}K_{p^+}^{\beta}\left(q_1(\chi)+q_2(\chi)\right)\right)^2 \\ &\leq \frac{1}{\nu_1}\left({}_{\alpha}^{\tau}K_{p^+}^{\beta}\left(q_1(\chi)q_2(\chi)\right)\right).\end{aligned}$$

Theorem 4.41 *Let (Δ,Σ,π) be a measure space with positive σ-finite measure. For $p,q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$. Suppose that there are two positive functions q_1 and q_2 on $[0,\infty)$ and $q_1,q_2 \in \mathfrak{U}(\mathfrak{p})$ such that $\chi > \kappa_1, (\mathfrak{L}q_1^p(\chi)) < \infty$ and $(\mathfrak{L}q_2^p(\chi)) < \infty$. If $0 < \mathfrak{h} \leq q_1(\zeta) \leq \mathfrak{H}, 0 < \mathfrak{m} \leq q_2(\zeta) \leq \mathfrak{M}$ and $\chi \in [\kappa_1,\kappa_2]$, then*

$$\left(\mathfrak{L}\left(q_1^p(\chi)\right)^{\frac{1}{p}}\right) + \left(\mathfrak{L}\left(q_2^p(\chi)\right)^{\frac{1}{p}}\right) \leq 2\left(\mathfrak{L}U^p\left(q_1(\chi),q_2(\chi)\right)\right)^{\frac{1}{p}}, \tag{4.36}$$

where

$$U^p\left(q_1(\chi),q_2(\chi)\right) = \max\left\{\nu_2\left[\left(1+\frac{\nu_2}{\nu_1}\right)q_1(\chi) - \nu_2 q_2(\chi)\right], \frac{\nu_1+\nu_2 q_2(\chi) - q_1(\chi)}{\nu_1}\right\}.$$

Proof By the supposition, we observe that

$$0 < \nu_1 \leq \nu_2 + \nu_1 - \frac{q_1(\zeta)}{q_2(\zeta)}$$

and

$$\nu_2 + \nu_1 - \frac{q_1(\zeta)}{q_2(\zeta)} \leq \nu_2.$$

From the above two inequalities, we obtain

$$q_2(\zeta) < \frac{(\nu_2+\nu_1)q_2(\zeta) - q_1(\zeta)}{\nu_1} \leq U\left(q_1(\chi),q_2(\chi)\right),$$

where

$$U^p\left(q_1(\chi),q_2(\chi)\right) = \max\left\{\nu_2\left[\left(1+\frac{\nu_2}{\nu_1}\right)q_1(\chi) - \nu_2 q_2(\chi)\right], \frac{\nu_1+\nu_2 q_2(\chi) - q_1(\chi)}{\nu_1}\right\}.$$

Also, from the given supposition

$$\frac{1}{\nu_2} \leq \frac{q_2(\zeta)}{q_1(\zeta)} \leq \frac{1}{\nu_1},$$

we have

$$\frac{1}{\nu_2} \leq \frac{1}{\nu_2} + \frac{1}{\nu_1} - \frac{q_2(\zeta)}{q_1(\zeta)} \tag{4.37}$$

and

$$\frac{1}{\nu_2} + \frac{1}{\nu_1} - \frac{q_2(\zeta)}{q_1(\zeta)} \leq \frac{1}{\nu_1}. \tag{4.38}$$

From (4.37) and (4.38), we obtain

$$\frac{1}{\nu_2} \leq \frac{(\frac{1}{\nu_1} + \frac{1}{\nu_2})q_1(\zeta) - q_2(\zeta)}{q_1(\zeta)} \leq \frac{1}{\nu_1}. \tag{4.39}$$

This implies that

$$\begin{aligned} q_1(\zeta) &\leq \nu_2\left(\frac{1}{\nu_1} + \frac{1}{\nu_2}\right)q_1(\zeta) - \nu_2 q_2(\zeta) \\ &\leq \nu_2\left[\left(\frac{\nu_2}{\nu_1} + 1\right)q_1(\zeta) - \nu_2 q_2(\zeta)\right] \\ &\leq U\bigl(q_1(\zeta), q_2(\zeta)\bigr), \end{aligned}$$

hence, we have

$$q_1^p(\zeta) \leq U^p\bigl(q_1(\zeta), q_2(\zeta)\bigr), \tag{4.40}$$

and

$$q_2^p(\zeta) \leq U^p\bigl(q_1(\zeta), q_2(\zeta)\bigr). \tag{4.41}$$

Multiplying both sides of the above inequality (4.40) with $\mathfrak{p}(\chi,\zeta)$ and integrating with respect to ζ over measure space Δ, we obtain

$$\int_\Delta \mathfrak{p}(\chi,\zeta) q_1^p(\zeta) \leq \int_\Delta \mathfrak{p}(\chi,\zeta) U^p\bigl(q_1(\zeta), q_2(\zeta)\bigr), \tag{4.42}$$

which can be written as

$$\mathfrak{L} q_1^p(\zeta) \leq \mathfrak{L} U^p\bigl(q_1(\zeta), q_2(\zeta)\bigr). \tag{4.43}$$

Using the same technique for inequality (4.41), we obtain

$$\mathfrak{L} q_1^p(\zeta) \leq \mathfrak{L} U^p\bigl(q_1(\zeta), q_2(\zeta)\bigr). \tag{4.44}$$

By adding the inequalities (4.43) and (4.44), we obtain the desired inequality. □

Corollary 4.42 *Applying Theorem 4.33 with $\Delta = (a,b), d\pi(\zeta) = d\zeta$ and $\mathfrak{p}(\chi,\zeta)$ defined by (3.6). Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}} = (I^{\alpha,k}_{a+;\mathfrak{g}}q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}} = (I^{\alpha,k}_{a+;\mathfrak{g}}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the following inequality*

$$\big(I^{\alpha,k}_{a+;\mathfrak{g}}\big(q_1^p(\chi)\big)^{\frac{1}{p}}\big) + \big(I^{\alpha,k}_{a+;\mathfrak{g}}\big(q_2^p(\chi)\big)^{\frac{1}{p}}\big) \le 2\big(I^{\alpha,k}_{a+;\mathfrak{g}}U^p\big(q_1(\chi),q_2(\chi)\big)\big)^{\frac{1}{p}}. \tag{4.45}$$

Remark 4.43 Applying Corollary 4.42 with $\mathfrak{g}(\chi) = \chi$ and the corresponding $\mathfrak{p}(\chi,\zeta)$ defined by (3.6), we have

$$\big(I^{\alpha}_{a+}\big(q_1^p(\chi)\big)^{\frac{1}{p}}\big) + \big(I^{\alpha}_{a+}\big(q_2^p(\chi)\big)^{\frac{1}{p}}\big) \le 2\big(I^{\alpha}_{a+}U^p\big(q_1(\chi),q_2(\chi)\big)\big)^{\frac{1}{p}}.$$

Example 4.44 If we take $\mathfrak{g}(\chi) = \log(\chi)$ and $k = 1$ in Corollary 4.42 and $\mathfrak{p}(\chi,\zeta)$ defined by (3.9), then (3.12) becomes

$$\big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\big(q_1^p(\chi)\big)^{\frac{1}{p}}\big) + \big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\big(q_2^p(\chi)\big)^{\frac{1}{p}}\big) \le 2\big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}U^p\big(q_1(\chi),q_2(\chi)\big)\big)^{\frac{1}{p}}.$$

Remark 4.45 Applying Theorem 4.33 with $\Delta = (a,b), d\pi(\zeta) = d\zeta$ and $\mathfrak{p}(\chi,\zeta)$ defined by (3.10). Substituting $(\mathfrak{L}q_1^p(\chi))^{\frac{1}{p}} = (I^{\alpha}_{a_+;\sigma;\eta}q_1^p(\chi))^{\frac{1}{p}}$ and $(\mathfrak{L}q_2^p(\chi))^{\frac{1}{p}} = (I^{\alpha}_{a_+;\sigma;\eta}q_2^p(\chi))^{\frac{1}{p}}$, we obtain the inequality for the Erdélyi–Köber fractional integral, i.e.,

$$\big(I^{\alpha}_{a_+;\sigma;\eta}\big(q_1^p(\chi)\big)^{\frac{1}{p}}\big) + \big(I^{\alpha}_{a_+;\sigma;\eta}\big(q_2^p(\chi)\big)^{\frac{1}{p}}\big) \le 2\big(I^{\alpha}_{a_+;\sigma;\eta}U^p\big(q_1(\chi),q_2(\chi)\big)\big)^{\frac{1}{p}}.$$

Remark 4.46 Taking $\beta > 0, \mathfrak{g}(\chi) = \frac{\chi^{\beta}}{\beta}$ and $k = 1$ in Corollary 4.42, we obtain the inequality for the Katugampola fractional integral operators in the literature [17] and the inequality takes the form

$$\big({}^{\tau}_{\alpha}K^{\beta}_{p^+}\big(q_1^p(\chi)\big)^{\frac{1}{p}}\big) + \big({}^{\tau}_{\alpha}K^{\beta}_{p^+}\big(q_2^p(\chi)\big)^{\frac{1}{p}}\big) \le 2\big({}^{\tau}_{\alpha}K^{\beta}_{p^+}U^p\big(q_1(\chi),q_2(\chi)\big)\big)^{\frac{1}{p}}.$$

Remark 4.47 Taking $\beta > 0, \mathfrak{g}(\chi) = \frac{(\chi-a)^{\beta}}{\beta}$ and $k = 1$ in Corollary 4.42, we obtain the inequality for the conformable fractional integral operators defined by Jarad et al. [1] and the inequality takes the form

$$\big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\big(q_1^p(\chi)\big)^{\frac{1}{p}}\big) + \big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}\big(q_2^p(\chi)\big)^{\frac{1}{p}}\big) \le 2\big({}^{\beta}_{\alpha}\mathfrak{J}^{\alpha}U^p\big(q_1(\chi),q_2(\chi)\big)\big)^{\frac{1}{p}}.$$

Remark 4.48 Taking $\beta > 0, \mathfrak{g}(\chi) = \frac{\chi^{\mu+\nu}}{\mu+\nu}$ and $k = 1$ in Corollary 4.42, we obtain the inequality for the conformable fractional integral operators defined by Khan et al. [3] and the inequality takes the form

$$\big({}^{\tau}_{\alpha}K^{\beta}_{p^+}\big(q_1^p(\chi)\big)^{\frac{1}{p}}\big) + \big({}^{\tau}_{\alpha}K^{\beta}_{p^+}\big(q_2^p(\chi)\big)^{\frac{1}{p}}\big) \le 2\big({}^{\tau}_{\alpha}K^{\beta}_{p^+}U^p\big(q_1(\chi),q_2(\chi)\big)\big)^{\frac{1}{p}}.$$

5 Concluding remarks

In recent years, many researchers have given the generalization of integral operators and constructed fruitful inequalities. It is always interesting and motivating for us to provide the generalization of all previous results. Motivated by the above, we presented certain elegant inequalities successfully that generalize the previous results. For this, we construct a class of functions that represent the integral transform with a general kernel. We prove a wide range of Pólya–Szegö- and Čebyšev-type inequalities involving a general kernel over a σ-finite measure. We extract the known results from our general results.

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Author contributions
S.I. was a major contributor to writing the manuscript, conceptualization, investigation and validation. M.S. dealt with the formal analysis, validation and supervision. M.A.K. dealt with the methodology, investigation, formal analysis and validation. G.R. performed conceptualization, formal analysis, and validation. K.N. performed the formal analysis, funding acquisition, validation, edition original draft preparation and writing of revised version. All authors read and approved the final manuscript.

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