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A simple proof for Imnang's algorithms

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Abstract
In this paper, a simple proof of the convergence of the recent iterative algorithm by relaxed (u, v)-cocoercive mappings due to Imnang (J. Inequal. Appl. 2013:249, 2013) is presented.

MSC: 47H09; 47H10

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1 Introduction and preliminaries

In this paper, a simple proof for the convergence of an iterative algorithm is presented that improves and refines the original proof.

Suppose that C is a nonempty closed convex subset of a real normed linear space E and E^* is its dual space. Suppose that $\langle .,. \rangle$ denotes the pairing between E and E^*. The normalized duality mapping $J : E \to E^*$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for each $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is called smooth if for all $x \in U$, there exists a unique functional $j_x \in E^*$ such that $\langle x, j_x \rangle = \|x\|$ and $\|j_x\| = 1$ (see [1]).

Recall that a mapping $f : C \to C$ is a contraction on C, if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, $\forall x, y \in C$. We use Π_C to denote the collection of all contractions on C, i.e., $\Pi_C = \{f | f : C \to C \text{ is a contraction}\}$.

For a map T from E into itself, we denote by $\mathrm{Fix(T)} := \{\mathrm{x} \in \mathrm{E} : \mathrm{x} = \mathrm{Tx}\}$, the fixed point set of T.

Recall the following well-known concepts:

(1) Suppose that C is a nonempty closed convex subset of a real Banach space E. A mapping $B : C \to E$ is called relaxed (u, v)-cocoercive [2], if there exist two constants $u, v > 0$ such that

$$\langle Bx - By, j(x - y) \rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2,$$

for all $x, y \in C$ and $j(x - y) \in J(x - y)$.

(2) Suppose that C is a nonempty closed convex subset of a real Banach space E and B is a self-mapping on C. If there exists a positive integer α such that

$$\|Bx - By\| \geq \alpha \|x - y\|$$

for all $x, y \in C$, then B is called α-expansive.

Lemma 1.1 ([2]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X with the 2-uniformly smooth constant K. Let Q_C be the sunny nonexpansive retraction from X onto C and let $A_i : C \to X$ be a relaxed (c_i, d_i)-cocoercive and L_i-Lipschitzian mapping for $i = 1, 2, 3$. Let $G : C \to C$ be a mapping defined by*

$$\begin{aligned} G(x) = Q_C\bigl[&Q_C\bigl(Q_C(x - \lambda_3 A_3 x) - \lambda_2 A_2 Q_C(x - \lambda_3 A_3 x)\bigr) \\ &- \lambda_1 A_1 Q_C\bigl(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x\bigr)\bigr]. \end{aligned}$$

If $\lambda_i \leq \frac{d_i - c_i L_i^2}{K^2 L_i^2}$ for all $i = 1, 2, 3$, then $G : C \to C$ is nonexpansive.

Lemma 1.2 ([3, Lemma 2.8]) *Suppose that C is a nonempty closed convex subset of a real Banach space X that is 2-uniformly smooth, and the mapping $A : C \to X$ is relaxed (c, d)-cocoercive and L_A-Lipschitzian. Then,*

$$\bigl\|(I - \lambda A)x - (I - \lambda A)y\bigr\|^2 \leq \|x - y\|^2 + 2\bigl(\lambda c L_A^2 - \lambda d + K^2 \lambda^2 L_A^2\bigr)\|x - y\|^2,$$

where $\lambda > 0$. In particular, when $d > cL_A^2$ and $\lambda \leq \frac{d - cL_A^2}{K^2 L_A^2}$, note $I - \lambda A$ is nonexpansive.

In this paper, using relaxed (u, v)-cocoercive mappings, a new proof for the iterative algorithm [2] is presented.

2 A simple proof for the theorem

Imnang [2] considered an iterative algorithm for finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of a variational inequality. Our argument will rely on the following lemma.

Lemma 2.1 *Suppose that C is a nonempty closed convex subset of a Banach space E. Suppose that $A : C \to E$ is a relaxed (m, v)-cocoercive mapping and ϵ-Lipschitz continuous with $v - m\epsilon^2 > 0$. Then, A is a $(v - m\epsilon^2)$-expansive mapping.*

Proof Since A is (m, v)-cocoercive and ϵ-Lipschitz continuous, for each $x, y \in C$ and $j(x - y) \in J(x - y)$, we have that

$$\begin{aligned} \bigl\langle Ax - Ay, j(x - y)\bigr\rangle &\geq (-m)\|Ax - Ay\|^2 + v\|x - y\|^2 \\ &\geq \bigl(-m\epsilon^2\bigr)\|x - y\|^2 + v\|x - y\|^2 \\ &= \bigl(v - m\epsilon^2\bigr)\|x - y\|^2 \geq 0, \end{aligned}$$

and hence

$$\|Ax - Ay\| \geq \left(\nu - m\epsilon^2\right)\|x - y\|,$$

therefore, A is $(\nu - m\epsilon^2)$-expansive. □

The following theorem is due to Imnang [2] that solves the viscosity iterative problem for a new general system of variational inequalities in Banach spaces:

Theorem 2.2 (*i.e.*, Theorem 3.1, from [2, §3, p.7]) *Suppose that X is a Banach space that is uniformly convex and 2-uniformly smooth with the 2-uniformly smooth constant K, C is a nonempty closed convex subset of X, and Q_C is a sunny nonexpansive retraction from X onto C. Assume that $A_i : C \to X$ is relaxed (c_i, d_i)-cocoercive and L_i-Lipschitzian with $0 < \lambda_i < \frac{d_i - c_i L_i^2}{K^2 L_i^2}$ for each $i = 1, 2, 3$. Suppose that f is a contraction mapping with the constant $\alpha \in (0, 1)$ and $S : C \to C$, a nonexpansive mapping such that $\Omega = F(S) \cap F(G) \neq \emptyset$, where G is defined as in Lemma* 1.1. *Suppose that $x_1 \in C$ and $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are the following sequences*:

$$\begin{cases} z_n = Q_C(x_n - \lambda_3 A_3 x_n), \\ y_n = Q_C(z_n - \lambda_2 A_2 z_n), \\ x_{n+1} = a_n f(x_n) + b_n x_n + (1 - a_n - b_n) S Q_C(y_n - \lambda_1 A_1 y_n, \end{cases}$$

where $\{a_n\}$ and $\{b_n\}$ are two sequences in $(0, 1)$ *such that*

(C1) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$.

Then, $\{x_n\}$ *converges strongly to* $q \in \Omega$, *which solves the following variational inequality*:

$$\left\langle q - f(q), J(q - p)\right\rangle \leq 0, \quad \forall f \in \Pi_C, p \in \Omega.$$

A Simple Proof Let $i = 1, 2, 3$. Consider Theorem 2.2 and the L_i-Lipschitz continuous and relaxed (c_i, d_i)-cocoercive mapping A_i in Theorem 2.2. From the condition that $0 < \lambda_i < \frac{d_i - c_i L_i^2}{K^2 L_i^2}$, we have that $0 < 1 + 2(\lambda_i c_i L_i^2 - \lambda_i d_i + K^2 \lambda_i^2 L_i^2) < 1$. Note that from Lemma 1.2, we have that $I - \lambda_i A_i$ is nonexpansive when $0 < 1 + 2(\lambda_i c_i L_i^2 - \lambda_i d_i + K^2 \lambda_i^2 L_i^2)$. Then, applying the coefficients $\alpha_i = 1 + 2(\lambda_i c_i L_i^2 - \lambda_i d_i + K^2 \lambda_i^2 L_i^2)$ in Lemma 1.2 we have that $I - \lambda_i A_i$ is an α_i-contraction, for each $i = 1, 2, 3$. Also, note that Q_C is nonexpansive and $I - \lambda_i A_i$ is an α_i-contraction, for each $i = 1, 2, 3$. Hence, using the proof of [2, Lemma 2.11], we conclude that

$$\begin{aligned} \left\|G(x) - G(y)\right\| &= \Big\|Q_C\Big[Q_C\big(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x\big) \\ &\quad - \lambda_1 A_1 Q_C\big(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x\big)\Big] \\ &\quad - Q_C\Big[Q_C\big(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y\big) \\ &\quad - \lambda_1 A_1 Q_C\big(Q_C(I - \lambda_3 A_3)y - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)y\big)\Big]\Big\| \\ &\leq \Big\|Q_C\big(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x\big) \\ &\quad - \lambda_1 A_1 Q_C\big(Q_C(I - \lambda_3 A_3)x - \lambda_2 A_2 Q_C(I - \lambda_3 A_3)x\big) \end{aligned}$$

$$\begin{aligned}
&- \big[Q_C\big(Q_C(I-\lambda_3A_3)y - \lambda_2A_2Q_C(I-\lambda_3A_3)y\big) \\
&- \lambda_1A_1Q_C\big(Q_C(I-\lambda_3A_3)y - \lambda_2A_2Q_C(I-\lambda_3A_3)y\big)\big]\big\| \\
= {}& \big\|(I-\lambda_1A_1)Q_C(I-\lambda_2A_2)Q_C(I-\lambda_3A_3)x \\
&- (I-\lambda_1A_1)Q_C(I-\lambda_2A_2)Q_C(I-\lambda_3A_3)y\big\| \\
\le {}& \alpha_1\alpha_2\alpha_3\|x-y\|,
\end{aligned}$$

and since $0 < \alpha_1\alpha_2\alpha_3 < 1$ then G is an α-contraction with $\alpha = \alpha_1\alpha_2\alpha_3$, hence from Banach's contraction principle $F(G)$ is a singleton set and hence, Ω is a singleton set, i.e., there exists an element $p \in X$ such that $\Omega = \{p\}$. Since $(d_i - c_iL_i^2) > 0$, from Lemma 2.1, A_i is $(d_i - c_iL_i^2)$-expansive, i.e.,

$$\|A_ix - A_iy\| \ge \big(d_i - c_iL_i^2\big)\|x-y\|, \tag{1}$$

in Theorem 2.2. The authors in [2, p.11] proved (see (3.12) in [2, p.11]) that

$$\lim_n \|A_3x_n - A_3p\| = 0, \tag{2}$$

for $x^* = p$. Now, put $x = x_n$ and $y = p$ in (1), and from (1) and (2), we have

$$\lim_n \|x_n - p\| = 0.$$

Hence, $x_n \to p$. As a result, one of the main claims of Theorem 2.2 is established (note $\Omega = \{p\}$).

Note that the main aims of Theorem 3.1 in [2] are $x_n \to p$ and

$$\big\langle q - f(q), J(q-p)\big\rangle \le 0, \quad \forall f \in \Pi_C, p \in \Omega.$$

Next, we show that the main aim of Theorem 3.1 in [2] can be concluded from the relations (3.12) in [2, page 11] and the proof in Theorem 2.2 can be simplified even further using the above. Note that the part of the proof between the relations (3.12) in [2, page 11] to the end of the proof of Theorem 3.1 can be removed from the proof. Indeed, since immediately from (3.12) in [2], we conclude that $x_n \to p$, i.e., the first aim of Theorem 3.1 is concluded. The second aim of the theorem, i.e.,

$$\big\langle q - f(q), J(q-p)\big\rangle \le 0, \quad \forall f \in \Pi_C, p \in \Omega,$$

is clear, because $p = q$ ($\Omega = \{p\}$) and $J(0) = \{0\}$. Consequently, the relations between (3.12) in [2, page 11] to the end of the proof of Theorem 3.1 in [2, page 11] can be removed. □

3 Discussion

In this paper, a simple proof for the convergence of an algorithm by relaxed (u, v)-cocoercive mappings due to Imnang is presented.

4 Conclusion

In this paper, a refinement of the proof of the results due to Imnang is given.

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