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Fixed points of completely positive maps and their dual maps

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Abstract

Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a row contraction and $\Phi_{\mathcal{A}}$ determined by $\mathcal{A}$ be a completely positive map on $\mathcal{B}(\mathcal{H})$. In this paper, we mainly consider fixed points of $\Phi_{\mathcal{A}}$ and its dual map $\Phi^{\dagger}_{\mathcal{A}}$. It is given that $\Phi_{\mathcal{A}}(X) \le X$ (or $\Phi_{\mathcal{A}}(X) \ge X$) implies $\Phi_{\mathcal{A}}(X) = X$ and $\Phi^{\dagger}_{\mathcal{A}}(X) = X$ when $X \in \mathcal{B}(\mathcal{H})$ is a compact operator. Some necessary conditions of $\Phi_{\mathcal{A}}(X) = X$ and $\Phi^{\dagger}_{\mathcal{A}}(X) = X$ are given.

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1 Introduction

Completely positive maps play an essential role in quantum information theory since they correspond to physical operations, see [7]. Recall that a quantum operation can be represented by a normal completely positive map, which is determined by an operator sequence, see [2, 3]. Hence, some problems about completely positive maps can be solved by researching operator sequences.

For the convenience of description, let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and $\mathcal{B}(\mathcal{K},\mathcal{H})$ be the set of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$ and abbreviate $\mathcal{B}(\mathcal{K},\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$ if $\mathcal{K} = \mathcal{H}$. $\mathcal{K}(\mathcal{H})$ is the set of compact operators on $\mathcal{H}$. Denote by J a finite or infinite countable index set. Let $\mathcal{A} = \{A_k\}_{k\in J} \subset \mathcal{B}(\mathcal{H})$. $\mathcal{A}$ is called a row contraction if $\sum_{k\in J} A_kA_k^* \le I$, where the series $\sum_{k\in J} A_kA_k^*$ is convergent in strong operator topology and A_k^* is the adjoint operator of A_k. We say that $\mathcal{A}$ is unital if $\sum_{k\in J} A_kA_k^* = I$ and trace preserving if $\sum_{k\in J} A_k^*A_k = I$.

To each row contraction $\mathcal{A} = \{A_k\}_{k\in J}$ one can associate a normal completely positive mapping $\Phi_{\mathcal{A}}$ on $\mathcal{B}(\mathcal{H})$,

$$\Phi_{\mathcal{A}}(X) = \sum_{k\in J} A_kXA_k^*, \quad \forall X \in \mathcal{B}(\mathcal{H}).$$

Then, we say that $\Phi_{\mathcal{A}}$ is a quantum operation on $\mathcal{B}(\mathcal{H})$ and each A_k is the operation element or the Kraus operator of $\Phi_{\mathcal{A}}$. $\mathcal{A}$ and $\Phi_{\mathcal{A}}$ are called self-adjoint if each A_k is self-adjoint. If a row contraction $\mathcal{A}$ also satisfies $\sum_{k\in J} A_k^*A_k \le I$, we can define a completely positive map $\Phi_{\mathcal{A}}^{\dagger}$ on $\mathcal{B}(\mathcal{H})$ as follows:

$$\Phi_{\mathcal{A}}^{\dagger}(X) = \sum_{k\in J} A_k^* X A_k, \quad \forall X \in \mathcal{B}(\mathcal{H}).$$

The map $\Phi_{\mathcal{A}}^{\dagger}$ is well defined and is called the dual operation of $\Phi_{\mathcal{A}}$. An operator $X \in \mathcal{B}(\mathcal{H})$ is said to be a fixed point of $\Phi_{\mathcal{A}}$ if $\Phi_{\mathcal{A}}(X) = X$. In fact, a fixed point $\Phi_{\mathcal{A}}$ means that it is not disturbed by the action of $\Phi_{\mathcal{A}}$. Denote by $\mathcal{B}(\mathcal{H})^{\Phi_{\mathcal{A}}}$ the set of fixed points of $\Phi_{\mathcal{A}}$.

Fixed points of completely positive maps were considered from different aspects since they are useful in the theory of quantum error correction, see [1, 4–6], and [8–11]. Li discussed fixed points of dual quantum operations on compact operators in [4] and given that the two fixed points sets of quantum operation and its dual operation are coincident under a certain condition. In [1], the authors noted that the positive fixed point $B \in \mathcal{B}(\mathcal{H})^{\Phi_{\mathcal{A}}}$ of $\Phi_{\mathcal{A}}$ and A_k commute if B has only discrete point spectra and $\Phi_{\mathcal{A}}$ is a self-adjoint quantum operation. However, the result does not necessary hold for a not self-adjoint quantum operation. Li generalized the result to the unital and trace-preserving quantum operation in [5], but B must be an operator when the spectra space is finite. In [4], the fixed points sets of $\Phi_{\mathcal{A}}$ and its dual map $\Phi_{\mathcal{A}}^{\dagger}$ were given by use of the properties of self-adjoint operators. It was given that the two sets were equivalent in compact operator space. Also, it was noted that $\Phi_{\mathcal{A}}(X) \geq X$ implied $\Phi_{\mathcal{A}}(X) = X$ under certain conditions. Popescu studied the inequality $\Phi_{\mathcal{A}}(X) \leq X$ and the equation $\Phi_{\mathcal{A}}(X) = X$ by use of the minimal isometric dilation and Poisson transforms in [5] and the canonical decompositions and lifting theorems were obtained to provide a description of all solutions of $\Phi_{\mathcal{A}}(X) \leq X$.

Inspired by the above results, we mainly consider fixed points of completely positive maps and their dual operations. For a given row contraction $\mathcal{A}$, we study the inequality $\Phi_{\mathcal{A}}(X) \leq X$ and the equation $\Phi_{\mathcal{A}}(X) = X$ on the set of all diagonalizable operators. It is given that $\Phi_{\mathcal{A}}(X) \leq X$ (or $\Phi_{\mathcal{A}}(X) \geq X$) implies $\Phi_{\mathcal{A}}(X) = X$ and $\Phi_{\mathcal{A}}^{\dagger}(X) = X$ when $X \in \mathcal{B}(\mathcal{H})$ is a compact operator. Simultaneously, an example is given to show that $\Phi_{\mathcal{A}}(X) = X$ does not necessarily imply $\Phi_{\mathcal{A}}^{\dagger}(X) = X$ when X is not compact. Some necessary conditions of $\Phi_{\mathcal{A}}(X) = X$ and $\Phi_{\mathcal{A}}^{\dagger}(X) = X$ are obtained.

2 Main result

In order to obtain the main results, we begin with some lemmas.

Lemma 1 ([8]) *Let Φ be a normal completely positive map on $\mathcal{B}(\mathcal{H})$ that is defined by*

$$\Phi(X) = \sum_{k\in J} A_k X A_k^*, \quad \forall \mathcal{B}(\mathcal{H}).$$

A positive operator $C \in \mathcal{B}(\mathcal{H})$ is a solution of the inequality $\Phi(X) \leq X$ (or $\Phi(X) = X$) if and only if there exists an operator sequence $\{B_k\}_{k\in J} \subset \mathcal{B}(\mathcal{H})$ such that $\sum_{k=1} B_k B_k^* \leq I$ (or $\sum_{k=1} B_k B_k^* = I$) and $A_k C^{\frac{1}{2}} = C^{\frac{1}{2}} B_k$ for any k.

Similar to Lemma 1, we give an equivalent condition of $\Phi(X) \geq X$.

Lemma 2 *Let Φ be a normal completely positive map on $\mathcal{B}(\mathcal{H})$ that is defined by*

$$\Phi(X) = \sum_{k\in J} A_k X A_k^*, \quad \forall \mathcal{B}(\mathcal{H}).$$

Then, an invertible and positive operator $C \in \mathcal{B}(\mathcal{H})$ is a solution of the inequality $\Phi(X) \geq X$ if and only if there exists an operator sequence $\{B_k\}_{k\in J} \subset \mathcal{B}(\mathcal{H})$ such that $\sum_{k\in J} B_k B_k^* \geq I$ and $C^{\frac{1}{2}} B_k = A_k C^{\frac{1}{2}}$ for any k.

Proof Suppose that C is an invertible and positive operator and also a solution of the inequality $\Phi(X) \geq X$. Define the operator B_k by setting $B_k = C^{-\frac{1}{2}} A_k C^{\frac{1}{2}}$ for any k. By direct computing, we have

$$\sum_{k\in J} B_k B_k^* = C^{-\frac{1}{2}} \sum_{k\in J} A_k A_k^* C^{-\frac{1}{2}} \leq C^{-1}.$$

That is to say, the operator series $\sum_{k\in J} B_k B_k^*$ is convergent in strong operator topology. From the definition of B_k, it is easy to obtain that $C^{\frac{1}{2}} B_k = A_k C^{\frac{1}{2}}$ and

$$\Phi(C) = \sum_{k\in J} A_k C A_k^* = C^{\frac{1}{2}} \sum_{k\in J} B_k B_k^* C^{\frac{1}{2}} \geq C.$$

Thus, $C^{\frac{1}{2}}(\sum_{k\in J} B_k B_k^* - I)C^{\frac{1}{2}} \geq 0$ and so $\sum_{k\in J} B_k B_k^* \geq I$.

On the contrary, suppose that $\{B_k\}_{k\in J} \subset \mathcal{B}(\mathcal{H})$ satisfies $\sum_{k=1} B_k B_k^* \geq I$ and $C^{\frac{1}{2}} B_k = A_k C^{\frac{1}{2}}$ for any k, then

$$\Phi(C) = \sum_{k\in J} A_k C A_k^* = C^{\frac{1}{2}} \sum_{k\in J} B_k B_k^* C^{\frac{1}{2}} \geq C.$$

The proof is completed. □

Lemma 3 *Let* $\dim \mathcal{H} < \infty$ *and* $\mathcal{A} = \{A_k\}_{k\in J} \subset \mathcal{B}(\mathcal{H})$ *be a row contraction. If* $\sum_{k\in J} A_k A_k^* = I$ *and* $\sum_{k\in J} A_k^* A_k \leq I$, *then* $\sum_{k\in J} A_k^* A_k = I$.

Proof Let τ be a faithful tracial state on $\mathcal{B}(\mathcal{H})$. This shows that $\tau(\sum_{k\in J} A_k A_k^*) = \tau(\sum_{k\in J} A_k^* A_k)$. That is to say $\tau(\sum_{k\in J} A_k A_k^* - \sum_{k\in J} A_k^* A_k) = 0$. This implies that $\sum_{k\in J} A_k^* A_k = \sum_{k\in J} A_k A_k^* = I$. □

Theorem 4 *Let* $\Phi_{\mathcal{A}}(I) = I$ *and* $\Phi_{\mathcal{A}}^{\dagger}(I) \leq I$. *If* $X \in \mathcal{B}(\mathcal{H})$ *is a compact and self-adjoint operator that satisfies* $\Phi_{\mathcal{A}}(X) \leq X$ *or* $\Phi_{\mathcal{A}}(X) \geq X$, *then* $\Phi_{\mathcal{A}}(X) = X$, $\Phi_{\mathcal{A}}^{\dagger}(X) = X$ *and* $X \in \mathcal{A}'$.

Proof (1) Suppose that $X \in \mathcal{B}(\mathcal{H})$ is a compact and self-adjoint operator with $\Phi_{\mathcal{A}}(X) \leq X$. Then, $\Phi_{\mathcal{A}}(\alpha + X) \leq \alpha + X$ holds for any real number α since $\Phi_{\mathcal{A}}(I) = I$. Without loss of generality, we may assume that X is an invertible and positive operator. According to the spectral theorem of compact normal operators, it is easy to show that the spectrum of X is at most countable and these spectral points can be arrayed as follows, $\lambda_1 > \lambda_2 > \cdots > \lambda_m$ (m is a positive integer or $+\infty$) and the dimension of the spectral projection space associated with λ_i is finite. It follows that $X = \sum_{i=1}^{m} \lambda_i P_i$, where P_i is the spectral projection associated with λ_i. From Lemma 1, there exists an operator sequence $\{B_k\}_{k\in J}$ with $\sum_{k\in J} B_k B_k^* \leq I$ such that $B_k X^{\frac{1}{2}} = X^{\frac{1}{2}} A_k$. Denote $\mathcal{H}_1 = R(P_1)$ and $\mathcal{H}_2 = \mathcal{H} \ominus \mathcal{H}_1$. Then, $X = \lambda_1 I_{\mathcal{H}_1} \oplus X_1$. It follows that $X^{\frac{1}{2}} = \lambda_1^{\frac{1}{2}} I_{\mathcal{H}_1} \oplus X_1^{\frac{1}{2}}$. A_k and B_k can be represented by

$$A_k = \begin{pmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{pmatrix} \quad \text{and} \quad B_k = \begin{pmatrix} B_{11}^k & B_{12}^k \\ B_{21}^k & B_{22}^k \end{pmatrix},$$

with respect to the space decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Therefore,

$$\begin{pmatrix} \lambda_1^{\frac{1}{2}} B_{11}^k & B_{12}^k X_1^{\frac{1}{2}} \\ \lambda_1^{\frac{1}{2}} B_{21}^k & B_{22}^k X_1^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \lambda_1^{\frac{1}{2}} A_{11}^k & \lambda_1^{\frac{1}{2}} A_{12}^k \\ X_1^{\frac{1}{2}} A_{21}^k & X_1^{\frac{1}{2}} A_{22}^k \end{pmatrix}.$$

This implies that $B_{11}^k = A_{11}^k$ and $A_{12}^k = \frac{1}{\sqrt{\lambda_1}} B_{12}^k X_1^{\frac{1}{2}}$ hold. According to $\sum_{k\in J} A_k A_k^* = I$ and $\sum_{k\in J} B_k B_k^* \leq I$, we have

$$\sum_{k\in J} A_{11}^k {A_{11}^k}^* + \sum_{k\in J} A_{12}^k {A_{12}^k}^* = \sum_{k\in J} A_{11}^k {A_{11}^k}^* + \sum_{k\in J} \frac{1}{\lambda_1} B_{12}^k X_1 {B_{12}^k}^* = I_{\mathcal{H}_1}$$

and

$$\sum_{k\in J} B_{11}^k {B_{11}^k}^* + \sum_{k\in J} B_{12}^k {B_{12}^k}^* \leq I_{\mathcal{H}_1}.$$

On the other hand, $0 \leq \frac{1}{\lambda_1} X_1 \leq I_{\mathcal{H}_2}$. Hence, $\sum_{k\in J} B_{12}^k (I_{\mathcal{H}_2} - \frac{1}{\lambda_1} X_1) {B_{12}^k}^* = 0$, and then $B_{12} = 0$. Therefore, $A_{12}^k = 0$ and $\sum_{k\in J} A_{11}^k {A_{11}^k}^* = I_{\mathcal{H}_1}$. From $\Phi_{\mathcal{A}}^{\dagger}(I) \leq I$, that is, $\sum_{k\in J} A_k^* A_k \leq I_{\mathcal{H}}$ then $\sum_{k\in J} {A_{11}^k}^* A_{11}^k \leq I_{\mathcal{H}_1}$. It follows from Lemma 3 that $\sum_{k\in J} {A_{11}^k}^* A_{11}^k = I_{\mathcal{H}_1}$ since $\mathcal{H}_1$ is a finite-dimensional space. Thus, $\sum_{k\in J} {A_{21}^k}^* A_{21}^k = 0$ and then $A_{21}^k = 0$. This shows that $A_k P_1 = P_1 A_k$, $\Phi_{\mathcal{A}}(P_1) = P_1$, $\Phi_{\mathcal{A}}^{\dagger}(P_1) = P_1$ and $\Phi_{\mathcal{A}}(X) = \lambda_1 \Phi_{\mathcal{A}}(P_1) \oplus \Phi_{\mathcal{A}}(X_1) \leq \lambda_1 P_1 \oplus X_1$. Therefore, $\Phi_{\mathcal{A}}(X_1) \leq X_1$. By induction, $X \in \mathcal{A}'$, $\Phi_{\mathcal{A}}(X) = X$ and $\Phi_{\mathcal{A}}^{\dagger}(X) = X$.

(2) If $\Phi_{\mathcal{A}}(X) \geq X$, the process is as above, the result holds by Lemma 2. The proof is completed. □

Similar to the proof of Theorem 4, we have the following result.

Theorem 5 ([4]) *Let $\Phi_{\mathcal{A}}(I) \leq I$ and $\Phi_{\mathcal{A}}^{\dagger}(I) \leq I$. If $X \in \mathcal{K}(\mathcal{H})$ satisfies $\Phi_{\mathcal{A}}(X) \geq X \geq 0$, then $\Phi_{\mathcal{A}}(X) = X$ and $X \in \mathcal{A}'$ hold.*

Corollary 6 ([1]) *Let $\dim \mathcal{H} < \infty$ and $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a unital and trace-preserving row contraction. Then, $\mathcal{B}(\mathcal{H})^{\Phi_{\mathcal{A}}} = \mathcal{A}'$.*

Proof As $\mathcal{A}$ is unital, it is natural that $\mathcal{A}' \subset \mathcal{B}(\mathcal{H})^{\Phi_{\mathcal{A}}}$ holds. We need only to prove that $\mathcal{B}(\mathcal{H})^{\Phi_{\mathcal{A}}} \subset \mathcal{A}'$. For any $X \in \mathcal{B}(\mathcal{H})^{\Phi_{\mathcal{A}}}$, then $X^* \in \mathcal{B}(\mathcal{H})^{\Phi_{\mathcal{A}}}$. Hence, we can assume that X is self-adjoint. Denote $\mathcal{H}_1 = P^X(0, \|X\|]$ and $\mathcal{H}_2 = [-\|X\|, 0]$. Then, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and X has the representation $X = X^+ \oplus (-X^-)$, where X^+ is invertible in $\mathcal{B}(\mathcal{H}_1)$. With respect to the space decomposition as above, the operator A_k can be expressed as $A_k = (A_{ij}^k)_{2\times 2}$ and then ${A_k}^* = ({A_{ji}^k}^*)_{2\times 2}$. It follows that

$$A_k X A_k^* = \begin{pmatrix} A_{11}^k X^+ {A_{11}^k}^* - A_{12}^k X^- {A_{12}^k}^* & A_{11}^k X^+ {A_{21}^k}^* - A_{12}^k X^- {A_{22}^k}^* \\ A_{21}^k X^+ {A_{11}^k}^* - A_{22}^k X^- {A_{12}^k}^* & A_{21}^k X^+ {A_{21}^k}^* - A_{22}^k X^- {A_{22}^k}^* \end{pmatrix}.$$

From $\Phi_{\mathcal{A}}(X) = X$, we obtain

$$\begin{cases} \sum_{k\in J} A_{11}^k X^+ {A_{11}^k}^* - \sum_{k\in J} A_{12}^k X^- {A_{12}^k}^* = X^+, \\ \sum_{k\in J} A_{21}^k X^+ {A_{21}^k}^* - \sum_{k\in J} A_{22}^k X^- {A_{22}^k}^* = -X^-, \end{cases}$$

whereas, $\sum_{k\in J} A_{12}^k X^- {A_{12}^k}^* \geq 0$, so $\sum_{k\in J} A_{11}^k X^+ {A_{11}^k}^* \geq X^+$. Combining this with Theorem 5, we have $X^+ \in \{A_{11}^k\}_{k\in J}'$ and $\sum_{k\in J} A_{11}^k X^+ {A_{11}^k}^* = X^+$. Furthermore, $\sum_{k\in J} A_{11}^k {A_{11}^k}^* = I_{\mathcal{H}_1}$ holds. Moreover, $\sum_{k\in J} A_k {A_k}^* = I_{\mathcal{H}}$ implies $\sum_{k\in J} A_{12}^k {A_{12}^k}^* = 0$ and hence $A_{12}^k = 0$. From Lemma 3 and $\sum_{k\in J} A_k^* A_k = I_{\mathcal{H}}$, we have $A_{21}^k = 0$ for any k. Hence, $\sum_{k\in J} A_{22}^k {A_{22}^k}^* = I_{\mathcal{H}_2}$. Combining $\sum_{k\in J} A_{22}^k X^- {A_{22}^k}^* \geq X^-$ with Theorem 4, it is easy to obtain $X^- \in \{A_{22}^k\}_{k\in J}'$, and then $X \in \mathcal{A}'$. The proof is completed. □

In Theorem 4, the result does not necessarily hold if X is not a compact operator.

Example 7 Let $\{e_1, e_2, \dots\}$ be a basis of an infinite Hilbert space $\mathcal{H}$ and S be the unilateral operator on $\mathcal{H}$. Then, $Se_i = e_{i+1}, \forall i \geq 1$. Suppose that $\mathcal{K} = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$. Define an operator A as follows,

$$A = \begin{pmatrix} S^* & 0 & 0 \\ \frac{1}{\sqrt{2}}(I - SS^*) & SS^* & \frac{1}{\sqrt{2}}(I - SS^*) \\ 0 & 0 & S^* \end{pmatrix}.$$

Then,

$$A^* = \begin{pmatrix} S & \frac{1}{\sqrt{2}}(I - SS^*) & 0 \\ 0 & SS^* & 0 \\ 0 & \frac{1}{\sqrt{2}}(I - SS^*) & S \end{pmatrix}.$$

By direct computing, it is easy to obtain that $AA^* = I_{\mathcal{K}}$ and $A^*A \leq I_{\mathcal{K}}$. Assume that $X \in \mathcal{B}(\mathcal{K})$ has the following matrix form,

$$X = \begin{pmatrix} I_{\mathcal{H}} & 0 & I_{\mathcal{H}} \\ 0 & \frac{3}{2} I_{\mathcal{H}} & 0 \\ 0 & 0 & I_{\mathcal{H}} \end{pmatrix}.$$

According to the matrix forms of A, A^*, X, $AXA^* = X$ holds, whereas,

$$AX = \begin{pmatrix} S^* & 0 & 0 \\ \frac{1}{\sqrt{2}}(I - SS^*) & \frac{3}{2}SS^* & \sqrt{2}(I - SS^*) \\ 0 & 0 & S^* \end{pmatrix},$$

$$XA = \begin{pmatrix} S^* & 0 & S^* \\ \frac{3}{\sqrt{2}}(I - SS^*) & \frac{3}{2}SS^* & \frac{3}{\sqrt{2}}(I - SS^*) \\ 0 & 0 & S^* \end{pmatrix}.$$

These show that $AX \neq XA$ and $A^*XA \neq X$.

Proposition 8 *Let $\Phi_{\mathcal{A}}(I) \leq I$ and $\Phi_{\mathcal{A}}^{\dagger}(I) \leq I$. Suppose that X is a positive operator with only at most a countable set of distinct eigenvalues $\{\lambda_i\}$ such that $X = \sum_i \lambda_i P_i$, where $P_i P_j = P_j P_i = 0$ and λ_i is strictly decreasing. If $\Phi_{\mathcal{A}}(X) \geq X$ and $\Phi_{\mathcal{A}}^{\dagger}(X) \geq X$, then $X \in \mathcal{A}'$ and $\Phi_{\mathcal{A}}(X) = \Phi_{\mathcal{A}}^{\dagger}(X) = X$.*

Proof Suppose that X is a positive operator with $X = \sum_i \lambda_i P_i$ and λ_i is strictly decreasing. Denote $\mathcal{H}_1 = P^X\{\lambda_1\}\mathcal{H}$ and $\mathcal{H}_2 = \mathcal{H} \ominus \mathcal{H}_1$. Then, $X = \lambda_1 I_{\mathcal{H}_1} \oplus X_1$. A_k and A_k^* have the following matrix forms,

$$A_k = \begin{pmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{pmatrix} \quad \text{and} \quad A_k^* = \begin{pmatrix} {A_{11}^k}^* & {A_{21}^k}^* \\ {A_{12}^k}^* & {A_{22}^k}^* \end{pmatrix}.$$

Therefore,

$$A_k X A_k^* = \begin{pmatrix} \lambda_1 A_{11}^k {A_{11}^k}^* + A_{12}^k X_1 {A_{12}^k}^* & \lambda_1 A_{11}^k {A_{21}^k}^* + A_{12}^k X_1 {A_{22}^k}^* \\ \lambda_1 A_{21}^k {A_{11}^k}^* + A_{22}^k X_1 {A_{12}^k}^* & \lambda_1 A_{21}^k {A_{21}^k}^* + A_{22}^k X_1 {A_{22}^k}^* \end{pmatrix}.$$

From $\sum_{k\in J} A_k X A_k^* \geq X$, we have

$$\lambda_1 I_{\mathcal{H}_1} \leq \sum_{k\in J} \lambda_1 A_{11}^k {A_{11}^k}^* + \sum_{k\in J} A_{12}^k X_1 {A_{12}^k}^* \leq \lambda_1 \left(\sum_{k\in J} A_{11}^k {A_{11}^k}^* + \sum_{k\in J} A_{12}^k {A_{12}^k}^* \right) \leq \lambda_1 \mathcal{H}_1.$$

If $X_1 = 0$, then $\sum_{k\in J} A_{11}^k {A_{11}^k}^* = I_{\mathcal{H}_1}$. It follows that $\sum_{k\in J} A_{12}^k {A_{12}^k}^* = 0$, hence $A_{12}^k = 0$. If $X_1 \neq 0$, then $X_1 < \lambda_1 I_{\mathcal{H}_2}$, which means $\lambda_1 I_{\mathcal{H}_2} - X_1$ is a positive and invertible operator. Therefore, $\sum_{k\in J} A_{12}^k {A_{12}^k}^* = 0$ and so $A_{12}^k = 0$. On the other hand, from $\Phi_{\mathcal{A}}^{\dagger}(X) \geq X$, we can obtain $A_{21}^k = 0$. That is, $A_k P_1 = P_1 A_k$, $\Phi_{\mathcal{A}}(P_1) = P_1$ and $\Phi_{\mathcal{A}}^{\dagger}(P_1) = P_1$. Meanwhile, $\Phi_{\mathcal{A}}(0 \oplus X_1) \geq 0 \oplus X_1$, $\Phi_{\mathcal{A}}^{\dagger}(0 \oplus X_1) \geq 0 \oplus X_1$. Continuing the above process, the result holds. The proof is completed. □

Theorem 9 *Let $\Phi_{\mathcal{A}}(I) \leq I$ and $\Phi_{\mathcal{A}}^{\dagger}(I) \leq I$. Suppose that X is a self-adjoint operator with only at most a countable set of distinct eigenvalues $\{\lambda_i\}$ and $|\lambda_i|$ can be arranged in decreasing order, where $|\lambda_i|$ means the absolute value of λ_i. If $\Phi_{\mathcal{A}}(X) = X$ and $\Phi_{\mathcal{A}}^{\dagger}(X) = X$, then $X \in \mathcal{A}'$.*

Proof Let $\mathcal{H}_1 = P^X[-\|X\|, 0)$, $\mathcal{H}_2 = P^X\{0\}$ and $\mathcal{H}_3 = P^X(0, \|X\|]$, where $P^X(\cdot)$ is the spectral measure of X. Then, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. X has the matrix form $X = X_1 \oplus 0 \oplus (-X_3)$, where X_1 and X_3 are injective and have dense ranges. Denote $A_k = (A_{ij}^k)_{3\times 3}$, then ${A_k}^* = ({A_{ji}^k}^*)_{3\times 3}$. By direct computing, we have

$$A_k X A_k^* = \begin{pmatrix} A_{11}^k X_1 A_{11}^k * - A_{13}^k X_3 {A_{13}^k}^* & A_{11}^k X_1 {A_{21}^k}^* - A_{13}^k X_3 {A_{23}^k}^* & A_{11}^k X_1 {A_{31}^k}^* - A_{13}^k X_3 {A_{33}^k}^* \\ A_{21}^k X_1 {A_{11}^k}^* - A_{23}^k X_3 {A_{13}^k}^* & A_{21}^k X_1 {A_{21}^k}^* - A_{23}^k X_3 {A_{23}^k}^* & A_{21}^k X_1 {A_{31}^k}^* - A_{23}^k X_3 {A_{33}^k}^* \\ A_{31}^k X_1 {A_{11}^k}^* - A_{33}^k X_3 {A_{13}^k}^* & A_{31}^k X_1 {A_{11}^k}^* - A_{33}^k X_3 {A_{13}^k}^* & A_{31}^k X_1 {A_{31}^k}^* - A_{33}^k X_3 {A_{33}^k}^* \end{pmatrix}.$$

From $\Phi_{\mathcal{A}}(X) = X$, it is easy to see that

$$\sum_{k\in J} A_{11}^k X_1 {A_{11}^k}^* - \sum_{k\in J} A_{13}^k X_3 {A_{13}^k}^* = X_1, \tag{1}$$

$$\sum_{k\in J} A_{31}^k X_1 {A_{31}^k}^* - \sum_{k\in J} A_{33}^k X_3 {A_{33}^k}^* = -X_3, \tag{2}$$

whereas,

$$A_k^* X A_k = \begin{pmatrix} {A_{11}^k}^* X_1 A_{11}^k - {A_{31}^k}^* X_3 A_{31}^k & {A_{11}^k}^* X_1 A_{12}^k - {A_{31}^k}^* X_3 A_{32}^k & {A_{11}^k}^* X_1 A_{13}^k - {A_{31}^k}^* X_3 A_{33}^k \\ {A_{12}^k}^* X_1 A_{11}^k - {A_{32}^k}^* X_3 A_{31}^k & {A_{12}^k}^* X_1 A_{12}^k - {A_{32}^k}^* X_3 A_{32}^k & {A_{12}^k}^* X_1 A_{13}^k - X_3 {A_{32}^k}^* A_{33}^k \\ {A_{13}^k}^* X_1 A_{11}^k - {A_{33}^k}^* X_3 A_{31}^k & {A_{13}^k}^* X_1 A_{12}^k - {A_{33}^k}^* X_3 A_{32}^k & {A_{13}^k}^* X_1 A_{13}^k - {A_{33}^k}^* X_3 A_{33}^k \end{pmatrix}.$$

From $\Phi_{\mathcal{A}}^{\dagger}(X) = X$, we can obtain

$$\sum_{k\in J} A_{11}^{k\,*} X_1 A_{11}^k - \sum_{k\in J} A_{31}^{k\,*} X_3 A_{31}^k = X_1, \tag{3}$$

$$\sum_{k\in J} A_{13}^{k\,*} X_1 A_{13}^k - \sum_{k\in J} A_{33}^{k\,*} X_3 A_{33}^k = -X_3. \tag{4}$$

As $\sum_{k\in J} A_{13}^k X_3 A_{13}^{k\,*} \geq 0$ and $\sum_{k\in J} A_{31}^k X_1 A_{31}^{k\,*} \geq 0$, combining Eq. (1) with Eq. (2), we have

$$\sum_{k\in J} A_{11}^k X_1 A_{11}^{k\,*} \geq X_1, \tag{5}$$

$$\sum_{k\in J} A_{33}^{k\,*} X_3 A_{33}^k \geq X_3. \tag{6}$$

Similarly, combining $\sum_{k\in J} A_{13}^{k\,*} X_1 A_{13}^k \geq 0$, $\sum_{k\in J} A_{31}^{k\,*} X_3 A_{31}^k \geq 0$ with Eqs. (3) and (4), the following equations hold,

$$\sum_{k\in J} A_{11}^{k\,*} X_1 A_{11}^k \geq X_1, \tag{7}$$

$$\sum_{k\in J} A_{33}^{k\,*} X_3 A_{33}^k \geq X_3. \tag{8}$$

It follows from Proposition 8, Eqs. (5), (7), (6), and (8) that

$$X_1 \in \left\{A_{11}^k, A_{11}^{k\,*}\right\}' \quad \text{and} \quad X_3 \in \left\{A_{33}^k, A_{33}^{k\,*}\right\}', \tag{9}$$

and

$$\sum_{k\in J} A_{11}^k X_1 A_{11}^{k\,*} = X_1, \qquad \sum_{k\in J} A_{33}^k X_3 A_{33}^{k\,*} = X_3.$$

Hence, $\sum_{k\in J} A_{13}^k X_3 A_{13}^{k\,*} = 0$. As X_3 is positive, injective, and also has dense range, hence $A_{13}^k = 0$. Similarly, $A_{31}^k = 0$. The operator X_1 is also a positive and injective operator with dense range, $\sum_{k\in J} A_{11}^k A_{11}^{k\,*} = I_{\mathcal{H}_1}$ from Eq. (5). According to $\sum_{k\in J} A_k A_k^* \leq I_{\mathcal{H}}$, then $\sum_{k\in J} A_{12}^k A_{12}^{k\,*} = 0$, and so $A_{12}^k = 0$ for any k. Similarly, $A_{21}^k = 0$. This shows that $A_k = A_{11}^k \oplus A_{22}^k \oplus A_{33}^k$. Combining Eq. (9) and the matrix forms of X and A_k, we have $A_k X = X A_k$ for any k. The proof is completed. □

If X has only two spectral points, we have the following result.

Theorem 10 *Let $\mathcal{A}$ be a unital operator sequence and X be a self-adjoint operator with only two spectral points. If $\Phi_{\mathcal{A}}(X) = X$, then $X \in \mathcal{A}'$.*

Proof Let λ_1, λ_2 be the two spectral points of X. Without loss of generality, suppose that $\lambda_1 > \lambda_2 > 0$ since $\Phi_{\mathcal{A}}(I) = I$. Denote $\mathcal{H}_1 = P^X\{\lambda_1\}\mathcal{H}$ and $\mathcal{H}_2 = P^X\{\lambda_2\}\mathcal{H}$, then $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}$. Hence, $X = \lambda_1 I_{\mathcal{H}_1} \oplus \lambda_2 I_{\mathcal{H}_2}$. Assume that A_k has the matrix form $A_k = (A_{ij}^k)_{2\times 2}$ with respect to the space decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. From $\Phi_{\mathcal{A}}(X) = X$, we have

$$\begin{aligned}
&\sum_{k\in J} A_k X A_k^* \\
&\quad = \begin{pmatrix} \lambda_1 \sum_{k\in J} A_{11}^k {A_{11}^k}^* + \lambda_2 \sum_{k\in J} A_{12}^k {A_{12}^k}^* & \lambda_1 \sum_{k\in J} A_{11}^k {A_{21}^k}^* + \lambda_2 \sum_{k\in J} A_{12}^k {A_{22}^k}^* \\ \lambda_1 \sum_{k\in J} A_{21}^k {A_{11}^k}^* + \lambda_2 \sum_{k\in J} A_{22}^k {A_{12}^k}^* & \lambda_1 \sum_{k\in J} A_{21}^k {A_{21}^k}^* + \lambda_2 \sum_{k\in J} A_{22}^k {A_{22}^k}^* \end{pmatrix} \\
&\quad = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.
\end{aligned}$$

This shows that $\lambda_1 \sum_{k\in J} A_{11}^k {A_{11}^k}^* + \lambda_2 \sum_{k\in J} A_{12}^k {A_{12}^k}^* = \lambda_1 I_{\mathcal{H}_1}$. That is,

$$\sum_{k\in J} A_{11}^k {A_{11}^k}^* + \frac{\lambda_2}{\lambda_1} \sum_{k\in J} A_{12}^k {A_{12}^k}^* = I_{\mathcal{H}_1}.$$

As $\mathcal{A}$ is unital, it is easy to obtain that

$$\sum_{k\in J} A_{11}^k {A_{11}^k}^* + \sum_{k\in J} A_{12}^k {A_{12}^k}^* = I_{\mathcal{H}_1}.$$

Hence, $\sum_{k\in J} A_{12}^k {A_{12}^k}^* = 0$ and then $A_{12}^k = 0$. Similarly, according to

$$\lambda_1 \sum_{k\in J} A_{21}^k {A_{21}^k}^* + \lambda_2 \sum_{k\in J} A_{22}^k {A_{22}^k}^* = \lambda_2 I_{\mathcal{H}_2}$$

and

$$\sum_{k\in J} A_{21}^k {A_{21}^k}^* + \sum_{k\in J} A_{22}^k {A_{22}^k}^* = I_{\mathcal{H}_2},$$

we have $A_{21}^k = 0$. Hence $X \in \mathcal{A}'$. The proof is completed. □

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Declarations

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The authors declare no competing interests.

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References

1. Arias, A., Gheondea, A., Gudder, S.: Fixed points of quantum operations. J. Math. Phys. **43**, 5872–5881 (2002)
2. Choi, M.D.: Completely positive linear maps on complex matrices. Linear Algebra Appl. **10**, 285–290 (1975)
3. Kraus, K.: General state changes in quantum theory. Ann. Phys. **64**, 311–355 (1971)
4. Li, Y.: Fixed points of dual quantum operations. Math. Phys. Anal. Geom. **382**, 172–179 (2011)
5. Li, Y.: Characterizations of fixed points of quantum operations. J. Math. Phys. **52**, 052103 (2011)
6. Magajna, B.: Fixed points of normal completely positive maps on $\mathcal{B}(\mathcal{H})$. J. Math. Anal. Appl. **389**, 1291–1302 (2012)
7. Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2010)
8. Popescu, G.: Similarity and ergodic theory of positive linear maps. J. Reine Angew. Math. **561**, 87–129 (2003)
9. Prunaru, B.: Toeplitz operators associated to commuting row contractions. J. Funct. Anal. **254**, 1626–1641 (2008)
10. Zhang, H.Y., Ji, G.X.: A note on fixed points of general quantum operations. Rep. Math. Phys. **70**, 111–117 (2012)
11. Zhang, H.Y., Xue, M.Z.: Fixed points of trace preserving completely positive maps. Linear Multilinear Algebra **64**(3), 404–411 (2016)