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On the characterization properties of certain hypergeometric functions in the open unit disk

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Abstract

Our purpose in the present investigation is to study certain geometric properties such as the close-to-convexity, convexity, and starlikeness of the hypergeometric function $z_1F_2(a;b,c;z)$ in the open unit disk. The usefulness of the main results for some familiar special functions like the modified Sturve function, the modified Lommel function, the modified Bessel function, and the ${}_0F_1(-;c;z)$ function are also mentioned. We further consider a boundedness property of the function ${}_1F_2(a;b,c;z)$ in the Hardy space of analytic functions. Several corollaries and special cases of the main results are also pointed out.

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1 Introduction and preliminaries

It is well known that special functions play important roles in geometric function theory, especially after the solution of the famous Bieberbach conjecture by De-Branges [10]. There exists an extensive literature that deals with the geometric properties of various special functions such as hypergeometric functions, confluent hypergeometric functons, Bessel's functions, Mittag–Leffler's function, Wright's function, Sturve's function, Lommel's function, Dini's function, and several other functions. Many researchers have determined sufficient conditions for the parameters involved in these special functions when they belong to certain classes of functions that are starlike, convex or close-to-convex. For instance, several sufficient conditions for the Gauss hypergeometric functions to be starlike or convex have been studied by Merkes and Scott [20], Lewis [19], Ruscheweyh and Singh [30], Miller and Mocanu [22], Silverman [32], Ponnusamy and Vuorinen [25], Küstner [17, 18], and Hästo *et al.* [16]. Most of the known results in this direction deal with the shifted hypergeometric function $z_2F_1(a,b;c;z)$ for real parameters a,b, and c. Recently, several authors have investigated the geometric properties of Bessel's functions [4, 5, 7, 8], Struve's functions [23, 36], Lommel's functions [9], Wright's function [26] (see also [13, 28]), and Mittag–Leffler's function [3].

The classical generalized hypergeometric function [2, 33] is defined by

$$ {}_pF_q(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;z)=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!}, $$

where the Pochhammer (or shifted factorial) symbols $(b_i)_n, i=1,2,\ldots,q$ are assumed to be nonzero and nonnegative integers. In this paper, our main aim is to study various several geometric properties of the function $z{}_1F_2(a;b,c;z)$, which is a particular case of the generalized hypergeometric function ${}_pF_q(z)$ (for $p=1$ and $q=2$) and is defined by

$$ {}_1F_2(a;b,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n(c)_n}\frac{z^n}{n!}, \tag{1} $$

where both the parameters b and c do not assume values of zero or a negative integer. By the ratio test, we see that the radius of convergence of this series (1) is infinity, and hence it is an entire function. However, in this paper we consider this function in the restricted domain $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, which is a suitable domain to be considered for studying its geometric properties. It can easily be verified that

$$ \frac{d}{dz}{}_1F_2(a;b,c;z)=\frac{a}{bc}{}_1F_2(a+1;b+1,c+1;z), \tag{2} $$

and further, the function $w(z)={}_1F_2(a;b,c;z)$ is a solution of the third-order homogeneous differential equation

$$ z^2w'''(z)+(b+c+1)zw''(z)+(bc-z)w'(z)-aw(z)=0. $$

Now, we briefly outline some of the useful notations. Let $\mathcal{H}$ denote the class of analytic functions defined on $\mathbb{D}$ and let the subclass $\mathcal{A}=\{f\in\mathcal{H}:f(0)=0=f'(0)-1\}$ consist of analytic functions in $\mathbb{D}$ having a Taylor-series expansion of the form

$$ f(z)=z+a_2z^2+a_3z^3+\cdots. $$

Evidently, the function $z{}_1F_2(a;b,c;z)$ belongs to the class $\mathcal{A}$. Our purpose in the present investigation is to study the geometric properties of the function $z{}_1F_2(a;b,c;z)$ that enables us to deduce the corresponding properties for functions like the modified Struve function, the modified Lommel function, the modified Bessel function, and the ${}_0F_1(-;c;z)$ function. We briefly give here the details of the special functions that stem from the function ${}_1F_2(a;b,c;z)$.

1.1 The modified Struve function

The well-known Struve function of order ν is defined by

$$ H_\nu(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\nu+3/2)\Gamma(n+3/2)}\left(\frac{z}{2}\right)^{2n+\nu-1}, $$

which is a particular solution of the nonhomogeneous Bessel differential equation defined by

$$z^2 w''(z) + zw'(z) + (z^2 - \nu^2)w = \frac{4(z/2)^{\nu+1}}{\sqrt{\pi}\Gamma(\nu + 1/2)}.$$

The modified Struve function $L_\nu(z)$ is defined by (see [37, p. 353])

$$\begin{aligned} L_\nu(z) &= -ie^{-i\nu\pi/2} H_\nu(iz) \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\nu+3/2)\Gamma(n+3/2)} \left(\frac{z}{2}\right)^{2n+\nu-1} \\ &= \frac{z^{\nu-1}}{\sqrt{\pi}\Gamma(\nu+3/2)2^{\nu-2}} {}_1F_2\left(1; \nu + \frac{3}{2}, \frac{3}{2}; \frac{z^2}{4}\right). \end{aligned}$$

The function $L_\nu(z)$ does not belong to the class $\mathcal{A}$, so we use the normalized form $\mathbb{L}_\nu(z)$ defined by

$$\mathbb{L}_\nu(z) = \sqrt{\pi}\Gamma(\nu + 3/2)2^{\nu-2} z^{2-\nu} L_\nu(z).$$

It is easy to see that

$$\mathbb{L}_\nu(z) = z {}_1F_2\left(1; \nu + \frac{3}{2}, \frac{3}{2}; \frac{z^2}{4}\right). \tag{3}$$

1.2 The modified Lommel function

The Lommel function of the first kind $S_{\mu,\nu}(z)$ is a particular solution of the nonhomogeneous Bessel differential equation (see for details [6] and [35])

$$z^2 w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = z^{\mu+1},$$

and can be expressed in terms of a hypergeometric series

$$S_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} {}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; \frac{-z^2}{4}\right),$$

where $\mu \pm \nu$ is a nonnegative odd integer. The modified Lommel function is defined by

$$\begin{aligned} T_{\mu,\nu}(z) &= -i^{1-\mu} S_{\mu,\nu}(iz) \\ &= \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} {}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; \frac{z^2}{4}\right). \end{aligned}$$

For more details on the modified Lommel functions, we refer to the works of [29] and Shafer [31]. The function $T_{\mu,\nu}(z)$ obviously does not belong to the class $\mathcal{A}$, so we use the normalized form $\mathbb{T}_{\mu,\nu}(z)$ defined by

$$\mathbb{T}_{\mu,\nu}(z) = (\mu - \nu + 1)(\mu + \nu + 1) z^{-\mu} T_{\mu,\nu}(z),$$

which has a series representation of the form:

$$\mathbb{T}_{\mu,\nu}(z) = z_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; \frac{z^2}{4}\right). \tag{4}$$

1.3 The modified Bessel function

For $a = b$ in (1), we obtain

$${}_0F_1(-;c;z) = \sum_{n=0}^{\infty} \frac{1}{(c)_n} \frac{z^n}{n!},$$

which is a solution of the second-order homogeneous differential equation defined by

$$zw''(z) + cw'(z) - w(z) = 0.$$

It may be noted that a relationship exists between the function ${}_0F_1$ and the modified Bessel function. Indeed, we have

$$z_0F_1\left(-;\nu+1;z^2/4\right) = 2^\nu\Gamma(\nu+1)z^{1-\nu}I_\nu(z) = \mathbb{I}_\nu(z), \tag{5}$$

where the modified Bessel function is defined by

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu-1}$$

and $\mathbb{I}_\nu(z)$ is the normalized modified Bessel function.

In order to have this paper reasonably self-contained, we give necessary details related to the consideration of geometric properties of functions analytic in the unit disk. We denote by $\mathcal{S}$, the class of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$, that is,

$$\mathcal{S} = \{f \in \mathcal{A} : f \text{ is one-to-one in } \mathbb{D}\}.$$

Recall the Bieberbach conjecture in which the Taylor coefficients of each function of the class $\mathcal{S}$ satisfy the inequality $|a_n| \le n$ for $n = 2, 3, \dots$, and only the rotations of the Koebe function $z/(1-z)^2$ provide the case of equality. In 1984, De-Branges [10] settled the Bieberbach conjecture by using the generalized hypergeometric functions. The exploitation of hypergeometric functions in the proof of the Bieberbach conjecture has provided new areas of interest to study various special functions from the viewpoint of geometric function theory. We need the following basic classes of functions in the present investigation. For $\beta < 1$, let

$$\mathcal{P}(\beta) = \left\{p : p \in \mathcal{H} \text{ with } p(0) = 1 \text{ and } \mathfrak{Re}\left\{p(z)\right\} > \beta, z \in \mathbb{D}\right\}$$

and

$$\mathcal{R} = \left\{f \in \mathcal{A} : \mathfrak{Re}\left\{f'(z)\right\} > 0, z \in \mathbb{D}\right\}.$$

A function $f \in \mathcal{S}$ is called starlike (with respect to the origin 0), denoted by $f \in \mathcal{S}^*$ if $tw \in f(\mathbb{D})$ for all $w \in f(\mathbb{D})$ and $t \in [0,1]$. A function $f \in \mathcal{S}$ that maps $\mathbb{D}$ onto a convex domain is called a convex function and the class of such functions is denoted by $\mathcal{K}$. For a given $0 \leq \alpha < 1$, a function $f \in \mathcal{S}$ is called a starlike function of order α, denoted by $\mathcal{S}^*(\alpha)$, if and only if

$$\mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{D}.$$

For a given $0 \leq \alpha < 1$, a function $f \in \mathcal{S}$ is called a convex function of order α, denoted by $\mathcal{K}(\alpha)$, if and only if

$$\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathbb{D}.$$

It is well known that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$. We recall [12] that the function $zg'(z)$ is starlike if and only if the function $g(z)$ is convex.

A function $f \in \mathcal{S}$ is said to be convex in the direction of the imaginary axis if $f(\mathbb{D})$ has a connected intersection with every line parallel to the imaginary axis. Given a convex function $g \in \mathcal{K}$ with $g(z) \neq 0$ and $\alpha < 1$, a function $f \in \mathcal{S}$, is called close-to-convex of order α with respect to the convex function g, denoted by $\mathcal{C}_g(\alpha)$, if and only if

$$\mathfrak{Re}\left(\frac{f'(z)}{g'(z)}\right) > \alpha, \quad z \in \mathbb{D}.$$

The class $\mathcal{C}_g(0)$ is the class of functions close-to-convex with respect to g. Geometrically, a function $f \in \mathcal{S}$ belongs to $\mathcal{C}$ if the complement E of the image-region $F = \{f(z) : |z| < 1\}$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays). The Noshiro–Warschawski theorem asserts that close-to-convex functions are univalent in $\mathbb{D}$, but not necessarily the converse. It is easy to verify that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C}$. For more details, one may refer to see [12].

2 Key lemmas

In order to establish our main results, we need the following results.

Lemma 1 ([14]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_1 = 1$, and that for $n \geq 2$ the sequence $\{a_n\}$ is a convex decreasing, i.e., $0 \geq a_{n+2} - a_{n+1} \geq a_{n+1} - a_n$, for all $n \in \mathbb{N}$. Then,*

$$\mathfrak{Re}\left(\sum_{n=1}^{\infty} a_n z^{n-1}\right) > 1/2 \quad (z \in \mathbb{D}). \tag{6}$$

It may be noted that each convex decreasing sequence also generates a convex null sequence. We recall that the sequence $a_0, a_1, \ldots$ of nonnegative numbers is called a convex null sequence if

$$\lim_{k\to\infty} a_k = 0 \quad \text{and} \quad a_0 - a_1 \geq a_1 - a_2 \geq \cdots \geq a_k - a_{k+1} \geq \cdots \geq 0.$$

For a convex null sequence $a_0 = 1, a_1, a_2, \dots$, we have instead of (6) the following inequality

$$\mathfrak{Re}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n z^n\right) > 0 \quad (z \in \mathbb{D}).$$

Lemma 2 ([14]) *If $A_n \geq 0$, $\{nA_n\}$ and $\{nA_n - (n+1)A_{n+1}\}$ are both nonincreasing, then the function $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$ is in $\mathcal{S}^*$.*

Lemma 3 ([24]) *Let $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$. Suppose that*

$$1 \geq 2A_2 \geq \cdots \geq (n+1)A_{n+1} \geq \cdots \geq 0$$

or

$$1 \leq 2A_2 \leq \cdots \leq (n+1)A_{n+1} \leq \cdots \leq 2,$$

then f is close-to-convex with respect to the convex function $-\log(1-z)$ in $\mathbb{D}$.

Lemma 4 ([24]) *Suppose that f is an odd function (i.e., of the form $f(z) = z + \sum_{n=2}^{\infty} A_{2n-1} z^{2n-1}$), such that*

$$1 \geq 3A_3 \geq \cdots \geq (2n+1)A_{2n+1} \geq \cdots \geq 0$$

or

$$1 \leq 3A_3 \leq \cdots \leq (2n+1)A_{2n+1} \leq \cdots \leq 2,$$

then f is close-to-convex with respect to the convex function $(1/2)\log((1+z)/(1-z))$.

Lemma 5 ([1]) *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that $A_1 = 1$ and*

$$(n+1)A_{n+1} \leq nA_n; (2n)^2 A_{2n} \leq (2n-1)^2 A_{2n-1} \quad \textit{for all } n \in \mathbb{N}.$$

Then, the functions defined by the series $\sum_{k=1}^{n} A_k z^k$ and $\sum_{n=1}^{\infty} A_n z^n$ are convex in the direction of the imaginary axis (see for details [1, Theorem 2.3.5, p. 34]).

Lemma 6 ([21]) *Let $\Omega \subset \mathbb{C}$, and suppose that $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfies the condition $\psi(is, t, u + iv; z) \notin \Omega$ for $z \in \mathbb{D}$ and for real s, t, u and v satisfying*

$$t \leq -\left(1 + s^2\right)/2 \quad \textit{and} \quad t + u \leq 0.$$

If $p(z)$ is analytic in $\mathbb{D}$, with $p(0) = 1$ and $\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $\mathfrak{Re}(p(z)) > 0$, for all $z \in \mathbb{D}$.

This lemma is a special case of Theorem 1 due to Miller and Mocanu in [21].

Lemma 7 ([34]) *For $\alpha < 1$, $\beta < 1$, we have*

$$\mathcal{P}(\alpha) * \mathcal{P}(\beta) \subset \mathcal{P}(\delta), \quad \textit{where } \delta = 1 - 2(1-\alpha)(1-\beta).$$

The value of δ is the best possible.

In this paper, we study certain geometric properties such as the close-to-convexity, starlikeness, and convexity of the function $z_1F_2(a;b,c;z)$. We also study the boundedness property of the function $z_1F_2(a;b,c;z)$ in the concluding section. Several special cases and corollaries of our main results are also pointed out.

3 Close-to-convexity of $z_1F_2(a;b,c;z)$

This section deals with conditions on the parameters a, b, and c so that the normalized function $z_1F_2(a;b,c;z)$ is close-to-convex and hence univalent in $\mathbb{D}$.

Theorem 1 *Let $a, b, c > 0$ and $a \leq bc/2$, then the function $z_1F_2(a;b,c;z)$ is close-to-convex with respect to the convex function $-\log(1-z)$.*

Proof Let $f(z) = z_1F_2(a;b,c;z)$. Then, $f(z) \in \mathcal{A}$, which has the standard normalized form $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$, where

$$A_n = \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}(n-1)!} \quad (n \geq 2 \text{ and } A_1 = 1).$$

Using the definition of the Pochhammer symbol, we have

$$A_{n+1} = \frac{(a+n-1)}{(b+n-1)(c+n-1)n} A_n. \tag{7}$$

We first note that the condition $a \leq bc/2$ implies that $2A_2 \leq A_1$. Now for $n \geq 2$, we obtain that

$$\begin{aligned} nA_n - (n+1)A_{n+1} &= A_n\left[n - \frac{(n+1)(a+n-1)}{(b+n-1)(c+n-1)n}\right] \\ &= \frac{A_n}{n(b+n-1)(c+n-1)} X(n), \end{aligned}$$

where

$$\begin{aligned} X(n) &= (b+n-1)(c+n-1)n^2 - (n+1)(a+n-1) \\ &= n^4 + n^3(b+c-2) + n^2(bc-b-c) - a - na + 1 \\ &= n^3(n-2) + n^2\big[n(b+c) - b - c\big] + \big[n(nbc - a) - a\big] + 1 \\ &\geq n^3(n-2) + n^2\big[n(b+c) - b - c\big] + \big[n(2na - a) - a\big] + 1 \quad (\text{in view of } bc \geq 2a) \\ &\geq 0 \quad (\text{for } n \geq 2). \end{aligned}$$

Thus, we have $X(n) \geq 0$ for all $n \geq 1$, provided that $a, b, c > 0$ and $a \leq bc/2$, and so the sequence $\{nA_n\}$ is nonincreasing. Applying Lemma 3, we conclude that the function $z_1F_2(a;b,c;z)$ is close-to-convex with respect to the convex function $-\log(1-z)$. □

Example 1 Setting $a = b$ in Theorem 1, it follows that for $c \geq 2$, the function $z_0F_1(-;c;z)$ is close-to-convex with respect to the convex function $-\log(1-z)$ in $\mathbb{D}$.

Corollary 1 *Let $a,b,c > -1$, $a+1 \leq (b+1)(c+1)/2$ and $a \neq 0$, then $z_1F_2'(a;b,c;z)$ is univalent in $\mathbb{D}$.*

Proof Differentiating (1) with respect z, we obtain

$$bcz_1F_2'(a;b,c;z) = az_1F_2(a+1;b+1,c+1;z). \tag{8}$$

Using (8) and Theorem 1, we deduce that $\frac{bc}{a}z_1F_2'(a;b,c;z)$ is close-to-convex with respect to $-\log(1-z)$ and hence the function $z_1F_2'(a;b,c;z)$ is univalent in $\mathbb{D}$. □

For $a = b$, Corollary 1 yields the result:

Corollary 2 *Let $c \geq 1$, then $z_0F_1'(-;c;z)$ is univalent in $\mathbb{D}$.*

Remark 1 In view of Biberbach's theorem, it is necessary that $|A_2| = |a/bc| \leq 2$ for f to belong to the class $\mathcal{S}$ and therefore if $a,b,c \in \mathbb{C}$, $b,c \neq 0,-1,-2,\ldots,$ then the function $z_1F_2(a;b,c;z)$ is not univalent whenever $|a| > 2|bc|$. Similarly for $|c| < 1/2$, the function $z_0F_1(-;c;z)$ is not univalent in $\mathbb{D}$. In particular, for $0 < c < 1/2$, the function $z_0F_1(-;c;z)$ is not univalent in $\mathbb{D}$. Therefore, in view of Example 1, the investigation of the univalence of the function $z_0F_1(-;c;z)$ for the case $1/2 \leq c < 2$ is still an open problem.

Theorem 2 *Let $a,b,c > 0$ and $a \leq 4bc/3$, then $z_1F_2(a;b,c;z^2/4)$ is close-to-convex with respect to $(1/2)\log((1+z)/(1-z))$.*

Proof Let $f(z) = z_1F_2(a;b;c;z^2/4) = z + \sum_{n=2}^{\infty} A_{2n-1}z^{2n-1}$, where

$$A_{2n-1} = \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}(n-1)!4^{n-1}} \quad (n \geq 2 \text{ and } A_1 = 1).$$

By means of the definition of the Pochhammer symbol, it is easy to see that

$$A_{2n+1} = \frac{(a+n-1)}{4n(b+n-1)(c+n-1)}A_{2n-1}.$$

We first note that the condition $a \leq 4bc/3$ implies that $3A_3 \leq A_1$. Now for $n \geq 2$, we estimate that

$$(2n-1)A_{2n-1} - (2n+1)A_{2n+1} = \frac{A_{2n-1}}{4n(b+n-1)(c+n-1)}Y(n),$$

where

$$\begin{aligned} Y(n) &= 4n(2n-1)(b+n-1)(c+n-1) - (2n+1)(a+n-1) \\ &= 8n^4 - 20n^3 + 14n^2 - 3n + 1 + 8cn^3 + 4cn - 12cn^2 + 8bn^3 + 4bn \\ &\quad - 12bn^2 + 8bcn^2 - 4bcn - 2an - a \end{aligned}$$

$$
\begin{aligned}
&\geq 8n^4 - 20n^3 + 14n^2 - 3n + 1 + 8cn^3 + 4cn - 12cn^2 + 8bn^3 + 4bn \\
&\quad - 12bn^2 + 6an^2 - 4bcn - 2an - a \\
&\geq 0 \quad (\text{in view of } 4bc \geq 3a).
\end{aligned}
$$

Thus, we have $Y(n) \geq 0$ for all $n \geq 1$, provided that $a, b, c > 0$ and $a \leq 4bc/3$ and so the sequence $\{nA_n\}$ is nonincreasing. Applying now Lemma 4, we assert that $z_1F_2(a; b, c; z^2/4)$ is close-to-convex with respect to the function $(1/2)\log((1 + z)/(1 - z))$. □

Example 2 Setting $a = 1, b = \nu + 3/2$, and $c = 3/2$ in Theorem 2 and using (3), we have the result that if $\nu \geq -1$, then $\mathbb{L}_\nu(z)$ is close-to-convex with respect to $(1/2)\log((1 + z)/(1 - z))$. Again, setting the parameter as in (4), we obtain the result that if $\mu \pm \nu$ is a nonnegative odd integer and $(\mu + 3)^2 - \nu^2 \geq 3$, then $\mathbb{T}_{\mu,\nu}(z)$ is close-to-convex with respect to $(1/2)\log((1 + z)/(1 - z))$.

Example 3 Setting $a = b$ in Theorem 2, we obtain the result that for $c \geq 3/4$, the function $z_0F_1(-; c; z^2/4)$ is close-to-convex with respect to $(1/2)\log((1 + z)/(1 - z))$. Using this result and (5), we have the assertion that for $\nu \geq -1/4$, the function $\mathbb{I}_\nu(z)$ is close-to-convex with respect to $(1/2)\log((1 + z)/(1 - z))$.

4 Starlikeness of $z_1F_2(a; b, c; z)$

In this section we determine the conditions on the parameters a, b, and c such that the function $z_1F_2(a; b, c; z)$ is not only close-to-convex with respect to $-\log(1 - z)$ but also starlike in $\mathbb{D}$. Let $\mathcal{KS}^*$ denote the family of functions in $\mathcal{A}$ that are close-to-convex with respect to $-\log(1 - z)$ and also starlike in $\mathbb{D}$.

Theorem 3 *Let $b, c > 0$, and*

$$
\frac{3bc}{5bc + 8b + 8c + 8} \leq a \leq bc/4.
$$

Then, the function $z_1F_2(a; b, c; z)$ is in the class $\mathcal{KS}^*$.

Proof In view of Lemma 2, it is sufficient to prove that $\{nA_n\}$ and $\{nA_n - (n + 1)A_{n+1}\}$ are nonincreasing sequences for all $n \geq 1$. The sequence $\{nA_n\}$ is nonincreasing under the condition that $a \leq bc/2$ and the same will be true here as $a \leq bc/4 < bc/2$. Therefore, it suffices to show that $nA_n - 2(n + 1)A_{n+1} + (n + 2)A_{n+2} \geq 0$ for all $n \geq 1$. Using (7), we have

$$
\begin{aligned}
&nA_n - 2(n + 1)A_{n+1} + (n + 2)A_{n+2} \geq 0 \\
\Leftrightarrow \quad & A_n\left[n - 2\frac{(n + 1)(a + n - 1)}{n(b + n - 1)(c + n - 1)}\right. \\
&\quad \left. + \frac{(n + 2)(a + n)(a + n - 1)}{(b + n)(b + n - 1)(c + n)(c + n - 1)n(n + 1)}\right] \geq 0.
\end{aligned}
$$

Therefore, it is sufficient to prove that

$$
n - 2\frac{(n + 1)(a + n - 1)}{n(b + n - 1)(c + n - 1)} \geq 0 \quad (\text{for all } n \geq 1). \tag{9}
$$

This is true for $n = 1$ as $a \le bc/4$. Also, (9) is true for $n = 2$ provided that

$$1 \ge \frac{3(a+1)}{2(b+1)(c+1)},$$

which holds true if we show that

$$1 \ge \frac{4a}{bc} \ge \frac{3(a+1)}{2(b+1)(c+1)},$$

and this inequality is valid under the hypothesis of the theorem. For $n \ge 3$, (9) is valid, if $Z(n) \ge 0$, where

$$\begin{aligned} Z(n) &= n^2(b+n-1)(c+n-1) - 2(n+1)(a+n-1) \\ &= n^4 + bn^3 + cn^3 - 2n^3 + bcn^2 - bn^2 - cn^2 - n^2 - 2an + 2 - 2a \\ &= \left(n^4 - 2n^3 - n^2 + 2\right) + n^2\left[n(b+c) - b - c\right] + \left[n(nbc - 2a) - 2a\right] \\ &\ge \left(n^4 - 2n^3 - n^2 + 2\right) + n^2\left[n(b+c) - b - c\right] + \left[n(4an - 2a) - 2a\right] \quad (\text{as } bc \ge 4a) \\ &\ge 0. \end{aligned}$$

This completes the proof. □

As before, if we set $a = b$ in Theorem 3, then we deduce the following result:

Corollary 3 *Let $c \ge 4$,then the function $z_0F_1(-;c;z)$ is in the class $\mathcal{KS}^*$.*

5 Convexity of $z_1F_2(a;b,c;z)$

Theorem 4 *Let $b, c > -1$, $abc \neq 0$ and*

$$-\frac{26 + 10b + 10c + 2bc}{29 + 13b + 13c + 5bc} \le a \le \frac{bc + b + c - 3}{4},$$

then ${}_1F_2(a;b,c;z)$ is a convex function in $\mathbb{D}$.

Proof Let $g(z) = {}_1F_2(a;b,c;z)$ and $f(z) = z_1F_2(a+1;b+1,c+1;z)$. Then, from relation (8), we have $f(z) = (bc/a)zg'(z)$. By taking logarithmic derivatives of both sides, we obtain the relation

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zg''(z)}{g'(z)}. \tag{10}$$

Using Theorem 3 and the hypothesis of Theorem 4, we infer that $f(z)$ is starlike and hence from (10), the function $g(z)) = {}_1F_2(a;b,c;z)$ is a convex function. □

Theorem 5 *Let $a, b, c > 0$ and $a \le bc/4$, then $z_1F_2(a;b,c;z)$ is convex in the direction of the imaginary axis.*

Proof We have already proved in Theorem 1 that $(n+1)A_{n+1} \le nA_n$ for all $n \ge 1$, provided that $a \le bc/4 < bc/2$. In view of Lemma 5, $z_1F_2(a;b,c;z)$ is convex in the direction of the imaginary axis provided that $(2n)^2A_{2n} \leq (2n-1)^2A_{2n-1}$ for all $n \geq 1$. Now,

$$
\begin{aligned}
(2n-1)^2A_{2n-1} - (2n)^2A_{2n} &= A_{2n-1}\left[(2n-1)^2 - 4n^2\frac{(a+2n-2)}{(b+2n-2)(c+2n-2)(2n-1)}\right] \\
&= \frac{A_{2n-1}}{(b+2n-2)(c+2n-2)(2n-1)}L(n),
\end{aligned}
$$

where

$$
\begin{aligned}
L(n) &= (2n-1)^3(b+2n-2)(c+2n-2) - 4n^2(a+2n-2) \\
&= \left(32n^5 - 112n^4 + 144n^3 - 92n^2 + 32n - 4\right) + b\left(16n^4 - 40n^3 + 36n^2 - 14n + 2\right) \\
&\quad + c\left(16n^4 - 40n^3 + 36n^2 - 14n + 2\right) + bc\left(8n^3 - 12n^2 + 6n - 1\right) - 4n^2a \\
&\geq 0 \quad (\text{as } bc \geq 4a).
\end{aligned}
$$

This completes the proof. □

Theorem 6 *Let c be a real number such that*

$$
c \geq \frac{3 - 3\beta + 2\beta^2}{2(1-\beta)}, \tag{11}
$$

and ${}_0F_1'(-;c;z) \neq 0$, then ${}_0F_1(-;c;z)$ is a convex function of order β $(0 \leq \beta < 1)$.

Proof Let $w(z) = {}_0F_1(-;c;z)$. If we put

$$
1 + \frac{zw''(z)}{w'(z)} = \beta + (1-\beta)p(z),
$$

then $p(z)$ is analytic in $\mathbb{D}$ and $p(0) = 1$. Therefore, to prove Theorem 6, we need to show that $\mathfrak{Re}(p(z)) > 0$ in $\mathbb{D}$. Since the function $w(z) = {}_0F_1(-;c;z)$ satisfies the differential equation

$$
zw''(z) + cw'(z) - w(z) = 0,
$$

we find that

$$
zp'(z) + (1-\beta)p^2(z) + p(z)\left[c - 2(1-\beta)\right] - \frac{z}{1-\beta} + 1 - \beta - c = 0.
$$

We may rewrite the above differential equation in the form of $\psi(p(z), zp'(z); z) = 0$. Now, for all real s and $t \leq -(1+s^2)/2$ and $z(= x + iy) \in \mathbb{D}$, we have

$$
\begin{aligned}
\mathfrak{Re}\psi(is, t; z) &= t - (1-\beta)s^2 - \frac{x}{1-\beta} + 1 - \beta - c \\
&\leq -\frac{(1+s^2)}{2} - (1-\beta)s^2 - \frac{x}{1-\beta} + 1 - \beta - c \\
&= -\frac{s^2}{2}(3-2\beta) - \left[\frac{x}{1-\beta} - \frac{1}{2} + \beta + c\right].
\end{aligned}
$$

Also,

$$\frac{x}{1-\beta}-\frac{1}{2}+\beta+c>\frac{-1}{1-\beta}-\frac{1}{2}+\beta+c\geq 0$$

for all $x \in (-1,1)$ and c satisfying the condition (11). Hence, we deduce that $\mathfrak{Re}\{\psi(is,t;z)\}<0$, and therefore by Lemma 6, it follows that $\mathfrak{Re}(p(z))>0$ in $\mathbb{D}$, which shows that the function ${}_0F_1(-;c;z)$ is convex of order β. □

If we use the identity

$$(c-1)z_0F_1'(-;c-1;z)=z_0F_1(-;c;z), \tag{12}$$

then from Theorem 6, we obtain the following result:

Corollary 4 *Let c be a real number such that*

$$c\geq\frac{5-5\beta+2\beta^2}{2(1-\beta)},$$

and ${}_0F_1'(-;c-1;z)\neq 0$, *then* $z_0F_1(-;c;z)\in\mathcal{S}^*(\beta)$ *for* $0\leq\beta<1$.

If we let $f(z)=z_0F_1(-;c;z)$ and $h(z)=f(z^2/4)/(z/4)$, then we have

$$\frac{zh'(z)}{h(z)}=2\frac{z^2}{4}\frac{f'(z^2/4)}{f(z^2/4)}-1. \tag{13}$$

This observation and Corollary 4 immediately yields the following result.

Corollary 5 *Let c be a real number such that*

$$c\geq\frac{5-5\beta+2\beta^2}{2(1-\beta)},$$

$1/2\leq\beta<1$, *and* ${}_0F_1'(-;c-1;z)\neq 0$, *then* $z_0F_1(-;c;z^2/4)\in\mathcal{S}^*(2\beta-1)$.

Theorem 7 *Let* $a,b,c>0$, $a\leq bc$, *and*

$$2bc(b+1)(c+1)\geq a\big[4(b+1)(c+1)-(a+1)\big], \tag{14}$$

then $\mathfrak{Re}\{{}_1F_2(a;b,c;z)\}>1/2$ *for* $z\in\mathbb{D}$.

Proof In view of Lemma 1, it is sufficient to show that the sequence

$$\{A_n\}_{n=1}^{\infty}=\left\{\frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}(n-1)!}\right\}_{n=1}^{\infty}\quad(A_1=1)$$

is a convex decreasing sequence. We first prove that the above sequence is a decreasing sequence, i.e.,

$$A_n-A_{n+1}=A_n\left[1-\frac{(a+n-1)}{(b+n-1)(c+n-1)n}\right]$$

$$= \frac{A_n}{n(b+n-1)(c+n-1)} M(n),$$

where

$$\begin{aligned} M(n) &= (b+n-1)(c+n-1)n - (a+n-1) \\ &= n^3 + n^2(b+c-2) + n(bc-b-c) - a + 1 \\ &= n^2(n-2) + 1 + n(b+c)(n-1) + (nbc - a) \\ &\geq 0 \quad (\text{In view of } bc \geq a). \end{aligned}$$

Thus, we have $M(n) \geq 0$ for all $n \geq 1$, provided that $a, b, c > 0$, and $a \leq bc$ and hence the sequence $\{A_n\}$ is nonincreasing. Next, we show that the sequence $\{A_n\}_{n=1}^{\infty}$ is convex decreasing, and for that we need to show that $A_n - 2A_{n+1} + A_{n+2} \geq 0$ for all $n \geq 1$. Using (7), we have

$$\begin{aligned} & A_n - 2A_{n+1} + A_{n+2} \geq 0 \\ \Leftrightarrow \quad & A_n \biggl[1 - 2\frac{(a+n-1)}{n(b+n-1)(c+n-1)} \\ & \quad + \frac{(a+n)(a+n-1)}{(b+n)(b+n-1)(c+n)(c+n-1)n(n+1)} \biggr] \geq 0. \end{aligned}$$

In view of (14), the above is true for $n = 1$. For $n \geq 2$, the difference between the first two terms is

$$1 - 2\frac{(a+n-1)}{n(b+n-1)(c+n-1)} = \frac{N(n)}{n(b+n-1)(c+n-1)},$$

where

$$\begin{aligned} N(n) &= n^3 + cn^2 + bn^2 - 2n^2 + bcn - bn - cn - n - 2a + 2 \\ &= \left(n^3 - 2n^2 - n + 2\right) + n\bigl[n(b+c) - b - c\bigr] + nbc - 2a. \end{aligned}$$

In view of $bc \geq a$, it is easy to see that $N(n) \geq 0$ for all $n \geq 2$, and hence applying Lemma 1, we obtain the desired result. □

Substituting $a = b$ in Theorem 7, we have

Corollary 6 *Let* $c \geq (1+\sqrt{7})/2$, *then* $\mathfrak{Re}\{{}_0F_1(-;c;z)\} > 1/2$ *for* $z \in \mathbb{D}$.

Remark 2 Applying the above Corollary 6 for positive half-integers greater than or equal to 2, we obtain the following inequality for $c = 5/2$:

$$\mathfrak{Re}\left({}_0F_1(-;5/2;z)\right) = \mathfrak{Re}\left(\frac{3}{8z}\left(2\cosh(2\sqrt{z}) - \frac{\sinh(2\sqrt{z})}{\sqrt{z}}\right)\right) > 1/2 \quad (z \in \mathbb{D}). \tag{15}$$

Similarly, for $c = 7/2$, Corollary 4.7 gives the inequality:

$$\mathfrak{Re}\left(\frac{3}{8z}\left(2\cosh(2\sqrt{z}) - \frac{\sinh(2\sqrt{z})}{\sqrt{z}}\right)\right) > 1/2 \quad (z \in \mathbb{D}). \tag{16}$$

6 Boundedness property of ${}_1F_2(a;b,c;z)$

Let $\mathcal{H}^\infty$ denotes the space of all bounded functions on $\mathcal{H}$. For the function $f \in \mathcal{H}$, set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p} \quad (0 < p < \infty)$$

and

$$M_\infty(r,f) = \max_{|z|\le r}\left|f(z)\right|.$$

The function f is said to belong to $\mathcal{H}^p$ and it is called the Hardy space $(0 < p \le \infty)$ if $M_p(r,f)$ is bounded for all $r \in [0,1)$. Evidently, we have the relationship ([11])

$$\mathcal{H}^\infty \subset \mathcal{H}^q \subset \mathcal{H}^p \quad \text{for } 0 < p < q < \infty.$$

The following results are widely known (see [15] and [27]) for the Hardy space ([11]):

$$\begin{aligned}\mathfrak{Re}\left(f'(z)\right) > 0 \quad &\Longrightarrow \quad f' \in \mathcal{H}^q \quad \text{for all } q < 1\\ &\Longrightarrow \quad f \in \mathcal{H}^{q/(1-q)} \quad \text{for all } 0 < q < 1.\end{aligned} \tag{17}$$

Theorem 8 *Let $a,b,c > 0$, $a \le bc$, $c > a-b-1$ and*

$$2bc(b+1)(c+1) \ge a\left[4(b+1)(c+1) - (a+1)\right].$$

If $f \in \mathcal{R}$, then the convolution $z{}_1F_2(a;b,c;z) * f(z)$ is in $\mathcal{H}^\infty \cap \mathcal{R}$.

Proof Let $f \in \mathcal{R}$, then $f' \in \mathcal{P}$. We define a function g by

$$g(z) = z{}_1F_2(a;b,c;z) * f(z) = \sum_{n=1}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}} \frac{z^n}{(n-1)!} a_n, \tag{18}$$

then

$$g'(z) = {}_1F_2(a;b,c;z) * f'(z). \tag{19}$$

Applying Theorem 7, we have $\mathfrak{Re}\{{}_1F_2(a;b,c;z)\} > 1/2$. Also, $f(z) \in \mathcal{R}$ implies that $\mathfrak{Re}(f'(z)) > 0$, and therefore from Lemma 7 it follows that $g \in \mathcal{R}$. Thus, in view of the first implication of (17), we have $g' \in \mathcal{H}^q$ for all $q < 1$. Further, from the second implication of (17), we have $g \in \mathcal{H}^{q/(1-q)}$ for all $0 < q < 1$, or equivalently, $g \in \mathcal{H}^p$ for all $0 < p < \infty$.

In view of the known bound for the Carathéodory functions in the unit disc [12], we note that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}$, then

$$|a_n| \le \frac{2}{n} \quad (n \ge 2).$$

By using this bound, we have in view of (18):

Using the well-known bound for the Carathéodory functions in the unit disc, we find that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}$ then

$$|a_n| \le \frac{2}{n} \quad (n \ge 2).$$

Using this fact we find that

$$\begin{aligned} \bigl|g(z)\bigr| &\le |z| + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}(n-1)!}|a_n||z|^n \\ &< 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}(n-1)!}\frac{2}{n} \\ &= 1 + 2\sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n(c)_n n!}\frac{1}{n+1} < 1 + 2\sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n(c)_n n!}\frac{1}{n}. \end{aligned} \tag{20}$$

Therefore, applying Raabe's test for convergence, we deduce that the series

$$\sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n(c)_n n!}\frac{1}{n} \tag{21}$$

converges absolutely for $|z| = 1$. This argument shows that the power series for $g(z)$ converges absolutely for $|z| = 1$. Furthermore, it is well known that [11, Theorem 3.11, p. 42] $g' \in \mathcal{H}^q$ implies continuity of g on $\overline{\mathbb{D}}$, the closure of $\mathbb{D}$. Finally, since the continuous function g on the compact set $\overline{\mathbb{D}}$ is bounded, $g(z)$ is a bounded analytic function in $\mathbb{D}$. Therefore, $g \in \mathcal{H}^\infty$ and this completes the proof. □

We conclude this paper by considering a special case of Theorem 8. Indeed, if we put $a = b$ in Theorem 8, we have the following corollary.

Corollary 7 *Let $c \ge \frac{1+\sqrt{7}}{2}$ and $f \in \mathcal{R}$, then the convolution $z{}_0F_1(-;c;z) * f(z)$ is in $\mathcal{H}^\infty \cap \mathcal{R}$.*

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D.B. and R.K.R. made the original draft preparation. S.M. contributed analysis of the mathematical computation. N.E.C reviewed and edited the manuscript. All authors read and approved the final manuscript.

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References

1. Acharya, A.P.: Univalence criteria for analytic functions and applications to hypergeometric functions. Ph. D. Thesis, Universität of Würzburg (1997)
2. Bailey, W.N.: Generalized Hypergeometric Series. Cambridge University Press, New York (1935)
3. Bansal, D., Prajapat, J.K.: Certain geometric properties of the Mittag-Leffler functions. Complex Var. Elliptic Equ. **61**(3), 338–350 (2016)
4. Baricz, Á.: Functional inequalities involving special functions II. J. Math. Anal. Appl. **327**(2), 1202–1213 (2007)
5. Baricz, Á.: Geometric properties of generalized Bessel functions. Publ. Math. (Debr.) **73**, 155–178 (2008)
6. Baricz, Á., Koumandos, S.: Turán type inequalities for some Lommel functions of the first kind. Proc. Edinb. Math. Soc. **59**(3), 569–579 (2016)
7. Baricz, Á., Kupán, P.A., Szász, R.: The radius of starlikeness of normalized Bessel function of first kind. Proc. Am. Math. Soc. **142**(6), 2019–2025 (2014)
8. Baricz, Á., Ponnusamy, S.: Starlikeness and convexity of generalized Bessel functions. Integral Transforms Spec. Funct. **21**(9), 641–653 (2010)
9. Cağlar, M., Deniz, E.: Partial sums of the normalized Lommel functions. Math. Inequal. Appl. **18**(3), 1189–1199 (2015)
10. De-Branges, L.: A proof of the Bieberbach conjecture. Acta Math. **154**, 137–152 (1985)
11. Duren, P.L.: Theory of H^p Spaces. A Series of Monographs and Textbook in Pure and Applied Mathematics, vol. 38. Academic Press, New York (1970)
12. Duren, P.L.: Univalent Functions. Springer, Berlin (1983)
13. Dziok, J., Raina, R.K.: Families of analytic functions associated with the Wright's generalized hypergeometric function. Demonstr. Math. **37**(3), 533–542 (2004)
14. Féjer, L.: Untersuchungen über Potenzreihen mit mehrfach monotoner Koeffizientenfolge. Acta Litterarum Sci. **8**, 89–115 (1936)
15. Hardy, G.H., Littlewood, J.E.: Theorems concerning mean values of analytic or harmonic functions. Q. J. Math. **12**, 221–256 (1941)
16. Hästö, P., Ponnusamy, S., Vuorinen, M.: Starlikeness of the Gaussian hypergeometric functions. Complex Var. Elliptic Equ. **55**, 173–184 (2010)
17. Küstner, R.: Mapping properties of hypergeometric functions and convolutions of starlike or convex functions of order α. Comput. Methods Funct. Theory **2**, 597–610 (2004)
18. Küstner, R.: On the order of starlikeness of the shifted Gauss hypergeometric function. J. Math. Anal. Appl. **334**(2), 1363–1385 (2007)
19. Lewis, J.: Applications of a convolution theorem to Jacobi polynomails. SIAM J. Math. Anal. **10**(6), 1110–1120 (1979)
20. Merkes, E.P., Scott, W.T.: Starlike hypergeometric functions. Proc. Am. Math. Soc. **12**, 885–888 (1961)
21. Miller, S.S., Mocanu, P.T.: Differential subordinations and inequalities in the complex plane. J. Differ. Equ. **67**, 199–211 (1987)
22. Miller, S.S., Mocanu, P.T.: Univalence of Gaussian and confluent hypergeometric functions. Proc. Am. Math. Soc. **110**(2), 333–342 (1990)
23. Orhan, H., Yagmur, N.: Geometric properties of generalized Struve functions. An. Ştiinţ. Univ. 'Al.I. Cuza' Iaşi, Mat. https://doi.org/10.2478/aicu-2014-0007
24. Ozaki, S.: On the theory of multivalent functions. Sci. Rep. Tokyo Bunrika Daigaku A **2**, 167–188 (1935)
25. Ponnusamy, S., Vuorinen, M.: Univalence and convexity properties for Gaussian hypergeometric functions. Rocky Mt. J. Math. **31**(1), 327–353 (2001)
26. Prajapat, J.K.: Certain geometric properties of the Wright functions. Integral Transforms Spec. Funct. **26**(3), 203–212 (2015)
27. Priwalow, I.I.: Randeigenschaften Analyticscher Funktionen. Deutscher Verlag Der Wissenschaften (1956)
28. Raina, R.K.: On univalent and starlike Wright's hypergeometric functions. Rend. Semin. Mat. Univ. Padova **95**, 11–22 (1996)
29. Rollinger, C.N.: Lommel functions with imaginary argument. Q. Appl. Math. **21**(4), 343–349 (1964)
30. Ruscheweyh, S., Singh, V.: On the order of starlikeness of hypergeometric functions. J. Math. Anal. Appl. **113**(1), 1–11 (1986)
31. Shafer, R.E.: Lommel functions of imaginary argument. Technical report, Lawrence Radiation Laboratory, Livermore (1964)
32. Silverman, H.: Starlike and convexity properties for hypergeometric functions. J. Math. Anal. Appl. **172**(2), 574–581 (1993)
33. Slater, L.J.: Generalized Hypergeometric Functions. Cambridge University Press, UK (1966)
34. Stankiewicz, J., Stankiewicz, Z.: Some applications of Hadamard convolutions in the theory of functions. Ann. Univ. Mariae Curie–Sklodowska **40**, 251–265 (1986)
35. Watson, G.N.: A Treatise on the Theory of Bessel Functions, 2nd edn. Cambridge University Press, Cambridge (1944)
36. Yagmur, N., Orhan, H.: Partial sums of generalized Struve functions. Miskolc Math. Notes **17**(1), 657–670 (2016)
37. Zhang, S., Jin, J.: Computation of Special Functions. Wiley Interscience Publication, New York (1996)