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Fractional version of Ostrowski-type inequalities for strongly p-convex stochastic processes via a k-fractional Hilfer–Katugampola derivative

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Abstract

In the present research, we introduce the notion of convex stochastic processes namely; strongly p-convex stochastic processes. We establish a generalized version of Ostrowski-type integral inequalities for strongly p-convex stochastic processes in the setting of a generalized k-fractional Hilfer–Katugampola derivative by using the Hölder and power-mean inequalities. By using our main results, we derived some known results as special cases and many well-known existing results are also recaptured. It is assumed that this research will offer new guidelines in fractional calculus.

Keywords: Convex stochastic processes; Hermite–Hadamard inequality; Ostrowski inequality

1 Introduction and preliminaries

The theory of inequalities has undergone rapid developments because of its widespread use in pure and applied mathematics. Recently, the role of fractional calculus made this area more interesting for researchers (see [1–3]). As classical convexity is being used in less applied problems, it is always appropriate to explore new versions of convexity. A consensus of the history of the Hermite–Hadamard integral inequality can be found in the literature [4]. In optimization and probability theory, the Hermite–Hadamard inequality has become a helpful tool [5].

In 1971, the research on convex stochastic processes began when Nagy [6] employed a characterization of measurable stochastic processes to resolve a generalization of the Cauchy functional equation. At the end of the twentieth century, Nikodem introduced the convex stochastic processes, and Skowronski derived several advanced results on convex stochastic processes that generalize further well-known properties, [7–9]. Further, Kotrys [10] presented the Hermite–Hadamard inequality for convex stochastic processes in 2012. Many studies have been done by several researchers on different classes of convex stochastic processes and also Hermite–Hadamard inequalities for convex stochastic

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processes in the literature [11–13]. The well-reputed Hermite–Hadamard inequality for convex stochastic processes is defined as follows:

Let $\eta : I \times \Omega \rightarrow \mathbb{R}$ be Jensen-convex and mean-square continuous in $I \times \Omega$, then

$$\eta\left(\frac{b_1 + b_2}{2}, \cdot\right) \leq \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \eta(y, \cdot)\, dy \leq \frac{\eta(b_1, \cdot) + \eta(b_2, \cdot)}{2} \quad \text{(a.e.)}, \tag{1.1}$$

for any $c_1, d_1 \in I$, $c_1 < d_1$.

In 2014, the authors of [14] considered Hermite–Hadamard integral-type inequalities for stochastic processes. Katugampola, considered a fractional integral operator that generalizes the Hadamard and Riemann–Liouville integrals into a single form and many authors use these results in the area of convexity, generalized convexity, and so on (see [15, 16]). Recently, different Hermite–Hadamard-type inequalities via fractional integrals have been presented [17–19].

The classical Ostrowski inequality was proposed in [20] and, recently, the Ostrowski inequality has attracted the attention of many researchers; many remarkable generalizations, extensions, variants, and applications can be found in the literature [21–29].

The purpose of this article is to develop some integral inequalities of Ostrowski-type for a strongly p-convex stochastic process using the generalized k-fractional Hilfer–Katugampola derivative.

Definition 1.1 ([19, 30]) A stochastic process is a family of random variables $\eta(\kappa)$ parameterized by $\kappa \in I$, where $I \subset \mathbb{R}$. When $I = \{1, 2, \ldots\}$, then $\eta(\kappa)$ is known as a stochastic process in discrete time. When $I = [0, \infty)$, then $\eta(\kappa)$ is a stochastic process in continuous time.

For any $\vartheta \in \Omega$ the function $I \ni \kappa \longmapsto \eta(\kappa, \vartheta)$ is termed a path of $\eta(\kappa)$.

Definition 1.2 ([19, 30]) A family $\mathbb{G}_k$ of α-fields on Ω parameterized by $k \in I$, where $I \subset \mathbb{R}$, is said to be a filtration if

$$\mathbb{G}_t \subset \mathbb{G}_k \subset \mathbb{G}$$

for any $\rho, \kappa \in I$ such that $\rho \leq \kappa$.

Definition 1.3 ([19, 30]) A stochastic process $\eta(\kappa)$ parameterized by $\kappa \in T$ is said to be martingale (supermartingale, submartingale) with respect to a filtration $\mathbb{G}_\kappa$ if

1) $\eta(\kappa)$ is integrable for each $\kappa \in I$;
2) $\eta(\kappa)$ is $\mathbb{G}_\kappa$-measurable for each $\kappa \in I$;
3) $\eta(\rho) = \mathbb{E}(\eta(\kappa)|\mathbb{G}_\rho)$ (respectively, $\leq$ or $\geq$) for every $\rho, \kappa \in I$ such that $\rho \leq \kappa$.

Definition 1.4 ([19, 31]) Consider (Ω, A, P) to be an arbitrary probability space and $I \subset \mathbb{R}$. A stochastic process $X : \Omega \rightarrow \mathbb{R}$ is termed

1) Stochastically continuous in I, if $\forall \kappa_\circ \in I$

$$P - \lim_{\kappa \rightarrow \kappa_\circ} \eta(\kappa, \cdot) = \eta(\kappa_\circ, \cdot),$$

where $P - \lim$ represents the limit in probability.

2) Mean-square continuous in I, if $\forall \kappa_\circ \in I$

$$P - \lim_{\kappa \to \kappa_\circ} \mathbb{E}\big(\eta(\kappa,\cdot) - \eta(\kappa_\circ,\cdot)\big) = 0,$$

where $\mathbb{E}(\eta(\kappa,\cdot))$ represent the expectation value of the random variable $\eta(\kappa,\cdot)$.

3) Increasing (decreasing) if $\forall b_1, b_2 \in I$, $b_1 < b_2$

$$\eta(b_1,\cdot) \le \eta(b_2,\cdot), \qquad \eta(b_1,\cdot) \ge \eta(b_2,\cdot) \quad \text{(a.e.)}.$$

4) It is said to be monotonic if it is increasing or decreasing.

5) If there exists a random variable $\eta'(\kappa,\cdot) : I \times \Omega \to \mathbb{R}$ then it is differentiable at a point $\kappa \in I$, such that

$$\eta'(\kappa,\cdot) = P - \lim_{\kappa \to \kappa_\circ} \frac{\eta(\kappa,\cdot) - \eta(\kappa_\circ,\cdot)}{\kappa - \kappa_\circ}.$$

A stochastic process $X : I \times \Omega \to \mathbb{R}$ is known as continuous (differentiable) if it is continuous (differentiable) at every point of I.

Definition 1.5 ([19, 31]) Let (Ω, A, P) be a probability space and $I \subset \mathbb{R}$ with $\mathbb{E}(\eta(\vartheta)^2) < \infty$ $\forall \theta \in I$. If $[\kappa_1, \kappa_2] \subset I$, $\kappa_1 = \vartheta_0 < \vartheta_1 < \theta_2 < \cdots < \vartheta_n = \kappa_2$ is a partition of $[\kappa_1, \kappa_2]$ and $\varphi \in [\vartheta_{\rho-1}, \vartheta_\rho]$ for $\rho = 1, 2, \ldots, n$. A random variable $Y : \Omega \to \mathbb{R}$ is said to be a mean-square integral of the process $\eta(\vartheta,\cdot)$ on $[\kappa_1, \kappa_2]$ if

$$\lim_{n \to \infty} \mathbb{E}\left[\sum_{\rho=1}^{\infty} \eta(\varphi_\rho,\cdot)(\vartheta_\rho, \vartheta_{\rho-1}) - Y(\cdot)\right]^2 = 0,$$

then

$$\int_{\kappa_1}^{\kappa_2} \eta(\vartheta,\cdot)\, d\vartheta = Y(\cdot) \quad \text{(a.e.)}.$$

A mean-square integral operator is increasing, thus,

$$\int_{\kappa_1}^{\kappa_2} \eta(\vartheta,\cdot)\, d\vartheta \le \int_{\kappa_1}^{\kappa_2} Y(\vartheta,\cdot) \quad \text{(a.e.)},$$

here $X(\vartheta,\cdot) \le Z(\vartheta,\cdot)$ in $[k_1, k_2]$.

Definition 1.6 ([7, 19]) Let (Ω, A, P) be a probability space and $I \subseteq R$. A stochastic process $\eta : I \times \Omega \to R$ is said to be a convex stochastic process, if

$$\eta\big(\xi b_1 + (1-\xi) b_2, \cdot\big) \le \xi \eta(b_1,\cdot) + (1-\xi)\eta(b_2,\cdot) \quad \text{(a.e.)} \tag{1.2}$$

holds for all $b_1, b_2 \in I$ and $\kappa \in [0, 1]$.

It is natural to view the new versions of convexity in stochastic processes settings, hence we introduce the strongly p-convex stochastic processes as follows:

Definition 1.7 Let $c : \Omega \to \mathbb{R}$ be a positive random variable. A stochastic process $\eta : I \times \Omega \to \mathbb{R}$ is said to be strongly p-convex with modulus $c(\cdot)$, if

$$\begin{aligned}
&\eta\bigl(\bigl[\xi b_1^p + (1-\xi)b_2^p\bigr]^{\frac{1}{p}}, \cdot\bigr) \\
&\quad \le \xi\eta(b_1,\cdot) + (1-\xi)\eta(b_2,\cdot) - c(\cdot)\xi(1-\xi)(b_2-b_1)^2 \quad \text{(a.e.)}
\end{aligned} \tag{1.3}$$

holds for all $b_1, b_2 \in I$ and $\kappa \in [0,1]$.

Remark 1.8 1) Taking $p = 1$ in the above definition, we obtain a strongly convex stochastic process [11].

2) Taking $c(\cdot) = 0$ in the above definition, we obtain *ap*-convex stochastic process [32].

3) Taking $p = 1$ and $c(\cdot) = 0$ in the above definition, we obtain a convex stochastic process [7].

Definition 1.9 ([33, 34]) Let $[b_1, b_2]$ be finite on the real axis $\mathbb{R} = (-\infty, \infty)$. Then, the Lebesgue measurable functions of η on $[b_1, b_2]$ of complex value are denoted by $M_q = (b_1, b_2)$.

$$Mq(b_1,b_2) = \left\{\eta : \|\eta_q\| = \sqrt[q]{\int \bigl|\eta(y)\bigr|^q \, dy} < +\infty\right\}, \quad 1 \le q < \infty.$$

For $q = 1$, one has that $M_q(b_1, b_2) = M(b_1, b_2)$.

Definition 1.10 ([34, 35]) The k-Gamma function is defined as;

$$\Gamma_\kappa(y) = \int_0^\infty \xi^{y-1} e^{-\frac{\xi^\kappa}{\kappa}} \, d\xi, \tag{1.4}$$

where, $y, \kappa > 0$. We can observe that

$$\Gamma_\kappa(y+\kappa) = y\Gamma_\kappa(y)$$

and

$$\Gamma_\kappa(y) = \kappa^{\frac{y}{\kappa}-1}\Gamma\left(\frac{y}{\kappa}\right).$$

Definition 1.11 ([34, 36]) The left-hand and right-hand generalized k-fractional integrals of order w with $z - 1 < w \le z$, $z \in \mathbb{N}$, $\kappa > 0$, $\xi > 0$, $w > 0$ are defined as

$$\bigl({}^{\xi}_{\kappa}I^{w}_{b_1}\eta\bigr)(y) = \frac{\xi^{1-\frac{w}{\kappa}}}{\kappa\Gamma_{\kappa(w)}} \int_{b_1}^{y} \bigl(y^p - x^p\bigr)^{\frac{w}{\kappa}-1} x^{\xi-1}\eta(x) \, dx, \quad y > b_1, \tag{1.5}$$

$$\bigl({}^{\xi}_{\kappa}I^{w}_{b_2}\eta\bigr)(y) = \frac{\xi^{1-\frac{w}{\kappa}}}{\kappa\Gamma_{\kappa(w)}} \int_{y}^{b_2} \bigl(x^p - y^p\bigr)^{\frac{w}{\kappa}-1} x^{\xi-1}\eta(x) \, dx, \quad y > b_2. \tag{1.6}$$

Definition 1.12 ([34, 37]) The left-hand and right-hand generalized k-fractional derivatives having order w are defined as;

$$ {}^{\xi}_{\kappa}D^{\phi}_{b_1}\eta(y) = \left(y^{1-\xi}\frac{d}{dy}\right)^{n}\left(\kappa^{n}\,{}^{\xi}_{.\kappa}I^{\kappa n-w}_{b_1}\eta\right), \quad y > b_1, \tag{1.7} $$

$$ {}^{\xi}_{\kappa}D^{\phi}_{b_2}\eta(y) = \left(y^{1-\xi}\frac{d}{dy}\right)^{n}\left(\kappa^{n}\,{}^{\xi}_{.\kappa}I^{\kappa n-w}_{b_2}\eta\right), \quad y > b_2. \tag{1.8} $$

Definition 1.13 ([34, 38]) Consider $z-1 < w \leq z$, $0 \leq \vartheta \leq 1$, $z \in \mathbb{N}$, $\kappa > 0$, $\xi > 0$ and $\eta \in M_q[(b_1, b_2)]$. Then, the left-hand and right-hand generalized k-fractional Hilfer–Katugampola derivatives are defined as;

$$ \left({}^{\xi}_{\kappa}D^{w,\vartheta}_{b_1}\eta\right)(y) = \left({}^{\xi}_{\kappa}I^{\vartheta(\kappa n-w)}_{b_1}\left(y^{1-\xi}\frac{d}{dy}\right)^{n}\left(\kappa^{n}\,{}^{\xi}_{.\kappa}I^{(1-\vartheta)(\kappa n-w)}_{b_1}\eta\right)\right)(y), \tag{1.9} $$

$$ \left({}^{\xi}_{\kappa}D^{w,\vartheta}_{b_2}\eta\right)(y) = \left({}^{\xi}_{\kappa}I^{\vartheta(\kappa n-w)}_{b_2}\left(y^{1-\xi}\frac{d}{dy}\right)^{n}\left(\kappa^{n}\,{}^{\xi}_{.\kappa}I^{(1-\vartheta)(\kappa n-w)}_{b_2}\eta\right)\right)(y), \tag{1.10} $$

where I is the integral presented in definition (1.6).

Lemma 1.14 ([34]) *Consider* $z-1 < w \leq z$, $0 \leq \vartheta \leq 1$, $z \in \mathbb{N}$, $\kappa > 0$, $\xi > 0$ *and* $\eta \in M_q[(b_1, b_2)]$, *then*

$$ \begin{aligned} {}^{\xi}_{\kappa}D^{w,\vartheta}_{b_1}\eta(y) &= \left({}^{\xi}_{\kappa}I^{\vartheta(\kappa n-w)}_{b_1}\left(y^{1-\xi}\frac{d}{dy}\right)^{n}\left(\kappa^{n}\,{}^{\xi}_{.\kappa}I^{(1-\vartheta)(\kappa n-w)}_{b_1}\eta\right)\right)(y) \\ &= \left({}^{\xi}_{\kappa}I^{\vartheta(\kappa n-w)}_{b_1}\left(y^{1-\xi}\frac{d}{dy}\right)^{n}\left(\kappa^{n}\,{}^{\xi}_{.\kappa}I^{\kappa n-w-\vartheta(\kappa n-w)}_{b_1}\right)\right)\eta(y) \\ &= \left({}^{\xi}_{\kappa}I^{\vartheta(\kappa n-w)}_{b_1}\left(y^{1-\xi}\frac{d}{dy}\right)^{n}\left(\kappa^{n}\,{}^{\xi}_{.\kappa}I^{\kappa n-\{w+\vartheta(\kappa n-w)\}}_{b_1}\eta\right)\right)(y) \\ &= \left({}^{\xi}_{\kappa}I^{\vartheta(\kappa n-w)}_{b_1}\,{}^{\xi}_{\kappa}D^{w+\vartheta(\kappa n-w)}_{b_1}\eta\right)(y) \quad (\textit{by using } (1.7)) \\ &= \left({}^{\xi}_{\kappa}I^{\phi-w}_{b_1}\,{}^{\xi}_{\kappa}D^{\phi}_{b_1}\eta\right)(y) \\ &= \left({}^{\xi}_{\kappa}I^{\phi-w}_{b_1}\eta^{(\phi)}\right)(y) \\ &= \frac{\xi^{1-\frac{\phi-w}{\kappa}}}{\kappa\Gamma_{\kappa}(\phi-w)}\int_{b_1}^{y}\left(y^{p}-x^{p}\right)^{\frac{\phi-w}{\kappa}-1}x^{\xi-1}\eta^{(\phi)}(x)\,dx \quad (\textit{by using } (1.5)), \end{aligned} \tag{1.11} $$

where $\phi = w + \vartheta(\kappa n - w)$, $w > 0$ *and* $\eta^{(\phi)}$ *is the derivative of* η *presented in* (1.7).

Thus, the generalized k-fractional Hilfer–Katugampola derivative can be presented as:

$$ \left({}^{\xi}_{\kappa}D^{w,\vartheta}_{b_1}\eta\right)(y) = \frac{\xi^{1-\frac{\phi-w}{\kappa}}}{\kappa\Gamma_{\kappa}(\phi-w)}\int_{b_1}^{y}\left(y^{p}-x^{p}\right)^{\frac{\phi-w}{\kappa}-1}x^{\xi-1}\eta^{(\phi)}(x)\,dx, \quad z > b_1, \tag{1.12} $$

$$ \left({}^{\xi}_{\kappa}D^{w,\vartheta}_{b_2}\eta\right)(y) = \frac{\xi^{1-\frac{\phi-w}{\kappa}}}{\kappa\Gamma_{\kappa}(\phi-w)}\int_{y}^{b_2}\left(x^{p}-y^{p}\right)^{\frac{\phi-w}{\kappa}-1}x^{\xi-1}\eta^{(\phi)}(x)\,dx, \quad z < b_2. \tag{1.13} $$

Definition 1.15 ([34]) The beta function denoted by B is defined as;

$$ B(b_1, b_2) = \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(b_1+b_2)} = \int_0^1 \xi^{b_1-1}(1-\xi)^{b_2-1}\,d\xi, \quad (b_1, b_2 > 0), \tag{1.14} $$

and the Gaussian hypergeometric function denoted by F_1 is defined as;

$$F_1(b_1,b_2;b_3,b) = \frac{1}{B(b_2,b_3-b_2)}\int_0^1 \xi^{b_2-1}(1-\xi)^{b_3-b_2-1}(1-b\xi)^{-b_1}\,d\xi,$$
$$\big(b_3>b_2>0, |b|<1\big), \tag{1.15}$$

where $\Gamma(b)=\int_0^\infty e^{-\xi}\xi^{b-1}\,d\xi$ is the well-known Euler Gamma function.

2 Ostrowski-type inequalities for a strongly p-convex stochastic process

The Ostrowski-type inequality for a strongly p-convex stochastic process in the setting of a generalized k-fractional Hilfer–Katugampola derivative is established in this section.

Lemma 2.1 *Let a differentiable stochastic process $\mu>0$ with $p\in\mathbb{R}\backslash\{0\}$ and $\eta^{(\mu)}:I\times\Omega\to\mathbb{R}$ defined on the I° such that $b_1,b_2\in I$ with $b_1<b_2$ and $\eta^{(\mu+1)}\in M([b_1,b_2])$, then the following inequality holds almost everywhere*:

$$\begin{aligned}
&\frac{(y^p-b_1^p)^\mu\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^\mu\eta^{(\mu)}(b_2,\cdot)}{p^\mu(b_2-b_1)}\\
&\quad-\frac{\Gamma_\kappa(\mu+\kappa)}{b_2-b_1}\big[\big({}^p_\kappa D^\mu_{b_1^+}\eta\big)(y,\cdot)+\big({}^p_\kappa D^\mu_{b_2^-}\eta\big)(y,\cdot)\big]\\
&=-\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\\
&\quad\times\int_0^1\xi^\mu\big(\xi b_1^p+(1-\xi)y^p\big)^{\frac{1-p}{p}}\eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\Big)\,d\xi\\
&\quad+\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\\
&\quad\times\int_0^1\xi^\mu\big(\xi b_2^p+(1-\xi)y^p\big)^{\frac{1-p}{p}}\eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\Big)\,d\xi.
\end{aligned}\tag{2.1}$$

Proof Integrating by parts, we can write

$$\begin{aligned}
&\int_0^1\xi^\mu\big(\xi b_1^p+(1-\xi)y^p\big)^{\frac{1-p}{p}}\eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\Big)\,d\xi\\
&=\bigg[\frac{p\xi^\mu\eta(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot)}{b_1^p-y^p}\bigg]_0^1-\int_0^1\frac{p\mu\xi^{\mu-1}\eta(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot)}{b_1^p-y^p}\,d\xi\\
&=\frac{p\eta(b_1,\cdot)}{b_1^p-y^p}-\int_0^1\frac{p\mu\xi^{\mu-1}\eta(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot)}{b_1^p-y^p}\,d\xi\\
&=\frac{p\eta(b_1,\cdot)}{b_1^p-y^p}-\frac{p^2\mu}{y^p-b_1^p}\int_y^{b_1}\bigg(\frac{y^p-w^p}{y^p-b_1^p}\bigg)^{\mu-1}\eta(w,\cdot)\frac{w^{p-1}}{y^p-b_1^p}\,dw\quad\Big[w=\sqrt[p]{\xi b_1^p+(1-\xi)y^p}\Big]\\
&=\frac{p\eta(b_1,\cdot)}{b_1^p-y^p}+\frac{p^2\mu}{(y^p-b_1^p)^{\mu+1}}\int_{b_1}^y\big(y^p-w^p\big)^{\mu-1}\eta(w,\cdot)w^{p-1}\,dw\\
&=\frac{p\eta(b_1,\cdot)}{b_1^p-y^p}+\frac{p^2\mu\Gamma_\kappa(\mu)}{p^{1-\mu}(y^p-b_1^p)^{\mu+1}}\frac{p^{1-\mu}}{\Gamma_\kappa(\mu)}\int_{b_1}^y w^{p-1}\big(y^p-w^p\big)^{\mu-1}\eta(w,\cdot)\,dw
\end{aligned}$$

$$
\begin{aligned}
&= \frac{p\eta(b_1,\cdot)}{b_1^p - y^p} + \frac{p^{1+\mu}\Gamma_\kappa(\mu+\kappa)}{(y^p - b_1^p)^{\mu+1}} \frac{p^{1-\mu}}{\Gamma_\kappa(\mu)} \int_{b_1}^{y} w^{p-1}\big(y^p - w^p\big)^{\mu-1}\eta(w,\cdot)\,dw \\
&= \frac{p\eta(b_1,\cdot)}{b_1^p - y^p} + \frac{p^{1+\mu}\Gamma_\kappa(\mu+\kappa)}{(y^p - b_1^p)^{\mu+1}} \big({}_\kappa^p D_{b_1^+}^{\mu}\eta\big)(y,\cdot). \qquad (2.2)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_0^1 \xi^\mu \big(\xi b_2^p + (1-\xi)y^p\big)^{\frac{1-p}{p}} \eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_2^p + (1-\xi)y^p},\cdot\Big)\,d\xi \\
&\quad= \frac{p\eta(b_2,\cdot)}{b_2^p - y^p} - \frac{p^{1+\mu}\Gamma_\kappa(\mu+\kappa)}{(b_2^p - y^p)^{\mu+1}} \big({}_\kappa^p D_{b_2^-}^{\mu}\eta\big)(y,\cdot). \qquad (2.3)
\end{aligned}
$$

For both sides of (2.2) and (2.3), multiplying by $\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}$ and $\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}$, respectively, we obtain

$$
\begin{aligned}
&\frac{(y^p - b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)} \int_0^1 \xi^\mu \big(\xi b_1^p + (1-\xi)y^p\big)^{\frac{1-p}{p}} \eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_1^p + (1-\xi)y^p},\cdot\Big)\,d\xi \\
&\quad= -\frac{(y^p - b_1^p)^{\mu}\eta(b_1,\cdot)}{p^{\mu}(b_2-b_1)} + \frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)} \big({}_\kappa^p D_{b_1^+}^{\mu}\eta\big)(y,\cdot), \qquad (2.4)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{(b_2^p - y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)} \int_0^1 \xi^\mu \big(\xi b_2^p + (1-\xi)y^p\big)^{\frac{1-p}{p}} \eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_2^p + (1-\xi)y^p},\cdot\Big)\,d\xi \\
&\quad= \frac{(b_2^p - y^p)^{\mu}\eta(b_2,\cdot)}{p^{\mu}(b_2-b_1)} - \frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)} \big({}_\kappa^p D_{b_2^-}^{\mu}\eta\big)(y,\cdot). \qquad (2.5)
\end{aligned}
$$

From (2.4) and (2.5), we obtain the inequality (2.1). □

Theorem 2.2 *For a differentiable stochastic process $b_1, b_2 \in I$ with $b_1 < b_2$, and $\eta^{(\mu)}: I \times \Omega \subset (0,\infty) \to \mathbb{R}$ on I° such that $\eta^{(\mu+1)} \in M([b_1,b_2])$ and $|\eta^{(\mu+1)}|$ a strongly p-convex stochastic process with modulus $c(\cdot)$ satisfying $|\eta^{(\mu+1)}(y,\cdot)| \le \mathbb{Q}$, $\forall y \in [b_1,b_2]$, the following inequality holds almost everywhere for all $y \in [b_1,b_2]$ and $p \in (0,\infty)$:*

$$
\begin{aligned}
&\bigg| \frac{(y^p - b_1^p)^{\mu}\eta^{(\mu)}(b_1,\cdot) + (b_2^p - y^p)^{\mu}\eta^{(\mu)}(b_2,\cdot)}{p^{\mu}(b_2-b_1)} \\
&\qquad - \frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)}\Big[\big({}_\kappa^p D_{b_1^+}^{\mu}\eta\big)(y,\cdot) + \big({}_\kappa^p D_{b_2^-}^{\mu}\eta\big)(y,\cdot)\Big]\bigg| \\
&\quad\le \frac{b_1^{1-p}}{p^{1+\mu}}\bigg[\frac{\mathbb{Q}}{\mu+1}\bigg(\frac{(y^p - b_1^p)^{\mu+1} + (b_2^p - y^p)^{\mu+1}}{(b_2-b_1)}\bigg) \\
&\qquad - \frac{c(\cdot)}{(\mu+2)(\mu+3)}\bigg(\frac{(y^p - b_1^p)^{\mu+1}(y-b_1)^2 + (b_2^p - y^p)^{\mu+1}(y-b_2)^2}{(b_2-b_1)}\bigg)\bigg], \qquad (2.6)
\end{aligned}
$$

and the following inequality holds almost everywhere for all $y \in (b_1, b_2)$ *and* $p \in (-\infty, 0) \cup (0, 1)$:

$$\begin{aligned} &\left| \frac{(y^p - b_1^p)^\mu \eta^{(\mu)}(b_1, \cdot) + (b_2^p - y^p)^\mu \eta^{(\mu)}(b_2, \cdot)}{p^\mu (b_2 - b_1)} \right. \\ &\quad \left. - \frac{\Gamma_\kappa(\mu + \kappa)}{(b_2 - b_1)} \left[\left({}_\kappa^p D_{b_1^+}^\mu \eta \right)(y, \cdot) + \left({}_\kappa^p D_{b_2^-}^\mu \eta \right)(y, \cdot) \right] \right| \\ &\quad \leq \frac{b_2^{1-p}}{p^{1+\mu}} \left[\frac{\mathbb{Q}}{\mu + 1} \left(\frac{(y^p - b_1^p)^{\mu+1} + (b_2^p - y^p)^{\mu+1}}{(b_2 - b_1)} \right) \right. \\ &\quad \left. - \frac{c(\cdot)}{(\mu + 2)(\mu + 3)} \left(\frac{(y^p - b_1^p)^{\mu+1}(y - b_1)^2 + (b_2^p - y^p)^{\mu+1}(y - b_2)^2}{(b_2 - b_1)} \right) \right]. \end{aligned} \tag{2.7}$$

Proof By using Lemma 2.1, to prove inequality (2.6) of Theorem 2.2 for a strongly p-convex stochastic process of $|\eta^{(\mu+1)}|$ yields

$$\begin{aligned} &\left| \frac{(y^p - b_1^p)^\mu \eta^{(\mu)}(b_1, \cdot) + (b_2^p - y^p)^\mu \eta^{(\mu)}(b_2, \cdot)}{p^\mu (b_2 - b_1)} \right. \\ &\quad \left. - \frac{\Gamma_\kappa(\mu + \kappa)}{(b_2 - b_1)} \left[\left({}_\kappa^p D_{b_1^+}^\mu \eta \right)(y, \cdot) + \left({}_\kappa^p D_{b_2^-}^\mu \eta \right)(y, \cdot) \right] \right| \\ &\quad \leq \frac{(y^p - b_1^p)^{\mu+1}}{p^{1+\mu}(b_2 - b_1)} \int_0^1 \xi^\mu \left(\xi b_1^p + (1 - \xi) y^p \right)^{\frac{1-p}{p}} \left| \eta^{(\mu+1)} \left(\sqrt[p]{\xi b_1^p + (1 - \xi) y^p}, \cdot \right) \right| d\xi \\ &\quad + \frac{(b_2^p - y^p)^{\mu+1}}{p^{1+\mu}(b_2 - b_1)} \int_0^1 \xi^\mu \left(\xi b_2^p + (1 - \xi) y^p \right)^{\frac{1-p}{p}} \left| \eta^{(\mu+1)} \left(\sqrt[p]{\xi b_2^p + (1 - \xi) y^p}, \cdot \right) \right| d\xi \\ &\quad \leq \frac{(y^p - b_1^p)^{\mu+1}}{p^{1+\mu}(b_2 - b_1)} \int_0^1 \xi^\mu \left(\xi b_1^p + (1 - \xi) y^p \right)^{\frac{1-p}{p}} \\ &\quad \times \left[\xi \left| \eta^{(\mu+1)}(b_1, \cdot) \right| + (1 - \xi) \left| \eta^{(\mu+1)}(y, \cdot) \right| - c(\cdot) \xi (1 - \xi)(y - b_1)^2 \right] d\xi \\ &\quad + \frac{(b_2^p - y^p)^{\mu+1}}{p^{1+\mu}(b_2 - b_1)} \int_0^1 \xi^\mu \left(\xi b_2^p + (1 - \xi) y^p \right)^{\frac{1-p}{p}} \\ &\quad \times \left[\xi \left| \eta^{(\mu+1)}(b_2, \cdot) \right| + (1 - \xi) \left| \eta^{(\mu+1)}(y, \cdot) \right| - c(\cdot) \xi (1 - \xi)(y - b_2)^2 \right] d\xi \quad \text{(a.e.)}. \end{aligned}$$

As $p \in (1, \infty)$, we can deduce that

$$\left(\xi b_1^p + (1 - \xi) y^p \right)^{\frac{1-p}{p}} \leq \left(\xi b_2^p + (1 - \xi) y^p \right)^{\frac{1-p}{p}} \leq b_1^{1-p}. \tag{2.8}$$

We proceed by simplifying

$$\begin{aligned} &\int_0^1 \xi^\mu \left[\xi \left| \eta^{(\mu+1)}(b_1, \cdot) \right| + (1 - \xi) \left| \eta^{(\mu+1)}(y, \cdot) \right| - c(\cdot) \xi (1 - \xi)(y - b_1)^2 \right] d\xi \\ &\quad = \left[\frac{\mathbb{Q}}{\mu + 1} - \frac{c(\cdot)}{(\mu + 2)(\mu + 3)} (y - b_1)^2 \right] \quad \text{(a.e.)}. \end{aligned}$$

Similarly,

$$\int_0^1 \xi^\mu\bigl[\xi\bigl|\eta^{(\mu+1)}(b_2,\cdot)\bigr| + (1-\xi)\bigl|\eta^{(\mu+1)}(y,\cdot)\bigr| - c(\cdot)\xi(1-\xi)(y-b_2)^2\bigr]\,d\xi$$
$$= \left[\frac{\mathbb{Q}}{\mu+1} - \frac{c(\cdot)}{(\mu+2)(\mu+3)}(y-b_2)^2\right] \quad \text{(a.e.)}.$$

The inequality (2.6) of Theorem 2.2 is proved.

Now, to prove inequality (2.7), we consider $p \in (-\infty,0)\cup(0,1)$ that yields

$$\bigl(\xi b_1^p + (1-\xi)y^p\bigr)^{\frac{1-p}{p}} \le \bigl(\xi b_2^p + (1-\xi)y^p\bigr)^{\frac{1-p}{p}} \le b_2^{1-p}. \tag{2.9}$$

This completes the proof. □

Theorem 2.3 *For a differentiable stochastic process $\delta,\lambda > 1$ with $\delta^{-1}+\lambda^{-1}=1$, $b_1,b_2\in I$ with $b_1<b_2$, and $\eta^{(\mu)}: I\times\Omega\subset(0,\infty)\to\mathbb{R}$ on I° such that $\eta^{(\mu+1)}\in M([b_1,b_2])$ and $|\eta^{(\mu+1)}|^\lambda$ a strongly p-convex stochastic process with modulus $c(\cdot)$ satisfying $|\eta^{(\mu+1)}(y,\cdot)|\le\mathbb{Q}$, $\forall y\in[b_1,b_2]$, the following inequality holds almost everywhere for all $y\in[b_1,b_2]$ and $p\in(0,\infty)$:*

$$\begin{aligned}&\left|\frac{(y^p-b_1^p)^\mu\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^\mu\eta^{(\mu)}(b_2,\cdot)}{p^\mu(b_2-b_1)}\right.\\ &\qquad\left.-\frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)}\bigl[\bigl({}_\kappa^pD_{b_1^+}^\mu\eta\bigr)(y,\cdot)+\bigl({}_\kappa^pD_{b_2^-}^\mu\eta\bigr)(y,\cdot)\bigr]\right|\\ &\quad\le\frac{b_1^{1-p}}{p^{1+\mu}(1+\delta\mu)^{\frac{1}{\delta}}}\left[\frac{(y^p-b_1^p)^{\mu+1}}{(b_2-b_1)}\left(\mathbb{Q}^\lambda-\frac{c(\cdot)}{6}(y-b_1)^2\right)^{\frac{1}{\lambda}}\right.\\ &\qquad\left.+\frac{(b_2^p-y^p)^{\mu+1}}{(b_2-b_1)}\left(\mathbb{Q}^\lambda-\frac{c(\cdot)}{6}(y-b_2)^2\right)^{\frac{1}{\lambda}}\right],\end{aligned}\tag{2.10}$$

and the following inequality holds almost everywhere for all $y\in(b_1,b_2)$ and $p\in(-\infty,0)\cup(0,1)$:

$$\begin{aligned}&\left|\frac{(y^p-b_1^p)^\mu\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^\mu\eta^{(\mu)}(b_2,\cdot)}{p^\mu(b_2-b_1)}\right.\\ &\qquad\left.-\frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)}\bigl[\bigl({}_\kappa^pD_{b_1^+}^\mu\eta\bigr)(y,\cdot)+\bigl({}_\kappa^pD_{b_2^-}^\mu\eta\bigr)(y,\cdot)\bigr]\right|\\ &\quad\le\frac{b_2^{1-p}}{p^{1+\mu}(1+\delta\mu)^{\frac{1}{\delta}}}\left[\frac{(y^p-b_1^p)^{\mu+1}}{(b_2-b_1)}\left(\mathbb{Q}^\lambda-\frac{c(\cdot)}{6}(y-b_1)^2\right)^{\frac{1}{\lambda}}\right.\\ &\qquad\left.+\frac{(b_2^p-y^p)^{\mu+1}}{(b_2-b_1)}\left(\mathbb{Q}^\lambda-\frac{c(\cdot)}{6}(y-b_2)^2\right)^{\frac{1}{\lambda}}\right].\end{aligned}\tag{2.11}$$

Proof From Lemma 2.1, (2.8), and Hölder's inequality to prove (2.10) of Theorem 2.3 yields

$$\begin{aligned}&\left|\frac{(y^p-b_1^p)^\mu\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^\mu\eta^{(\mu)}(b_2,\cdot)}{p^\mu(b_2-b_1)}\right.\\ &\qquad\left.-\frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)}\bigl[\bigl({}_\kappa^pD_{b_1^+}^\mu\eta\bigr)(y,\cdot)+\bigl({}_\kappa^pD_{b_2^-}^\mu\eta\bigr)(y,\cdot)\bigr]\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\int_0^1 \xi^{\mu}\left(\xi b_1^p+(1-\xi)y^p\right)^{\frac{1-p}{p}}\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\right)\right|d\xi \\
&\quad+\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\int_0^1 \xi^{\mu}\left(\xi b_2^p+(1-\xi)y^p\right)^{\frac{1-p}{p}}\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\right)\right|d\xi \\
&\leq \frac{b_1^{1-p}(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\int_0^1 \xi^{\mu}\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\right)\right|d\xi \\
&\quad+\frac{b_1^{1-p}(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\int_0^1 \xi^{\mu}\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\right)\right|d\xi \\
&\leq \frac{b_1^{1-p}(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)} \\
&\quad\times\left(\int_0^1 \xi^{\delta\mu}\,d\xi\right)^{\frac{1}{\delta}}\left(\int_0^1\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\right)\right|^{\lambda}d\xi\right)^{\frac{1}{\lambda}} \\
&\quad+\frac{b_1^{1-p}(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)} \\
&\quad\times\left(\int_0^1 \xi^{\delta\mu}\,d\xi\right)^{\frac{1}{\delta}}\left(\int_0^1\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\right)\right|^{\lambda}d\xi\right)^{\frac{1}{\lambda}}\quad \text{(a.e.)}.
\end{aligned}
$$

As $|\eta^{(\mu+1)}|^{\lambda}$ is a strongly p-convex stochastic process and $|\eta^{(\mu+1)}(y,\cdot)|\leq\mathbb{Q}$ for all $y\in[b_1,b_2]$, we have

$$
\begin{aligned}
&\int_0^1\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\right)\right|^{\lambda}d\xi \\
&\leq\int_0^1\left[\xi\left|\eta^{(\mu+1)}(b_1,\cdot)\right|^{\lambda}+(1-\xi)\left|\eta^{(\mu+1)}(y,\cdot)\right|^{\lambda}-c(\cdot)\xi(1-\xi)(y-b_1)^2\right]d\xi \\
&\leq\mathbb{Q}^{\lambda}-\frac{c(\cdot)}{6}\left(y-b_1^2\right)\quad\text{(a.e.)},
\end{aligned}
$$

and

$$
\int_0^1\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\right)\right|^{\lambda}d\xi\leq\mathbb{Q}^{\lambda}-\frac{c(\cdot)}{6}(y-b_2)^2\quad\text{(a.e.)}.
$$

The remaining proof is simple. □

Theorem 2.4 *For a differentiable stochastic process $\delta,\lambda>1$ with $\delta^{-1}+\lambda^{-1}=1$, $b_1,b_2\in I$ with $b_1<b_2$, and $\eta^{(\mu)}:I\times\Omega\subset(0,\infty)\to\mathbb{R}$ on I° such that $\eta^{(\mu+1)}\in M([b_1,b_2])$ and $|\eta^{(\mu+1)}|^{\lambda}$ a strongly p-convex stochastic process with modulus $c(\cdot)$ satisfying $|\eta^{(\mu+1)}(y,\cdot)|\leq\mathbb{Q}$, $\forall y\in[b_1,b_2]$, the following inequality holds almost everywhere for all $y\in[b_1,b_2]$ and $p\in(0,\infty)$:*

$$
\begin{aligned}
&\left|\frac{(y^p-b_1^p)^{\mu}\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^{\mu}\eta^{(\mu)}(b_2,\cdot)}{p^{\mu}(b_2-b_1)}\right. \\
&\quad\left.-\frac{\Gamma_{\kappa}(\mu+\kappa)}{(b_2-b_1)}\left[\left({}_{\kappa}^{p}D_{b_1^+}^{\mu}\eta\right)(y,\cdot)+\left({}_{\kappa}^{p}D_{b_2^-}^{\mu}\eta\right)(y,\cdot)\right]\right| \\
&\leq\frac{b_1^{1-p}}{p^{1+\mu}}\left[\frac{(y^p-b_1^p)^{\mu+1}}{(b_2-b_1)}\left(\frac{\mathbb{Q}^{\lambda}}{1+\lambda\mu}-\frac{c(\cdot)}{(\lambda\mu+2)(\lambda\mu+3)}(y-b_1)^2\right)^{\frac{1}{\lambda}}\right.
\end{aligned}
$$

$$+\frac{(b_2^p-y^p)^{\mu+1}}{(b_2-b_1)}\left(\frac{\mathbb{Q}^\lambda}{1+\lambda\mu}-\frac{c(\cdot)}{(\lambda\mu+2)(\lambda\mu+3)}(y-b_2)^2\right)^{\frac{1}{\lambda}}\Bigg], \tag{2.12}$$

and the following inequality holds almost everywhere for all $y\in(b_1,b_2)$ *and* $p\in(-\infty,0)\cup(0,1)$:

$$\begin{aligned}&\Bigg|\frac{(y^p-b_1^p)^\mu\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^\mu\eta^{(\mu)}(b_2,\cdot)}{p^\mu(b_2-b_1)}\\&\quad-\frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)}\big[\big({}_\kappa^pD^\mu_{b_1^+}\eta\big)(y,\cdot)+\big({}_\kappa^pD^\mu_{b_2^-}\eta\big)(y,\cdot)\big]\Bigg|\\&\le\frac{b_2^{1-p}}{p^{1+\mu}}\Bigg[\frac{(y^p-b_1^p)^{\mu+1}}{(b_2-b_1)}\left(\frac{\mathbb{Q}^\lambda}{1+\lambda\mu}-\frac{c(\cdot)}{(\lambda\mu+2)(\lambda\mu+3)}(y-b_1)^2\right)^{\frac{1}{\lambda}}\\&\quad+\frac{(b_2^p-y^p)^{\mu+1}}{(b_2-b_1)}\left(\frac{\mathbb{Q}^\lambda}{1+\lambda\mu}-\frac{c(\cdot)}{(\lambda\mu+2)(\lambda\mu+3)}(y-b_2)^2\right)^{\frac{1}{\lambda}}\Bigg].\end{aligned}\tag{2.13}$$

Proof From Lemma 2.1, (2.8), and the power-mean inequality to prove (2.12) of Theorem 2.4 yields

$$\begin{aligned}&\Bigg|\frac{(y^p-b_1^p)^\mu\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^\mu\eta^{(\mu)}(b_2,\cdot)}{p^\mu(b_2-b_1)}\\&\quad-\frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)}\big[\big({}_\kappa^pD^\mu_{b_1^+}\eta\big)(y,\cdot)+\big({}_\kappa^pD^\mu_{b_2^-}\eta\big)(y,\cdot)\big]\Bigg|\\&\le\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\int_0^1\xi^\mu\big(\xi b_1^p+(1-\xi)y^p\big)^{\frac{1-p}{p}}\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\right)\right|d\xi\\&\quad+\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\int_0^1\xi^\mu\big(\xi b_2^p+(1-\xi)y^p\big)^{\frac{1-p}{p}}\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\right)\right|d\xi\\&\le\frac{b_1^{1-p}(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\int_0^1\xi^\mu\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\right)\right|d\xi\\&\quad+\frac{b_1^{1-p}(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\int_0^1\xi^\mu\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\right)\right|d\xi\\&\le\frac{b_1^{1-p}(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\left(\int_0^1\xi^{\lambda\mu}\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\right)\right|^\lambda d\xi\right)^{\frac{1}{\lambda}}\\&\quad+\frac{b_1^{1-p}(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\left(\int_0^1\xi^{\lambda\mu}\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\right)\right|^\lambda d\xi\right)^{\frac{1}{\lambda}}\quad\text{(a.e.)}.\end{aligned}$$

As $|\eta^{(\mu+1)}|^\lambda$ is a strongly p-convex stochastic process and $|\eta^{(\mu+1)}(y,\cdot)|\le\mathbb{Q}$ for all $y\in[b_1,b_2]$, we obtain

$$\begin{aligned}&\int_0^1\xi^{\lambda\mu}\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\right)\right|^\lambda d\xi\\&\le\int_0^1\xi^{\lambda\mu}\big[\xi\big|\eta^{(\mu+1)}(b_1,\cdot)\big|^\lambda+(1-\xi)\big|\eta^{(\mu+1)}(y,\cdot)\big|^\lambda-c(\cdot)\xi(1-\xi)(y-b_1)^2\big]d\xi\\&=\frac{\mathbb{Q}^\lambda}{\lambda\mu+1}-\frac{c(\cdot)}{(\lambda\mu+2)(\lambda\mu+3)}(y-b_1)^2\quad\text{(a.e.)}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_0^1 \xi^{\lambda\mu}\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\right)\right|^{\lambda} d\xi\\&\quad\le \frac{\mathbb{Q}^{\lambda}}{\lambda\mu+1}-\frac{c(\cdot)}{(\lambda\mu+2)(\lambda\mu+3)}(y-b_2)^2 \quad \text{(a.e.)}.\end{aligned}$$

By combining all the above inequalities we obtain our desired result. □

Theorem 2.5 *Let the differentiable stochastic process $\delta,\lambda>1$ with $\delta^{-1}+\lambda^{-1}=1$, $b_1,b_2\in I$ with $b_1<b_2$, and $\eta^{(\mu)}: I\times\Omega\subset(0,\infty)\to\mathbb{R}$ on I° such that $\eta^{(\mu+1)}\in M([b_1,b_2])$ and $|\eta^{(\mu+1)}|^{\lambda}$ a strongly p-convex stochastic process with modulus $c(\cdot)$ satisfying $|\eta^{(\mu+1)}(y,\cdot)|\le\mathbb{Q}$, $\forall y\in[b_1,b_2]$, the following inequality holds almost everywhere for all $y\in[b_1,b_2]$ and $p\in(0,\infty)$:*

$$\begin{aligned}&\Bigg|\frac{(y^p-b_1^p)^{\mu}\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^{\mu}\eta^{(\mu)}(b_2,\cdot)}{p^{\mu}(b_2-b_1)}\\&\qquad-\frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)}\Big[\big({}_{\kappa}^{p}D_{b_1^+}^{\mu}\eta\big)(y,\cdot)+\big({}_{\kappa}^{p}D_{b_2^-}^{\mu}\eta\big)(y,\cdot)\Big]\Bigg|\\&\quad\le\frac{(y^p-b_1^p)^{\mu+1}+(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\left[\frac{b_1^{\delta(1-p)}}{\delta(\delta\mu+1)}+\frac{\mathbb{Q}^{\lambda}}{\lambda}\right]\\&\qquad-\frac{c(\cdot)\lambda}{6}\left[\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}(y-b_1)^2+\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}(y-b_2)^2\right], \qquad (2.14)\end{aligned}$$

and the following inequality holds almost everywhere for all $y\in(b_1,b_2)$ and $p\in(-\infty,0)\cup(0,1)$:

$$\begin{aligned}&\Bigg|\frac{(y^p-b_1^p)^{\mu}\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^{\mu}\eta^{(\mu)}(b_2,\cdot)}{p^{\mu}(b_2-b_1)}\\&\qquad-\frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)}\Big[\big({}_{\kappa}^{p}D_{b_1^+}^{\mu}\eta\big)(y,\cdot)+\big({}_{\kappa}^{p}D_{b_2^-}^{\mu}\eta\big)(y,\cdot)\Big]\Bigg|\\&\quad\le\frac{(y^p-b_1^p)^{\mu+1}+(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\left[\frac{b_2^{\delta(1-p)}}{\delta(\delta\mu+1)}+\frac{\mathbb{Q}^{\lambda}}{\lambda}\right]\\&\qquad-\frac{c(\cdot)\lambda}{6}\left[\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}(y-b_1)^2+\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}(y-b_2)^2\right]. \qquad (2.15)\end{aligned}$$

Proof The Young's inequality is $mn\le\frac{1}{\delta}m^{\delta}+\frac{1}{\lambda}n^{\lambda}$, $m,n\ge0$, $\delta,\lambda>1$, $\delta^{-1}+\lambda^{-1}=1$. By using Lemma 2.1, to prove (2.14) of Theorem 2.5, and taking the definition of a strongly p-convex stochastic process of $|\eta^{(\mu+1)}|^{\lambda}$ yields

$$\begin{aligned}&\Bigg|\frac{(y^p-b_1^p)^{\mu}\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^{\mu}\eta^{(\mu)}(b_2,\cdot)}{p^{\mu}(b_2-b_1)}\\&\qquad-\frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)}\Big[\big({}_{\kappa}^{p}D_{b_1^+}^{\mu}\eta\big)(y,\cdot)+\big({}_{\kappa}^{p}D_{b_2^-}^{\mu}\eta\big)(y,\cdot)\Big]\Bigg|\\&\quad\le\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\\&\qquad\times\int_0^1\left(\frac{1}{\delta}\left|\xi^{\mu}\left(\xi b_1^p+(1-\xi)y^p\right)^{\frac{1-p}{p}}\right|^{\delta}+\frac{1}{\lambda}\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\right)\right|^{\lambda}\right)d\xi\end{aligned}$$

$$
\begin{aligned}
&+\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\\
&\times\int_0^1\left(\frac{1}{\delta}\left|\xi^{\mu}\left(\xi b_2^p+(1-\xi)y^p\right)^{\frac{1-p}{p}}\right|^{\delta}+\frac{1}{\lambda}\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\right)\right|^{\lambda}\right)d\xi\\
\le{}&\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\left[\int_0^1\left(\frac{\xi^{\delta\mu}}{\delta}\left|\left(\xi b_1^p+(1-\xi)y^p\right)^{\frac{1-p}{p}}\right|^{\delta}\right.\right.\\
&\left.\left.+\frac{1}{\lambda}\left\{\xi\left|\eta^{(\mu+1)}(b_1,\cdot)\right|^{\lambda}+(1-\xi)\left|\eta^{(\mu+1)}(y,\cdot)\right|^{\lambda}-c(\cdot)\xi(1-\xi)(y-b_1)^2\right\}\right)d\xi\right]\\
&+\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\left[\int_0^1\left(\frac{\xi^{\delta\mu}}{\delta}\left|\left(\xi b_2^p+(1-\xi)y^p\right)^{\frac{1-p}{p}}\right|^{\delta}\right.\right.\\
&\left.\left.+\frac{1}{\lambda}\left\{\xi\left|\eta^{(\mu+1)}(b_2,\cdot)\right|^{\lambda}+(1-\xi)\left|\eta^{(\mu+1)}(y,\cdot)\right|^{\lambda}-c(\cdot)\xi(1-\xi)(y-b_2)^2\right\}\right)d\xi\right]\\
\le{}&\frac{(y^p-b_1^p)^{\mu+1}+(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\left[\frac{b_1^{\delta(1-p)}}{\delta(\delta\mu+1)}+\frac{\mathbb{Q}^{\lambda}}{\lambda}\right]\\
&-\frac{c(\cdot)}{6\lambda}\left[\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}(y-b_1)^2+\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}(y-b_2)^2\right]\quad\text{(a.e.)}.
\end{aligned}
$$

Continuing in the same way, we can prove (2.15). □

Theorem 2.6 *For a differentiable stochastic process $\delta,\lambda>1$ with $\delta^{-1}+\lambda^{-1}=1$, $b_1,b_2\in I$ with $b_1<b_2$, and $\eta^{(\mu)}:I\times\Omega\subset(0,\infty)\to\mathbb{R}$ on I° such that $\eta^{(\mu+1)}\in M([b_1,b_2])$ and $|\eta^{(\mu+1)}|^{\lambda}$ a strongly p-convex stochastic process with modulus $c(\cdot)$ satisfying $|\eta^{(\mu+1)}(y,\cdot)|\le\mathbb{Q}$, $\forall y\in[b_1,b_2]$, the following inequality holds almost everywhere for all $y\in[b_1,b_2]$ and $p\in(0,\infty)$:*

$$
\begin{aligned}
&\left|\frac{(y^p-b_1^p)^{\mu}\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^{\mu}\eta^{(\mu)}(b_2,\cdot)}{p^{\mu}(b_2-b_1)}\right.\\
&\left.-\frac{\Gamma_{\kappa}(\mu+\kappa)}{(b_2-b_1)}\left[\left({}_{\kappa}^{p}D_{b_1^+}^{\mu}\eta\right)(y,\cdot)+\left({}_{\kappa}^{p}D_{b_2^-}^{\mu}\eta\right)(y,\cdot)\right]\right|\\
\le{}&\frac{(y^p-b_1^p)^{\mu+1}+(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\left[\frac{b_1^{1-p}\delta}{(\mu+1)}+\lambda\mathbb{Q}\right]\\
&-\frac{c(\cdot)\lambda}{6}\left[\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}(y-b_1)^2+\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}(y-b_2)^2\right],
\end{aligned}
\tag{2.16}
$$

and the following inequality holds almost everywhere for all $y\in(b_1,b_2)$ and $p\in(-\infty,0)\cup(0,1)$:

$$
\begin{aligned}
&\left|\frac{(y^p-b_1^p)^{\mu}\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^{\mu}\eta^{(\mu)}(b_2,\cdot)}{p^{\mu}(b_2-b_1)}\right.\\
&\left.-\frac{\Gamma_{\kappa}(\mu+\kappa)}{(b_2-b_1)}\left[\left({}_{\kappa}^{p}D_{b_1^+}^{\mu}\eta\right)(y,\cdot)+\left({}_{\kappa}^{p}D_{b_2^-}^{\mu}\eta\right)(y,\cdot)\right]\right|\\
\le{}&\frac{(y^p-b_1^p)^{\mu+1}+(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\left[\frac{b_2^{1-p}\delta}{(\mu+1)}+\lambda\mathbb{Q}\right]\\
&-\frac{c(\cdot)\lambda}{6}\left[\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}(y-b_1)^2+\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}(y-b_2)^2\right].
\end{aligned}
\tag{2.17}
$$

Proof To prove this result, we will use the following inequality:

$$m^{\delta}n^{\lambda} \leq \delta m + \lambda n, \qquad m,n \geq 0, \qquad \delta,\lambda > 0, \qquad \delta + \lambda = 1.$$

By using (2.1) and taking the definition of a strongly p-convex stochastic process of $|\eta^{(\mu+1)}|^{\lambda}$ yields

$$\begin{aligned}
&\left|\frac{(y^p - b_1^p)^{\mu}\eta^{(\mu)}(b_1,\cdot) + (b_2^p - y^p)^{\mu}\eta^{(\mu)}(b_2,\cdot)}{p^{\mu}(b_2 - b_1)}\right.\\
&\quad\left. - \frac{\Gamma_{\kappa}(\mu+\kappa)}{(b_2 - b_1)}\left[\left({}_{\kappa}^{p}D_{b_1^+}^{\mu}\eta\right)(y,\cdot) + \left({}_{\kappa}^{p}D_{b_2^-}^{\mu}\eta\right)(y,\cdot)\right]\right|\\
&\leq \frac{(y^p - b_1^p)^{\mu+1}}{p^{1+\mu}(b_2 - b_1)}\\
&\quad\times \int_0^1 \left[\xi^{\mu}\left(\xi b_1^p + (1-\xi)y^p\right)^{\frac{1-p}{p}}\right]^{\delta}\left[\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_1^p + (1-\xi)y^p},\cdot\right)\right|\right]^{\lambda} d\xi\\
&\quad+ \frac{(b_2^p - y^p)^{\mu+1}}{p^{1+\mu}(b_2 - b_1)}\\
&\quad\times \int_0^1 \left[\xi^{\mu}\left(\xi b_2^p + (1-\xi)y^p\right)^{\frac{1-p}{p}}\right]^{\delta}\left[\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_2^p + (1-\xi)y^p},\cdot\right)\right|\right]^{\lambda} d\xi\\
&\leq \frac{(y^p - b_1^p)^{\mu+1}}{p^{1+\mu}(b_2 - b_1)}\\
&\quad\times \left[\int_0^1 \xi^{\mu}\left(\xi b_1^p + (1-\xi)y^p\right)^{\frac{1-p}{p}} d\xi\right.\\
&\quad\left.+ \int_0^1 \lambda\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_1^p + (1-\xi)y^p},\cdot\right)\right| d\xi\right]\\
&\quad+ \frac{(b_2^p - y^p)^{\mu+1}}{p^{1+\mu}(b_2 - b_1)}\\
&\quad\times \left[\int_0^1 \xi^{\mu}\left(\xi b_2^p + (1-\xi)y^p\right)^{\frac{1-p}{p}} d\xi\right.\\
&\quad\left.+ \int_0^1 \lambda\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_2^p + (1-\xi)y^p},\cdot\right)\right| d\xi\right]\\
&\leq \frac{(y^p - b_1^p)^{\mu+1}}{p^{1+\mu}(b_2 - b_1)}\left[\int_0^1 b_1^{1-p}\delta\xi^{\mu}\, d\xi\right.\\
&\quad\left.+ \int_0^1 \lambda\left\{\xi\left|\eta^{(\mu+1)}(b_1,\cdot)\right| + (1-\xi)\left|\eta^{(\mu+1)}(y,\cdot)\right| - c(\cdot)\xi(1-\xi)(y-b_1)^2\right\} d\xi\right]\\
&\quad+ \frac{(b_2^p - y^p)^{\mu+1}}{p^{1+\mu}(b_2 - b_1)}\left[\int_0^1 b_1^{1-p}\delta\xi^{\mu}\, d\xi\right.\\
&\quad\left.+ \int_0^1 \lambda\left\{\xi\left|\eta^{(\mu+1)}(b_2,\cdot)\right| + (1-\xi)\left|\eta^{(\mu+1)}(y,\cdot)\right| - c(\cdot)\xi(1-\xi)(y-b_2)^2\right\} d\xi\right]\\
&\leq \frac{(y^p - b_1^p)^{\mu+1} + (b_2^p - y^p)^{\mu+1}}{p^{1+\mu}(b_2 - b_1)}\left[\frac{b_1^{1-p}\delta}{(\mu+1)} + \lambda\mathbb{Q}\right]\\
&\quad- \frac{c(\cdot)\lambda}{6}\left[\frac{(y^p - b_1^p)^{\mu+1}}{p^{1+\mu}(b_2 - b_1)}(y-b_1)^2 + \frac{(b_2^p - y^p)^{\mu+1}}{p^{1+\mu}(b_2 - b_1)}(y-b_2)^2\right] \quad \text{(a.e.)}.
\end{aligned}$$

Continuing in the same manner, we can also prove (2.17). □

Theorem 2.7 *For a differentiable stochastic process $\delta,\lambda > 1$ with $\delta^{-1}+\lambda^{-1}=1$, $b_1,b_2 \in I$ with $b_1 < b_2$, and $\eta^{(\mu)}: I\times\Omega \subset (0,\infty)\to\mathbb{R}$ on I° such that $\eta^{(\mu+1)} \in M([b_1,b_2])$ and $|\eta^{(\mu+1)}|^\lambda$ a strongly p-convex stochastic process with modulus $c(\cdot)$ satisfying $|\eta^{(\mu+1)}(y,\cdot)| \le \mathbb{Q}$, $\forall y \in [b_1,b_2]$, the following inequality holds almost everywhere for all $y\in[b_1,b_2]$ and $p\in(0,\infty)$:*

$$\begin{aligned}&\biggl|\frac{(y^p-b_1^p)^\mu\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^\mu\eta^{(\mu)}(b_2,\cdot)}{p^\mu(b_2-b_1)}\\&\quad-\frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)}\bigl[\bigl({}_\kappa^p D^\mu_{b_1^+}\eta\bigr)(y,\cdot)+\bigl({}_\kappa^p D^\mu_{b_2^-}\eta\bigr)(y,\cdot)\bigr]\biggr|\\&\quad\le\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\\&\quad\quad\times\bigl[\bigl(\varPi_1(b_1,y;p)\bigr)^{\frac{1}{\delta}}+\bigl(\varPi_2(b_1,y;p)\bigr)^{\frac{1}{\delta}}\bigr]\biggl(\frac{\mathbb{Q}^\lambda}{2}-\frac{c(\cdot)}{12}(y-b_1)^2\biggr)^{\frac{1}{\lambda}},\end{aligned}\tag{2.18}$$

and the following inequality holds almost everywhere for all $y\in(b_1,b_2)$ and $p\in(-\infty,0)\cup(0,1)$:

$$\begin{aligned}&\biggl|\frac{(y^p-b_1^p)^\mu\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^\mu\eta^{(\mu)}(b_2,\cdot)}{p^\mu(b_2-b_1)}\\&\quad-\frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)}\bigl[\bigl({}_\kappa^p D^\mu_{b_1^+}\eta\bigr)(y,\cdot)+\bigl({}_\kappa^p D^\mu_{b_2^-}\eta\bigr)(y,\cdot)\bigr]\biggr|\\&\quad\le\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\\&\quad\quad\times\bigl[\bigl(\varPi_3(b_2,y;p)\bigr)^{\frac{1}{\delta}}+\bigl(\varPi_4(b_2,y;p)\bigr)^{\frac{1}{\delta}}\bigr]\biggl(\frac{\mathbb{Q}^\lambda}{2}-\frac{c(\cdot)}{12}(y-b_2)^2\biggr)^{\frac{1}{\lambda}}.\end{aligned}\tag{2.19}$$

Here,

$$\varPi_1(b_1,y;p)=\begin{cases}\frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{b_1}{y})^p)]}{y^{\delta(p-1)}(\delta\mu+1)(\delta\mu+2)}, & p\in(-\infty,0)\cup(0,1),\\ \frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{y}{b_1})^p)]}{b_1^{\delta(p-1)}(\delta\mu+1)(\delta\mu+2)}, & p\in(1,\infty),\end{cases}$$

$$\varPi_2(b_1,y;p)=\begin{cases}\frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{b_1}{y})^p)]}{y^{\delta(p-1)}(\delta\mu+2)}, & p\in(-\infty,0)\cup(0,1),\\ \frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{y}{b_1})^p)]}{b_1^{\delta(p-1)}(\delta\mu+2)}, & p\in(1,\infty),\end{cases}$$

$$\varPi_3(b_2,y;p)=\begin{cases}\frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{b_2}{y})^p)]}{y^{\delta(p-1)}(\delta\mu+1)(\delta\mu+2)}, & p\in(-\infty,0)\cup(0,1),\\ \frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{y}{b_2})^p)]}{b_2^{\delta(p-1)}(\delta\mu+1)(\delta\mu+2)}, & p\in(1,\infty),\end{cases}$$

and

$$\varPi_4(b_2,y;p)=\begin{cases}\frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{b_2}{y})^p)]}{y^{\delta(p-1)}(\delta\mu+2)}, & p\in(-\infty,0)\cup(0,1),\\ \frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{y}{b_2})^p)]}{b_2^{\delta(p-1)}(\delta\mu+2)}, & p\in(1,\infty).\end{cases}$$

Proof By using Lemma 2.1, to prove the first part of Theorem 2.7 and from the Hölder–İşcan inequality yields

$$\begin{aligned}
&\bigg|\frac{(y^p-b_1^p)^\mu\eta^{(\mu)}(b_1,\cdot)+(b_2^p-y^p)^\mu\eta^{(\mu)}(b_2,\cdot)}{p^\mu(b_2-b_1)}\\
&\quad-\frac{\Gamma_\kappa(\mu+\kappa)}{(b_2-b_1)}\big[\big({}_\kappa^p D_{b_1^+}^\mu\eta\big)(y,\cdot)+\big({}_\kappa^p D_{b_2^-}^\mu\eta\big)(y,\cdot)\big]\bigg|\\
&\quad\le\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\int_0^1\xi^\mu\big(\xi b_1^p+(1-\xi)y^p\big)^{\frac{1-p}{p}}\Big|\eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\Big)\Big|\,d\xi\\
&\qquad+\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\int_0^1\xi^\mu\big(\xi b_2^p+(1-\xi)y^p\big)^{\frac{1-p}{p}}\Big|\eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\Big)\Big|\,d\xi\\
&\quad\le\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\bigg[\bigg(\int_0^1\xi^{\delta\mu}(1-\xi)\big(\xi b_1^p+(1-\xi)y^p\big)^{\delta(\frac{1-p}{p})}\,d\xi\bigg)^{\frac{1}{\delta}}\\
&\qquad\times\bigg(\int_0^1(1-\xi)\Big|\eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\Big)\Big|^\lambda\,d\xi\bigg)^{\frac{1}{\lambda}}\\
&\qquad+\bigg(\int_0^1\xi^{\delta\mu+1}\big(\xi b_1^p+(1-\xi)y^p\big)^{\delta(\frac{1-p}{p})}\,d\xi\bigg)^{\frac{1}{\delta}}\\
&\qquad\times\bigg(\int_0^1\xi\Big|\eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\Big)\Big|^\lambda\,d\xi\bigg)^{\frac{1}{\lambda}}\bigg]\\
&\qquad+\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\bigg[\bigg(\int_0^1\xi^{\delta\mu}(1-\xi)\big(\xi b_2^p+(1-\xi)y^p\big)^{\delta(\frac{1-p}{p})}\,d\xi\bigg)^{\frac{1}{\delta}}\\
&\qquad\times\bigg(\int_0^1(1-\xi)\Big|\eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\Big)\Big|^\lambda\,d\xi\bigg)^{\frac{1}{\lambda}}\\
&\qquad+\bigg(\int_0^1\xi^{\delta\mu+1}\big(\xi b_2^p+(1-\xi)y^p\big)^{\delta(\frac{1-p}{p})}\,d\xi\bigg)^{\frac{1}{\delta}}\\
&\qquad\times\bigg(\int_0^1\xi\Big|\eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\Big)\Big|^\lambda\,d\xi\bigg)^{\frac{1}{\lambda}}\bigg]\\
&\quad\le\frac{(y^p-b_1^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\\
&\qquad\times\bigg[\big(\Pi_1(b_1,y;p)\big)^{\frac{1}{\delta}}\bigg(\int_0^1(1-\xi)\Big|\eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\Big)\Big|^\lambda\,d\xi\bigg)^{\frac{1}{\lambda}}\\
&\qquad+\big(\Pi_2(b_1,y;p)\big)^{\frac{1}{\delta}}\bigg(\int_0^1\xi\Big|\eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_1^p+(1-\xi)y^p},\cdot\Big)\Big|^\lambda\,d\xi\bigg)^{\frac{1}{\lambda}}\bigg]\\
&\qquad+\frac{(b_2^p-y^p)^{\mu+1}}{p^{1+\mu}(b_2-b_1)}\\
&\qquad\times\bigg[\big(\Pi_3(b_2,y;p)\big)^{\frac{1}{\delta}}\bigg(\int_0^1(1-\xi)\Big|\eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\Big)\Big|^\lambda\,d\xi\bigg)^{\frac{1}{\lambda}}\\
&\qquad+\big(\Pi_4(b_2,y;p)\big)^{\frac{1}{\delta}}\bigg(\int_0^1\xi\Big|\eta^{(\mu+1)}\Big(\sqrt[p]{\xi b_2^p+(1-\xi)y^p},\cdot\Big)\Big|^\lambda\,d\xi\bigg)^{\frac{1}{\lambda}}\bigg]\quad\text{(a.e.)}.
\end{aligned}$$

As $|\eta^{(\mu+1)}|^{\lambda}$ is a strongly p-convex stochastic process and $|\eta^{(\mu+1)}(y,\cdot)| \leq \mathbb{Q}$ for all $y \in [b_1, b_2]$, we obtain

$$\begin{aligned}
&\int_0^1 (1-\xi)\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_1^p + (1-\xi)y^p},\cdot\right)\right|^{\lambda} d\xi \\
&\quad\leq \int_0^1 (1-\xi)\left[\xi\left|\eta^{(\mu+1)}(b_1,\cdot)\right|^{\lambda} + (1-\xi)\left|\eta^{(\mu+1)}(y,\cdot)\right|^{\lambda} - c(\cdot)\xi(1-\xi)(y-b_1)^2\right] d\xi \\
&\quad= \frac{\mathbb{Q}^{\lambda}}{2} - \frac{c(\cdot)}{12}(y-b_1)^2 \quad \text{(a.e.)}, \\
&\int_0^1 \xi\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_1^p + (1-\xi)y^p},\cdot\right)\right|^{\lambda} d\xi \\
&\quad\leq \int_0^1 \xi\left[\xi\left|\eta^{(\mu+1)}(b_1,\cdot)\right|^{\lambda} + (1-\xi)\left|\eta^{(\mu+1)}(y,\cdot)\right|^{\lambda} - c(\cdot)\xi(1-\xi)(y-b_1)^2\right] d\xi \\
&\quad= \frac{\mathbb{Q}^{\lambda}}{2} - \frac{c(\cdot)}{12}(y-b_1)^2 \quad \text{(a.e.)},
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1 (1-\xi)\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_2^p + (1-\xi)y^p},\cdot\right)\right|^{\lambda} d\xi \\
&\quad\leq \frac{\mathbb{Q}^{\lambda}}{2} - \frac{c(\cdot)}{12}(y-b_2)^2 \quad \text{(a.e.)}, \\
&\int_0^1 \xi\left|\eta^{(\mu+1)}\left(\sqrt[p]{\xi b_2^p + (1-\xi)y^p},\cdot\right)\right|^{\lambda} d\xi \\
&\quad\leq \frac{\mathbb{Q}^{\lambda}}{2} - \frac{c(\cdot)}{12}(y-b_2)^2 \quad \text{(a.e.)}.
\end{aligned}$$

We now have the result that

$$\begin{aligned}
\Pi_1(b_1,y;p) &= \int_0^1 \xi^{\delta\mu}(1-\xi)\left(\xi b_1^p + (1-\xi)y^p\right)^{\delta(\frac{1-p}{p})} d\xi \\
&= \begin{cases} \frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{b_1}{y})^p)]}{y^{\delta(p-1)}(\delta\mu+1)(\delta\mu+2)}, & p \in (-\infty,0)\cup(0,1), \\ \frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{y}{b_1})^p)]}{b_1^{\delta(p-1)}(\delta\mu+1)(\delta\mu+2)}, & p \in (1,\infty), \end{cases}
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
\Pi_2(b_1,y;p) &= \int_0^1 \xi^{\delta\mu+1}\left(\xi b_1^p + (1-\xi)y^p\right)^{\delta(\frac{1-p}{p})} d\xi \\
&= \begin{cases} \frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{b_1}{y})^p)]}{y^{\delta(p-1)}(\delta\mu+2)}, & p \in (-\infty,0)\cup(0,1), \\ \frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{y}{b_1})^p)]}{b_1^{\delta(p-1)}(\delta\mu+2)}, & p \in (1,\infty), \end{cases}
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
\Pi_3(b_2,y;p) &= \int_0^1 \xi^{\delta\mu}(1-\xi)\left(\xi b_2^p + (1-\xi)y^p\right)^{\delta(\frac{1-p}{p})} d\xi \\
&= \begin{cases} \frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{b_2}{y})^p)]}{y^{\delta(p-1)}(\delta\mu+1)(\delta\mu+2)}, & p \in (-\infty,0)\cup(0,1), \\ \frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{y}{b_2})^p)]}{b_2^{\delta(p-1)}(\delta\mu+1)(\delta\mu+2)}, & p \in (1,\infty), \end{cases}
\end{aligned} \tag{2.22}$$

$$\Pi_4(b_2,y;p) = \int_0^1 \xi^{\delta\mu+1}\bigl(\xi b_2^p + (1-\xi)y^p\bigr)^{\delta(\frac{1-p}{p})}\,d\xi \tag{2.23}$$
$$= \begin{cases} \frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{b_2}{y})^p)]}{y^{\delta(p-1)}(\delta\mu+2)}, & p\in(-\infty,0)\cup(0,1),\\ \frac{[2F_1(\delta(1-\frac{1}{p}),\delta\mu+1,\delta\mu+3,1-(\frac{y}{b_2})^p)]}{b_2^{\delta(p-1)}(\delta\mu+2)}, & p\in(1,\infty).\end{cases}$$

Combining all above inequalities, we obtain the desired result (2.18) and (2.19). □

3 Conclusion

In the present note, we introduced the notion of a strongly p-convex stochastic process. We established Ostrowski-type inequalities for a strongly p-convex stochastic process. Also, we established some integral inequalities of Ostrowski-type via the generalized k-fractional Hilfer–Katugampola derivative.

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