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Some inequalities for *cr*-log-*h*-convex functions

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Abstract

The main purpose of this paper is to study certain inequalities for *cr*-log-*h*-convex functions with an interval value. To this end, we first give a definition of *cr*-log-*h*-convexity of interval-valued functions under the *cr*-order and study some properties of such functions. On this basis, we establish the Jensen-, Hermite–Hadamard-, and Fejér-type inequalities for *cr*-log-*h*-convex functions, and discuss some special cases. In addition, we give some numerical examples to illustrate the accuracy of the results obtained.

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1 Introduction

The convexity of a function is a classical concept that plays an important fundamental role in fields such as game theory and optimal control, but many practical problems encounter functions that are not the classical class of convex functions, but a weaker class of functions than convex functions. The study of generalized convexity of functions is therefore a meaningful and interesting area of work, and one that has received a great deal of scholarly attention. Many new weaker generalized convexities have been proposed to meet the needs of practical problems. For example, Noor et al. [1] gave the definition of *a* log-*h*-convex function, which is a more general form of several kinds of convex functions. Moreover, they established integral inequalities for log-*h*-convex functions and studied the basic properties of such functions. It is well known that the Hermite–Hadamard inequality is an equivalent form of a convex function, so generalized convexity is a necessary condition for establishing a Hermite–Hadamard-type inequality. The generalized convex functions and their related inequalities have been fruitfully studied in the last decade, and the interested reader is referred to [2–13].

On the other hand, interval analysis is a useful tool for measuring uncertainty problems. It has a long history dating back to Archimedes' measurement of π, but it was through Moore's [14] first application of interval analysis to automated error analysis that the study of it was taken seriously. Since 2013, Costa et al. [15], Flores-Franulič et al. [16], Chalco-Cano et al. [17] and others have extended many classical integral inequalities to interval-valued functions and fuzzy-valued functions. In particular, Guo et al. [18] gave the definition of an interval log-h-convex function and established the corresponding integral inequality by using the interval-inclusion relation. In 2021, Muhammad [19] introduced the definition of an h-convex interval-valued function by using Kulisch–Miranker order, and proved some inequalities of this kind of convex functions. However, both of these order relations are partial, meaning that any two intervals may be incomparable. It is therefore an interesting task to find a suitable order for the study of inequalities related to interval-valued functions. In 2014, Bhunia [20] used the center radius of the interval to establish a new rank relation, the cr-order. This is a full-order relationship, meaning that any two intervals can be compared using this order. The main aim of this paper is therefore to investigate the generalized convexity of interval-valued functions using the cr-order and to establish the inequalities associated with that convexity.

The paper is organized as follows: after the preliminaries in Sect. 2, we introduce the concept of cr-log-h-convexity, and discuss some of its fascinating characteristics in Sect. 3, then some Jensen-type, Hermite–Hadamard-type, and Fejér-type inequalities for cr-log-h-convex functions are proved. Also, some numerical examples are given to illustrate the accuracy of the results obtained. We end with Sect. 4 giving some conclusions and suggestions for future work.

2 Preliminaries

Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}^+$ the set of all positive real numbers. The set of all closed intervals on $\mathbb{R}$ is denoted by $\mathbb{R}_{\mathcal{I}}$. For $[\underline{a},\overline{a}] \in \mathbb{R}_{\mathcal{I}}$, if $\underline{a} > 0$, then $[\underline{a},\overline{a}]$ is called a positive interval. The set of all positive intervals is denoted by $\mathbb{R}_{\mathcal{I}}^+$.

For any $\lambda \in \mathbb{R}$, $a = [\underline{a},\overline{a}]$, $b = [\underline{b},\overline{b}] \in \mathbb{R}_{\mathcal{I}}$, the Minkowski addition and scalar multiplication of intervals are defined by

$$a + b = [\underline{a},\overline{a}] + [\underline{b},\overline{b}] = [\underline{a} + \underline{b},\overline{a} + \overline{b}];$$

and

$$\lambda a = \lambda[\underline{a},\overline{a}] = \begin{cases} [\lambda\underline{a},\ \lambda\overline{a}], & \lambda > 0, \\ [0,0], & \lambda = 0, \\ [\lambda\overline{a},\lambda\underline{a}], & \lambda < 0. \end{cases}$$

Let $a = [\underline{a},\overline{a}] \in \mathbb{R}_{\mathcal{I}}$, $a_c = \frac{\overline{a}+\underline{a}}{2}$ is called the center of a and $a_r = \frac{\overline{a}-\underline{a}}{2}$ is called the radius of a. Then, $a = [\underline{a},\overline{a}]$ can also be presented in center-radius form as

$$a = \left\langle \frac{\overline{a} + \underline{a}}{2}, \frac{\overline{a} - \underline{a}}{2} \right\rangle = \langle a_c, a_r\rangle.$$

Definition 2.1 ([20]) Let $a = [\underline{a},\overline{a}] = \langle a_c, a_r\rangle$, $b = [\underline{b},\overline{b}] = \langle b_c, b_r\rangle \in \mathbb{R}_{\mathcal{I}}$, then the center-radius order (for brevity, cr-order) relation is defined as

$$a \preceq_{cr} b \Leftrightarrow \begin{cases} a_c < b_c, & \text{if } a_c \neq b_c, \\ a_r \leq b_r, & \text{if } a_c = b_c. \end{cases}$$

Obviously, for any two intervals $a,\ b \in \mathbb{R}_{\mathcal{I}}$, either $a \preceq_{cr} b$ or $b \preceq_{cr} a$.

Let $f : [a, b] \to \mathbb{R}_{\mathcal{I}}$, then f is called an interval-valued function. For more basic notations and properties of interval-valued functions, see [14, 21–23]. In particular, the concept of Riemann integrals for interval-valued functions is given in [21]. The set of all Riemann integrable interval-valued functions on $[a, b]$ is denoted by $\mathcal{IR}_{([a,b])}$.

Moreover, we have

Theorem 2.2 ([21]) *Let $f : [a, b] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function given by $f = [\underline{f}, \overline{f}]$. Then, the f is Riemann integrable on $[a, b]$ if and only if $\underline{f}$ and $\overline{f}$ are Riemann integrable on $[a, b]$ and*

$$\int_a^b f(x)\,dx = \left[\int_a^b \underline{f}(x)\,dx, \int_a^b \overline{f}(x)\,dx\right].$$

Theorem 2.3 ([23]) *Let $f, g : [a, b] \to \mathbb{R}_{\mathcal{I}}^{+}$ be interval-valued functions given by $f = [\underline{f}, \overline{f}]$ and $g = [\underline{g}, \overline{g}]$. If $f, g \in \mathcal{IR}_{([a,b])}$, and $f(x) \preceq_{cr} g(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)\,dx \preceq_{cr} \int_a^b g(x)\,dx.$$

3 Main results

In this section, we mainly introduce a new concept of *cr*-log-*h*-convexity for interval-valued functions and investigate its properties. On this basis, we establish the inequalities associated with the *cr*-log-*h*-convex function.

First, we recall the concept of log-*h*-convexity for real functions in [1].

Definition 3.1 ([1]) Let $h : [0, 1] \to \mathbb{R}^{+}$ be a function. We say that $f : [a, b] \to \mathbb{R}^{+}$ is a log-*h*-convex function, if for all $x, y \in [a, b]$ and $t \in [0, 1]$,

$$f\big(tx + (1-t)y\big) \le \big[f(x)\big]^{h(t)}\big[f(y)\big]^{h(1-t)}.$$

h is called a supermultiplicative function if for any $\vartheta, t \in [0, 1]$,

$$h(\vartheta t) \ge h(\vartheta)h(t).$$

Now, we generalize this concept to interval-valued functions.

Definition 3.2 Let $f : [a, b] \to \mathbb{R}_{\mathcal{I}}^{+}$ be an interval-valued function with $f = [\underline{f}, \overline{f}]$, and $h : [0, 1] \to \mathbb{R}^{+}$ be a nonnegative function. Then, f is said to be *cr*-log-*h*-convex on $[a, b]$ if

$$f\big(tx_1 + (1-t)x_2\big) \preceq_{cr} \big[f(x_1)\big]^{h(t)}\big[f(x_2)\big]^{h(1-t)},$$

for each $t \in (0, 1)$ and any $x_1, x_2 \in [a, b]$.

Denote by $SX(cr\text{-}\log\text{-}h, [a, b], \mathbb{R}_{\mathcal{I}}^{+})$ the set of all *cr*-log-*h*-convex functions on $[a, b]$.

Remark 3.3 In Definition 3.2,

- If $\underline{f} = \overline{f}$, then f is reduced to a log-*h*-convex function defined in Definition 3.1.

- If $h(t) = t$, then f is called a cr-log-convex function, i.e.,

$$f\big(tx_1 + (1-t)x_2\big) \preceq_{cr} \big[f(x_1)\big]^{t}\big[f(x_2)\big]^{1-t}.$$

- If $h(t) = t^s$, $s \in (0,1]$, then f is called a cr-log-s-convex function, i.e.,

$$f\big(tx_1 + (1-t)x_2\big) \preceq_{cr} \big[f(x_1)\big]^{t^s}\big[f(x_2)\big]^{(1-t)^s}.$$

- If $h(t) = 1$, then f is called a cr-log-P function, i.e.,

$$f\big(tx_1 + (1-t)x_2\big) \preceq_{cr} f(x_1)f(x_2).$$

Proposition 3.4 *Let $f : [a,b] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function given by $f = [\underline{f},\overline{f}] = \langle f_c,f_r\rangle$. If f_c and f_r are* log*-h-convex on $[a,b]$, then f is a cr-*log*-h-convex function on $[a,b]$.*

Proof Since f_c and f_r are log-h-convex on $[a,b]$, then for each $t \in (0,1)$ and any $x_1,x_2 \in [a,b]$, we have

$$f_c\big(tx_1 + (1-t)x_2\big) \le \big[f_c(x_1)\big]^{h(t)}\big[f_c(x_2)\big]^{h(1-t)},$$

and

$$f_r\big(tx_1 + (1-t)x_2\big) \le \big[f_r(x_1)\big]^{h(t)}\big[f_r(x_2)\big]^{h(1-t)}.$$

Now, if $f_c(tx_1 + (1-t)x_2) \neq [f_c(x_1)]^{h(t)}[f_c(x_2)]^{h(1-t)}$, then for each $t \in (0,1)$ and every $x_1,x_2 \in [a,b]$,

$$f_c\big(tx_1 + (1-t)x_2\big) < \big[f_c(x_1)\big]^{h(t)}\big[f_c(x_2)\big]^{h(1-t)},$$

that is,

$$f\big(tx_1 + (1-t)x_2\big) \preceq_{cr} \big[f(x_1)\big]^{h(t)}\big[f(x_2)\big]^{h(1-t)}.$$

Otherwise, for each $t \in (0,1)$ and all $x_1,x_2 \in [a,b]$, we have

$$f_r\big(tx_1 + (1-t)x_2\big) \le \big[f_r(x_1)\big]^{h(t)}\big[f_r(x_2)\big]^{h(1-t)},$$

that is,

$$f\big(tx_1 + (1-t)x_2\big) \preceq_{cr} \big[f(x_1)\big]^{h(t)}\big[f(x_2)\big]^{h(1-t)}.$$

The proof is completed by combining the above inequations. □

Proposition 3.5 *Let $f : [a,b] \to \mathbb{R}^+_{\mathcal{I}}$ be an interval-valued function given by $f = [\underline{f},\overline{f}] = \langle f_c,f_r\rangle$ and $f \in SX(cr\text{-}\log\text{-}h,[a,b],\mathbb{R}^+_{\mathcal{I}})$, $h : [0,1] \to \mathbb{R}^+$ is a multiplicative function. Then, for any $x_1,x_2,x_3 \in [0,1]$ and $x_1 < x_2 < x_3$,*

$$\big[f(x_2)\big]^{h(x_3-x_1)} \preceq_{cr} \big[f(x_1)\big]^{h(x_3-x_2)}\big[f(x_3)\big]^{h(x_2-x_1)}.$$

Proof Let $x_1, x_2, x_3 \in [0,1]$ satisfy the assumptions, then

$$h\left(\frac{x_3 - x_2}{x_3 - x_1}\right) = \frac{h(x_3 - x_2)}{h(x_3 - x_1)}, \qquad h\left(\frac{x_2 - x_1}{x_3 - x_1}\right) = \frac{h(x_2 - x_1)}{h(x_3 - x_1)}.$$

Let $t = \frac{x_3 - x_2}{x_3 - x_1}$, then $x_2 = tx_1 + (1-t)x_3$. Since $f \in SX(cr\text{-}\log\text{-}h, [a,b], \mathbb{R}_{\mathcal{I}}^+)$ and h is a multiplicative function, then

$$\begin{aligned} f(x_2) &= f\big(tx_1 + (1-t)x_3\big) \preceq_{cr} \big[f(x_1)\big]^{h(\frac{x_3-x_2}{x_3-x_1})}\big[f(x_3)\big]^{h(\frac{x_2-x_1}{x_3-x_1})} \\ &\preceq_{cr} \big[f(x_1)\big]^{\frac{h(x_3-x_2)}{h(x_3-x_1)}}\big[f(x_3)\big]^{\frac{h(x_2-x_1)}{h(x_3-x_1)}}. \end{aligned}$$

Thus,

$$\big[f(x_2)\big]^{h(x_3-x_1)} \preceq_{cr} \big[f(x_1)\big]^{h(x_3-x_2)}\big[f(x_3)\big]^{h(x_2-x_1)}.$$

The proof is completed. □

In what follows, we give a Jensen-type inequality for cr-log-h-convex functions.

Theorem 3.6 *Let $f : [a,b] \to \mathbb{R}_{\mathcal{I}}^+$ be an interval-valued function and $h : [0,1] \to \mathbb{R}^+$ be a nonnegative supermultiplicative function. If $f \in SX(cr\text{-}\log\text{-}h, [a,b], \mathbb{R}_{\mathcal{I}}^+)$, then*

$$f\left(\frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i x_i\right) \preceq_{cr} \prod_{i=1}^{n}\big[f(x_i)\big]^{h(\frac{p_i}{\mathcal{P}_n})}, \tag{3.1}$$

where $p_i \geq 0$ but not all 0, $x_i \in [a,b]$, $i = 1,2,\ldots,n$, and $\mathcal{P}_n = \sum_{i=1}^{n} p_i$.

Proof Let $f \in SX(cr\text{-}\log\text{-}h, [a,b], \mathbb{R}_{\mathcal{I}}^+)$. If $n = 2$, according to Definition 3.2, we have

$$f\left(\frac{p_1}{\mathcal{P}_2}x_1 + \frac{p_2}{\mathcal{P}_2}x_2\right) \preceq_{cr} \big[f(x_1)\big]^{h(\frac{p_1}{\mathcal{P}_2})}\big[f(x_2)\big]^{h(\frac{p_2}{\mathcal{P}_2})}.$$

Assume that (3.1) holds for $n = k$, that is

$$f\left(\frac{1}{\mathcal{P}_k}\sum_{i=1}^{k} p_i x_i\right) \preceq_{cr} \prod_{i=1}^{k}\big[f(x_i)\big]^{h(\frac{p_i}{\mathcal{P}_k})}.$$

Now, let us prove that (3.1) is valid when $n = k + 1$,

$$\begin{aligned} &f\left(\frac{1}{\mathcal{P}_{k+1}}\sum_{i=1}^{k+1} p_i x_i\right) \\ &\quad = f\left(\frac{1}{\mathcal{P}_{k+1}}\sum_{i=1}^{k-1} p_i x_i + \frac{p_k + p_{k+1}}{\mathcal{P}_{k+1}}\left(\frac{p_k x_k}{p_k + p_{k+1}} + \frac{p_{k+1}x_{k+1}}{p_k + p_{k+1}}\right)\right) \\ &\quad \preceq_{cr} \prod_{i=1}^{k-1} h\left(\frac{p_i}{\mathcal{P}_{k+1}}\right) f(x_i)\left[f\left(\frac{p_k x_k}{p_k + p_{k+1}} + \frac{p_{k+1}x_{k+1}}{p_k + p_{k+1}}\right)\right]^{h(\frac{p_k + p_{k+1}}{\mathcal{P}_{k+1}})} \end{aligned}$$

$$\preceq_{cr} \prod_{i=1}^{k-1} h\left(\frac{p_i}{\mathcal{P}_{k+1}}\right) f(x_i)\Big[\big[f(x_k)\big]^{h(\frac{p_k}{p_k+p_{k+1}})}\big[f(x_{k+1})\big]^{h(\frac{p_{k+1}}{p_k+p_{k+1}})}\Big]^{h(\frac{p_k+p_{k+1}}{\mathcal{P}_{k+1}})}$$

$$\preceq_{cr} \prod_{i=1}^{k-1} h\left(\frac{p_i}{\mathcal{P}_{k+1}}\right) f(x_i)\Big[\big[f(x_k)\big]^{h(\frac{p_k}{\mathcal{P}_{k+1}})}\big[f(x_{k+1})\big]^{h(\frac{p_{k+1}}{\mathcal{P}_{k+1}})}\Big]$$

$$= \prod_{i=1}^{k+1} h\left(\frac{p_i}{\mathcal{P}_{k+1}}\right) f(x_i).$$

This completes the proof. □

Remark 3.7 In Theorem 3.6,

- If $\underline{f} = \overline{f}$, then we obtain Theorem 12 of [18].
- If $h(t) = t$, we obtain the Jensen-type inequality for cr-log-convex functions:

$$f\left(\frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i x_i\right) \preceq_{cr} \prod_{i=1}^{n}\big[f(x_i)\big]^{\frac{p_i}{\mathcal{P}_n}}.$$

- If $h(t) = t^s$, $s \in (0,1]$, we obtain the Jensen-type inequality for cr-log-s-convex functions:

$$f\left(\frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i x_i\right) \preceq_{cr} \prod_{i=1}^{n}\big[f(x_i)\big]^{(\frac{p_i}{\mathcal{P}_n})^s}.$$

- If $h(t) = 1$, we obtain the Jensen-type inequality for cr-log-P functions:

$$f\left(\frac{1}{\mathcal{P}_n}\sum_{i=1}^{n} p_i x_i\right) \preceq_{cr} \prod_{i=1}^{n} f(x_i).$$

Next, we prove the Hermite–Hadamard-type inequality for cr-log-h-convex functions.

Theorem 3.8 *Let $f : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$ be an interval-valued function such that $f = [\underline{f}, \overline{f}]$ and $f \in \mathcal{IR}_{([a,b])}$, and let $h : [0,1] \to \mathbb{R}^{+}$ be a nonnegative function with $h(\frac{1}{2}) \neq 0$. If $f \in SX(cr\text{-}\log\text{-}h, [a,b], \mathbb{R}_{\mathcal{I}}^{+})$, then*

$$\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2h(\frac{1}{2})}} \preceq_{cr} \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \preceq_{cr} \big[f(a)f(b)\big]^{\int_0^1 h(t)\,dt}.$$

Proof Since $f \in SX(cr\text{-}\log\text{-}h, [a,b], \mathbb{R}_{\mathcal{I}}^{+})$, we have

$$f\left(\frac{x+y}{2}\right) \preceq_{cr} \big[f(x)f(y)\big]^{h(\frac{1}{2})}.$$

That is,

$$\frac{1}{h(\frac{1}{2})}\ln f\left(\frac{x+y}{2}\right) \preceq_{cr} \ln f(x) + \ln f(y).$$

Let $x = ta + (1-t)b$, $y = (1-t)a + tb$, $t \in [0,1]$, then

$$\frac{1}{h(\frac{1}{2})} \ln f\left(\frac{a+b}{2}\right) \preceq_{cr} \ln f\big(ta + (1-t)b\big) + \ln f\big((1-t)a + tb\big). \tag{3.2}$$

Integrating (3.2) on $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{h(\frac{1}{2})} \ln f\left(\frac{a+b}{2}\right) \\
&\quad\preceq_{cr} \left[\int_0^1 \ln f\big(ta + (1-t)b\big)\,dt + \int_0^1 \ln f\big((1-t)a + tb\big)\,dt\right] \\
&\quad= \left[\int_0^1 \big(\ln \underline{f}\big(ta + (1-t)b\big) + \ln \underline{f}\big((1-t)a + tb\big)\big)\,dt,\right. \\
&\qquad\left.\int_0^1 \big(\ln \overline{f}\big(ta + (1-t)b\big) + \ln \overline{f}\big((1-t)a + tb\big)\big)\,dt\right] \\
&\quad= \left[\int_b^a \ln \underline{f}(x)\frac{dx}{a-b} + \int_a^b \ln \underline{f}(x)\frac{dx}{b-a}, \int_b^a \ln \overline{f}(x)\frac{dx}{a-b} + \int_a^b \ln \overline{f}(x)\frac{dx}{b-a}\right] \\
&\quad= \left[2\int_a^b \ln \underline{f}(x)\frac{dx}{b-a}, 2\int_a^b \ln \overline{f}(x)\frac{dx}{b-a}\right] \\
&\quad= \frac{2}{b-a}\int_a^b \ln f(x)\,dx.
\end{aligned} \tag{3.3}$$

Similarly, since $f \in SX(cr\text{-}\log\text{-}h, [a,b], \mathbb{R}_{\mathcal{I}}^{+})$, one has

$$f\big(ta + (1-t)b\big) \preceq_{cr} \big[f(a)\big]^{h(t)}\big[f(b)\big]^{h(1-t)}.$$

That is,

$$\ln f\big(ta + (1-t)b\big) \preceq_{cr} h(t)\ln f(a) + h(1-t)\ln f(b). \tag{3.4}$$

Integrating (3.4) on $[0,1]$, we have

$$\frac{1}{b-a}\int_a^b \ln f(x)\,dx \preceq_{cr} \big[\ln f(a)f(b)\big]\int_0^1 h(t)\,dt. \tag{3.5}$$

The proof is completed by combining (3.3) with (3.5). □

Remark 3.9 In Theorem 3.8,

- It is clear that if $\underline{f} = \overline{f}$, then we obtain Theorem 4.3 of [1].
- If $h(t) = t$, we obtain the Hermite–Hadamard-type inequality for cr-log-convex functions:

$$f\left(\frac{a+b}{2}\right) \preceq_{cr} \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \preceq_{cr} \sqrt{f(a)f(b)}.$$

- If $h(t) = 1$, we obtain the Hermite–Hadamard-type inequality for cr-log-P-functions:

$$\sqrt{f\left(\frac{a+b}{2}\right)} \preceq_{cr} \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \preceq_{cr} \big[f(a)f(b)\big].$$

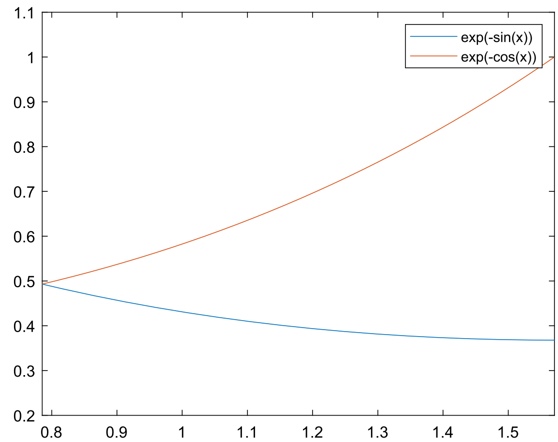

Figure 1 Illustration of Example 3.10: the function $\underline{f}$ is the blue line and the function $\overline{f}$ is the red line

- If $h(t) = t^s$, $s \in (0,1]$, we obtain the Hermite–Hadamard-type inequality for cr-log-s-convex functions:

$$\left[f\left(\frac{a+b}{2}\right)\right]^{2^{s-1}} \preceq_{cr} \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \preceq_{cr} \left[f(a)f(b)\right]^{\frac{1}{s+1}}.$$

Example 3.10 Let $f : [\frac{\pi}{4}, \frac{\pi}{2}] \to \mathbb{R}_{\mathcal{I}}^{+}$ be an interval-valued function given by $f(x) = [e^{-\sin x}, e^{-\cos x}]$, the images can be seen in Fig. 1. Suppose that $h(t) = t$ for all $t \in [0,1]$, then

$$\begin{aligned}
&\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2h(\frac{1}{2})}} = \left[e^{-\sin\frac{3}{8}\pi}, e^{-\cos\frac{3}{8}\pi}\right] = [0.3970, 0.6820],\\
&\exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] = \left[e^{-\frac{2\sqrt{2}}{\pi}}, e^{\frac{2\sqrt{2}-4}{\pi}}\right] = [0.4064, 0.6887],\\
&\left[f(a)f(b)\right]^{\int_0^1 h(t)\,dt} = \left[e^{-\frac{2+\sqrt{2}}{4}}, e^{-\frac{\sqrt{2}}{4}}\right] = [0.4259, 0.7022].
\end{aligned}$$

Since

$$[0.3970, 0.6820] \preceq_{cr} [0.4064, 0.6887] \preceq_{cr} [0.4259, 0.7022].$$

Hence, Theorem 3.8 is verified.

As a further extension, we establish the Fejér-type inequality for cr-log-h-convex functions.

Theorem 3.11 *Let $f : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$ be an interval-valued function such that $f = [\underline{f}, \overline{f}]$ and $f \in \mathcal{IR}_{([a,b])}$, and let $h : [0,1] \to \mathbb{R}^{+}$ be a nonnegative function with $h(\frac{1}{2}) \neq 0$. $\xi : [a,b] \to$*

$\mathbb{R}^+$ *is a function symmetric about* $\frac{a+b}{2}$. *If* $f \in SX(cr\text{-}\log\text{-}h, [a,b], \mathbb{R}^+_{\mathcal{I}})$, *then*

$$\begin{aligned} &\frac{1}{2(b-a)h(\frac{1}{2})} \ln f\left(\frac{a+b}{2}\right) \int_a^b \xi(x)\,dx \\ &\quad \preceq_{cr} \frac{1}{b-a} \int_a^b \bigl[\ln f(x)\bigr]\xi(x)\,dx \\ &\quad \preceq_{cr} \ln\bigl[f(a)f(b)\bigr] \int_0^1 h(t)\xi\bigl((1-t)a+tb\bigr)\,dt. \end{aligned} \tag{3.6}$$

Proof Since $f \in SX(cr\text{-}\log\text{-}h, [a,b], \mathbb{R}^+_{\mathcal{I}})$, we have

$$\ln f\bigl((1-t)a+tb\bigr) \preceq_{cr} h(1-t)\ln f(a) + h(t)\ln f(b), \tag{3.7}$$

$$\ln f\bigl(ta+(1-t)b\bigr) \preceq_{cr} h(t)\ln f(a) + h(1-t)\ln f(b). \tag{3.8}$$

Multiplying (3.7) and (3.8) by $\xi((1-t)a+tb)$ and $\xi(ta+(1-t)b)$, respectively, we have

$$\begin{aligned} &\bigl[\ln f\bigl((1-t)a+tb\bigr)\bigr]\xi\bigl((1-t)a+tb\bigr) \\ &\quad \preceq_{cr} \bigl[h(1-t)\ln f(a) + h(t)\ln f(b)\bigr]\xi\bigl((1-t)a+tb\bigr), \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\bigl[\ln f\bigl(ta+(1-t)b\bigr)\bigr]\xi\bigl(ta+(1-t)b\bigr) \\ &\quad \preceq_{cr} \bigl[h(t)\ln f(a) + h(1-t)\ln f(b)\bigr]\xi\bigl(ta+(1-t)b\bigr). \end{aligned} \tag{3.10}$$

Adding (3.9) to (3.10), and integrating the result on $[0,1]$, one has

$$\begin{aligned} &\int_0^1 \bigl[\ln f\bigl((1-t)a+tb\bigr)\bigr]\xi\bigl((1-t)a+tb\bigr)\,dt + \int_0^1 \bigl[\ln f\bigl(ta+(1-t)b\bigr)\bigr]\xi\bigl(ta+(1-t)b\bigr)\,dt \\ &\quad \preceq_{cr} \int_0^1 \ln f(a)\bigl[h(t)\xi\bigl(ta+(1-t)b\bigr) + h(1-t)\xi\bigl((1-t)a+tb\bigr)\bigr]\,dt \\ &\qquad + \int_0^1 \ln f(b)\bigl[h(1-t)\xi\bigl(ta+(1-t)b\bigr) + h(t)\xi\bigl((1-t)a+tb\bigr)\bigr]\,dt \\ &\quad = 2\ln f(a) \int_0^1 h(t)\xi\bigl((1-t)a+tb\bigr)\,dt + 2\ln f(b) \int_0^1 h(t)\xi\bigl(ta+(1-t)b\bigr)\,dt \\ &\quad = 2\bigl[\ln f(a)f(b)\bigr] \int_0^1 h(t)\xi\bigl((1-t)a+tb\bigr)\,dt. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_0^1 \bigl[\ln f\bigl((1-t)a+tb\bigr)\bigr]\xi\bigl((1-t)a+tb\bigr)\,dt \\ &\quad = \int_0^1 \bigl[\ln f\bigl(ta+(1-t)b\bigr)\bigr]\xi\bigl(ta+(1-t)b\bigr)\,dt \\ &\quad = \biggl[\int_0^1 \bigl[\ln \underline{f}\bigl(ta+(1-t)b\bigr)\bigr]\xi\bigl(ta+(1-t)b\bigr)\,dt, \end{aligned}$$

$$\int_0^1 \big[\ln \overline{f}\big(ta+(1-t)b\big)\big]\xi\big(ta+(1-t)b\big)\,dt\Big]$$
$$= \frac{1}{b-a}\int_a^b \big[\ln f(x)\big]\xi(x)\,dx,$$

which gives the second inequality in (3.6).

On the other hand, since $f \in SX(cr\text{-}\log\text{-}h, [a,b], \mathbb{R}_{\mathcal{I}}^{+})$, $\xi > 0$ and there is symmetry about $\frac{a+b}{2}$, multiplying (3.2) by $\xi(ta+(1-t)b) = \xi((1-t)a+tb)$ and integrating the result on $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{h(\frac{1}{2})}\ln f\Big(\frac{a+b}{2}\Big)\int_0^1 \xi\big((1-t)a+tb\big)\,dt \\
&\quad\preceq_{cr} \int_0^1 \big[\ln f\big(ta+(1-t)b\big)\big]\xi\big(ta+(1-t)b\big)\,dt \\
&\qquad + \int_0^1 \big[\ln f\big((1-t)a+tb\big)\big]\xi\big((1-t)a+tb\big)\,dt.
\end{aligned}$$

Since

$$\int_0^1 \xi\big((1-t)a+tb\big)\,dt = \frac{1}{b-a}\int_a^b \xi(x)\,dx,$$

and

$$\begin{aligned}
&\int_0^1 \big[\ln f\big(ta+(1-t)b\big)\big]\xi\big(ta+(1-t)b\big)\,dt \\
&\quad= \int_0^1 \big[\ln f\big((1-t)a+tb\big)\big]\xi\big((1-t)a+tb\big)\,dt \\
&\quad= \frac{1}{b-a}\int_a^b \big[\ln f(x)\big]\xi(x)\,dx,
\end{aligned}$$

we obtain

$$\frac{1}{2h(\frac{1}{2})}\ln f\Big(\frac{a+b}{2}\Big)\int_a^b \xi(x)\,dx \preceq_{cr} \int_a^b \big[\ln f(x)\big]\xi(x)\,dx.$$

The proof is therefore completed. □

Remark 3.12 Similarly, we can obtain particular results for *cr*-log-convex functions, *cr*-log-P-functions, and *cr*-log-*s*-convex functions by taking special multiplicative functions in Theorem 3.11.

4 Conclusions

This work investigates inequalities related to convexity for interval-valued functions. First, under the full-order relation, we give a definition of the *cr*-log-*h*-convex function and study some of its induced properties. On this basis, we establish Jensen-, Hermite–Hadamard-, and Fejér-type inequalities for *cr*-log-*h*-convex functions, and discuss some special cases of them. Moreover, numerical examples are given to verify the accuracy of the results developed. With the techniques and ideas developed in this work, it is possible to further investigate other types of convex inequalities, with possible applications to problems such as optimization and differential equations with convex shapes associated with them.

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Declarations

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The authors declare no competing interests.

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