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Inclusion relations for some classes of analytic functions involving Pascal distribution series

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Abstract

In this article, we introduce and discuss some new subclasses of functions that are analytic in the open unit disc and involve the Pascal distribution series. Moreover, inclusion relations and integral preserving properties of these subclasses are determined.

Keywords: Pascal operator; Analytic functions; Convex functions; Concave functions; Inclusion relations; Bernardi integral operator

1 Introduction

Let $\mathcal{B}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

that are analytic in the open unit disc $\mathcal{E} := \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{B}$ is said to be in the class $\mathcal{M}(\beta)$ if it satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} < \beta, \quad (\beta > 1, z \in \mathcal{E}). \tag{1.2}$$

A function $f \in \mathcal{B}$ is said to be in the class $\mathcal{N}(\beta)$ if it satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta, \quad (\beta > 1, z \in \mathcal{E}). \tag{1.3}$$

The classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ were introduced and studied by Nishiwaki and Owa [23] and Owa and Nishiwaki [24] (see also [25] and [27]). It follows from (1.2) and (1.3) that, for a function $f \in \mathcal{B}$, we have the equivalence

$$f(z) \in \mathcal{N}(\beta) \quad \Leftrightarrow \quad zf'(z) \in \mathcal{M}(\beta). \tag{1.4}$$

For the functions $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = \sum_{k=2}^{\infty} a_{k,j} z^k,$$

let $f_1 * f_2$ denote the Hadamard product (or convolution) of f_1 and f_2, which is defined by

$$(f_1 * f_2)(z) = f_1(z) * f_2(z) = \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k.$$

It is well known that

$$z(f * g)' = f * zg' = g * zf'. \tag{1.5}$$

A variable x is said to be Pascal distribution if it takes the values $0, 1, 2, 3, \ldots$ with probabilities

$$(1-q)^n, \frac{qn(1-q)^n}{1!}, \frac{q^2 n(n+1)(1-q)^n}{2!}, \frac{q^3 n(n+1)(n+2)(1-q)^n}{3!}, \ldots,$$

respectively, where n and q are called the parameters, and thus

$$P(x = r) = \binom{r+n-1}{n-1} q^r (1-q)^n, \quad r = 0, 1, 2, \ldots.$$

Recently, a power series whose coefficients are probabilities of Pascal distribution was introduced by El-Deeb et al. [6] as follows:

$$\Theta_q^n(z) = (1-q)^n z + \sum_{k=2}^{\infty} \binom{k+n-2}{n-1} q^{k-1} (1-q)^n z^k \quad (z \in \mathcal{E}),$$

where $n \in \mathbb{Z}^+$, $0 \le q \le 1$. Note that, by using the ratio test, we deduce that the radius of convergence of the power series shown above is infinity. For $n \in \mathbb{Z}^+$, $0 \le q \le 1$, we consider the Pascal operator

$$\Omega_q^n : \mathcal{B} \to \mathcal{B},$$

which is defined as follows:

$$\begin{aligned} \Omega_q^n f(z) &= f_{q,n}(z) * f(z), \\ &= z + \sum_{k=2}^{\infty} \binom{k+n-2}{n-1} q^{k-1} a_k z^k \quad (z \in \mathcal{E}), \end{aligned}$$

where

$$f_{q,n}(z) = \frac{\Theta_q^n(z)}{(1-q)^n} = z + \sum_{k=2}^{\infty} \binom{k+n-2}{n-1} q^{k-1} z^k.$$

Now, we define the operator $\mathcal{I}_q^n : \mathcal{B} \to \mathcal{B}$ analogous to the Pascal operator Ω_q^n as follows:

$$\mathcal{I}_q^n f(z) = f_{q,n}^{(-1)}(z) * f(z) \quad (z \in \mathcal{E}) \tag{1.6}$$

and

$$f_{q,n}(z) * f_{q,n}^{(-1)}(z) = \frac{z}{1-z}. \tag{1.7}$$

By using the two operators Ω_q^n and $\mathcal{I}_q^n$, we define and study the properties of the following new classes of analytic functions:

$$\mathcal{M}_{\Omega,n,q}(\beta) = \{f \in \mathcal{B} : \Omega_q^n f(z) \in \mathcal{M}(\beta), \beta > 1, z \in \mathcal{E}\}, \tag{1.8}$$

$$\mathcal{N}_{\Omega,n,q}(\beta) = \{f \in \mathcal{B} : \Omega_q^n f(z) \in \mathcal{N}(\beta), \beta > 1, z \in \mathcal{E}\}, \tag{1.9}$$

$$\mathcal{M}_{\mathcal{I},n,q}(\beta) = \{f \in \mathcal{B} : \mathcal{I}_q^n f(z) \in \mathcal{M}(\beta), \beta > 1, z \in \mathcal{E}\}, \tag{1.10}$$

$$\mathcal{N}_{\mathcal{I},n,q}(\beta) = \{f \in \mathcal{B} : \mathcal{I}_q^n f(z) \in \mathcal{N}(\beta), \beta > 1, z \in \mathcal{E}\}. \tag{1.11}$$

Many fields have used the Pascal distribution, including communications, health, climatology, demographics, and engineering (see [13]). Geometric function theory has recently focused on the geometric properties of analytic functions associated with the Pascal distributions. According to the work of El-Deeb et al. [6], several studies have established a connection between the Pascal distribution series and some classes of normalized analytic functions. Following this, Bulboaca and Murugusundaramoorthy [4], Murugusundaramoorthy and Yalcın [22] and Murugusundaramoorthy [21] established some sufficient conditions for the Pascal distribution series to be in certain subclasses of analytic functions. Subsequently, Amourah et al. [1] constructed a new subclass of analytic bi-univalent functions defined by means of the Pascal distribution series and provided estimates for the first two coefficients of Taylor–Maclaurin series for functions in this class. Numerous recent investigations have investigated the properties of various subclasses of analytic functions defined by the Pascal distribution series (see, for example, [3–5, 7–12, 14, 15, 26, 28]). The purpose of this article is to obtain some inclusion relations for functions in the classes $\mathcal{M}_{\Omega,n,q}(\beta)$, $\mathcal{N}_{\Omega,n,q}(\beta)$, $\mathcal{M}_{\mathcal{I},n,q}(\beta)$, and $\mathcal{N}_{\mathcal{I},n,q}(\beta)$. In addition, it discusses the integral operator associated with these classes of functions.

2 Main results

To prove our main results, we shall need the following lemma.

Lemma 1 ([19, 20]) *Let $u = u_1 + iu_2, v = v_1 + iv_2$ and let $\phi(u,v)$ be a complex valued function satisfying*:

(i) *$\phi(u,v)$ is continuous in a domain $D \subset \mathbb{C} \times \mathbb{C}$*;

(ii) *$(1,0) \in D$ and $\Re\{\phi(1,0)\} > 0$*;

(iii) *$\Re\{\phi(iu_2, v_1)\} \leq 0$ when $v_1 \leq \frac{-(1+u_2^2)}{2}$*.

Let $T(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be an analytic function in $\mathcal{E}$ such that $(T(z), zT'(z)) \in D$ and $\Re\{\phi(T(z), zT'(z))\} > 0$ for all $z \in \mathcal{E}$. Then $\Re\{T(z)\} > 0$.

Proposition 2 $z(\Omega_q^n f(z))' = n\Omega_q^{n+1} f(z) - (n-1)\Omega_q^n f(z)$.

Proof Since

$$\Omega_q^{n+1}f(z) = z + \sum_{k=2}^{\infty}\binom{k+n-1}{n}q^{k-1}a_k z^k,$$

then

$$\begin{aligned}
\Omega_q^{n+1}f(z) &= z + \sum_{k=2}^{\infty}\frac{(k+n-1)(k+n-2)!}{n(n-1)!(k-1)!}q^{k-1}a_k z^k,\\
&= z + \frac{1}{n}\sum_{k=2}^{\infty}k\binom{k+n-2}{n-1}q^{k-1}a_k z^k + \frac{n-1}{n}\sum_{k=2}^{\infty}\binom{k+n-2}{n-1}q^{k-1}a_k z^k,\\
&= \frac{1}{n}\left\{z + \sum_{k=2}^{\infty}k\binom{k+n-2}{n-1}q^{k-1}a_k z^k\right\}\\
&\quad + \frac{n-1}{n}\left\{z + \sum_{k=2}^{\infty}\binom{k+n-2}{n-1}q^{k-1}a_k z^k\right\},\\
&= \frac{1}{n}z\bigl(\Omega_q^n f(z)\bigr)' + \frac{n-1}{n}\Omega_q^n f(z),
\end{aligned}$$

which is equivalent to

$$n\Omega_q^{n+1}f(z) = z\bigl(\Omega_q^n f(z)\bigr)' + (n-1)\Omega_q^n f(z).$$

This ends the proof. □

Proposition 3 $z(\mathcal{I}_q^{n+1}f(z))' = n(\mathcal{I}_q^n f(z)) - (n-1)(\mathcal{I}_q^{n+1}f(z))$.

Proof From Proposition 2, replacing $f(z)$ by $\mathcal{I}_q^n f(z)$, we get

$$z\bigl(\Omega_q^n\bigl(\mathcal{I}_q^n f(z)\bigr)\bigr)' = n\Omega_q^{n+1}\bigl(\mathcal{I}_q^n f(z)\bigr) - (n-1)\Omega_q^n\bigl(\mathcal{I}_q^n f(z)\bigr),$$

using (1.5), we have

$$z\bigl(\mathcal{I}_q^{n+1}f(z)\bigr)' = n\bigl(\mathcal{I}_q^n f(z)\bigr) - (n-1)\bigl(\mathcal{I}_q^{n+1}f(z)\bigr).$$ □

In Theorem 4, we obtain the containment relation $\mathcal{M}_{\Omega,n+1,q}(\beta) \subset \mathcal{M}_{\Omega,n,q}(\beta)$.

Theorem 4 *Let f be an analytic function defined by* (1.1). *If $f \in \mathcal{M}_{\Omega,n+1,q}(\beta)$, then for all $\beta > 1, n \in \mathbb{Z}^+$, and $z \in \mathcal{E}$, we have $f \in \mathcal{M}_{\Omega,n,q}(\beta)$.*

Proof Let $f(z) \in \mathcal{M}_{\Omega,n+1,q}(\beta)$. We have to show that

$$\Re\left\{\frac{z(\Omega_q^n f(z))'}{\Omega_q^n f(z)}\right\} < \beta, \quad \beta > 1,$$

or, equivalently,

$$\Re\left\{\frac{\beta - \frac{z(\Omega_q^n f(z))'}{\Omega_q^n f(z)}}{\beta - 1}\right\} > 0. \tag{2.1}$$

Let

$$\frac{z(\Omega_q^n f(z))'}{\Omega_q^n f(z)} = \beta - (\beta - 1)T(z), \tag{2.2}$$

where $T(z) = 1 + b_1 z + b_2 z^2 + \cdots$. Using Proposition 2 and (2.2), we have

$$\frac{n\Omega_q^{n+1} f(z) - (n-1)\Omega_q^n f(z)}{\Omega_q^n f(z)} = \beta - (\beta - 1)T(z),$$

or, equivalently,

$$\frac{\Omega_q^{n+1} f(z)}{\Omega_q^n f(z)} = \frac{1}{n}\big\{\beta - 1 + n - (\beta - 1)T(z)\big\}, \tag{2.3}$$

differentiating logarithmically both sides, we get

$$\frac{\beta - \frac{z(\Omega_q^{n+1} f(z))'}{\Omega_q^{n+1} f(z)}}{\beta - 1} = T(z) + \frac{zT'(z)}{\beta - 1 + n - (\beta - 1)T(z)}. \tag{2.4}$$

Now, we form the function $\phi(u, v)$ by taking $u = T(z)$ and $v = zT'(z)$, therefore

$$\phi(u, v) = u + \frac{v}{\beta - 1 + n - (\beta - 1)u}. \tag{2.5}$$

We note that the function $\phi(u, v)$ fulfills conditions (i) and (ii) of Lemma 1, where $D = (\mathcal{C} - \{\frac{\beta+n-1}{\beta-1}\}) \times \mathcal{C}$. To prove condition (iii), we have

$$\begin{aligned}
\Re\big\{\phi(iu_2, v_1)\big\} &= \Re\left\{\frac{v_1}{\beta + n - 1 - (\beta - 1)iu_2}\right\}, \\
&= \frac{v_1(\beta + n - 1)}{\{(\beta - 1 + n)^2 + (\beta - 1)^2 u_2^2\}} \\
&\le \frac{-(1 + u_2^2)(\beta + n - 1)}{2\{(\beta - 1 + n)^2 + (\beta - 1)^2 u_2^2\}} \\
&\le 0,
\end{aligned}$$

where $v_1 \le \frac{-(1+u_2^2)}{2}$ and $(iu_2, v_1) \in D$. Therefore, the function $\phi(u, v)$ fulfills all conditions of Lemma 1, which shows that if $\Re\{\phi(T(z), zT'(z))\} > 0$, then $\Re\{T(z)\} > 0$. Hence (2.1) holds, which means $f(z) \in \mathcal{M}_{\Omega,n,q}(\beta)$. This ends the proof. □

In the following theorem, an inclusion relation between the classes $\mathcal{M}_{\mathcal{I},n,q}(\beta)$ and $\mathcal{M}_{\mathcal{I},n+1,q}(\beta)$ is obtained.

Theorem 5 *Let f be an analytic function defined by* (1.1). *If $f \in \mathcal{M}_{\mathcal{I},n,q}(\beta)$, then for all $\beta > 1, n \in \mathbb{Z}^+$, and $z \in \mathcal{E}$, we have $f \in \mathcal{M}_{\mathcal{I},n+1,q}(\beta)$.*

Proof Let $f(z) \in \mathcal{M}_{\mathcal{I},n,q}(\beta)$. We have to show that

$$\Re\left\{\frac{z(\mathcal{I}_q^{n+1}f(z))'}{\mathcal{I}_q^{n+1}f(z)}\right\} < \beta, \quad \beta > 1,$$

or, equivalently,

$$\Re\left\{\frac{\beta - \frac{z(\mathcal{I}_q^{n+1}f(z))'}{\mathcal{I}_q^{n+1}f(z)}}{\beta - 1}\right\} > 0. \tag{2.6}$$

Let

$$\frac{z(\mathcal{I}_q^{n+1}f(z))'}{\mathcal{I}_q^{n+1}f(z)} = \beta - (\beta - 1)T(z), \tag{2.7}$$

where $T(z) = 1 + b_1 z + b_2 z^2 + \cdots$. Using Proposition 3 and (2.7), we have

$$\frac{n(\mathcal{I}_q^{n}f(z)) - (n-1)(\mathcal{I}_q^{n+1}f(z))}{\mathcal{I}_q^{n+1}f(z)} = \beta - (\beta - 1)T(z),$$

or, equivalently,

$$\frac{\mathcal{I}_q^{n}f(z)}{\mathcal{I}_q^{n+1}f(z)} = \frac{1}{n}\left\{\beta + n - 1 - (\beta - 1)T(z)\right\}, \tag{2.8}$$

using logarithmic differentiation, we have

$$\frac{\beta - \frac{z(\mathcal{I}_q^{n}f(z))'}{\mathcal{I}_q^{n}f(z)}}{\beta - 1} = T(z) + \frac{zT'(z)}{n - 1 + \beta - (\beta - 1)T(z)}. \tag{2.9}$$

Now, we form the function $\phi(u, v)$ by taking $u = T(z)$ and $v = zT'(z)$, therefore

$$\phi(u, v) = u + \frac{v}{n - 1 + \beta - (\beta - 1)u}.$$

We note that the function $\phi(u, v)$ fulfills conditions (i) and (ii) of Lemma 1, where $D = (\mathbb{C} - \{\frac{n-1+\beta}{\beta-1}\}) \times \mathbb{C}$. To prove condition (iii), we have

$$\begin{aligned}
\Re\left\{\phi(iu_2, v_1)\right\} &= \Re\left\{iu_2 + \frac{v_1}{n - 1 + \beta - (\beta - 1)iu_2}\right\} \\
&= \frac{\{n - 1 + \beta\}v_1}{(n - 1 + \beta)^2 + (\beta - 1)^2 u_2^2} \\
&\leq \frac{-(1 + u_2^2)(n - 1 + \beta)}{2\{(n - 1 + \beta)^2 + (\beta - 1)^2 u_2^2\}} \\
&\leq 0,
\end{aligned}$$

where $v_1 \leq \frac{-(1+u_2^2)}{2}$ and $(iu_2, v_1) \in D$. Therefore, the function $\phi(u, v)$ fulfills all conditions of Lemma 1, which shows that if $\Re\{\phi(T(z), zT'(z))\} > 0$, then $\Re\{T(z)\} > 0$, which means that $f \in \mathcal{M}_{\mathcal{I},n+1,q}(\beta)$. This ends the proof. □

Theorem 6 gives an inclusion property between the classes $\mathcal{N}_{\Omega,n+1,q}(\beta)$ and $\mathcal{N}_{\Omega,n,q}(\beta)$.

Theorem 6 *Let f be an analytic function given by* (1.1). *If $f \in \mathcal{N}_{\Omega,n+1,q}(\beta)$, where $\beta > 1$, $n \in \mathbb{Z}^+$, and $z \in \mathcal{E}$, then we have $f \in \mathcal{N}_{\Omega,n,q}$.*

Proof Let $f(z) \in \mathcal{N}_{\Omega,n+1,q}(\beta)$. From (1.9), we get

$$\Omega_q^{n+1} f(z) \in \mathcal{N}(\beta).$$

By using (1.4), we get

$$z\big(\Omega_q^{n+1} f(z)\big)' \in \mathcal{M}(\beta).$$

From (1.5), we have

$$\Omega_q^{n+1}\big(zf'(z)\big) \in \mathcal{M}(\beta),$$

which is equivalent to

$$zf'(z) \in \mathcal{M}_{\Omega,n+1,q}(\beta).$$

By using Theorem 4, we have

$$zf'(z) \in \mathcal{M}_{\Omega,n,q}(\beta),$$

which is equivalent to

$$\Omega_q^{n}\big(zf'(z)\big) \in \mathcal{M}(\beta).$$

From (1.5) and (1.4), we get

$$z\big(\Omega_q^{n} f(z)\big)' \in \mathcal{M}(\beta) \quad \Leftrightarrow \quad \Omega_q^{n} f(z) \in \mathcal{N}(\beta),$$

which means $f(z) \in \mathcal{N}_{\Omega,n,q}(\beta)$. This ends the proof. □

In Theorem 7, the containment relation $\mathcal{N}_{\mathcal{I},n,q}(\beta) \subset \mathcal{N}_{\mathcal{I},n+1,q}(\beta)$ is obtained.

Theorem 7 *Let f be an analytic function given by* (1.1). *If $f \in \mathcal{N}_{\mathcal{I},n,q}(\beta)$, where $\beta > 1$, $n \in \mathbb{Z}^+$, and $z \in \mathcal{E}$, then we have $f \in \mathcal{N}_{\mathcal{I},n+1,q}(\beta)$.*

Proof Let $f(z) \in \mathcal{N}_{\mathcal{I},n,q}(\beta)$. From (1.11), we get

$$\mathcal{I}_q^{n} f(z) \in \mathcal{N}(\beta).$$

By using (1.4), we have

$$z\big(\mathcal{I}_q^{n} f(z)\big)' \in \mathcal{M}(\beta).$$

From (1.5), we have

$$\mathcal{I}_q^n\big(zf'(z)\big) \in \mathcal{M}(\beta),$$

which is equivalent to

$$zf'(z) \in \mathcal{M}_{\mathcal{I},n,q}(\beta).$$

By using Theorem 5, we get

$$zf'(z) \in \mathcal{M}_{\mathcal{I},n+1,q}(\beta),$$

which means that

$$\mathcal{I}_q^{n+1}\big(zf'(z)\big) \in \mathcal{M}(\beta).$$

From (1.4) and (1.5), we have

$$z\big(\mathcal{I}_q^{n+1}f(z)\big)' \in \mathcal{M}(\beta) \quad \Leftrightarrow \quad \mathcal{I}_q^{n+1}f(z) \in \mathcal{N}(\beta),$$

which means that $f(z) \in \mathcal{N}_{\mathcal{I},n+1,q}(\beta)$. This ends the proof. □

3 Integral operator

For the function $f \in \mathcal{B}$, Bernardi [2] in 1969 introduced the following operator:

$$L_\gamma\big(f(z)\big) = \frac{\gamma+1}{z}\int_0^z t^{\gamma-1}f(t)\,dt, \quad \gamma > -1. \tag{3.1}$$

The operator $L_1(\gamma = 1)$ was studied earlier by Libera [16, 17] and Liviningston [18]. From (3.1), it is not difficult to demonstrate the following relations:

$$z\big(\Omega_q^n L_\gamma\big(f(z)\big)\big)' = (\gamma+1)\Omega_q^n f(z) - \gamma\,\Omega_q^n L_\gamma\big(f(z)\big) \tag{3.2}$$

and

$$z\big(\mathcal{I}_q^n L_\gamma\big(f(z)\big)\big)' = (\gamma+1)\mathcal{I}_q^n f(z) - \gamma\,\mathcal{I}_q^n L_\gamma\big(f(z)\big). \tag{3.3}$$

The following theorem proves that the integral operator L_γ preserves class $\mathcal{M}_{\Omega,n,q}(\beta)$ properties.

Theorem 8 *Let f be an analytic function given by (1.1). If $f \in \mathcal{M}_{\Omega,n,q}(\beta)$ for $\beta > 1, \gamma > -1, n \in \mathbb{Z}^+$, and $z \in \mathcal{E}$, then we have $L_\gamma(f(z)) \in \mathcal{M}_{\Omega,n,q}(\beta)$.*

Proof Let $f(z) \in \mathcal{M}_{\Omega,n,q}(\beta)$. We have to show that

$$\Re\left\{\frac{z(\Omega_q^n L_\gamma(f(z)))'}{\Omega_q^n L_\gamma(f(z))}\right\} < \beta, \quad \beta > 1,$$

or, equivalently,

$$\Re\left\{\frac{\beta-\frac{z(\Omega_q^n L_\gamma(f(z)))'}{\Omega_q^n L_\gamma(f(z))}}{\beta-1}\right\}>0,\quad \beta>1. \tag{3.4}$$

Let

$$\frac{z(\Omega_q^n L_\gamma(f(z)))'}{\Omega_q^n L_\gamma(f(z))}=\beta-(\beta-1)T(z), \tag{3.5}$$

where $T(z)=1+b_1z+b_2z^2+\cdots$. Using (3.2) and (3.5), we get

$$\frac{\Omega_q^n f(z)}{\Omega_q^n L_\gamma(f(z))}=\frac{1}{\gamma+1}\big\{\gamma+\beta-(\beta-1)T(z)\big\}, \tag{3.6}$$

differentiating logarithmically both sides, we have

$$\frac{\beta-\frac{z(\Omega_q^n f(z))'}{\Omega_q^n f(z)}}{\beta-1}=T(z)+\frac{zT'(z)}{\gamma+\beta-(\beta-1)T(z)}. \tag{3.7}$$

Now, we form the function $\phi(u,v)$ by taking $u=T(z), v=zT'(z)$, where $u=u_1+iu_2, v=v_1+iv_2$, and $u_1,u_2,v_1,v_2\in\mathcal{R}$. Therefore

$$\phi(u,v)=u+\frac{v}{\gamma+\beta-(\beta-1)u}. \tag{3.8}$$

We note that the function $\phi(u,v)$ satisfies conditions (i) and (ii) of Lemma 1, where $D=(\mathcal{C}-\{\frac{\gamma+\beta}{\beta-1}\})\times\mathcal{C}$. To prove condition (iii), we have

$$\begin{aligned}\Re\big\{\phi(iu_2,v_1)\big\}&=\Re\left\{iu_2+\frac{v_1}{\gamma+\beta-(\beta-1)iu_2}\right\}\\&=\frac{(\gamma+\beta)v_1}{(\gamma+\beta)^2+(\beta-1)^2u_2^2},\\&\le\frac{-(\gamma+\beta)(1+u_2^2)}{2\{(\gamma+\beta)^2+(\beta-1)^2u_2^2\}}\\&\le 0,\end{aligned}$$

where $v_1\le\frac{-(1+u_2^2)}{2}$ and $(iu_2,v_1)\in D$. Therefore, the function $\phi(u,v)$ satisfies all conditions of Lemma 1, which shows that if $\Re\{\phi(T(z),zT'(z))\}>0$, then $\Re\{T(z)\}>0$. Hence (3.4) holds, which means $L_\gamma(f(z))\in\mathcal{M}_{\Omega,n,q}(\beta)$. This finishes the proof. □

Theorem 9 proves that integral operator L_γ preserves class $\mathcal{M}_{\mathcal{I},n,q}(\beta)$ properties.

Theorem 9 *For $\beta>1$ and $\gamma>-1$, if $f(z)\in\mathcal{M}_{\mathcal{I},n,q}(\beta)$, then $L_\gamma(f(z))\in\mathcal{M}_{\mathcal{I},n,q}(\beta)$.*

Proof Let $f(z)\in\mathcal{M}_{\mathcal{I},n,q}(\beta)$. We have to show that

$$\Re\left\{\frac{z(\mathcal{I}_q^n L_\gamma(f(z)))'}{\mathcal{I}_q^n L_\gamma(f(z))}\right\}<\beta,\quad \beta>1,$$

or, equivalently,

$$\Re\left\{\frac{\beta-\frac{z(\mathcal{I}_q^n L_\gamma(f(z)))'}{\mathcal{I}_q^n L_\gamma(f(z))}}{\beta-1}\right\}>0, \quad \beta>1. \tag{3.9}$$

Let

$$\frac{z(\mathcal{I}_q^n L_\gamma(f(z)))'}{\mathcal{I}_q^n L_\gamma(f(z))}=\beta-(\beta-1)T(z), \tag{3.10}$$

where $T(z)=1+b_1z+b_2z^2+\cdots$. Using (3.3) and (3.10), we have

$$\frac{\mathcal{I}_q^n f(z)}{\mathcal{I}_q^n L_\gamma(f(z))}=\frac{1}{\gamma+1}\big\{(\gamma+\beta)-(\beta-1)T(z)\big\}, \tag{3.11}$$

taking logarithmic differentiation, we have

$$\frac{\beta-\frac{z(\mathcal{I}_q^n f(z))'}{\mathcal{I}_q^n f(z)}}{\beta-1}=T(z)+\frac{zT'(z)}{\gamma+\beta-(\beta-1)T(z)}. \tag{3.12}$$

Now, we form the function $\phi(u,v)$ by taking $u=T(z)$ and $v=zT'(z)$, therefore

$$\phi(u,v)=u+\frac{v}{\gamma+\beta-(\beta-1)u}.$$

We note that the function $\phi(u,v)$ fulfills conditions (i) and (ii) of Lemma 1, where $D=(\mathbb{C}-\{\frac{\gamma+\beta}{\beta-1}\})\times\mathbb{C}$. To prove condition (iii), we have

$$\begin{aligned}\Re\big\{\phi(iu_2,v_1)\big\}&=\Re\left\{iu_2+\frac{v_1}{\gamma+\beta-(\beta-1)iu_2}\right\}\\&=\frac{(\gamma+\beta)v_1}{(\gamma+\beta)^2+(\beta-1)^2u_2^2}\\&\leq\frac{-(\gamma+\beta)(1+u_2^2)}{2\{(\gamma+\beta)^2+(\beta-1)^2u_2^2\}}\\&\leq 0,\end{aligned}$$

where $v_1\leq\frac{-(1+u_2^2)}{2}$ and $(iu_2,v_1)\in D$. Therefore, the function $\phi(u,v)$ fulfills all conditions of Lemma 1, which shows that if $\Re\{\phi(T(z),zT'(z))\}>0$, then $\Re\{T(z)\}>0$, which means that $L_\gamma(f(z))\in\mathcal{N}_{\Omega,n,q}(\beta)$. This ends the proof. □

The preserving property of the class $\mathcal{N}_{\Omega,n,q}(\beta)$ under the integral operator L_γ is proved in the following theorem.

Theorem 10 *For $\beta>1$ and $\gamma>-1$, if $f(z)\in\mathcal{N}_{\Omega,n,q}(\beta)$, then $L_\gamma(f(z))\in\mathcal{N}_{\Omega,n,q}(\beta)$.*

Proof Let $f(z)\in\mathcal{N}_{\Omega,n,q}(\beta)$. From (1.9), we get

$$\Omega_q^n f(z)\in\mathcal{N}(\beta).$$

By using (1.4), we have

$$z\big(\Omega_q^n f(z)\big)' \in \mathcal{M}(\beta),$$

applying (1.5), we have

$$\Omega_q^n\big(zf'(z)\big) \in \mathcal{M}(\beta),$$

which means that

$$zf'(z) \in \mathcal{M}_{\Omega,n,q}(\beta).$$

By using Theorem 8, we have

$$L_\gamma\big(zf'(z)\big) \in \mathcal{M}_{\Omega,n,q}(\beta),$$

by using (1.5), we have

$$z\big(L_\gamma\big(f(z)\big)\big)' \in \mathcal{M}_{\Omega,n,q}(\beta).$$

Applying again (1.4), we get

$$L_\gamma\big(f(z)\big) \in \mathcal{N}_{\Omega,m,q}(\beta),$$

which ends the proof. □

Theorem 11 proves that integral operator L_γ preserves class $\mathcal{N}_{\mathcal{I},n,q}(\beta)$ properties.

Theorem 11 *For $\beta > 1$ and $\gamma > -1$, if $f(z) \in \mathcal{N}_{\mathcal{I},n,q}(\beta)$, then $L_\gamma(f(z)) \in \mathcal{N}_{\mathcal{I},n,q}(\beta)$.*

Proof Let $f(z) \in \mathcal{N}_{\mathcal{I},n,q}(\beta)$. From (1.11), we get

$$\mathcal{I}_q^n f(z) \in \mathcal{N}(\beta).$$

By using (1.4), we have

$$z\big(\mathcal{I}_q^n\big(f(z)\big)\big)' \in \mathcal{M}(\beta),$$

(1.5) gives

$$\mathcal{I}_q^n(zf'(z) \in \mathcal{M}(\beta),$$

which means that

$$zf'(z) \in \mathcal{M}_{\mathcal{I},n,q}(\beta).$$

By using Theorem 9, we get

$$L_\gamma\big(zf'(z)\big) \in \mathcal{M}_{\mathcal{I},n,q}(\beta),$$

(1.5) gives

$$z\big(L_\gamma\big(f(z)\big)\big)' \in \mathcal{M}_{\mathcal{I},n,q}(\beta).$$

Using again (1.4), we have

$$L_\gamma\big(f(z)\big) \in \mathcal{N}_{\mathcal{I},n,q}(\beta),$$

which ends the proof. □

4 Conclusion

There are several known results on connections between various subclasses of analytic and univalent functions using the Pascal distribution series (see, for example, [4, 6, 22]). In the present work, we have constructed some new subclasses of analytic functions in the open unit disc using the Pascal distribution series. In addition, inclusion relations and integral preserving properties of these subclasses are studied. However, one can extend this work to new subclasses of analytic functions.

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Declarations

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The authors declare no competing interests.

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Authors' contributions The primary analysis and initial draft were prepared by AL, the Project Administrator. Together, AOB and BMA conceptualized, analyzed all the results, made necessary improvements, and wrote the original draft. All authors read and approved the final manuscript.

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