Luangboon et al. *Journal of Inequalities and Applications* (2022) 2022:157
https://doi.org/10.1186/s13660-022-02896-9

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On generalizations of some integral inequalities for preinvex functions via (p,q)-calculus

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Abstract

In this paper, we establish some new (p,q)-integral inequalities of Simpson's second type for preinvex functions. Many results given in this paper provide generalizations and extensions of the results given in previous research. Moreover, some examples are given to illustrate the investigated results.

MSC: 05A30; 26A51; 26D10; 26D15

Keywords: Simpson's second type; Preinvex function; (p,q)-calculus; (p,q)-differentiable function; (p,q)-integrable function

1 Introduction

Mathematical inequalities have been applied in the fields of both pure and applied mathematics [1–6]. Such inequalities have been continuously improved because they can be widely applied in those areas. One of the interesting functions employed to study the inequalities is a convex function defined as follows: A function $f:[a,b]\to\mathbb{R}$ is convex if the inequality

$$f\big(tx+(1-t)y\big)\leq tf(x)+(1-t)f(y)$$

holds for all $x,y\in[a,b]$ and $t\in[0,1]$.

A preinvex function is a generalization of the classical convex function that is defined as follows: A function f on the invex set $\mathcal{K}\subset\mathbb{R}$ is preinvex with respect to $\xi:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ if the inequality

$$f\big(y+t\xi(x,y)\big)\leq(1-t)f(y)+tf(x)$$

holds for all $x,y\in\mathcal{K}$ and $t\in[0,1]$. For $\xi(x,y)=x-y$, the preinvex functions reduce to the convex functions.

Simpson type inequalities are the most well-known inequalities associated with convex and preinvex functions. Simpson's rules are techniques for the numerical integration and the numerical estimation of definite integrals, revealed by T. Simpson (1710–1761). Two famous Simpson's rules are as follows:

1) Simpson's quadrature formula (Simpson's 1/3 rule) is formulated as follows:

$$\int_a^b f(x)\,dx \approx \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],$$

see [7] for more details.

2) Simpson's second formula (Simpson's 3/8 rule) is formulated as follows:

$$\int_a^b f(x)\,dx \approx \frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right],$$

see [8] for more details.

The error estimation for Simpson's quadrature formula known as Simpson's inequality is stated as follows.

Theorem 1.1 ([7]) *If $f : [a,b] \to \mathbb{R}$ is a four times continuously differentiable function on (a,b) and*

$$\|f^{(4)}\|_\infty = \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty,$$

then

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^5.$$

The error estimation for Simpson's second formula is stated as follows.

Theorem 1.2 ([8]) *If $f : [a,b] \to \mathbb{R}$ is a four times continuously differentiable function on (a,b) and*

$$\|f^{(4)}\|_\infty = \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty,$$

then

$$\left|\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{6480}\|f^{(4)}\|_\infty (b-a)^5.$$

Quantum calculus (briefly called q-calculus) is the study of calculus without limits. At the beginning of the q-calculus study, L. Euler (1707–1783) introduced Newton's infinite series. Then, F. H. Jackson [9, 10] relied on the concept of L. Euler to define the q-derivative and q-integral (also called q-Jackson derivative and q-Jackson integral) of a continuous function on the interval $(0,\infty)$ in 1910. The main objective of q-calculus is to obtain the q-analogues of mathematical objects recaptured by taking $q \to 1$. The topic of q-calculus has become an interesting topic for many researchers because it has applications in various areas of mathematics and physics, see [11–17] for more details and the references cited therein.

In 2013, J. Tariboon and S. K. Ntouyas [18] defined the new q-derivative and q-integral of a continuous function on a finite interval. Furthermore, they investigated the existence and uniqueness results of initial value problems for first and second order impulsive q-difference equations. In recent years, the q-calculus has been studied in various inequalities such as Hermite–Hadamard, Hermite–Hadamard-like, Ostrowski, Fejér, Hanh, and Simpson inequalities, see [19–27] and the references cited therein for more details. Especially, Simpson type inequalities have been also studied by using q-calculus for convex and preinvex functions by many researchers, see [28–39] and the references cited therein for more details.

Post quantum calculus (briefly called (p,q)-calculus) is the generalization of q-calculus. The (p,q)-calculus was firstly introduced by R. Chakrabarti and R. A. Jagannathan [40] in 1991. Then, Tunç et al. [41, 42] presented new (p,q)-calculus of a continuous function on a finite interval in 2016. The (p,q)-calculus includes two-parameter quantum calculus (p and q-numbers) which is independent. It is generally known that q-calculus cannot be got by taking q by q/p in q-calculus, but it can be obtained by taking $p = 1$ in (p,q)-calculus. Moreover, the classical formula can be gained by taking $q \to 1$. In the past few years, the topic of (p,q)-calculus has become an interesting topic for many researchers, and the results of (p,q)-calculus can be found in [43–49] and the references cited therein.

In 2020, S. Erden et al. [50] presented integral inequalities of Simpson's second type inequalities for convex functions via q-calculus. They obtained more general results on Simpson's second type quantum integral inequalities. By taking $q \to 1$, they obtained classical results on Simpson's 3/8 formula.

In 2020, Y. M. Chu et al. [51] presented some integral inequalities for preinvex functions via (p,q)-calculus. They obtained more general results on (p,q)-integral inequalities.

Motivated by the above mentioned reports, we establish some new integral inequalities related to Simpson's second type inequalities for preinvex functions via (p,q)-calculus. Many results given in this paper provide generalizations and extensions of other results given in previous papers. Moreover, we give some examples to show the investigated results.

The rest of the paper is organized as follows. In Sect. 2, we give some basic knowledge and notation. In Sect. 3, we give Simpson's second type inequalities via (p,q)-calculus for preinvex function. In Sect. 4, we display some special cases and some examples of our main results. In the final section, we summarize our results.

2 Preliminaries

In this section, we give basic knowledge used in our work. Throughout this paper, let $[a,b] \subseteq \mathbb{R}$ be an interval with $a < b$ and $0 < q < p \leq 1$ be constants. The definitions of (p,q)-derivative and (p,q)-integral are given in [41, 42]. The (p,q)-number is given by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad p \neq q.$$

If $p = 1$, then $[n]_{p,q}$ is reduced to $[n]_q$, which is a quantum number.

Definition 2.1 ([41, 42]) *If $f : [a,b] \to \mathbb{R}$ is a continuous function, then the (p,q)-derivative of function f at $x \in [a,b]$ is defined by*

$$\begin{aligned} {}_aD_{p,q}f(x) &= \frac{f(px+(1-p)a) - f(qx+(1-q)a)}{(p-q)(x-a)}, \quad x \neq a, \\ {}_aD_{p,q}f(a) &= \lim_{x\to a} {}_aD_{p,q}f(x). \end{aligned} \tag{2.1}$$

The function f is said to be the (p,q)-differentiable function on $[a,b]$ if ${}_aD_{p,q}f(x)$ exists for all $x \in [a,b]$.

In Definition 2.1, if $p = 1$, then ${}_aD_{1,q}f(x) = {}_aD_qf(x)$, and (2.1) reduces to

$$\begin{aligned} {}_aD_qf(x) &= \frac{f(x) - f(qx+(1-q)a)}{(1-q)(x-a)}, \quad x \neq a, \\ {}_aD_qf(a) &= \lim_{x\to a} {}_aD_qf(x), \end{aligned} \tag{2.2}$$

which is the q-derivative of function f defined on $[a,b]$, see [52–54] for more details. In addition, if $a = 0$, then ${}_0D_qf(x) = D_qf(x)$, and (2.2) reduces to

$$\begin{aligned} D_qf(x) &= \frac{f(x) - f(qx)}{(1-q)(x)}, \quad x \neq 0, \\ D_qf(a) &= \lim_{x\to 0} D_qf(x), \end{aligned} \tag{2.3}$$

which is the q_a-derivative of function f defined on $[0,b]$, see [55] for more details.

Example 2.1 *Define function $f : [a,b] \to \mathbb{R}$ by $f(x) = x^2 + C$, where C is a constant. Applying Definition 2.1 for $x \neq a$, we have*

$$\begin{aligned} {}_aD_{p,q}\bigl(x^2 + C\bigr) &= \frac{[(px+(1-p)a)^2 + C] - [(qx+(1-q)a)^2 + C]}{(p-q)(x-a)} \\ &= \frac{(p+q)x^2 + 2ax[1-(p+q)] + a^2[(p+q)-2]}{(x-a)} \\ &= \frac{(p+q)(x-a)^2 + 2a(x-a)}{(x-a)} \\ &= [2]_{p,q}(x-a) + 2a. \end{aligned} \tag{2.4}$$

If $p = 1$, then (2.4) is reduced to $D_qf(x) = (1+q)(x-a) + 2a$. Furthermore, if $p = 1$, $a = x$, and $q \to 1$, then (2.4) reduces to the classical derivative.

Definition 2.2 ([41, 42]) *If $f : [a,b] \to \mathbb{R}$ is a continuous function, then the (p,q)-integral of function f at $x \in [a,b]$ is defined by*

$$\int_a^b f(x)_a \, d_{p,q}x = (p-q)(b-a) \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}} b + \left(1 - \frac{q^j}{p^{j+1}}\right) a\right). \tag{2.5}$$

The function f is said to be the (p,q)-integrable function on $[a,b]$ if $\int_a^b f(x)_a \, d_{p,q}x$ exists for all $x \in [a,b]$.

If $a = 0$, then (2.5) is the (p,q)-integral on $[0,b]$, which can be expressed as follows:

$$\int_0^b f(x)\,d_{p,q}x = (p-q)b\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}f\left(\frac{q^j}{p^{j+1}}b\right). \tag{2.6}$$

In addition, if $p = 1$, then (2.6) reduces to

$$\int_0^b f(x)\,d_q x = (1-q)b\sum_{j=0}^{\infty}q^j f\left(q^j b\right), \tag{2.7}$$

which is the q-Jackson integral of function f defined on $[0,b]$, see [55] for more details.

Example 2.2 *Define function* $f : [a,b] \to \mathbb{R}$ *by* $f(x) = Ax^2 + Bx + C$, *where* A, B, *and* C *are constants. Applying Definition 2.2, we have*

$$\begin{aligned}
\int_a^b f(t)_a\,d_{p,q}t &= \int_a^b \left(Ax^2 + Bx + C\right)_a\,d_{p,q}t \\
&= A(p-q)(b-a)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}\left(\frac{q^j}{p^{j+1}}b + \left(1-\frac{q^j}{p^{j+1}}\right)a\right)^2 \\
&\quad + B(p-q)(b-a)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}\left(\frac{q^j}{p^{j+1}}b + \left(1-\frac{q^j}{p^{j+1}}\right)a\right) \\
&\quad + C(p-q)(b-a)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}} \\
&= \frac{A(b-a)([2]_{p,q}(b-a)^2 + 2a[3]_{p,q}(b-a) + [2]_{p,q}[3]_{p,q}a^2)}{[2]_{p,q}[3]_{p,q}} \\
&\quad + \frac{B(b-a)(b-a(1-p-q))}{[2]_{p,q}} + C(b-a).
\end{aligned} \tag{2.8}$$

If $p = 1$, *then, in* (2.8), $[2]_{p,q}$ *and* $[3]_{p,q}$ *are reduced by* $[2]_q$ *and* $[3]_q$, *respectively. Furthermore, if* $p = 1$ *and* $q \to 1$, *then* (2.8) *reduces to the classical integration.*

Theorem 2.1 ([41]) *If* $f, g : [a,b]$ *are continuous functions,* $c \in [a,b]$ *and* $e \in \mathbb{R}$, *then the following identities hold*:

(i) $\int_a^b (f(t) + g(t))_a\,d_{p,q}t = \int_a^b f(t)_a\,d_{p,q}t + \int_a^b g(t)_a\,d_{p,q}t$;
(ii) $\int_a^b ef(t)_a\,d_{p,q}t = e\int_a^b f(t)_a\,d_{p,q}t$;
(iii) $\int_c^b f(t)_a\,d_{p,q}t = \int_a^b f(t)_a\,d_{p,q}t - \int_a^c f(t)_a\,d_{p,q}t$.

Lemma 2.1 ([41]) *For* $\alpha \in \mathbb{R}\backslash\{-1\}$, *the following expression holds*:

$$\int_a^b (t-a)^{\alpha}{}_a\,d_{p,q}t = \frac{1}{[\alpha+1]_{p,q}}(b-a)^{\alpha+1}. \tag{2.9}$$

Theorem 2.2 ([42]) *If* $f, g : [a,b] \to \mathbb{R}$ *are continuous functions and* $r > 0$ *with* $1/s + 1/r = 1$, *then*

$$\int_a^b \left|f(t)g(t)\right|_a\,d_{p,q}t \le \left(\int_a^b \left|f(t)\right|^r{}_a\,d_{p,q}t\right)^{1/r}\left(\int_a^b \left|g(t)\right|^s{}_a\,d_{p,q}t\right)^{1/s}. \tag{2.10}$$

Theorem 2.3 ([56]) *If $f:[a,b]\to\mathbb{R}$ is a convex differentiable function on $[a,b]$, then the (p,q)-Hermite–Hadamard inequalities are as follows*:

$$f\left(\frac{qa+pb}{[2]_{p,q}}\right)\le\frac{1}{p(b-a)}\int_a^{pb+(1-p)a}f(t)_a\,d_{p,q}t\le\frac{qf(a)+pf(b)}{[2]_{p,q}}. \tag{2.11}$$

3 Main results

In 2020, Y. M. Chu et al. [51] presented a generalization of some (p,q)-integral inequalities for preinvex function. Unfortunately, the results of the lemma and theorems are incorrect in the proofs. Here, we will show the errors of Theorem 1 in [51].

Statement 3.1 (Theorem 1, [51]) *If $f:[a,a+\xi(b,a)]$ is a (p,q)-differentiable function on $(a,a+\xi(b,a))$ with $\xi(b,a)>0$ such that $|{}_aD_{p,q}f|$ is a preinvex function and (p,q)-integrable function on $[a,a+\xi(b,a)]$, where $\frac{7}{8}\le q<p\le 1$, then*

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{3a+\xi(b,a)}{3}\right)+3f\left(\frac{3a+2\xi(b,a)}{3}\right)+f\bigl(a+\xi(b,a)\bigr)\right]\right.\\&\quad\left.-\frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)}f(t)_a\,d_{p,q}t\right|\\&\quad\le(b-a)\bigl[M_1(p,q)\bigl|{}_aD_{p,q}f(a)\bigr|+M_2(p,q)\bigl|{}_aD_{p,q}f(b)\bigr|\bigr],\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}M_1(p,q)&=\frac{1}{6912q^2[2]_{p,q}[3]_{p,q}}\bigl[768q^5+13{,}488q^4-13{,}056pq^4-14{,}256q^3+27{,}744pq^3\\&\quad-13{,}056p^2q^3-11{,}016q^2-14{,}256pq^2+39{,}264p^2q^2-11{,}016p^2+11{,}016q\\&\quad-11{,}016pq-14{,}256p^2q+14{,}256p^3q+11{,}016p-13{,}824p^3q^2\bigr];\\M_2(p,q)&=\frac{1}{6912q^2[2]_{p,q}[3]_{p,q}}\bigl[768q^4+768pq^3+11{,}016q^2-10{,}752p^2q^2-11{,}016q\\&\quad+11{,}016pq-11{,}016p+11{,}016p^2\bigr].\end{aligned}$$

Example 3.1 *The $f:[0,1]\to\mathbb{R}$ is defined by $f(x)=2x+5$. Then $|{}_aD_{p,q}f(x)|=|{}_aD_{p,q}(2x+5)|=2$ is a (p,q)-integrable function on $[0,1]$. Applying Statement 3.1 with $p=1$, $q=\frac{9}{10}$, and $\xi(b,a)=b-a$, the left-hand side of (3.1) becomes*

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{3a+\xi(b,a)}{3}\right)+3f\left(\frac{3a+2\xi(b,a)}{3}\right)+f\bigl(a+\xi(b,a)\bigr)\right]\right.\\&\quad\left.-\frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)}f(t)_a\,d_{p,q}t\right|\\&\quad=\left|\frac{1}{8}\left[3f\left(\frac{3\cdot0+(1-0)}{3}\right)+3f\left(\frac{3\cdot0+2(1-0)}{3}\right)+f\bigl(0+(1-0)\bigr)\right]\right.\\&\quad\left.-\frac{1}{1\cdot(1-0)}\int_0^{0+1\cdot(1-0)}f(2x+5)_0\,d_{1,\frac{9}{10}}x\right|\\&\quad=\left|\frac{1}{8}[17+19+7]-\frac{115}{19}\right|\approx0.67763158,\end{aligned}$$

and the right-hand side of (3.1) *becomes*

$$
\begin{aligned}
&(b-a)\left[M_1(p,q)\left|{}_aD_{p,q}f(a)\right| + M_2(p,q)\left|{}_aD_{p,q}f(b)\right|\right] \\
&\quad = (1-0)\left[M_1\left(1,\frac{9}{10}\right)\left|{}_0D_{1,\frac{9}{10}}f(0)\right| + M_2\left(1,\frac{9}{10}\right)\left|{}_0D_{1,\frac{9}{10}}f(1)\right|\right] \\
&\quad = (1-0)\left[\frac{161}{3907}(2) + \frac{39}{880}(2)\right] \approx 0.17105263.
\end{aligned}
$$

This implies that

$$0.67763158 \not\leq 0.17105263.$$

Therefore, Statement 3.1 *is not correct.*

The established Statement 3.1 gives the result involving (p,q)-integral identity as follows.

Statement 3.2 (Lemma 1, [51]) *If $f : [a, a+\xi(b,a)]$ is a (p,q)-differentiable function on $(a, a+\xi(b,a))$ with $\xi(b,a) > 0$ such that ${}_aD_{p,q}f$ is a (p,q)-integrable function on $[a, a+\xi(b,a)]$, where $\frac{7}{8} \leq q < p \leq 1$, then*

$$
\begin{aligned}
&\frac{1}{8}\left[3f\left(\frac{3a+\xi(b,a)}{3}\right) + 3f\left(\frac{3a+2\xi(b,a)}{3}\right) + f\left(a+\xi(b,a)\right)\right] \\
&\qquad - \frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)} f(t)_a\,d_{p,q}t \\
&\quad = \xi(b,a)\int_0^1 \varphi(t)\,{}_aD_{p,q}f\left(a+t\xi(b,a)\right)d_{p,q}t, \qquad (3.2)
\end{aligned}
$$

where

$$
\varphi(t) = \begin{cases} qt - \frac{1}{8}, & t \in [0, \frac{1}{3}); \\ qt - \frac{1}{2}, & t \in [\frac{1}{3}, \frac{2}{3}); \\ qt - \frac{7}{8}, & t \in [\frac{2}{3}, 1]. \end{cases}
$$

In the following, we provide a modified version involving (p,q)-integral identity for the preinvex function of Statement 3.2.

Theorem 3.1 *If $f : [a,b] \to \mathbb{R}$ is a (p,q)-differentiable function on $(a, a+\xi(b,a))$ with $\xi(b,a) > 0$ such that ${}_aD_{p,q}f$ is a (p,q)-integrable function on $[a, a+\xi(b,a)]$, where $\frac{7}{8} \leq q < p \leq 1$, then*

$$
\begin{aligned}
&\frac{1}{8}\left[f(a) + 3f\left(\frac{3a+\xi(b,a)}{3}\right) + 3f\left(\frac{3a+2\xi(b,a)}{3}\right) + f\left(a+\xi(b,a)\right)\right] \\
&\qquad - \frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)} f(t)_a\,d_{p,q}t \\
&\quad = \xi(b,a)\int_0^1 \varphi(t)\,{}_aD_{p,q}f\left(a+t\xi(b,a)\right)d_{p,q}t, \qquad (3.3)
\end{aligned}
$$

where

$$\varphi(t) = \begin{cases} qt - \frac{1}{8}, & t \in [0, \frac{1}{3}); \\ qt - \frac{1}{2}, & t \in [\frac{1}{3}, \frac{2}{3}); \\ qt - \frac{7}{8}, & t \in [\frac{2}{3}, 1]. \end{cases}$$

Proof It is not difficult to see that

$$\int_0^1 \varphi(t)_a D_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t = Q_1 + Q_2 + Q_3, \tag{3.4}$$

where

$$Q_1 = \int_0^{1/2} \left(qt - \frac{1}{8}\right)_a D_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t,$$

$$Q_2 = \int_{1/3}^{2/3} \left(qt - \frac{1}{2}\right)_a D_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t,$$

and

$$Q_3 = \int_{2/3}^{1} \left(qt - \frac{7}{8}\right)_a D_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t.$$

By Definition 2.1, we obtain

$$\begin{aligned} {}_aD_{p,q} f\big(a + t\xi(b,a)\big) &= \frac{f(p(a + t\xi(b,a)) + (1-p)a) - f(q(a + t\xi(b,a)) + (1-q)a)}{(p-q)((a + t\xi(b,a)) - a)} \\ &= \frac{f(a + pt\xi(b,a)) - f(a + qt\xi(b,a))}{t(p-q)\xi(b,a)}. \end{aligned} \tag{3.5}$$

By Definition 2.2, Theorem 2.1, and (3.5), we have

$$\begin{aligned} Q_1 &= \int_0^{1/3} \left(qt - \frac{1}{8}\right)_a D_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t \\ &= \int_0^{1/3} qt\,{}_aD_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t - \frac{1}{8} \int_0^{1/3} {}_aD_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t \\ &= \int_0^{1/3} q\frac{f(a + pt\xi(b,a)) - f(a + qt\xi(b,a))}{(p-q)\xi(b,a)}\, d_{p,q}t \\ &\quad - \frac{1}{8} \int_0^{1/3} \frac{f(a + pt\xi(b,a)) - f(a + qt\xi(b,a))}{t(p-q)\xi(b,a)}\, d_{p,q}t \\ &= \frac{1}{3\xi(b,a)} \left[\sum_{j=0}^{\infty} \frac{q^{j+1}}{p^{j+1}} f\left(a + \frac{q^j}{3p^j}\xi(b,a)\right) - \sum_{j=0}^{\infty} \frac{q^{j+1}}{p^{j+1}} f\left(a + \frac{q^{j+1}}{3p^{j+1}}\xi(b,a)\right) \right] \\ &\quad - \frac{1}{8\xi(b,a)} \left[\sum_{j=0}^{\infty} f\left(a + \frac{q^j}{3p^j}\xi(b,a)\right) - \sum_{j=0}^{\infty} f\left(a + \frac{q^{j+1}}{3p^{j+1}}\xi(b,a)\right) \right] \\ &= \frac{1}{3\xi(b,a)} \left[\frac{q}{p} \sum_{j=0}^{\infty} \frac{q^j}{p^j} f\left(a + \frac{q^j}{3p^j}\xi(b,a)\right) - \sum_{j=1}^{\infty} \frac{q^j}{p^j} f\left(a + \frac{q^j}{3p^j}\xi(b,a)\right) \right] \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{8\xi(b,a)}\left[\sum_{j=0}^{\infty} f\left(a + \frac{q^j}{3p^j}\xi(b,a)\right) - \sum_{j=1}^{\infty} f\left(a + \frac{q^j}{3p^j}\xi(b,a)\right)\right] \\
&= \frac{1}{3\xi(b,a)}\left[\frac{q}{p} f\left(\frac{3a+\xi(b,a)}{3}\right) - \frac{p-q}{p}\sum_{j=1}^{\infty}\frac{q^j}{p^j} f\left(a + \frac{q^j}{3p^j}\xi(b,a)\right)\right] \\
&- \frac{1}{8\xi(b,a)}\left[f\left(\frac{3a+\xi(b,a)}{3}\right) - f(a)\right] \\
&= \frac{q}{3p\xi(b,a)} f\left(\frac{3a+\xi(b,a)}{3}\right) - \frac{p-q}{3p\xi(b,a)}\sum_{j=0}^{\infty}\frac{q^j}{p^j} f\left(a + \frac{q^j}{3p^j}\xi(b,a)\right) \\
&+ \frac{p-q}{3p\xi(b,a)} f\left(\frac{3a+\xi(b,a)}{3}\right) - \frac{1}{8\xi(b,a)}\left[f\left(\frac{3a+\xi(b,a)}{3}\right) - f(a)\right] \\
&= \frac{5}{24\xi(b,a)} f\left(\frac{3a+\xi(b,a)}{3}\right) + \frac{1}{8}\frac{f(a)}{\xi(b,a)} - \frac{p-q}{3p\xi(b,a)}\sum_{j=0}^{\infty}\frac{q^j}{p^j} f\left(a + \frac{q^j}{3p^j}\xi(b,a)\right) \\
&= \frac{5}{24\xi(b,a)} f\left(\frac{3a+\xi(b,a)}{3}\right) + \frac{1}{8}\frac{f(a)}{\xi(b,a)} - \frac{1}{\xi(b,a)}\int_0^{\frac{1}{3}} f\big(a + pt\xi(b,a)\big)\, d_{p,q}t. \qquad (3.6)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
Q_2 &= \int_{1/3}^{2/3}\left(qt - \frac{1}{2}\right) {}_aD_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t \\
&= \int_0^{2/3}\left(qt - \frac{1}{2}\right) {}_aD_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t \\
&\quad - \int_0^{1/3}\left(qt - \frac{1}{2}\right) {}_aD_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t \\
&= \frac{1}{6\xi(b,a)} f\left(\frac{3a+\xi(b,a)}{3}\right) + \frac{1}{6\xi(b,a)} f\left(\frac{3a+2\xi(b,a)}{3}\right) \\
&\quad - \frac{1}{\xi(b,a)}\int_{1/3}^{2/3} f\big(a + pt\xi(b,a)\big)\, d_{p,q}t, \qquad (3.7)
\end{aligned}
$$

and

$$
\begin{aligned}
Q_3 &= \int_{2/3}^{1}\left(qt - \frac{7}{8}\right) {}_aD_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t \\
&= \int_0^{1}\left(qt - \frac{7}{8}\right) {}_aD_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t \\
&\quad - \int_0^{2/3}\left(qt - \frac{7}{8}\right) {}_aD_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t \\
&= \frac{5}{24\xi(b,a)} f\left(\frac{3a+2\xi(b,a)}{3}\right) + \frac{1}{8\xi(b,a)} f\big(a + \xi(b,a)\big) \\
&\quad - \frac{1}{\xi(b,a)}\int_{2/3}^{1} f\big(a + pt\xi(b,a)\big)\, d_{p,q}t. \qquad (3.8)
\end{aligned}
$$

Substituting (3.6), (3.7), and (3.8) in (3.4), we have

$$\begin{aligned}
&\int_0^1 \varphi(t)_a D_{p,q} f\big(a + t\xi(b,a)\big)\, d_{p,q}t \\
&= Q_1 + Q_2 + Q_3 \\
&= \frac{1}{8}\left[f(a) + 3f\left(\frac{3a+\xi(b,a)}{3}\right) + 3f\left(\frac{3a+2\xi(b,a)}{3}\right) + f\big(a+\xi(b,a)\big)\right] \\
&\quad - \frac{1}{\xi(b,a)} \int_0^1 f\big(a + pt\xi(b,a)\big)_a\, d_{p,q}t \\
&= \frac{1}{8}\left[f(a) + 3f\left(\frac{3a+\xi(b,a)}{3}\right) + 3f\left(\frac{3a+2\xi(b,a)}{3}\right) + f\big(a+\xi(b,a)\big)\right] \\
&\quad - \frac{1}{p\xi^2(b,a)} \int_a^{a+p\xi(b,a)} f(t)_a\, d_{p,q}t.
\end{aligned}$$

Multiplying the above equality with $\xi(b,a)$, we obtain the required (p,q)-integral identity. Therefore, the proof is completed. □

Corollary 3.1 *If $f : [a,b] \to \mathbb{R}$ is a (p,q)-differentiable function on (a,b) such that ${}_aD_{p,q}f$ is a (p,q)-integrable function on $[a,b]$, where $\frac{7}{8} \le q < p \le 1$, then*

$$\begin{aligned}
&\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(t)_a\, d_{p,q}t \\
&= (b-a)\int_0^1 \varphi(t)_a D_{p,q} f\big((1-t)a + tb\big)\, d_{p,q}t, \qquad (3.9)
\end{aligned}$$

where

$$\varphi(t) = \begin{cases} qt - \frac{1}{8}, & t \in [0, \frac{1}{3}); \\ qt - \frac{1}{2}, & t \in [\frac{1}{3}, \frac{2}{3}); \\ qt - \frac{7}{8}, & t \in [\frac{2}{3}, 1]. \end{cases}$$

Remark 3.1 *If $p = 1$, then (3.9) reduces to*

$$\begin{aligned}
&\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{1}{(b-a)} \int_a^{b} f(x)_a\, d_{q}x \\
&= (b-a)\int_0^1 \varphi(t)_a D_{q} f\big((1-t)a + tb\big)\, d_{q}t, \qquad (3.10)
\end{aligned}$$

where

$$\varphi(t) = \begin{cases} qt - \frac{1}{8}, & t \in [0, \frac{1}{3}); \\ qt - \frac{1}{2}, & t \in [\frac{1}{3}, \frac{2}{3}); \\ qt - \frac{7}{8}, & t \in [\frac{2}{3}, 1], \end{cases}$$

which appeared in [50]. *In addition, if* $q \to 1$, *then* (3.10) *reduces to*

$$\begin{aligned}&\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{(b-a)}\int_a^b f(x)\,dx\\&\quad=(b-a)\int_0^1 \varphi(t)f'\big((1-t)a+tb\big)\,dt,\end{aligned}$$

where

$$\varphi(t)=\begin{cases}t-\frac{1}{8}, & t\in[0,\frac{1}{3});\\ t-\frac{1}{2}, & t\in[\frac{1}{3},\frac{2}{3});\\ t-\frac{7}{8}, & t\in[\frac{2}{3},1],\end{cases}$$

which appeared in [57].

Theorem 3.2 *If* $f:[a,b]\to\mathbb{R}$ *is a continuous function such that* $|{}_aD_{p,q}f|$ *is a preinvex function and a* (p,q)*-integrable function on* $[a,b]$, *where* $\frac{7}{8}\le q<1$, *then*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{3a+\xi(b,a)}{3}\right)+3f\left(\frac{3a+2\xi(b,a)}{3}\right)+f\big(a+\xi(b,a)\big)\right]\right.\\&\qquad\left.-\frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)} f(t)_a\,d_{p,q}t\right|\\&\quad\le\xi(b,a)\big[M_1(p,q)\big|{}_aD_{p,q}f(a)\big|+M_2(p,q)\big|{}_aD_{p,q}f(b)\big|\big],\end{aligned}\tag{3.11}$$

where

$$\begin{aligned}M_1(p,q)&=\frac{1}{6912q^2[2]_{p,q}[3]_{p,q}}\big[768q^5+13{,}488q^4-13{,}056pq^4-14{,}256q^3+27{,}744pq^3\\&\quad-13{,}056p^2q^3-11{,}016q^2-14{,}256pq^2+39{,}264p^2q^2-11{,}016p^2+11{,}016q\\&\quad-11{,}016pq-14{,}256p^2q+14{,}256p^3q+11{,}016p-13{,}824p^3q^2\big];\\M_2(p,q)&=\frac{1}{6912q^2[2]_{p,q}[3]_{p,q}}\big[768q^4+768pq^3+11{,}016q^2-10{,}752p^2q^2-11{,}016q\\&\quad+11{,}016pq-11{,}016p+11{,}016p^2\big].\end{aligned}$$

Proof Using Theorem 3.1, Lemma 2.1, Definition 2.2, and the preinvexity of $|{}_aD_{p,q}f|$, we have

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{3a+\xi(b,a)}{3}\right)+3f\left(\frac{3a+2\xi(b,a)}{3}\right)+f\big(a+\xi(b,a)\big)\right]\right.\\&\qquad\left.-\frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)} f(t)_a\,d_{p,q}t\right|\\&\quad=\left|\xi(b,a)\int_0^1\varphi(t)_aD_{p,q}f\big(a+t\xi(b,a)\big)\,d_{p,q}t\right|\\&\quad=\xi(b,a)\left|\int_0^{1/3}\left(qt-\frac{1}{8}\right){}_aD_{p,q}f\big(a+t\xi(b,a)\big)\,d_{p,q}t\right.\end{aligned}$$

$$
\begin{aligned}
&+\int_{1/3}^{2/3}\left(qt-\frac{1}{2}\right){}_aD_{p,q}f\bigl(a+t\xi(b,a)\bigr)_a\,d_{p,q}t\\
&+\int_{2/3}^{1}\left(qt-\frac{7}{8}\right){}_aD_{p,q}f\bigl(a+t\xi(b,a)b\bigr)_a\,d_{p,q}t\biggr|\\
\le{}&\xi(b,a)\biggl[\int_0^{1/3}\left|qt-\frac{1}{8}\right|\bigl|{}_aD_{p,q}f\bigl(a+t\xi(b,a)\bigr)\bigr|\,d_{p,q}t\\
&+\int_{1/3}^{2/3}\left|qt-\frac{1}{2}\right|\bigl|{}_aD_{p,q}f\bigl(a+t\xi(b,a)\bigr)\bigr|_a\,d_{p,q}t\\
&+\int_{2/3}^{1}\left|qt-\frac{7}{8}\right|\bigl|{}_aD_{p,q}f\bigl(a+t\xi(b,a)\bigr)\bigr|_a\,d_{p,q}t\biggr]\\
\le{}&\xi(b,a)\int_0^{1/3}\left|qt-\frac{1}{8}\right|\bigl[(1-t)\bigl|{}_aD_{p,q}f(a)\bigr|+t\bigl|{}_aD_{p,q}f(b)\bigr|\bigr]\,d_{p,q}t\\
&+\xi(b,a)\int_{1/3}^{2/3}\left|qt-\frac{1}{2}\right|\bigl[(1-t)\bigl|{}_aD_{p,q}f(a)\bigr|+t\bigl|{}_aD_{p,q}f(b)\bigr|\bigr]_a\,d_{p,q}t\\
&+\xi(b,a)\int_{2/3}^{1}\left|qt-\frac{7}{8}\right|\bigl[(1-t)\bigl|{}_aD_{p,q}f(a)\bigr|+t\bigl|{}_aD_{p,q}f(b)\bigr|\bigr]_a\,d_{p,q}t\\
={}&\xi(b,a)\biggl(\bigl|{}_aD_{p,q}f(a)\bigr|\int_0^{1/3}(1-t)\left|qt-\frac{1}{8}\right|\,d_{p,q}t+\bigl|{}_aD_{p,q}f(b)\bigr|\int_0^{1/3}t\left|qt-\frac{1}{8}\right|\,d_{p,q}t\biggr)\\
&+\xi(b,a)\biggl(\bigl|{}_aD_{p,q}f(a)\bigr|\int_{1/3}^{2/3}(1-t)\left|qt-\frac{1}{2}\right|_a\,d_{p,q}t\\
&+\bigl|{}_aD_{p,q}f(b)\bigr|\int_{1/3}^{2/3}t\left|qt-\frac{1}{2}\right|_a\,d_{p,q}t\biggr)\\
&+\xi(b,a)\biggl(\bigl|{}_aD_{p,q}f(a)\bigr|\int_{2/3}^{1}(1-t)\left|qt-\frac{7}{8}\right|_a\,d_{p,q}t+\bigl|{}_aD_{p,q}f(b)\bigr|\int_{2/3}^{1}t\left|qt-\frac{7}{8}\right|_a\,d_{p,q}t\biggr)\\
={}&\xi(b,a)\biggl(\frac{1}{6912q^2[2]_{p,q}[3]_{p,q}}\bigl[480q^5+192pq^4+56q^4+192p^2q^3+272pq^3\\
&-216q^3-288p^3q^2+528p^2q^2-216pq^2-27q^2+216p^3q-216p^2q-27pq\\
&+27q-27p^2+27p\bigr]\bigl|{}_aD_{p,q}f(a)\bigr|\\
&+\frac{160q^4+160pq^3-96p^2q^2+27q^2+27pq-27q+27p^2-27p}{6912q^2[2]_{p,q}[3]_{p,q}}\bigl|{}_aD_{p,q}f(b)\bigr|\biggr)\\
&+\xi(b,a)\biggl(\frac{1}{108q^2[2]_{p,q}[3]_{p,q}}\bigl[6q^5-48pq^4+48q^4-48p^2q^3+102pq^3-54q^3\\
&-54p^3q^2+138p^2q^2-54pq^2-27q^2+54p^3q-54p^2q-27pq+27q-27p^2\\
&+27p\bigr]\bigl|{}_aD_{p,q}f(a)\bigr|\\
&+\frac{6q^4+6pq^3-30p^2q^2+27q^2+27pq-27q+27p^2-27p}{108q^2[2]_{p,q}[3]_{p,q}}\bigl|{}_aD_{p,q}f(b)\bigr|\biggr)\\
&+\xi(b,a)\biggl(\frac{1}{6912q^2[2]_{p,q}[3]_{p,q}}\bigl[-96q^5-10{,}176pq^4+10{,}360q^4-10{,}176p^2q^3\\
&+20{,}944pq^3-10{,}584q^3-10{,}080p^3q^2+29{,}904p^2q^2-10{,}584pq^2-9261q^2\\
&+10{,}584p^3q-10{,}584p^2q-9261pq+9261q-9261p^2+9261p\bigr]\bigl|{}_aD_{p,q}f(a)\bigr|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{6912q^2[2]_{p,q}[3]_{p,q}}\big[224q^4+224pq^3-8736p^2q^2+9261q^2+9261pq-9261q \\
&+9261p^2-9261p\big]\big|{}_aD_{p,q}f(b)\big|\bigg) \\
&=\xi(b,a)\bigg(\frac{1}{6912q^2[2]_{p,q}[3]_{p,q}}\big[768q^5+13{,}488q^4-13{,}056pq^4-14{,}256q^3 \\
&+27{,}744pq^3-13{,}056p^2q^3-11{,}016q^2-14{,}256pq^2+39{,}264p^2q^2-13{,}824p^3q^2 \\
&+11{,}016q-11{,}016pq-14{,}256p^2q+14{,}256p^3q+11{,}016p-11{,}016p^2\big]\big|{}_aD_{p,q}f(a)\big| \\
&+\frac{1}{6912q^2[2]_{p,q}[3]_{p,q}}\big[768q^4+768pq^3+11{,}016q^2-10{,}752p^2q^2-11{,}016q \\
&+11{,}016pq-11{,}016p+11{,}016p^2\big]\big|{}_aD_{p,q}f(b)\big|\bigg) \\
&=\xi(b,a)\big[M_1(p,q)\big|{}_aD_{p,q}f(a)\big|+M_2(p,q)\big|{}_aD_{p,q}f(b)\big|\big],
\end{aligned}
$$

which completes the proof. □

Corollary 3.2 *If $f:[a,b]\to\mathbb{R}$ is a continuous function such that $|{}_aD_{p,q}f|$ is a convex function and a (p,q)-integrable function on $[a,b]$, where $\frac{7}{8}\le q<1$, then*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a}f(x)_a\,d_{p,q}x\right| \\
&\quad\le(b-a)\big[M_1(p,q)\big|{}_aD_{p,q}f(a)\big|+M_2(p,q)\big|{}_aD_{p,q}f(b)\big|\big],
\end{aligned}
\tag{3.12}
$$

where $M_i(p,q)$, $i=1,2$, are given in Theorem 3.2.

Remark 3.2 *If $p=1$, then (3.12) reduces to*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^{b}f(x)_a\,d_qx\right| \\
&\quad\le(b-a)\bigg(\frac{768q^3+432q^2+432q+168}{6912[2]_q[3]_q}\big|{}_aD_qf(a)\big| \\
&\quad+\frac{768q^2+768q+264}{6912[2]_q[3]_q}\big|{}_aD_qf(b)\big|\bigg),
\end{aligned}
\tag{3.13}
$$

which appeared in [50]. In addition, if $q\to1$, then (3.13) reduces to

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^{b}f(x)\,dx\right| \\
&\quad\le\frac{25(b-a)}{576}\big[\big|f'(a)\big|+\big|f'(b)\big|\big],
\end{aligned}
$$

which appeared in [57].

Theorem 3.3 *If $f:[a,b]\to\mathbb{R}$ is a continuous function such that $|{}_aD_{p,q}f|^s$ is a preinvex function and a (p,q)-integrable function on $[a,b]$, where $\frac{7}{8}\le q<1$ and $r,s>1$ with $1/r+$*

$1/s = 1$, *then*

$$\left|\frac{1}{8}\left[f(a) + 3f\left(\frac{3a+\xi(b,a)}{3}\right) + 3f\left(\frac{3a+2\xi(b,a)}{3}\right) + f\left(a+\xi(b,a)\right)\right] - \frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)} f(t)_a\,d_{p,q}t\right|$$
$$\leq \xi(b,a)\left\{\left(\frac{[3^{r+1}+(8q-3)^{r+1}]}{24^{r+1}q[r+1]_{p,q}}\right)^{1/r}\left(\frac{(3q+3p-1)|_aD_{p,q}f(a)|^s + |_aD_{p,q}f(b)|^s}{9[2]_{p,q}}\right)^{1/s}\right.$$
$$+\left(\frac{[(3-2q)^{r+1}+(4q-3)^{r+1}]}{6^{r+1}q[r+1]_{p,q}}\right)^{1/r}\left(\frac{(q+p-1)|_aD_{p,q}f(a)|^s + |_aD_{p,q}f(b)|^s}{3[2]_{p,q}}\right)^{1/s}$$
$$+\left(\frac{[(21-16q)^{r+1}+(24q-21)^{r+1}]}{24^{r+1}q[r+1]_{p,q}}\right)^{1/r}$$
$$\left.\times\left(\frac{(3q+3p-5)|_aD_{p,q}f(a)|^s + 5|_aD_{p,q}f(b)|^s}{9[2]_{p,q}}\right)^{1/s}\right\}. \tag{3.14}$$

Proof Using Theorem 3.1 and the Hölder inequality, we have

$$\left|\frac{1}{8}\left[f(a) + 3f\left(\frac{3a+\xi(b,a)}{3}\right) + 3f\left(\frac{3a+2\xi(b,a)}{3}\right) + f\left(a+\xi(b,a)\right)\right] - \frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)} f(t)_a\,d_{p,q}t\right|$$
$$= \left|\xi(b,a)\int_0^1 \varphi(t)_aD_{p,q}f\left(a+t\xi(b,a)\right)d_{p,q}t\right|$$
$$= \xi(b,a)\left|\int_0^{1/3}\left(qt-\frac{1}{8}\right)_aD_{p,q}f\left(a+t\xi(b,a)\right)d_{p,q}t\right.$$
$$+\int_{1/3}^{2/3}\left(qt-\frac{1}{2}\right)_aD_{p,q}f\left(a+t\xi(b,a)\right)_a\,d_{p,q}t$$
$$\left.+\int_{2/3}^{1}\left(qt-\frac{7}{8}\right)_aD_{p,q}f\left(a+t\xi(b,a)b\right)_a\,d_{p,q}t\right|$$
$$\leq \xi(b,a)\left[\int_0^{1/3}\left|qt-\frac{1}{8}\right|\left|_aD_{p,q}f\left(a+t\xi(b,a)\right)\right|d_{p,q}t\right.$$
$$+\int_{1/3}^{2/3}\left|qt-\frac{1}{2}\right|\left|_aD_{p,q}f\left(a+t\xi(b,a)\right)\right|_a\,d_{p,q}t$$
$$\left.+\int_{2/3}^{1}\left|qt-\frac{7}{8}\right|\left|_aD_{p,q}f\left(a+t\xi(b,a)\right)\right|_a\,d_{p,q}t\right]$$
$$\leq \xi(b,a)\left\{\left(\int_0^{1/3}\left|qt-\frac{1}{8}\right|^r d_{p,q}t\right)^{1/r}\left(\int_0^{1/3}\left|_aD_{p,q}f\left(a+t\xi(b,a)\right)\right|^s d_{p,q}t\right)^{1/s}\right.$$
$$+\left(\int_{1/3}^{2/3}\left|qt-\frac{1}{2}\right|^r{}_a\,d_{p,q}t\right)^{1/r}\left(\int_{1/3}^{2/3}\left|_aD_{p,q}f\left(a+t\xi(b,a)\right)\right|^s{}_a\,d_{p,q}t\right)^{1/s}$$
$$\left.+\left(\int_{2/3}^{1}\left|qt-\frac{7}{8}\right|^r{}_a\,d_{p,q}t\right)^{1/r}\left(\int_{2/3}^{1}\left|_aD_{p,q}f\left(a+t\xi(b,a)\right)\right|^s{}_a\,d_{p,q}t\right)^{1/s}\right\}. \tag{3.15}$$

From the case when $a = 0$ of Lemma 2.1, it follows that

$$\begin{aligned}
\int_0^{1/3}\left|qt-\frac{1}{8}\right|^r d_{p,q}t &= \int_0^{1/8q}\left(\frac{1}{8}-qt\right)^r d_{p,q}t+\int_{1/8q}^{1/3}\left(qt-\frac{1}{8}\right)^r d_{p,q}t \\
&= (-1)^{r+1}q^r\int_{1/8q}^{0}\left(t-\frac{1}{8q}\right)^r d_{p,q}t+q^r\int_{1/8q}^{1/3}\left(t-\frac{1}{8q}\right)^r d_{p,q}t \\
&= \frac{[3^{r+1}+(8q-3)^{r+1}]}{24^{r+1}q[r+1]_{p,q}}, \qquad (3.16) \\
\int_{1/3}^{2/3}\left|qt-\frac{1}{2}\right|^r d_{p,q}t &= \int_{1/3}^{1/2q}\left(\frac{1}{2}-qt\right)^r d_{p,q}t+\int_{1/2q}^{2/3}\left(qt-\frac{1}{2}\right)^r d_{p,q}t \\
&= (-1)^{r+1}q^r\int_{1/2q}^{1/3}\left(t-\frac{1}{2q}\right)^r d_{p,q}t+q^r\int_{1/2q}^{2/3}\left(t-\frac{1}{2q}\right)^r d_{p,q}t \\
&= \frac{[(3-2q)^{r+1}+(4q-3)^{r+1}]}{6^{r+1}q[r+1]_{p,q}}, \qquad (3.17)
\end{aligned}$$

and

$$\begin{aligned}
\int_{2/3}^{1}\left|qt-\frac{7}{8}\right|^r d_{p,q}t &= \int_{2/3}^{7/8q}\left(\frac{7}{8}-qt\right)^r d_{p,q}t+\int_{7/8q}^{1}\left(qt-\frac{7}{8}\right)^r d_{p,q}t \\
&= (-1)^{r+1}q^r\int_{7/8q}^{2/3}\left(t-\frac{7}{8q}\right)^r d_{p,q}t+q^r\int_{7/8q}^{1}\left(t-\frac{7}{8}\right)^r d_{p,q}t \\
&= \frac{[(21-16q)^{r+1}+(24q-21)^{r+1}]}{24^{r+1}q[r+1]_{p,q}}. \qquad (3.18)
\end{aligned}$$

From the case when $a = 0$ of Lemma 2.1 and the preinvexity of $|{}_aD_{p,q}f|^s$, we find that

$$\begin{aligned}
&\int_0^{1/3}\left|{}_aD_{p,q}f\big(a+t\xi(b,a)\big)\right|^s{}_a\, d_{p,q}t \\
&\quad\le \left|{}_aD_{p,q}f(a)\right|^s\int_0^{1/3}(1-t)\,d_{p,q}t+\left|{}_aD_{p,q}f(b)\right|^s\int_0^{1/3}t\,d_{p,q}t \\
&\quad= \frac{(3q+3p-1)|{}_aD_{p,q}f(a)|^s+|{}_aD_{p,q}f(b)|^s}{9[2]_{p,q}}, \qquad (3.19) \\
&\int_{1/3}^{2/3}\left|{}_aD_{p,q}f\big(a+t\xi(b,a)\big)\right|^s{}_0\, d_{p,q}t \\
&\quad\le \left|{}_aD_{p,q}f(a)\right|^s\int_{1/3}^{2/3}(1-t)_a\,d_{p,q}t+\left|{}_aD_{p,q}f(b)\right|^s\int_{1/3}^{2/3}t_a\,d_{p,q}t \\
&\quad= \frac{(q+p-1)|{}_aD_{p,q}f(a)|^s+|{}_aD_{p,q}f(b)|^s}{3[2]_{p,q}}, \qquad (3.20)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{2/3}^{1}\left|{}_aD_{p,q}f\big(a+t\xi(b,a)\big)\right|^s{}_0\, d_{p,q}t \\
&\quad\le \left|{}_aD_{p,q}f(a)\right|^s\int_{2/3}^{1}(1-t)_a\,d_{p,q}t+\left|{}_aD_{p,q}f(b)\right|^s\int_{2/3}^{1}t_a\,d_{p,q}t
\end{aligned}$$

$$= \frac{(3q+3p-5)|_aD_{p,q}f(a)|^s + 5|_aD_{p,q}f(b)|^s}{9[2]_{p,q}}. \tag{3.21}$$

Substituting (3.16) to (3.21) in (3.15), we obtain the required result. Therefore, the proof is completed. □

Corollary 3.3 *If $f:[a,b]\to\mathbb{R}$ is a continuous function such that $|_aD_{p,q}f|^s$ is a convex function and a (p,q)-integrable function on $[a,b]$, where $\frac{7}{8}\le q<1$ and $r,s>1$ with $1/r+1/s=1$, then*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_ad_{p,q}x\right| \\
&\le (b-a)\left\{\left(\frac{[3^{r+1}+(8q-3)^{r+1}]}{24^{r+1}q[r+1]_{p,q}}\right)^{1/r}\left(\frac{(3q+3p-1)|_aD_{p,q}f(a)|^s+|_aD_{p,q}f(b)|^s}{9[2]_{p,q}}\right)^{1/s}\right. \\
&\quad+\left(\frac{[(3-2q)^{r+1}+(4q-3)^{r+1}]}{6^{r+1}q[r+1]_{p,q}}\right)^{1/r}\left(\frac{(q+p-1)|_aD_{p,q}f(a)|^s+|_aD_{p,q}f(b)|^s}{3[2]_{p,q}}\right)^{1/s} \\
&\quad+\left(\frac{[(21-16q)^{r+1}+(24q-21)^{r+1}]}{24^{r+1}q[r+1]_{p,q}}\right)^{1/r} \\
&\quad\left.\times\left(\frac{(3q+3p-5)|_aD_{p,q}f(a)|^s+5|_aD_{p,q}f(b)|^s}{9[2]_{p,q}}\right)^{1/s}\right\}.
\end{aligned} \tag{3.22}$$

Remark 3.3 *If $p=1$, then (3.22) reduces to*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^{b} f(x)\,{}_ad_{q}x\right| \\
&\le (b-a)\left\{\left(\frac{[3^{r+1}+(8q-3)^{r+1}]}{24^{r+1}q[r+1]_{q}}\right)^{1/r}\left(\frac{(3q+2)|_aD_{q}f(a)|^s+|_aD_{q}f(b)|^s}{9[2]_{q}}\right)^{1/s}\right. \\
&\quad+\left(\frac{[(3-2q)^{r+1}+(4q-3)^{r+1}]}{6^{r+1}q[r+1]_{q}}\right)^{1/r}\left(\frac{q|_aD_{q}f(a)|^s+|_aD_{q}f(b)|^s}{3[2]_{q}}\right)^{1/s} \\
&\quad+\left(\frac{[(21-16q)^{r+1}+(24q-21)^{r+1}]}{24^{r+1}q[r+1]_{q}}\right)^{1/r} \\
&\quad\left.\times\left(\frac{(3q-2)|_aD_{q}f(a)|^s+5|_aD_{q}f(b)|^s}{9[2]_{q}}\right)^{1/s}\right\},
\end{aligned} \tag{3.23}$$

which appeared in [50]. *In addition, if $q\to1$, then* (3.23) *reduces to*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^{b} f(x)\,dx\right| \\
&\le (b-a)\left\{\left(\frac{[3^{r+1}+5^{r+1}]}{24^{r+1}(r+1)}\right)^{1/r}\left(\frac{5|f'(a)|^s+|f'(b)|^s}{18}\right)^{1/s}\right. \\
&\quad+\left(\frac{2}{6^{r+1}(r+1)}\right)^{1/r}\left(\frac{|f'(a)|^s+|f'(b)|^s}{6}\right)^{1/s} \\
&\quad\left.+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{1/r}\left(\frac{|f'(a)|^s+5|f'(b)|^s}{18}\right)^{1/s}\right\},
\end{aligned}$$

which appeared in [57].

Theorem 3.4 *If $f : [a, b] \to \mathbb{R}$ is a continuous function such that $|{}_aD_{p,q}f|^s$ is a convex function and a (p,q)-integrable function on $[a,b]$, where $\frac{7}{8} \le q < 1$ and $r, s > 1$ with $1/r + 1/s = 1$, then*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\, d_{p,q}x\right| \\
&\le (b-a)\left\{\left(\frac{[3^{r+1} + (8q-3)^{r+1}]}{24^{r+1}q[r+1]_{p,q}}\right)^{1/r}\left(\frac{(q+p-1)|{}_aD_{p,q}f(a)|^s + |{}_aD_{p,q}f(\frac{2a+b}{3})|^s}{3[2]_{p,q}}\right)^{1/s}\right. \\
&\quad + \left(\frac{[(3-2q)^{r+1} + (4q-3)^{r+1}]}{6^{r+1}q[r+1]_{p,q}}\right)^{1/r} \\
&\quad \times \left(\frac{(q+p-1)|{}_aD_{p,q}f(\frac{2a+b}{3})|^s + |{}_aD_{p,q}f(\frac{a+2b}{3})|^s}{3[2]_{p,q}}\right)^{1/s} \\
&\quad + \left(\frac{[(21-16q)^{r+1} + (24q-21)^{r+1}]}{24^{r+1}q[r+1]_{p,q}}\right)^{1/r} \\
&\quad \left.\times \left(\frac{(q+p-1)|{}_aD_{p,q}f(\frac{a+2b}{3})|^s + |{}_aD_{p,q}f(b)|^s}{3[2]_{p,q}}\right)^{1/s}\right\}.
\end{aligned} \tag{3.24}$$

Proof Using Corollary 3.1 and the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\, d_{p,q}x\right| \\
&= (b-a)\int_0^1 \varphi(t)_a D_{p,q}f\big((1-t)a + tb\big)\, d_{p,q}t \\
&\le (b-a)\left[\int_0^{\frac{1}{3}} \left|qt - \frac{1}{8}\right| \left|{}_aD_{p,q}f\big((1-t)a+tb\big)\right|_0 d_{p,q}t\right. \\
&\quad + \int_{1/3}^{2/3} \left|qt - \frac{1}{2}\right| \left|{}_aD_{p,q}f\big((1-t)a+tb\big)\right|_a d_{p,q}t \\
&\quad \left. + \int_{2/3}^{1} \left|qt - \frac{7}{8}\right| \left|{}_aD_{p,q}f\big((1-t)a+tb\big)\right|_a d_{p,q}t\right] \\
&\le (b-a)\left\{\left(\int_0^{1/3}\left|qt - \frac{1}{8}\right|^r d_{p,q}t\right)^{1/r}\left(\int_0^{1/3}\left|{}_aD_{p,q}f\big((1-t)a+tb\big)\right|^s d_{p,q}t\right)^{1/s}\right. \\
&\quad + \left(\int_{1/3}^{2/3}\left|qt - \frac{1}{2}\right|^r{}_a\, d_{p,q}t\right)^{1/r}\left(\int_{1/3}^{2/3}\left|{}_aD_{p,q}f\big((1-t)a+tb\big)\right|^s{}_a\, d_{p,q}t\right)^{1/s} \\
&\quad \left. + \left(\int_{2/3}^{1}\left|qt - \frac{7}{8}\right|^r{}_a\, d_{p,q}t\right)^{1/r}\left(\int_{2/3}^{1}\left|{}_aD_{p,q}f\big((1-t)a+tb\big)\right|^s{}_a\, d_{p,q}t\right)^{1/s}\right\}.
\end{aligned} \tag{3.25}$$

From the case when $a = 0$ of Lemma 2.1, it follows that

$$\int_0^{1/3}\left|qt - \frac{1}{8}\right|^r d_{p,q}t = \frac{[3^{r+1} + (8q-3)^{r+1}]}{24^{r+1}q[r+1]_{p,q}}, \tag{3.26}$$

$$\int_{1/3}^{2/3}\left|qt - \frac{1}{2}\right|^r d_{p,q}t = \frac{[(3-2q)^{r+1} + (4q-3)^{r+1}]}{6^{r+1}q[r+1]_{p,q}}, \tag{3.27}$$

and

$$\int_{2/3}^{1}\left|qt-\frac{7}{8}\right|^{r}d_{p,q}t=\frac{[(21-16q)^{r+1}+(24q-21)^{r+1}]}{24^{r+1}q[r+1]_{p,q}}. \tag{3.28}$$

Using Definition 2.2, it is not difficult to show that

$$\begin{aligned}
&\int_{0}^{1/3}\left|{}_{a}D_{p,q}f\big((1-t)a+tb\big)\right|^{s}d_{p,q}t\\
&=(p-q)\left(\frac{1}{3}-0\right)\sum_{j=0}^{\infty}\left|{}_{a}D_{p,q}f\left(\left(1-\frac{q^{j}}{3p^{j+1}}\right)a+\frac{q^{j}}{3p^{j+1}}b\right)\right|^{s}\\
&=\frac{1}{3}(p-q)\sum_{j=0}^{\infty}\left|{}_{a}D_{p,q}f\left(a-\frac{q^{j}}{3p^{j+1}}a+\frac{q^{j}}{3p^{j+1}}b+\frac{q^{j}}{p^{j+1}}a-\frac{q^{j}}{p^{j+1}}a\right)\right|^{s}\\
&=\frac{1}{3}(p-q)\sum_{j=0}^{\infty}\frac{q^{j}}{p^{j+1}}\left|{}_{a}D_{p,q}f\left(\left(1-\frac{q^{j}}{p^{j+1}}\right)a+\left(\frac{2a+b}{3}\right)\frac{q^{j}}{p^{j+1}}\right)\right|^{s}\\
&=\frac{1}{3}(p-q)(1-0)\sum_{j=0}^{\infty}\frac{q^{j}}{p^{j+1}}\left|{}_{a}D_{p,q}f\left(\left(1-\frac{q^{j}}{p^{j+1}}\right)a+\left(\frac{2a+b}{3}\right)\frac{q^{j}}{p^{j+1}}\right)\right|^{s}\\
&=\frac{1}{3}\int_{0}^{1}\left|{}_{a}D_{p,q}f\left((1-t)a+\left(\frac{2a+b}{3}\right)t\right)\right|^{s}d_{p,q}t.
\end{aligned}$$

From the case when $a=0$ of Lemma 2.1 and the convexity of $|{}_{a}D_{p,q}f|^{s}$, we have

$$\begin{aligned}
&\int_{0}^{1/3}\left|{}_{a}D_{p,q}f\big((1-t)a+tb\big)\right|^{s}d_{p,q}t\\
&\leq\frac{1}{3}\left[\left|{}_{a}D_{p,q}f(a)\right|^{s}\int_{0}^{1}(1-t)\,d_{p,q}t+\left|{}_{a}D_{p,q}f\left(\frac{2a+b}{3}\right)\right|^{s}\int_{0}^{1}t\,d_{p,q}t\right]\\
&=\frac{(q+p-1)|{}_{a}D_{p,q}f(a)|^{s}+|{}_{a}D_{p,q}f(\frac{2a+b}{3})|^{s}}{3[2]_{p,q}}.
\end{aligned} \tag{3.29}$$

Similarly, we obtain

$$\begin{aligned}
&\int_{1/3}^{2/3}\left|{}_{a}D_{p,q}f\big((1-t)a+tb\big)\right|^{s}d_{p,q}t\\
&=\frac{(q+p-1)|{}_{a}D_{p,q}f(\frac{2a+b}{3})|^{s}+|{}_{a}D_{p,q}f(\frac{a+2b}{3})|^{s}}{3[2]_{p,q}}
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
&\int_{2/3}^{1}\left|{}_{a}D_{p,q}f\big((1-t)a+tb\big)\right|^{s}d_{p,q}t\\
&=\frac{(q+p-1)|{}_{a}D_{p,q}f(\frac{a+2b}{3})|^{s}+|{}_{a}D_{p,q}f(b)|^{s}}{3[2]_{p,q}}.
\end{aligned} \tag{3.31}$$

Substituting (3.26) to (3.31) in (3.25), we obtain the required result. Therefore, the proof is completed. □

Remark 3.4 *If $p = 1$, then (3.24) reduces to*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)_a\,d_qx\right|$$
$$\leq (b-a)\left\{\left(\frac{[3^{r+1}+(8q-3)^{r+1}]}{24^{r+1}q[r+1]_q}\right)^{1/r}\left(\frac{q|_aD_qf(a)|^s+|_aD_qf(\frac{2a+b}{3})|^s}{3[2]_q}\right)^{1/s}\right.$$
$$+\left(\frac{[(3-2q)^{r+1}+(4q-3)^{r+1}]}{6^{r+1}q[r+1]_q}\right)^{1/r}\left(\frac{q|_aD_qf(\frac{2a+b}{3})|^s+|_aD_qf(\frac{a+2b}{3})|^s}{3[2]_q}\right)^{1/s}$$
$$\left.+\left(\frac{[(21-16q)^{r+1}+(24q-21)^{r+1}]}{24^{r+1}q[r+1]_q}\right)^{1/r}\left(\frac{q|_aD_qf(\frac{a+2b}{3})|^s+|_aD_qf(b)|^s}{3[2]_q}\right)^{1/s}\right\}, \tag{3.32}$$

which appeared in [50]. *In addition, if $q \to 1$, then* (3.32) *reduces to*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)\left\{\left(\frac{[3^{r+1}+5^{r+1}]}{24^{r+1}(r+1)}\right)^{1/r}\left(\frac{|f'(a)|^s+|f'(\frac{2a+b}{3})|^s}{6}\right)^{1/s}\right.$$
$$+\left(\frac{2}{6^{r+1}(r+1)}\right)^{1/r}\left(\frac{|f'(\frac{2a+b}{3})|^s+|f'(\frac{a+2b}{3})|^s}{6}\right)^{1/s}$$
$$\left.+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{1/r}\left(\frac{|f'(\frac{a+2b}{3})|^s+|f'(b)|^s}{6}\right)^{1/s}\right\}, \tag{3.33}$$

which appeared in [57].

Theorem 3.5 *If $f:[a,b]\to\mathbb{R}$ is a continuous function such that $|_aD_{p,q}f|^s$ is a convex function and a (p,q)-integrable function on $[a,b]$, where $\frac{7}{8}\leq q<1$ and $s\geq 1$, then*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{3a+\xi(b,a)}{3}\right)+3f\left(\frac{3a+2\xi(b,a)}{3}\right)+f\big(a+\xi(b,a)\big)\right]\right.$$
$$\left.-\frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)} f(t)_a\,d_{p,q}t\right|$$
$$\leq \xi(b,a)\left\{\big(\psi_1(p,q)\big)^{1-1/s}\big(\psi_2(p,q)\big|_aD_{p,q}f(a)\big|^s+\psi_3(p,q)\big|_aD_{p,q}f(b)\big|^s\big)^{1/s}\right.$$
$$+\big(\psi_4(p,q)\big)^{1-1/s}\big(\psi_5(p,q)\big|_aD_{p,q}f(a)\big|^s+\psi_6(p,q)\big|_aD_{p,q}f(b)\big|^s\big)^{1/s}$$
$$\left.+\big(\psi_7(p,q)\big)^{1-1/s}\big(\psi_8(p,q)\big|_aD_{p,q}f(a)\big|^s+\psi_9(p,q)\big|_aD_{p,q}f(b)\big|^s\big)^{1/s}\right\}, \tag{3.34}$$

where $\psi_i(p,q)$, $i=1,2,\dots,9$, are defined by

$$\psi_1(p,q)=\frac{20q^2-12pq+9q+9p-9}{288q[2]_{p,q}};$$

$$\psi_2(p,q)=\frac{1}{6912q^2[2]_{p,q}[3]_{p,q}}\big[480q^5+192pq^4+56q^4+192p^2q^3+272pq^3-216q^3$$
$$-288p^3q^2+528p^2q^2-216pq^2-27q^2+216p^3q-216p^2q-27pq+27q$$
$$-27p^2+27p\big];$$

$$\psi_3(p,q) = \frac{160q^4 + 160pq^3 - 96p^2q^2 + 27q^2 + 27pq - 27q + 27p^2 - 27p}{6912q^2[2]_{p,q}[3]_{p,q}};$$

$$\psi_4(p,q) = \frac{q^2 - 9pq + 9q + 9p - 9}{18q[2]_{p,q}};$$

$$\begin{aligned}\psi_5(p,q) = {}& \frac{1}{108q^2[2]_{p,q}[3]_{p,q}}\bigl[6q^5 - 48pq^4 + 48q^4 - 48p^2q^3 + 102pq^3 - 54q^3 - 54p^3q^2 \\ &+ 138p^2q^2 - 54pq^2 - 27q^2 + 54p^3q - 54p^2q - 27pq + 27q - 27p^2 + 27p\bigr];\end{aligned}$$

$$\psi_6(p,q) = \frac{6q^4 + 6pq^3 - 30p^2q^2 + 27q^2 + 27pq - 27q + 27p^2 - 27p}{108q^2[2]_{p,q}[3]_{p,q}};$$

$$\psi_7(p,q) = \frac{-4q^2 - 420pq + 441q + 441p - 441}{288q[2]_{p,q}};$$

$$\begin{aligned}\psi_8(p,q) = {}& \frac{1}{6912q^2[2]_{p,q}[3]_{p,q}}\bigl[-96q^5 - 10{,}176pq^4 + 10{,}360q^4 - 10{,}176p^2q^3 \\ &+ 20{,}944pq^3 - 10{,}584q^3 - 10{,}080p^3q^2 + 29{,}904p^2q^2 - 10{,}584pq^2 - 9261q^2 \\ &+ 10{,}584p^3q - 10{,}584p^2q - 9261pq + 9261q - 9261p^2 + 9261p\bigr];\end{aligned}$$

$$\psi_9(p,q) = \frac{224q^4 + 224pq^3 - 8736p^2q^2 + 9261q^2 + 9261pq - 9261q + 9261p^2 - 9261p}{6912q^2[2]_{p,q}[3]_{p,q}}.$$

Proof Using Theorem 3.1, the Hölder inequality, and the preinvexity of $|{}_aD_{p,q}f|^s$, we have

$$\begin{aligned}&\biggl|\frac{1}{8}\biggl[f(a) + 3f\biggl(\frac{3a+\xi(b,a)}{3}\biggr) + 3f\biggl(\frac{3a+2\xi(b,a)}{3}\biggr) + f\bigl(a+\xi(b,a)\bigr)\biggr] \\ &\quad - \frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)} f(t)_a\,d_{p,q}t\biggr| \\ &= \biggl|\xi(b,a)\int_0^1 \varphi(t)_aD_{p,q}f\bigl(a+t\xi(b,a)\bigr)\,d_{p,q}t\biggr| \\ &= \xi(b,a)\biggl|\int_0^{1/3}\biggl(qt-\frac{1}{8}\biggr)_aD_{p,q}f\bigl(a+t\xi(b,a)\bigr)\,d_{p,q}t \\ &\quad + \int_{1/3}^{2/3}\biggl(qt-\frac{1}{2}\biggr)_aD_{p,q}f\bigl(a+t\xi(b,a)\bigr)_a\,d_{p,q}t \\ &\quad + \int_{2/3}^{1}\biggl(qt-\frac{7}{8}\biggr)_aD_{p,q}f\bigl(a+t\xi(b,a)b\bigr)_a\,d_{p,q}t\biggr| \\ &\le \xi(b,a)\biggl[\int_0^{1/3}\biggl|qt-\frac{1}{8}\biggr|\bigl|{}_aD_{p,q}f\bigl(a+t\xi(b,a)\bigr)\bigr|\,d_{p,q}t \\ &\quad + \int_{1/3}^{2/3}\biggl|qt-\frac{1}{2}\biggr|\bigl|{}_aD_{p,q}f\bigl(a+t\xi(b,a)\bigr)\bigr|_a\,d_{p,q}t \\ &\quad + \int_{2/3}^{1}\biggl|qt-\frac{7}{8}\biggr|\bigl|{}_aD_{p,q}f\bigl(a+t\xi(b,a)\bigr)\bigr|_a\,d_{p,q}t\biggr] \\ &\le (b-a)\biggl(\int_0^{1/3}\biggl|qt-\frac{1}{8}\biggr|\,d_{p,q}t\biggr)^{1-1/s}\biggl(\int_0^{1/3}\biggl|qt-\frac{1}{8}\biggr|\bigl|{}_aD_{p,q}f\bigl(a+t\xi(b,a)\bigr)\bigr|^s\,d_{p,q}t\biggr)^{1/s} \\ &\quad + (b-a)\biggl(\int_{1/3}^{2/3}\biggl|qt-\frac{1}{2}\biggr|\,d_{p,q}t\biggr)^{1-1/s}\biggl(\int_{1/3}^{2/3}\biggl|qt-\frac{1}{2}\biggr|\bigl|{}_aD_{p,q}f\bigl(a+t\xi(b,a)\bigr)\bigr|^s\,d_{p,q}t\biggr)^{1/s}\end{aligned}$$

$$
\begin{aligned}
&+(b-a)\left(\int_{2/3}^{1}\left|qt-\frac{7}{8}\right|d_{p,q}t\right)^{1-1/s}\left(\int_{2/3}^{1}\left|qt-\frac{7}{8}\right|\left|{}_aD_{p,q}f\left(a+t\xi(b,a)\right)\right|^s d_{p,q}t\right)^{1/s}\\
\le{}&(b-a)\left(\int_{0}^{1/3}\left|qt-\frac{1}{8}\right|d_{p,q}t\right)^{1-1/s}\\
&\times\left(\left|{}_aD_{p,q}f(a)\right|^s\int_{0}^{1/3}(1-t)\left|qt-\frac{1}{8}\right|d_{p,q}t+\left|{}_aD_{p,q}f(b)\right|^s\int_{0}^{1/3}t\left|qt-\frac{1}{8}\right|d_{p,q}t\right)^{1/s}\\
&+(b-a)\left(\int_{1/3}^{2/3}\left|qt-\frac{1}{2}\right|d_{p,q}t\right)^{1-1/s}\\
&\times\left(\left|{}_aD_{p,q}f(a)\right|^s\int_{1/3}^{\frac{2}{3}}(1-t)\left|qt-\frac{1}{2}\right|d_{p,q}t+\left|{}_aD_{p,q}f(b)\right|^s\int_{1/3}^{2/3}t\left|qt-\frac{1}{2}\right|d_{p,q}t\right)^{1/s}\\
&+(b-a)\left(\int_{2/3}^{1}\left|qt-\frac{7}{8}\right|d_{p,q}t\right)^{1-1/s}\\
&\times\left(\left|{}_aD_{p,q}f(a)\right|^s\int_{2/3}^{1}(1-t)\left|qt-\frac{7}{8}\right|d_{p,q}t+\left|{}_aD_{p,q}f(b)\right|^s\int_{2/3}^{1}t\left|qt-\frac{7}{8}\right|d_{p,q}t\right)^{1/s}.
\end{aligned}
$$

Using Definition 2.2, Theorem 2.1, and Lemma 2.1, we have

$$
\begin{aligned}
\psi_1(p,q)&=\int_0^{1/3}\left|qt-\frac{1}{8}\right|d_{p,q}t\\
&=\frac{20q^2-12pq+9q+9p-9}{288q[2]_{p,q}};\\
\psi_2(p,q)&=\int_0^{1/3}\left|qt-\frac{1}{8}\right|(1-t)\,d_{p,q}t\\
&=\frac{1}{6912q^2[2]_{p,q}[3]_{p,q}}\big[480q^5+192pq^4+56q^4+192p^2q^3+272pq^3-216q^3\\
&\quad-288p^3q^2+528p^2q^2-216pq^2-27q^2+216p^3q-216p^2q-27pq+27q\\
&\quad-27p^2+27p\big];\\
\psi_3(p,q)&=\int_0^{1/3}\left|qt-\frac{1}{8}\right|t\,d_{p,q}t\\
&=\frac{160q^4+160pq^3-96p^2q^2+27q^2+27pq-27q+27p^2-27p}{6912q^2[2]_{p,q}[3]_{p,q}};\\
\psi_4(p,q)&=\int_{1/3}^{2/3}\left|qt-\frac{1}{2}\right|d_{p,q}t\\
&=\frac{q^2-9pq+9q+9p-9}{18q[2]_{p,q}};\\
\psi_5(p,q)&=\int_{1/3}^{2/3}\left|qt-\frac{1}{2}\right|(1-t)\,d_{p,q}t\\
&=\frac{1}{108q^2[2]_{p,q}[3]_{p,q}}\big[6q^5-48pq^4+48q^4-48p^2q^3+102pq^3-54q^3-54p^3q^2\\
&\quad+138p^2q^2-54pq^2-27q^2+54p^3q-54p^2q-27pq+27q-27p^2+27p\big];
\end{aligned}
$$

$$\begin{aligned}
\psi_6(p,q) &= \int_{1/3}^{2/3}\left|qt-\frac{1}{2}\right|t\,d_{p,q}t \\
&= \frac{6q^4+6pq^3-30p^2q^2+27q^2+27pq-27q+27p^2-27p}{108q^2[2]_{p,q}[3]_{p,q}}; \\
\psi_7(p,q) &= \int_{2/3}^{1}\left|qt-\frac{7}{8}\right|d_{p,q}t \\
&= \frac{-4q^2-420pq+441q+441p-441}{288q[2]_{p,q}}; \\
\psi_8(p,q) &= \int_{2/3}^{1}\left|qt-\frac{7}{8}\right|(1-t)\,d_{p,q}t \\
&= \frac{1}{6912q^2[2]_{p,q}[3]_{p,q}}\bigl[-96q^5-10{,}176pq^4+10{,}360q^4-10{,}176p^2q^3 \\
&\quad +20{,}944pq^3-10{,}584q^3-10{,}080p^3q^2+29{,}904p^2q^2-10{,}584pq^2-9261q^2 \\
&\quad +10{,}584p^3q-10{,}584p^2q-9261pq+9261q-9261p^2+9261p\bigr]; \\
\psi_9(p,q) &= \int_{1/3}^{1}\left|qt-\frac{7}{8}\right|t\,d_{p,q}t \\
&= \frac{224q^4+224pq^3-8736p^2q^2+9261q^2+9261pq-9261q+9261p^2-9261p}{6912q^2[2]_{p,q}[3]_{p,q}}.
\end{aligned}$$

Hence, we gain (3.35). Therefore, the proof is completed. □

Corollary 3.4 *If $f:[a,b]\to\mathbb{R}$ is a continuous function such that $|{}_aD_{p,q}f|^s$ is a convex function and a (p,q)-integrable function on $[a,b]$, where $\frac{7}{8}\le q<1$ and $s\ge 1$, then*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a}f(x)_a\,d_{p,q}x\right| \\
&\quad\le (b-a)\Bigl\{\bigl(\psi_1(p,q)\bigr)^{1-1/s}\bigl(\psi_2(p,q)\bigl|{}_aD_{p,q}f(a)\bigr|^s+\psi_3(p,q)\bigl|{}_aD_{p,q}f(b)\bigr|^s\bigr)^{1/s} \\
&\qquad+\bigl(\psi_4(p,q)\bigr)^{1-1/s}\bigl(\psi_5(p,q)\bigl|{}_aD_{p,q}f(a)\bigr|^s+\psi_6(p,q)\bigl|{}_aD_{p,q}f(b)\bigr|^s\bigr)^{1/s} \\
&\qquad+\bigl(\psi_7(p,q)\bigr)^{1-1/s}\bigl(\psi_8(p,q)\bigl|{}_aD_{p,q}f(a)\bigr|^s+\psi_9(p,q)\bigl|{}_aD_{p,q}f(b)\bigr|^s\bigr)^{1/s}\Bigr\},
\end{aligned}\tag{3.35}$$

where $\psi_i(p,q)$, $i=1,2,\dots,9$, are given in Theorem 3.5.

Remark 3.5 *If $p=1$, then (3.35) reduces to*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^{b}f(x)_a\,d_qx\right| \\
&\quad\le (b-a)\left\{\left(\frac{20q-3}{288[2]_q}\right)^{1-1/s}\right. \\
&\qquad\times\left(\frac{480q^3+248q^2+248q-3}{6912[2]_q[3]_q}\bigl|{}_aD_qf(a)\bigr|^s+\frac{160q^2+160q-69}{6912[2]_q[3]_q}\bigl|{}_aD_qf(b)\bigr|^s\right)^{1/s} \\
&\qquad+\left(\frac{q}{18[2]_q}\right)^{1-1/s}\left(\frac{6q^3+3}{108[2]_q[3]_q}\bigl|{}_aD_qf(a)\bigr|^s+\frac{6q^2+6q-3}{108[2]_q[3]_q}\bigl|{}_aD_qf(b)\bigr|^s\right)^{1/s}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{21-4q}{288[2]_q}\right)^{1-1/s}\\
&\times\left(\frac{-96q^3+184q^2+148q-21}{6912[2]_q[3]_q}\left|{}_aD_qf(a)\right|^s\right.\\
&\left.+\frac{224q^2+224q+525}{6912[2]_q[3]_q}\left|{}_aD_qf(b)\right|^s\right)^{1/s}\Bigg\},
\end{aligned}
\tag{3.36}
$$

which appeared in [50]. *In addition, if $q\to 1$, then* (3.36) *reduces to*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
&\leq(b-a)\left\{\left(\frac{17}{576}\right)^{1-1/s}\left(\frac{973|f'(a)|^s+251|f'(b)|^s}{41{,}472}\right)^{1/s}\right.\\
&\left.+\left(\frac{1}{36}\right)^{1-1/s}\left(\frac{|f'(a)|^s+|f'(b)|^s}{2}\right)^{1/s}+\left(\frac{17}{576}\right)^{1-1/s}\left(\frac{251|f'(a)|^s+973|f'(b)|^s}{41{,}472}\right)^{1/s}\right\},
\end{aligned}
$$

which appeared in [57].

4 Examples

In this section, we give some examples of our main results.

Example 4.1 *Define function $f:[0,1]\to\mathbb{R}$ by $f(x)=2x+5$. Then $|{}_aD_{p,q}f(x)|=|{}_aD_{p,q}(2x+5)|=2$ is a convex function and a (p,q)-integrable function on $[0,1]$. Applying Corollary* 3.2 *with $p=1$ and $q=\frac{9}{10}$, the left-hand side of* (3.12) *becomes*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a}f(x)_a\,d_{p,q}x\right|\\
&=\left|\frac{1}{8}\left[f(0)+3f\left(\frac{2\cdot 0+1}{3}\right)+3f\left(\frac{0+2\cdot 1}{3}\right)+f(1)\right]\right.\\
&\quad\left.-\frac{1}{1\cdot(1-0)}\int_0^{1\cdot 1+(1-1)\cdot 0}(2x+5)_0\,d_{1,\frac{9}{10}}x\right|\\
&=\left|\frac{1}{8}[5+17+19+7]-\frac{115}{19}\right|\approx 0.05263158,
\end{aligned}
$$

and the right-hand side of (3.12) *becomes*

$$
\begin{aligned}
&(b-a)\left[M_1(p,q)\left|{}_aD_{p,q}f(a)\right|+M_2(p,q)\left|{}_aD_{p,q}f(b)\right|\right]\\
&=(1-0)\left[M_1\left(1,\frac{9}{10}\right)\left|{}_aD_{1,\frac{9}{10}}f(0)\right|+M_2\left(1,\frac{9}{10}\right)\left|{}_aD_{1,\frac{9}{10}}f(1)\right|\right]\\
&=(1-0)\left[\frac{161}{3907}(2)+\frac{39}{880}(2)\right]\approx 0.17105263.
\end{aligned}
$$

It is clear that

$$0.05263158\leq 0.17105263,$$

which demonstrates the result described in Corollary 3.2.

Example 4.2 *Define function* $f:[0,1]\to\mathbb{R}$ *by* $f(x)=2x+1$. *Then* $|{}_aD_{p,q}f(x)|^s=|{}_aD_{p,q}(1-x)|^s=2^s$ *is a convex function and a* (p,q)*-integrable function on* $[0,1]$. *Applying Corollary 3.3 with* $p=1$, $q=\frac{9}{10}$, $r=2$, *and* $s=2$, *the left-hand side of (3.22) by using Example 4.1 becomes*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a}f(x)_a\,d_{p,q}x\right|\\&\quad=\left|\frac{1}{8}[5+17+19+7]-\frac{115}{19}\right|\approx 0.05263158,\end{aligned}$$

and the right-hand side of (3.22) becomes

$$\begin{aligned}&(b-a)\left\{\left(\frac{[3^{r+1}+(8q-3)^{r+1}](p-q)}{24^{r+1}q(p^{r+1}-q^{r+1})}\right)^{1/r}\right.\\&\quad\times\left(\frac{(3q+3p-1)|{}_aD_{p,q}f(a)|^s+|{}_aD_{p,q}f(b)|^s}{9[2]_{p,q}}\right)^{1/s}\\&\quad+\left(\frac{[(3-2q)^{r+1}+(4q-3)^{r+1}](p-q)}{6^{r+1}q(p^{r+1}-q^{r+1})}\right)^{1/r}\\&\quad\times\left(\frac{(q+p-1)|{}_aD_{p,q}f(a)|^s+|{}_aD_{p,q}f(b)|^s}{3[2]_{p,q}}\right)^{1/s}\\&\quad+\left(\frac{[(21-16q)^{r+1}+(24q-21)^{r+1}](p-q)}{24^{r+1}q(p^{r+1}-q^{r+1})}\right)^{1/r}\\&\quad\left.\times\left(\frac{(3q+3p-5)|{}_aD_{p,q}f(a)|^s+5|{}_aD_{p,q}f(b)|^s}{9[2]_{p,q}}\right)^{1/s}\right\}\\&\quad=(1-0)\left\{\left(\frac{13}{4336}\right)^{1/2}\left(\frac{4}{3}\right)^{1/2}+\left(\frac{8}{243}\right)^{1/2}\left(\frac{4}{3}\right)^{1/2}+\left(\frac{216}{2533}\right)^{1/2}\left(\frac{4}{3}\right)^{1/2}\right\}\\&\quad\approx 0.60993243.\end{aligned}$$

It is clear that

$$0.05263158\le 0.60993243,$$

which demonstrates the result described in Theorem 3.3.

5 Conclusion

In this work, we used (p,q)-calculus to establish new integral inequalities related to Simpson's second type inequalities for preinvex functions. The presented results in this study generalize and extend some previous inequalities in the literature of Simpson's second type inequalities. Moreover, some examples were given to show the investigated results.

Acknowledgements
This research was supported by the Fundamental Fund of Khon Kaen University. The first author is supported by Development and Promotion of Science and Technology talents project (DPST), Thailand. We would like to thank anonymous referees for comments which are helpful for improvement in this paper.

Funding
This research was funded by National Science, Research and Innovation Fund (NSRF), and King Mongkut's University of Technology North Bangkok with Contract no. KMUTNB-FF-66-11.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author contribution
WL was a major contributor to writing the manuscript, conceptualization, investigation, and validation. KN performed the formal analysis, funding acquisition, validation, edition of the original draft preparation, and writing a revised version. JT dealt with the formal analysis, validation, and supervision. SKN dealt with the methodology, investigation, formal analysis, and validation. HB performed conceptualization, formal analysis, and validation. All authors read and approved the final manuscript.

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Publisher's Note

Received: 23 June 2021 Accepted: 30 November 2022 Published online: 15 December 2022

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