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Nonlinear operators between neutrosophic normed spaces and Fréchet differentiation

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Abstract

The article focuses on the introduction of neutrosophic continuity and neutrosophic boundedness, which is a fair extension of intuitionistic fuzzy continuity and intuitionistic fuzzy boundedness, respectively. The article further advances to illustrate the Fréchet derivative of nonlinear operators between neutrosophic normed spaces (NNS). Examples have been provided in compliance with the theory with the aid of some standard sequence spaces.

Keywords: Bounded Linear operator; Fréchet derivative; Continuity; Neutrosophic normed space

1 Introduction

The researchers persistently come across distinct mathematical problems that quite often fail to get solved or analyzed by the classical theory of nonlinear analysis. A handful of the arduous problems can be reduced to operator equations to analyze them, and the Fréchet derivative plays a crucial role in solving such problems. The extension of crisp sets was put forth by Zadeh [1], wherein each element had a degree of membership indicating the extent to which an element belongs in a set. This notion is a strong mathematical tool to deal with the complexity of uncertainty in the form of vagueness in several problems arising in the area of science and engineering. It has valuable applications in areas such as population dynamics [2], computer programming [3], chaos control [4, 5]. As an extension of fuzzy sets, Atanassov [6] proposed the concept of intuitionistic fuzzy sets in 1986, which incorporated the degree of non-membership and hesitant function along with the degree of membership. A few papers depicting convergence in the setting of intuitionistic fuzzy sets were defined by Mursaleen et al. [7, 8], where he analyzed the statistical and ideal convergence in intuitionistic fuzzy topological space. Recently, Khan et al. [9] studied the continuous and bounded linear operators in neutrosophic normed spaces.

In the ultimate presence of quantum gravitational effects, the theory of the Heisenberg uncertainty principle must be generalized on the basis of fuzzy structure spacetime. These facts were mainly encouraged by string theory and non-commutative geometry. Quantum gravity robustly explained the spacetime points in a fuzzy way. So, the contrariety of determining the exact location of articles gives spacetime a vague structure [4, 10, 11]. Due to this ambiguous structure, the position space depiction of quantum mechanics may break down. Therefore, in this manner, a generalized normed space of quasi-position eigenfunction is specifically required [12]. It is monitored that in the quantum gravity regime, the most basic idea of spacetime in self-identity causes diffusion on the wave packet profile. The formed fact originates in the quantum fluctuation of spacetime, best described as fuzzy spacetime.

Smarandache et al. [13] projected another idea, which can be said to be a neutrosophic set by counting a transitional membership function, which was expected to be a formal setting trying to compute the truth, indeterminacy, and falsehood. Later on in [14], he furthermore studied the contrariness between intuitionistic fuzzy set and neutrosophic set, as well as the relations between these two sets [10]. In the generated neutrosophic set, the components T (truth), S (indeterminacy), and U (falsity) have values in between $]0^-, 1^+[$.

Bera and Mahapatra [15, 16] imported the neutrosophic soft normed linear space and defined convexity, metric, and Cauchy sequence on it. Recently, Kirişci & Şimşek [12] examined the statistical convergence on *NNS*, while Khan et al. [17] evaluated some results of *NNS* via the Fibonacci matrix. By reviewing the literature, we can conclude that there has been exponential growth in the study of neutrosophic theory in dissimilar fields by various researchers. There are a number of conditions where the norm of a vector is impossible to get, and the concept of neutrosophic norm [18–20] looks more authentic in such cases. These theories can best deal with situations by modeling the inexactness through the neutrosophic norm.

This presented paper resolves and studies the Fréchet derivative of nonlinear operators between *NNS*, which may produce a helpful functional tool to explain the operator equations. Likewise, particular and universal discordance analysis of solutions of operator mathematical equations are determinate over this concept. So, here we have tried a proper approach of nonlinear functional analysis of operator equations by applying Fréchet derivative.

2 Preliminaries

Definition 2.1 ([6]) Let $F \neq \phi$ be the intuitionistic fuzzy set, and $F \subseteq Y$ is an ordered triplet defined by

$$F = \{\langle \eta, \mathbf{T}(\eta), \mathbf{U}(\eta)\rangle : \eta \in Y\},$$

where $\mathbf{T}(\eta), \mathbf{U}(\eta) : Y \to [0,1]$ represent the degree of membership and degree of non-membership, respectively, in such a way that

$$0 \leq \mathbf{T}(\eta) + \mathbf{U}(\eta) \leq 1$$

$1 - \mathbf{T}(\eta) - \mathbf{U}(\eta)$ is also said to be degree of hesitancy. The intuitionistic fuzzy components $\mathbf{T}(\eta)$, $\mathbf{U}(\eta)$ and degree of hesitancy depend on each other.

Definition 2.2 ([14]) Let $A \neq \phi$, $A \subseteq Y$. Then,

$$A_{NS} = \{\langle \eta, \mathbf{T}(\eta), \mathbf{S}(\eta), \mathbf{U}(\eta)\rangle : \eta \in Y\},$$

where $\mathbf{T}(\eta), \mathbf{S}(\eta), \mathbf{U}(\eta) : Y \to [0,1]$ represent the degree of membership, degree of indeterminacy, and degree of nonmembership, respectively, in such a way that

$$0 \le \mathbf{T}(\eta) + \mathbf{S}(\eta) + \mathbf{U}(\eta) \le 3.$$

The neutrosophic components $\mathbf{T}(\eta)$, $\mathbf{S}(\eta)$ and $\mathbf{U}(\eta)$ are independent of each other.

Definition 2.3 ([21]) A continuous t-norm is a binary operation $\star : [0,1] \times [0,1] \to [0,1]$ with the following conditions: (i) $\star$ is associative and commutative; (ii) $\star$ is continuous; (iii) $\rho_1 \star 1 = \rho_1$, $\forall \rho_1 \in [0,1]$; (iv) $\rho_1 \star \rho_2 \le \rho_3 \star \rho_4$ whenever $\rho_1 \le \rho_3$ and $\rho_2 \le \rho_4$, for each $\rho_1, \rho_2, \rho_3, \rho_4 \in [0,1]$.

Definition 2.4 ([21]) A continuous t-conorm is a binary operation $\circ : [0,1] \times [0,1] \to [0,1]$ with the following conditions (i) $\circ$ is associative and commutative; (ii) $\circ$ is continuous; (iii) $\rho_1 \circ 0 = \rho_1$, $\forall \rho_1 \in [0,1]$; (iv) $\rho_1 \circ \rho_2 \le \rho_3 \circ \rho_4$ whenever $\rho_1 \le \rho_3$ and $\rho_2 \le \rho_4$, for each $\rho_1, \rho_2, \rho_3, \rho_4 \in [0,1]$.

Definition 2.5 ([22]) Consider Y be a linear space and $\mathcal{M} = \{\langle \eta, \mathbf{T}(\eta), \mathbf{S}(\eta), \mathbf{U}(\eta)\rangle : \eta \in Y\}$ be a normed space in such a way that $\mathcal{M} : Y \times R^+ \to [0,1]$. Let $\bullet$ and $\star$ be continuous t-norm and continuous t-conorm, respectively. Then, the four-tuple $(Y, \mathcal{M}, \bullet, \star)$ is called neutrosophic normed space (NNS) if it satisfies the following axioms, $\forall \eta, y, z \in Y$ and $d, t > 0$;

(i) $0 \le \mathbf{T}(\eta,t), \mathbf{S}(\eta,t), \mathbf{U}(\eta,t) \le 1$,
(ii) $0 \le \mathbf{T}(\eta,t) + \mathbf{S}(\eta,t) + \mathbf{U}(\eta,t) \le 3$,
(iii) $\mathbf{T}(\eta,t) = 0$ for $t \le 0$,
(iv) $\mathbf{T}(\eta,t) = 1$ for $t > 0$ iff $\eta = 0$
(v) $\mathbf{T}(c\eta,t) = \mathbf{T}(\eta, \frac{t}{|c|})$ $\forall$ $c \neq 0$, $t > 0$,
(vi) $\mathbf{T}(\eta,t) \bullet \mathbf{T}(y,d) \le \mathbf{T}(\eta + y, t + d)$
(vii) $\mathbf{T}(\eta, \bullet)$ is continuous nondecreasing function for $t > 0$, $\lim_{t\to\infty} \mathbf{T}(\eta,t) = 1$
(viii) $\mathbf{S}(\eta,t) = 1$ for $t \le 0$,
(ix) $\mathbf{S}(\eta,t) = 0$ for $t > 0$ iff $\eta = 0$
(x) $\mathbf{S}(c\eta,t) = \mathbf{S}(\eta, \frac{t}{|c|})$, $\forall$ $c \neq 0$
(xi) $\mathbf{S}(\eta,t) \star \mathbf{S}(y,d) \ge \mathbf{S}(\eta + y, t + d)$,
(xii) $\mathbf{S}(\eta, \star)$ is continuous nonincreasing function, $\lim_{t\to\infty} \mathbf{S}(\eta, \star) = 0$,
(xiii) $\mathbf{U}(\eta,t) = 1$ for $t \le 0$,
(xiv) $\mathbf{U}(\eta,t) = 0$ for $t > 0$ iff $\eta = 0$,
(xv) $\mathbf{U}(c\eta,t) = \mathbf{U}(\eta, \frac{t}{|c|})$ $\forall c \neq 0$
(xvi) $\mathbf{U}(\eta,t) \star \mathbf{U}(y,d) \ge \mathbf{U}(\eta + y, t + d)$,
(xvii) $\mathbf{U}(\eta, \star)$ is continuous nonincreasing function, $\lim_{t\to\infty} \mathbf{U}(\eta,t) = 0$.

In this case, $\mathcal{M}$ is called neutrosophic norm (*NN*).

Since all the components lie between $[0,1]$, therefore, if the indetreminacy is zero, then still neutrosophic components are more flexible and more general then fuzzy and intuitionistic fuzzy components. If all the neutrosohic components on a vector space satisfy all conditions of Definition 2.5, then we say that $(Y, \mathcal{M}, \bullet, \star)$ is *NNS*, where $\mathcal{M} = \{\langle \eta, \mathbf{T}(\eta), \mathbf{S}(\eta), \mathbf{U}(\eta)\rangle : \eta \in Y\}$.

Neutrosophic normed space is more general than norm space. This is illustrated by proper example

Let $(Y, \mathcal{M}, \bullet, \star)$ be NNS. Assume that $\eta \bullet y = \eta y$ and $\eta \star y = \eta + y - \eta y$ for all $\eta, y \in Y$ and $t > 0$ with the condition

$$\mathbf{T}(\eta, t) > 0 \quad \text{and} \quad \mathbf{S}(\eta, t) < 1, \qquad \mathbf{U}(\eta, t) < 1 \quad \Longrightarrow \quad \eta = 0 \quad \text{for all } t > 0.$$

Let $\|\eta\|_\beta = \inf\{t > 0 : \mathbf{T}(\eta, t) \geq \beta \text{ and } \mathbf{S}(\eta, t) \leq 1 - \beta, \mathbf{U}(\eta, t) \leq 1 - \beta\}$, $\forall \beta \in (0,1)$. Then, $\{\| \cdot \|_\beta : \beta \in (0,1)\}$ is an ascending family of norms on Y. These norms are said to be β-norms on Y compatible to neutrosophic norm $(\mathbf{T}, \mathbf{S}, \mathbf{U})$.

3 Neutrosophic continuity

Definition 3.1 ([23]) Let $(Y, \mathbf{T}_1, \mathbf{S}_1, \mathbf{U}_1, \bullet, \star)$ and $(W, \mathbf{T}', \mathbf{S}', \mathbf{U}' \bullet, \star)$ be two *NNS*, and consider a mapping h from $(Y, \mathbf{T}_1, \mathbf{S}_1, \mathbf{U}_1, \bullet, \star) \to (W, \mathbf{T}', \mathbf{S}', \mathbf{U}' \bullet, \star)$. Then,

(i) ξ is called strong neutrosophic limit of h at some $\eta_0 \in Y$ if for each $\epsilon > 0$, $\exists$ some $\delta = \delta(\epsilon) > 0$ such that

$$\begin{aligned}
&\mathbf{T}'\big(h(\eta) - \xi, \epsilon\big) \geq \mathbf{T}_1(\eta - \eta_0, \delta), \\
&\mathbf{S}'\big(h(\eta) - \xi, \epsilon\big) \leq \mathbf{S}_1(\eta - \eta_0, \delta), \\
&\mathbf{U}'\big(h(\eta) - \xi, \epsilon\big) \leq \mathbf{U}_1(\eta - \eta_0, \delta).
\end{aligned}$$

In this case, we write *(Strong Neutrosophic-SN)*-$\lim_{\eta \to \eta_0} h(\eta) = \xi$, which also means that

$$\begin{aligned}
&\lim_{\mathbf{T}_1(\eta - \eta_0, t) \to 1} \mathbf{T}'\big(h(\eta) - \xi, t\big) = 1, (SN) \quad \text{and} \\
&\lim_{\mathbf{S}_1(\eta - \eta_0, t) \to 0} \mathbf{S}'\big(h(\eta) - \xi, t\big) = 0\ (SN), \\
&\lim_{\mathbf{U}_1(\eta - \eta_0, t) \to 0} \mathbf{U}'\big(h(\eta) - \xi, t\big) = 0\ (SN)
\end{aligned}$$

or

$$\begin{cases}
\mathbf{T}_1(h(\eta) - \xi, t) = \xi, (SN) \text{ as } \mathbf{T}'(\eta - \eta_0, t) \to 1, \text{ and } \mathbf{S}_1(h(\eta) - \xi, t) = \xi, (SN) \text{ as} \\
\mathbf{S}'(\eta - \eta_0, t) \to 0\ \mathbf{U}_1(h(\eta) - \xi, t) = \xi, (SN) \text{ as } \mathbf{U}'(\eta - \eta_0, t) \to 0
\end{cases}$$

$\forall t > 0$.

(ii) ξ is called weak neutrosophic limit of h at some $\eta_0 \in Y$ if for given $\epsilon > 0$ & $\beta \in (0,1)$, $\delta = \delta(\epsilon, \beta) > 0$ s.t

$$\begin{aligned}
&\mathbf{T}_1(\eta - \eta_0, \delta) \geq \beta \quad \Longrightarrow \quad \mathbf{T}'\big(h(\eta) - \xi, \epsilon\big) \quad \text{and} \\
&\mathbf{S}'(\eta - \eta_0, \delta) \leq 1 - \beta \quad \Longrightarrow \quad \mathbf{S}_1\big(h(\eta) - \xi, \epsilon\big), \\
&\mathbf{U}'(\eta - \eta_0, \delta) \leq 1 - \beta \quad \Longrightarrow \quad \mathbf{U}_1\big(h(\eta) - \xi, \epsilon\big).
\end{aligned}$$

In this case, we write *(Weak Neutrosophic-WN)*-$\lim_{\eta\to\eta_0} h(\eta) = \xi$, which also means that

$$\lim_{\mathbf{T}_1(\eta-\eta_0,t)\to 1} \mathbf{T}'\big(h(\eta)-\xi,t\big) = \xi,(WN) \quad \text{and}$$

$$\lim_{\mathbf{S}_1(\eta-\eta_0,t)\to 0} \mathbf{S}'\big(h(\eta)-\xi,t\big) = \xi(SN),$$

$$\lim_{\mathbf{U}_1(\eta-\eta_0,t)\to 0} \mathbf{U}'\big(h(\eta)-\xi,t\big) = \xi(SN).$$

or

$$\begin{cases} \mathbf{T}'(h(\eta)-\xi,t)=\xi,(WN) \text{ as } \mathbf{T}_1(\eta-\eta_0,t)\to 1, \text{ and } \mathbf{S}'(h(\eta)-\xi,t)=\xi,(WN) \text{ as} \\ \mathbf{S}_1(\eta-\eta_0,t)\to 0 \ \mathbf{U}'(h(\eta)-\xi,t)=\xi,(WN) \text{ as } \mathbf{U}_1(\eta-\eta_0,t)\to 0 \end{cases}$$

for all $t > 0$.

Definition 3.2 Let $h_n : (Y,\mathbf{T}_1,\mathbf{S}_1,\mathbf{U}_1,\bullet,\star) \to (W,\mathbf{T}',\mathbf{S}',\mathbf{U}',\bullet,\star)$ be a sequence of functions. Then Ψ_n is called pointwise neutrosophic convergent on Y to a function Ψ w.r.t $(\mathbf{T},\mathbf{S},\mathbf{U})$ if for each $\eta\in Y$, the sequence $(\Psi_n(\eta))$ is convergent to $\Psi(\eta)$ w.r.t $(\mathbf{T},\mathbf{S},\mathbf{U})$.

Definition 3.3 ([12]) Suppose $(Y,\mathcal{M},\bullet,\star)$ is an *NNS*. A sequence $m=(m_i)$ is said to be a Cauchy sequence w.r.t $\mathcal{M}$, if for each $\epsilon>0$ & $t>0$, $\exists\ d\in\mathbb{N}$ s.t $\mathbf{T}(m_i-m_k,t)>1-\epsilon$, $\mathbf{S}(m_i-m_k,t)<\epsilon$ and $\mathbf{U}(m_i-m_k,t)<\epsilon$ for all $i,k\geq d$.

Definition 3.4 Suppose $(Y,\mathbf{T}_1,\mathbf{S}_1,\mathbf{U}_1,\bullet,\star)$ and $(W,\mathbf{T}',\mathbf{S}',\mathbf{U}',\bullet,\star)$ are two *NNS*. A mapping h from $(Y,\mathbf{T}_1,\mathbf{S}_1,\mathbf{U}_1,\bullet,\star)\to(W,\mathbf{T}',\mathbf{S}',\mathbf{U}'\bullet,\star)$ is called neutrosophic continuous at $\rho_0\in Y$ if for any given $\epsilon>0$, $\exists\ t=t(b,\epsilon)$, $d=d(b,\epsilon)\in(0,1)$ such that $\forall\rho\in Y$ and for all $b\in(0,1)$,

$$\begin{aligned} \mathbf{T}_1(\rho-\rho_0,t)>1-\gamma \quad &\Longrightarrow \quad \mathbf{T}'\big(\Psi(\rho)-\Psi(\rho_0),\epsilon\big)>1-b, \\ \mathbf{S}_1(\rho-\rho_0,t)<\gamma \quad &\Longrightarrow \quad \mathbf{S}'\big(\Psi(\rho)-\Psi(\rho_0),\epsilon\big)<b, \\ \mathbf{U}_1(\rho-\rho_0,t)<\gamma \quad &\Longrightarrow \quad \mathbf{U}'\big(\Psi(\rho)-\Psi(\rho_0),\epsilon\big)<b. \end{aligned}$$

Proposition 3.1 $(SN)-\lim \implies (WN)-\lim$ *but the contrary need not hold. Further,* $(WN)-\lim=(SN)-\lim$ *whenever* $(SN)-\lim$ *exists.*

Proof It is simple to show. Now, we prove that $(WN)-\lim$ does not imply $(SN)-\lim$ in regular. □

Example 3.1 Consider $Y=W=\mathbb{R}$

$$\mathbf{T}_1(\eta,t)=\begin{cases} \frac{t}{t+\|\eta\|} & \text{if } t>\|\eta\|, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathbf{T}'(\eta,t)=\begin{cases} 1, & \text{if } t>\|\eta\|, \\ 0, & \text{if } t\leq\|\eta\|; \end{cases}$$

and

$$\mathbf{S}_1(\eta,t)=\begin{cases}\frac{\|\eta\|}{t+\|\eta\|} & \text{if } t>\|\eta\|,\\ 1 & \text{otherwise;}\end{cases}$$

$$\mathbf{S}'(\eta,t)=\begin{cases}1, & \text{if } t\leq\|\eta\|,\\ 0, & \text{if } t>\|\eta\|;\end{cases}$$

$$\mathbf{U}_1(\eta,t)=\begin{cases}\frac{\|\eta\|}{t} & \text{if } t>\|\eta\|,\\ 1 & \text{otherwise;}\end{cases}$$

$$\mathbf{U}'(\eta,t)=\begin{cases}1, & \text{if } t\leq\|\eta\|,\\ 0, & \text{if } t>\|\eta\|.\end{cases}$$

Let the function h from $(\mathbb{R},\mathbf{T}_1,\mathbf{S}_1,\mathbf{U}_1,\bullet,\star)$ onto $(\mathbb{R},\mathbf{T}',\mathbf{S}',\mathbf{U}',\bullet,\star)$ be defined by $h(\eta)=\eta$. Then $(WN)-\lim_{\eta\to 0}h(\eta)=0$. However, $(SN)-\lim_{\eta\to 0}h(\eta)$ does not exist. Because, for $\|\eta\|=\epsilon$ there is no $\delta>0$ satisfying the conditions

$$\mathbf{T}'(\eta,\epsilon)=0\geq\mathbf{T}_1(\eta,\delta)=\frac{\|\delta\|}{\delta+\|\eta\|}=\frac{\|\delta\|}{\delta+\|\epsilon\|}$$

and

$$\mathbf{S}'(\eta,\epsilon)=0\leq\mathbf{S}_1(\eta,\delta)=\frac{\|\epsilon\|}{\delta+\|\epsilon\|},$$

$$\mathbf{U}'(\eta,\epsilon)=0\leq\mathbf{U}_1(\eta,\delta)=\frac{\|\epsilon\|}{\delta+\|\epsilon\|}.$$

Now, we assign the strong and weak neutrosophic continuity of mappings between NNS.

Definition 3.5 ([24]) Suppose $(Y,\mathcal{M},\bullet,\star)$ is an NNS. & $V\subset Y$. Then V is called neutrosophic open subset Y if for every $\eta\in V$, there exists some $t>0$ and $\beta\in(0,1)$ such that $\mathcal{B}(\eta,\beta,t)\subseteq V$, where

$$\mathcal{B}(\eta,\beta,t):=\big\{y:\mathbf{T}(\eta-y,t)>1-\beta \text{ and } \mathbf{S}(\eta-y,t)<\beta,\mathbf{U}(\eta-y,t)<\beta\big\}.$$

Definition 3.6 Suppose $(Y,\mathbf{T}_1,\mathbf{S}_1,\mathbf{U}_1,\bullet,\star)$ and $(W,\mathbf{T}',\mathbf{S}',\mathbf{U}',\bullet,\star)$ are two *NNS*. A mapping $h:Y\to W$. Then h is called

(i) weakly neutrosophic continuous at $\eta_0\in Y$ if for given $\epsilon>0$ and $\beta\in(0,1)$ $\exists$ some $\delta=\delta(\epsilon,\beta)>0$ s.t

$$\mathbf{T}_1(\eta-\eta_0,\delta)\geq\beta \quad\Longrightarrow\quad \mathbf{T}'\big(h(\eta)-h(\eta_0),\epsilon\big) \quad \text{and}$$

$$\mathbf{S}_1(\eta-\eta_0,\delta)\leq 1-\beta \quad\Longrightarrow\quad \mathbf{S}'(h(\eta)-h(\eta_0,\epsilon),$$

$$\mathbf{U}_1(\eta-\eta_0,\delta)\leq 1-\beta \quad\Longrightarrow\quad \mathbf{U}'(h(\eta)-h(\eta_0,\epsilon),$$

for all $\eta\in Y$.

(*ii*) strongly neutrosophic continuous at $\eta_0 \in Y$ if for given $\epsilon > 0$ and $\beta \in (0,1)$ $\exists$ some $\delta = \delta(\epsilon,\beta) > 0$ such that

$$\mathbf{T}'\big(h(\eta) - h(\eta_0),\epsilon\big) \geq \mathbf{T}_1(\eta - \eta_0,\delta) \quad \text{and} \quad \mathbf{S}'(h(\eta) - h(\eta_0,\epsilon) \leq \mathbf{S}_1(\eta - \eta_0,\delta),$$
$$\mathbf{U}'(h(\eta) - h(\eta_0,\epsilon) \leq \mathbf{U}_1(\eta - \eta_0,\delta),$$

for all $\eta \in Y$.

(*iii*) Let h be linear. Then h is said be weakly neutrosophic bounded (for short, WNB) on Y if for given $\beta \in (0,1)$ $\exists$ some, $n_\beta > 0$ such that

$$\mathbf{T}_1\left(\eta, \frac{t}{n_\beta}\right) \geq \beta \quad \Longrightarrow \quad \mathbf{T}'\big(h(\eta),t\big) \geq \beta \quad \text{and}$$
$$\mathbf{S}_1\left(\eta, \frac{t}{n_\beta}\right) \leq 1-\beta \quad \Longrightarrow \quad \mathbf{S}'\big(h(\eta),t\big) \leq 1-\beta,$$
$$\mathbf{U}_1\left(\eta, \frac{t}{n_\beta}\right) \leq 1-\beta \quad \Longrightarrow \quad \mathbf{U}'\big(h(\eta),t\big) \leq 1-\beta,$$

for all $\eta \in Y$ and $t > 0$. Let $E'(Y,W)$ indicate the set of all WNB linear operators.

(*iv*) Let h be linear. Then h is called weakly neutrosophic bounded (for short, SNB) on Y if for given $\beta \in (0,1)$, $\exists$ some, $K > 0$ such that

$$\mathbf{T}'\big(h(\eta),t\big) \geq \mathbf{T}_1\left(\eta, \frac{t}{K}\right) \quad \text{and} \quad \mathbf{S}'\big(h(\eta),t\big) \leq \mathbf{S}_1\left(\eta, \frac{t}{K}\right),$$
$$\mathbf{U}'\big(h(\eta),t\big) \leq \mathbf{U}_1\left(\eta, \frac{t}{K}\right),$$

for all $\eta \in Y$ and $t > 0$. Let $E(Y,W)$ denote the set of all SNB linear operators.

Theorem 3.1 *Suppose* $(Y,\mathbf{T}_1,\mathbf{S}_1,\mathbf{U}_1,\bullet,\star)$ *and* $(W,\mathbf{T}',\mathbf{S}',\mathbf{U}',\bullet,\star)$ *are two NNS and* $h: Y \to W$ *be a linear mapping. Then h is strongly* (*weakly*) *neutrosophic continuous if it is strongly* (*weakly*) *neutrosophic bounded.*

Definition 3.7 Consider $(Y,\mathbf{T}_1,\mathbf{S}_1,\mathbf{U}_1,\bullet,\star)$ be an NNS. Then, a sequence η_p is called

(*i*) weakly neutrosophic convergent to $\eta \in Y$ if and only if, for every $\epsilon > 0$ & $\beta \in (0,1)$, $\exists$ some, $p_0 = p_0(\beta,\epsilon)$ such that

$$\mathbf{T}_1(\eta_p - \eta_0,\epsilon) \geq \beta \quad \& \quad \mathbf{S}_1(\eta_p - \eta_0,\epsilon) \leq 1-\beta,$$
$$\mathbf{U}_1(\eta - \eta_0,\epsilon) \leq 1-\beta \quad \text{for all } \eta \geq \eta_0.$$

In this case, we write $\eta_p \xrightarrow{WN} \eta$.

(*ii*) strongly neutrosophic convergent to $\eta \in Y$ if and only if, for every $\beta \in (0,1)$, there exists some $p_0 = p_0(\epsilon)$ such that

$$\mathbf{T}_1(\eta_p - \eta,t) \geq 1-\beta \quad \text{and} \quad \mathbf{S}_1(\eta_p - \eta_0,t) \leq \beta, \qquad \mathbf{U}_1(\eta - \eta_0,t) \leq \beta, \quad \forall t > 0,$$

we write $\eta_p \xrightarrow{SN} \eta$.

Accordingly, we can define an SN(WN)-Cauchy sequence, SN(WN)-closure of a subset and an SN(WN)-complete NNS.

Theorem 3.2 *If a sequence* (η_p) *is SN-convergent then it is WN-convergent to the same limit, but not conversely.*

Example 3.2 It is easy to show that SN-convergence implies WN-convergence. Following is an example to depict that the converse of the statement may not be true. Consider c_0, the Banach space of all sequences $\eta = \eta_p$ convergent to zero with the sup-norm $\|\eta\|_\infty = \sup_p \|\eta\|$, and recognize the neutrosophic norm

$$\mathbf{T}_1(\eta,t) = \begin{cases} \frac{t-\|\eta\|_\infty}{t+\|\eta\|_\infty} & \text{if } t > \|\eta\|_\infty, \\ 0 & \text{if } t \leq \|\eta\|_\infty; \end{cases}$$

and

$$\mathbf{S}_1(\eta,t) = \begin{cases} \frac{2\|\eta\|_\infty}{t+\|\eta\|_\infty} & \text{if } t < \|\eta\|_\infty, \\ 1 & \text{if } t \leq \|\eta\|_\infty; \end{cases}$$

$$\mathbf{U}_1(\eta,t) = \begin{cases} \frac{2\|\eta\|_\infty}{t+\|\eta\|_\infty} & \text{if } t < \|\eta\|_\infty, \\ 1 & \text{if } t \leq \|\eta\|_\infty. \end{cases}$$

on Y. We can find β-norms of neutrosophic norm $(\mathbf{T}_1, \mathbf{S}_1, \mathbf{U}_1)$ since it satisfies the condition (1.1.1). Thus,

$$\mathbf{T}_1(\eta,t) \geq \beta \iff \frac{t-\|\eta\|_\infty}{t+\|\eta\|_\infty} \geq \beta \iff \frac{1+\beta}{1-\beta}\|\eta\|_\infty \leq t,$$

and

$$\mathbf{S}_1(\eta,t) \leq 1-\beta \iff \frac{2\|\eta\|_\infty}{t+\|\eta\|_\infty} \leq 1-\beta \iff \frac{1+\beta}{1-\beta}\|\eta\|_\infty \leq t,$$

$$\mathbf{U}_1(\eta,t) \leq 1-\beta \iff \frac{2\|\eta\|_\infty}{t+\|\eta\|_\infty} \leq 1-\beta \iff \frac{1+\beta}{1-\beta}\|\eta\|_\infty \leq t.$$

This shows that

$$\|\eta\|_\beta = \inf\bigl\{t > 0 : \mathbf{T}_1(\eta,t) \geq \beta \text{ and } \mathbf{S}_1(\eta,t) \leq 1-\beta, \mathbf{U}_1(\eta,t) \leq 1-\beta\bigr\} = \|\eta\|_\infty.$$

Now, we show that the sequence $\eta = (\eta_p) = (\frac{1}{p})_{p=1}^\infty$ is WN-convergent but not SN-convergent. Since each $\|\cdot\|_\beta$ is equivalent to $\|\cdot\|_\infty$, clearly (η_p) is WN-convergent to 0. However, this convergence is not uniform in β. In fact, for given $\epsilon > 0$

$$\|\eta\|_\beta = \frac{1+\beta}{1-\beta}\|\eta\|_\infty < \epsilon \iff \frac{1+\beta}{(1-\beta)\epsilon} < 1.$$

Since $\|\eta\|_\infty = 1$ for $\eta = (\frac{1}{p})_{p=1}^\infty$. But this is not possible, since $\frac{1+\beta}{(1-\beta)\epsilon} \to \infty$ as $\beta \to 1$.

3.1 Neutrosophic Fréchet derivative (NFD)

Definition 3.8 Let $(Y, \mathbf{T}_1, \mathbf{S}_1, \mathbf{U}_1, \bullet, \star)$ and $(W, \mathbf{T}', \mathbf{S}', \mathbf{U}', \bullet, \star)$ be two *NNS*, $H \subseteq Y$ be an neutrosophic open subset, and $h : H \to W$ probably nonlinear. Then h is called strong(weak) neutrosophic Fréchet differentiable at $\eta_0 \in H$ if there exists a strongly (weakly) neutrosophic bounded linear operator T from $(Y, \mathbf{T}_1, \mathbf{S}_1, \mathbf{U}_1, \bullet, \star)$ to $(W, \mathbf{T}', \mathbf{S}', \mathbf{U}', \bullet, \star)$ such that for every $t > 0$

$$\lim_{\mathbf{T}_1(a,t)\to 1} \mathbf{T}'\left(\frac{h(\eta_0 + a) - h(\eta_0) - Ta}{1 - \mathbf{T}_1(a,t)}, t\right) = 1 \ \big(SN(WN)\big),$$
$$\lim_{\mathbf{S}_1(a,t)\to 0} \mathbf{S}'\left(\frac{h(\eta_0 + a) - h(\eta_0) - Ta}{1 - \mathbf{S}_1(a,t)}, t\right) = 0 \ \big(SN(WN)\big),$$
$$\lim_{\mathbf{U}_1(a,t)\to 0} \mathbf{U}'\left(\frac{h(\eta_0 + a) - h(\eta_0) - Ta}{1 - \mathbf{U}_1(a,t)}, t\right) = 0 \ \big(SN(WN)\big).$$

In this condition, the operator T is said to be strong (weak) neutrosophic, or shot, SN(WN)-Fréchet derivative of h at η_0 and is indicated by $\mathcal{D}_{SN(WN)}h[\eta_0]$ is called SN(WN)-Fréchet differentiable on U if it is SN(WN)-Fréchet differentiable at every point of H. In this condition, $\eta \to \mathcal{D}_{SN(WN)}h[\eta]$ is a basis from H to $(\mathcal{F}(Y, W)\mathcal{F}'(Y, W))$, denoted by $\mathcal{D}_{SN(WN)}h$.

Theorem 3.3 *A strongly (weakly) neutrosophic bounded linear operator $\mathcal{A}$ is SN(WN)-Fréchet differentiable at every point η_0 & $\mathcal{D}_{SN(WN)}\mathcal{A}[\eta_0] = \mathcal{A}$.*

Proof This is explicit since

$$\mathbf{T}'\left(\frac{\mathcal{A}(\eta_0 + a) - \mathcal{A}(\eta_0) - \mathcal{A}a}{1 - \mathbf{T}_1(a,t)}, t\right) = \mathbf{T}'(0, t) = 1,$$

and

$$\mathbf{S}'\left(\frac{\mathcal{A}(\eta_0 + a) - \mathcal{A}(\eta_0) - \mathcal{A}a}{\mathbf{S}_1(a,t)}, t\right) = \mathbf{S}'(0, t) = 0,$$
$$\mathbf{U}'\left(\frac{\mathcal{A}(\eta_0 + a) - \mathcal{A}(\eta_0) - \mathcal{A}a}{\mathbf{U}_1(a,t)}, t\right) = \mathbf{U}'(0, t) = 0,$$

$\forall t > 0$. □

Proposition 3.2 *If h is SN(WN)-Fréchet differentiable at $\eta_0 \in H$, then it is strong (weak) neutrosophic continuous at η_0.*

Proof We take the following inequalities. For given $t > 0$,

$$\begin{aligned}\mathbf{T}'\big(h(\eta) - h(\eta_0), t\big) &= \mathbf{T}'\big(h(\eta) - h(\eta_0) - Ta + Ta, t\big) \\ &\geq \mathbf{T}'\big(h(\eta) - h(\eta_0) - Ta, t\big(1 - \mathbf{T}_1(a,t)\big)\big) \bullet (1 - \mathbf{T}'\big(Ta, t\mathbf{T}_1(a,t)\big) \\ &= \mathbf{T}'\left(\frac{h(\eta) - h(\eta_0) - Ta}{(1 - \mathbf{T}_1(a,t))}, t\right) \bullet \mathbf{T}'\left(\frac{Ta}{\mathbf{T}_1(a,t)}, t\right),\end{aligned}$$

and

$$\begin{aligned}
\mathbf{S}'\big(h(\eta)-h(\eta_0),t\big) &= \mathbf{S}'\big(h(\eta)-h(\eta_0)-Ta+Ta,t\big)\\
&\le \mathbf{S}'\big(h(\eta)-h(\eta_0)-Ta,t\big(1-\mathbf{S}_1(a,t)\big)\big)\star\big(1-\mathbf{S}_1\big(Ta,t\mathbf{S}'(a,t)\big)\big)\\
&= \mathbf{S}'\left(\frac{h(\eta)-h(\eta_0)-Ta}{(\mathbf{S}_1(a,t))},t\right)\star\mathbf{S}'\left(\frac{Ta}{1-\mathbf{S}_1(a,t)},t\right),\\
\mathbf{U}'\big(h(\eta)-h(\eta_0),t\big) &= \mathbf{U}'\big(h(\eta)-h(\eta_0)-Ta+Ta,t\big)\\
&\le \mathbf{U}'\big(h(\eta)-h(\eta_0)-Ta,t\big(1-\mathbf{U}_1(a,t)\big)\big)\star\big(1-\mathbf{U}_1\big(Ta,t\mathbf{U}'(a,t)\big)\big)\\
&= \mathbf{U}'\left(\frac{h(\eta)-h(\eta_0)-Ta}{(\mathbf{U}_1(a,t))},t\right)\star\mathbf{U}'\left(\frac{Ta}{1-\mathbf{U}_1(a,t)},t\right).
\end{aligned}$$

Since h is SN(WN)-Fréchet differentiable at $\eta_0\in H$, it follows that

$$\mathbf{T}'\big(h(\eta)-h(\eta_0),t\big)\ge 1\bullet\mathbf{T}'\left(\frac{Ta}{\mathbf{T}_1(a,t)},t\right),$$

and

$$\begin{aligned}
\mathbf{S}'\big(h(\eta)-h(\eta_0),t\big) &\le 0\star\mathbf{S}'\left(\frac{Ta}{1-\mathbf{S}_1(a,t)},t\right),\\
\mathbf{U}'\big(h(\eta)-h(\eta_0),t\big) &\le 0\star\mathbf{U}'\left(\frac{Ta}{1-\mathbf{U}_1(a,t)},t\right),
\end{aligned}$$

where $T=\mathcal{D}_{SN(WN)}h[\eta_0]$. Therefore, h is strong(weak) neutrosophic continuous. □

Theorem 3.4 *Suppose* $(Y,\mathbf{T}_1,\mathbf{S}_1,\mathbf{U}_1,\bullet,\star)$ *and* $(W,\mathbf{T}',\mathbf{S}',\mathbf{U}',\bullet,\star)$ *are two NNS,* $H\subseteq Y$ *be an neutrosophic open subset, and* $h:H\to W$*. If* h *is SN-Fréchet differentiable at some* $\eta_0\in H$*, then it is WN-Fréchet differentiable at some* η_0 *with the same derivative but not conversely. The proof of the above theorem follows directly from the Proposition 3.2. For the converse part, let us consider the following example.*

Example 3.3 Let $Y=W=\ell_p$ $(1\le p<\infty)$, the Banach space of all absolutely p-summable sequences with the norm $\|\eta\|_p=(\sum_n|\eta|^p)^{\frac{1}{p}}$. Define the functions

$$\begin{aligned}
\mathbf{T}_1(\eta,t) &= \begin{cases}\frac{t^2}{t^2+2\|\eta\|_p^p} & \text{if } t>0,\\ 0 & \text{if } t<0;\end{cases}\\
\mathbf{T}'(\eta,t) &= \begin{cases}1, & \text{if } t>0 \text{ and } t^2>\|\eta\|_p^p,\\ 0, & \text{if } t<0 \text{ and } t^2\le\|\eta\|_p^p;\end{cases}
\end{aligned}$$

and

$$\mathbf{S}_1(\eta,t)=\begin{cases}\frac{2\|\eta\|_p^p}{t^2+2\|\eta\|_p^p} & \text{if } t>0,\\ 0 & \text{if } t<0;\end{cases}$$

$$\mathbf{S}'(\eta,t)=\begin{cases}0, & \text{if } t>0 \text{ and } t^2>\|\eta\|_p^p,\\ 1, & \text{if } t<0 \text{ and } t^2\le\|\eta\|_p^p;\end{cases}$$

$$\mathbf{U}_1(\eta,t)=\begin{cases}\frac{2\|\eta\|_p^p}{t^2+2\|\eta\|_p^p} & \text{if } t>0,\\ 0 & \text{if } t<0;\end{cases}$$

$$\mathbf{U}'(\eta,t)=\begin{cases}0, & \text{if } t>0 \text{ and } t^2>\|\eta\|_p^p,\\ 1, & \text{if } t<0 \text{ and } t^2\le\|\eta\|_p^p.\end{cases}$$

Then $(Y,\mathbf{T}_1,\mathbf{S}_1,\mathbf{U}_1)$ and $(W,\mathbf{T}',\mathbf{S}',\mathbf{U}')$ are neutrosophic norms on ℓ_p. Consider the shift operator $G(\eta)=G(\{\eta_1,\eta_2...\})=\{0,\eta_1,\eta_2...\}$ on ℓ_p. It is simple to see that linear operator G is weakly bounded hence is weakly continuous from $(\ell_p,\mathbf{T}_1,\mathbf{S}_1,\mathbf{U}_1,\bullet,\star)$ into $(\ell_p,\mathbf{T}',\mathbf{S}',\mathbf{U}',\bullet,\star)$ but not strongly continuous; since G is linear, we get from Theorem 3.2 that $G=\mathcal{D}_{WN}G[\eta]$ for all $\eta\in\ell_p$, while $\mathcal{D}_{SN}G[\eta]$ does not exist.

Definition 3.9 A subset $\mathcal{B}$ in *NNS* $(Y,\mathbf{T},\mathbf{S},\mathbf{U},\star,\circ)$ is called SN(WN)-compact if each sequence of elements of $\mathcal{B}$ has an SN(WN)-convergent subsequence.

Definition 3.10 $(Y,\mathbf{T}_1,\mathbf{S}_1,\mathbf{U}_1,\bullet,\star)$ and $(W,\mathbf{T}',\mathbf{S}',\mathbf{U}',\bullet,\star)$ be two *NNS* and $h:Y\to W$. Then h is said to be SN(WN)-compact if for every neutrosophic bounded subset $\mathcal{B}$ of Y, the subset $h(\mathcal{B})$ is relatively SN(WN)-compact, i.e., the closure of $h(\mathcal{B})$ is SN(WN)-compact.

Remark 3.1 By the same way as in the proof of [25, Theorem 5], we can prove that h is SN(WN)-compact if it maps every neutrosophic bounded sequence onto a sequence which has an SN(WN)-convergent subsequence. Therefore, an SN-compact operator is WN-compact but not conversely. For example, the identity operator on $(c_0,\mathbf{T},\mathbf{S},\mathbf{U},\bullet,\star)$, in Example 3.3 is not SN-compact while it is WN-compact. Because $(\frac{1}{p})_{p=1}^{\infty}$ cannot have SN-convergent subsequence.

Theorem 3.5 $(Y,\mathbf{T}_1,\mathbf{S}_1,\mathbf{U}_1,\bullet,\star)$ *and* $(W,\mathbf{T}',\mathbf{S}',\mathbf{U}',\bullet,\star)$ *be two WN-complete NNS and* $h:Y\to W$ *be nonlinear WN-compact operator. Let, for some* $\eta_0\in Y$, $\mathcal{D}_{WN}h[\eta_0]=\mathcal{A}$ *exist. Then the linear operator* $\mathcal{A}$ *is also WN-compact.*

Proof Suppose $\eta_p\subset(Y,\mathbf{T}_1,\mathbf{S}_1,\mathbf{U}_1,\bullet,\star)$ is an arbitrary neutrosophic bounded sequence. ∃ some $t_0>0$ and $k\in(0,1)$ such that $\mathbf{T}_1(\eta_p,\frac{t_0}{2})\ge 1-k$ and $\mathbf{S}_1(\eta_p,\frac{t_0}{2})\le k$, $\mathbf{U}_1(\eta_p,\frac{t_0}{2})\le k$, for every +ve integer p. Let the sequence $(\eta_0+\eta_p)_{k=1}^{\infty}$, and let us show that it is neutrosophic bounded. If we take $1-p_1=\mathbf{T}_1(\eta_p,\frac{t_0}{2})\bullet 1-p$ and $p_1=\mathbf{S}_1(\eta_p,\frac{t_0}{2})\star p$, $p_1=\mathbf{U}_1(\eta_p,\frac{t_0}{2})\star p$ then

$$\mathbf{T}_1(\eta_0+\eta_p,t_0)\ge\mathbf{T}_1\left(\eta_0,\frac{t_0}{2}\right)\bullet\mathbf{T}_1\left(\eta_p,\frac{t_0}{2}\right)>\mathbf{T}_1\left(\eta_0,\frac{t_0}{2}\right)\bullet(1-p)=1-p_1,$$

and

$$\mathbf{S}_1(\eta_0+\eta_p,t_0)\le\mathbf{S}_1\left(\eta_0,\frac{t_0}{2}\right)\star\mathbf{S}_1\left(\eta_p,\frac{t_0}{2}\right)<\mathbf{S}_1\left(\eta_0,\frac{t_0}{2}\right)\star p=p_1,$$

$$\mathbf{U}_1(\eta_0+\eta_p,t_0)\le\mathbf{U}_1\left(\eta_0,\frac{t_0}{2}\right)\star\mathbf{U}_1\left(\eta_p,\frac{t_0}{2}\right)<\mathbf{U}_1\left(\eta_0,\frac{t_0}{2}\right)\star p=p_1$$

for every positive integer p. Rest of the proof can be done on the same lines as in [26]. $\square$

4 Conclusion

The present paper introduces the Fréchet derivative of nonlinear operators between *NNS* and the boundedness of linear operators between neutrosophic normed spaces. Current work is an increase and extension of the work of Mursaleen et al. [27], i.e., in an *NNS* which is more regular than the *IFNS*. So that, one may await it to be a more practical, modest work in the domain of neutrosophic topology in modeling the inaccuracy and ambiguity of several subjects arising in many fields of engineering, economics, and science.

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Author contributions
VAK carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. MDK participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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References

1. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)
2. Barros, L.C., Bassanezi, R.C., Tonelli, P.A.: Fuzzy modelling in population dynamics. Ecol. Model. **128**, 27–33 (2000)
3. Giles, R.: A computer program for fuzzy reasoning. Fuzzy Sets Syst. **4**, 221–234 (1980)
4. Feng, G., Chen, G.: Adaptative control of discrete-time chaotic systems: a fuzzy control approach. Chaos Solitons Fractals **253**, 459–467 (2005)
5. Fradkov, A.L., Evans, R.J.: Control of chaos: methods and applications in engineering. Chaos Solitons Fractals **29**, 33–56 (2005)
6. Atanassov, K.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**, 87–96 (1986)
7. Mursaleen, M., Mohiuddine, S.A.: Statistical convergence of double sequences in intuitionistic fuzzy normed spaces. Chaos Solitons Fractals **41**(5), 2414–2421 (2009)
8. Mursaleen, M., Mohiuddine, S.A., Edely, O.H.H.: On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces. Comput. Math. Appl. **59**(2), 603–611 (2010)
9. Khan, V.A., Esi, A., Ahmad, M., Khan, M.D.: Continuous and bounded linear operators in neutrosophic normed spaces. J. Intell. Fuzzy Syst. **40**, 11063–11070 (2021). https://doi.org/10.3233/JIFS-202189
10. Hong, L., Sun, J.Q.: Bifurcations of fuzzy nonlinear dynamical systems. Commun. Nonlinear Sci. Numer. Simul. **1**, 1–12 (2006)
11. Katrasas, A.K.: Fuzzy topological vector spaces. Fuzzy Sets Syst. **12**, 143–154 (1984)
12. Kirişci, M., Şimşek, N.: Neutrosophic normed spaces and statistical convergence. J. Anal. **28**, 1059–1073 (2020)
13. Smarandache, F.: A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability: Neutrosophic Logic: Neutrosophy, Neutrosophic Set, Neutrosophic Probability (2003)
14. Smarandache, F.: Neutrosophic set-a generalization of the intuitionistic fuzzy set. Int. J. Pure Appl. Math. **24**(3), 287 (2005)
15. Bera, T., Mahapatra, N.K.: On neutrosophic soft linear spaces. Fuzzy Inf. Eng. **9**(3), 299–324 (2017)
16. Bera, T., Mahapatra, N.K.: Neutrosophic soft normed linear space. Neutrosophic Sets Syst. **23**(1), 1–6 (2018)
17. Khan, V.A., Daud, K., Mobeen, A.: A some results of neutrosophic normed spaces VIA Fibonacci matrix. UPB Sci. Bull., Ser. A **83**(2), 1–12 (2021)
18. Park, J.H.: Intuitionistic fuzzy metric space. Chaos Solitons Fractals **22**, 1039–1046 (2004)
19. Saadati, R.: A note on some results on the IF-normed spaces. Chaos Solitons Fractals **41**, 206–213 (2009)
20. Alimohammady, M., Roohi, M.: Compactness in fuzzy minimal spaces. Chaos Solitons Fractals **28**, 906–912 (2006)

21. Saadati, R., Park, J.H.: On the intuitionistic fuzzy topological spaces. Chaos Solitons Fractals **27**, 331–344 (2006)
22. Muralikrishna, P., Kumar, D.S.: Neutrosophic approach on normed linear space. Neutrosophic Sets Syst. **30**(1), 1–18 (2019)
23. Salama, A.A., Smarandache, F., Kromov, V.: Neutrosophic closed set and neutrosophic continuous functions. Neutrosophic Sets Syst. **4**, Article 2 (2014)
24. Kirişci, M., Şimşek, N.: Neutrosophic metric spaces. arXiv:1907.00798. arXiv preprint
25. Kaleva, O., Seikkala, S.: On fuzzy metric spaces. Fuzzy Sets Syst. **12**, 215–229 (1984)
26. Schweizer, B., Sklar, A.: Statistical metric spaces. Pac. J. Math. **10**, 313–334 (1960)
27. Mursaleen, M., Mohiuddine, S.A.: Nonlinear operators between intuitionistic fuzzy normed spaces and Fréchet derivative. Chaos Solitons Fractals **42**, 1010–1015 (2009)