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Prequasiideal of the type weighted binomial matrices in the Nakano sequence space of soft functions with some applications

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Abstract

Consider the space of weighted binomial matrices in the Nakano sequence space of soft functions. We have offered some geometric and topological structures of the multiplication operator acting on this space and its associated operator ideal. The existence of a fixed point of the Kannan contraction operator in this prequasioperator ideal is confirmed. Finally, we discuss many applications of solutions to nonlinear stochastic dynamical matrix systems and illustrative examples of our findings.

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Keywords: Binomial matrix; Nakano sequence space; Extended *s*-soft functions; Multiplication mapping; Kannan contraction mapping

1 Introduction

The study of uncertainty has been greatly helped by probability theory, fuzzy set theory, soft sets, and rough sets. However, there are problems with these ideas that must be considered, see [1–5] for more information and examples from real life. Since the book [6] on the Banach Fixed-Point Theorem came out, many mathematicians have looked into how the theorem could be expanded and how it could be used in different situations. The nonlinear analysis exploits the Banach contraction principle as a strong tool [7, 8]. Kannan [9] presented a group of mappings with the same actions at fixed locations as contractions. This collection is somewhat discontinuous. In Reference, [10], an explanation of Kannan operators in modular vector spaces was attempted. Only this one attempt was ever made as [11–15] show that much attention has been paid to the *s*-number mapping ideal and the multiplication operator hypothesis in functional analysis. Bakery and Mohamed [16] offered the idea of a prequasinorm on the Nakano sequence space with a variable exponent that fell somewhere in the interval $(0,1]$. For the normed sequence spaces and related topics, the reader can refer to the textbooks [17] and [18]. They discussed the conditions that must be met to generate a prequasi-Banach and closed space when it is endowed with a specified prequasinorm, as well as the Fatou property of various prequasinorms on it. They also determined a fixed point for Kannan prequasinorm contraction mappings on it, in addition to the ideal of prequasi-Banach mappings derived from s-numbers in this sequence space. Both of these ideals were established. In addition, several fixed-point findings of Kannan nonexpansive mappings on a generalized Cesàro backward difference sequence space of a nonabsolute type were discovered in [19]. Assume that $\mathbb{R}$ is the set of real numbers and $\mathbb{N}$ is the set of nonnegative integers. We denote the collection of all nonempty bounded subsets of $\mathbb{R}$ by $\mathfrak{B}(\mathbb{R})$ and $\mathbb{E}$ is the set of parameters. By $\mathbb{R}(\mathbb{A})^*$ and $\mathbb{R}(\mathbb{A})$, we indicate the set of nonnegative and all soft real numbers (corresponding to $\mathbb{A}$), where $\mathbb{A} \subset \mathbb{E}$. The additive identity and multiplicative identity in $\mathbb{R}(\mathbb{A})$ are denoted by $\widetilde{0}$ and $\widetilde{1}$, respectively. For more details on the arithmetic operations on $\mathbb{R}(\mathbb{A})$, see [20]. Let $\mu : \mathbb{R}(\mathbb{A}) \times \mathbb{R}(\mathbb{A}) \to \mathbb{R}(\mathbb{A})^*$, where $\mu(\widetilde{f},\widetilde{g}) = |\widetilde{f} - \widetilde{g}|$, for all $\widetilde{f},\widetilde{g} \in \mathbb{R}(\mathbb{A})$. Assume $\widetilde{\rho} : \mathbb{R}(\mathbb{A}) \times \mathbb{R}(\mathbb{A}) \to \mathbb{R}^+$ is defined by

$$\widetilde{\rho}(\widetilde{f},\widetilde{g}) = \max_{\lambda \in A} \mu(\widetilde{f},\widetilde{g})(\lambda).$$

The binomial formula is defined by

$$(u+v)^l = \sum_{z=0}^{l} \binom{l}{z} u^z v^{l-z},$$

where u and v are nonnegative real numbers, and $l \in \mathbb{N}$. Given that the proof of many fixed-point theorems in a given space requires either growing the space itself or expanding the self-mapping that acts on it, both of these options are viable, we have constructed the space, $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$, which is the domain of a weighted binomial matrix in the Nakano sequence space of soft functions $[\ell^{\mathcal{P}}(w)]_\tau$, where the weighted binomial matrix, $E_{u,v} = ((\lambda_{u,v})_{lz}(q))$, is defined as:

$$(\lambda_{u,v})_{lz}(q) = \begin{cases} \frac{A(l,z)q_{l,z}}{(u+v)^l}, & 0 \le z \le l, \\ 0, & z > l, \end{cases}$$

where $q_{l,z} \in (0,\infty)$, for all $l,z \in \mathbb{N}$ and $A(l,z) = \binom{l}{z} u^z v^{l-z}$.

We have offered some geometric and topological structures of a multiplication operator acting on $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ and operators ideal of type $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$. A fixed point of the Kannan contraction operator that exists in this prequasioperator ideal is confirmed. Finally, we discuss many applications of solutions to nonlinear stochastic dynamical matrix systems and illustrative examples of our findings.

2 Preliminaries and definitions

Here, we discuss the background of our study and what it means.

The spaces of null, bounded, and r-absolutely summable sequences of reals are indicated by c_0, ℓ_∞, and ℓ_r, respectively. We denote the spaces of every bounded and finite rank linear mappings from an infinite-dimensional Banach space $\mathcal{G}$ into an infinite-dimensional Banach space $\mathcal{V}$ by $\mathbb{D}(\mathcal{G},\mathcal{V})$ and $\mathbb{F}(\mathcal{G},\mathcal{V})$, respectively. If $\mathcal{G} = \mathcal{V}$, then we denote them by $\mathbb{D}(\mathcal{G})$ and $\mathbb{F}(\mathcal{G})$, respectively. We also denote the spaces of approximable and compact bounded linear operators from $\mathcal{G}$ into $\mathcal{V}$ by $\mathcal{A}(\mathcal{G},\mathcal{V})$ and $\mathcal{K}(\mathcal{G},\mathcal{V})$, respectively. The ideal of bounded, approximable, and compact mappings between each two infinite-dimensional Banach spaces will be indicated by $\mathbb{D}$, $\mathcal{A}$, and $\mathcal{K}$, respectively.

Lemma 2.1 ([21]) *Assume $w_b > 0$ and $x_b, z_b \in \mathbb{R}$ for every $b \in \mathbb{N}$ and $\hbar = \max\{1, \sup_b w_b\}$, then*

$$|x_b + z_b|^{w_b} \leq 2^{\hbar-1}\big(|x_b|^{w_b} + |z_b|^{w_b}\big). \tag{1}$$

If $\mathcal{E}^{\mathfrak{S}}$ is a linear space of sequences of soft functions, and $[p]$ describes an integral part of the real number p:

Definition 2.2 ([22]) The space $\mathcal{E}^{\mathfrak{S}}$ is called a private sequence space of soft functions ($\mathfrak{psssf}$) if the following settings are satisfied:

(a1) Suppose $b \in \mathbb{N}$, then $\widetilde{e_b} \in \mathcal{E}^{\mathfrak{S}}$, where $\widetilde{e_b} = (\widetilde{0}, \widetilde{0}, \ldots, \widetilde{1}, \widetilde{0}, \widetilde{0}, \ldots)$, while $\widetilde{1}$ displays at the b^{th} place;

(a2) Assume $\widetilde{f} = (\widetilde{f_b}) \in \omega^{\mathfrak{S}}$, $|\widetilde{g}| = (|\widetilde{g_b}|) \in \mathcal{E}^{\mathfrak{S}}$ and $|\widetilde{f_b}| \leq |\widetilde{g_b}|$, with $b \in \mathbb{N}$, then $|\widetilde{f}| \in \mathcal{E}^{\mathfrak{S}}$;

(a3) $(|\widetilde{h_{[\frac{b}{2}]}}|)_{b=0}^{\infty} \in \mathcal{E}^{\mathfrak{S}}$ if $(|\widetilde{h_b}|)_{b=0}^{\infty} \in \mathcal{E}^{\mathfrak{S}}$.

Definition 2.3 ([23]) An s-number is a function $s : \mathbb{D}(\mathcal{G}, \mathcal{V}) \to \mathbb{R}^{+\mathbb{N}}$ that sorts every $V \in \mathbb{D}(\mathcal{G}, \mathcal{V})$ a $(s_d(V))_{d=0}^{\infty}$ verifies the next settings:

(1) $\|V\| = s_0(V) \geq s_1(V) \geq s_2(V) \geq \cdots \geq 0$, for every $V \in \mathbb{D}(\mathcal{G}, \mathcal{V})$;

(2) $s_d(VYW) \leq \|V\| s_d(Y) \|W\|$ if for all $W \in \mathbb{D}(\mathcal{G}_0, \mathcal{G})$, $Y \in \mathbb{D}(\mathcal{G}, \mathcal{V})$, and $V \in \mathbb{D}(\mathcal{V}, \mathcal{V}_0)$, where $\mathcal{G}_0$ and $\mathcal{V}_0$ are arbitrary Banach spaces;

(3) $s_{l+d-1}(V_1 + V_2) \leq s_l(V_1) + s_d(V_2)$ if for all $V_1, V_2 \in \mathbb{D}(\mathcal{G}, \mathcal{V})$ and $l, d \in \mathbb{N}$;

(4) assume $V \in \mathbb{D}(\mathcal{G}, \mathcal{V})$ and $\gamma \in \mathbb{R}$, then $s_d(\gamma V) = |\gamma| s_d(V)$;

(5) assume $\operatorname{rank}(V) \leq d$, then $s_d(V) = 0$ for every $V \in \mathbb{D}(\mathcal{G}, \mathcal{V})$;

(6) $s_{l \geq a}(I_a) = 0$ or $s_{l<a}(I_a) = 1$, where I_a marks the unit mapping on the a-dimensional Hilbert space ℓ_2^a.

Some examples of s-numbers:

(i) The bth approximation number is defined as

$$\alpha_b(X) = \inf\big\{\|X - Y\| : Y \in \mathbb{D}(\mathcal{G}, \mathcal{V}) \text{ and } \operatorname{rank}(Y) \leq b\big\};$$

(ii) The bth Kolmogorov number is defined as $d_b(X) = \inf_{\dim J \leq b} \sup_{\|f\| \leq 1} \inf_{g \in J} \|Xf - g\|$.

Notations 2.4 ([24])

$$\begin{aligned}
&\widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}} := \big\{\widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}}(\mathcal{G}, \mathcal{V})\big\}, \quad \text{where } \widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}}(\mathcal{G}, \mathcal{V}) := \big\{V \in \mathbb{D}(\mathcal{G}, \mathcal{V}) : \big(\big(\widetilde{s_j(V)}\big)_{j=0}^{\infty} \in \mathcal{E}^{\mathfrak{S}}\big\}, \\
&\widetilde{\mathbb{D}^\alpha}_{\mathcal{E}^{\mathfrak{S}}} := \big\{\widetilde{\mathbb{D}^\alpha}_{\mathcal{E}^{\mathfrak{S}}}(\mathcal{G}, \mathcal{V})\big\}, \quad \text{where } \widetilde{\mathbb{D}^\alpha}_{\mathcal{E}^{\mathfrak{S}}}(\mathcal{G}, \mathcal{V}) := \big\{V \in \mathbb{D}(\mathcal{G}, \mathcal{V}) : \big(\big(\widetilde{\alpha_j(V)}\big)_{j=0}^{\infty} \in \mathcal{E}^{\mathfrak{S}}\big\}, \\
&\widetilde{\mathbb{D}^d}_{\mathcal{E}^{\mathfrak{S}}} := \big\{\widetilde{\mathbb{D}^d}_{\mathcal{E}^{\mathfrak{S}}}(\mathcal{G}, \mathcal{V})\big\}, \quad \text{where } \widetilde{\mathbb{D}^d}_{\mathcal{E}^{\mathfrak{S}}}(\mathcal{G}, \mathcal{V}) := \big\{V \in \mathbb{D}(\mathcal{G}, \mathcal{V}) : \big(\big(\widetilde{d_j(V)}\big)_{j=0}^{\infty} \in \mathcal{E}^{\mathfrak{S}}\big\}, \\
&\big(\widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}}\big)^{\gamma} := \big\{\big(\widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}}\big)^{\gamma}(\mathcal{G}, \mathcal{V})\big\}, \quad \text{where} \\
&\big(\widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}}\big)^{\gamma}(\mathcal{G}, \mathcal{V}) \\
&\quad := \big\{V \in \mathbb{D}(\mathcal{G}, \mathcal{V}) : \big(\big(\widetilde{\gamma_b(V)}\big)_{b=0}^{\infty} \in \mathcal{E}^{\mathfrak{S}} \text{ and } \big\|V - \widetilde{\rho}\big(\widetilde{\gamma_b(V)}, \widetilde{0}\big)I\big\| = 0, \text{for all } b \in \mathbb{N}\big\}.
\end{aligned}$$

Theorem 2.5 ([22]) *If the linear sequence space $\mathcal{E}^{\mathfrak{S}}$ is a $\mathfrak{psssf}$ then $\widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}}$ is an operator ideal.*

If $\widetilde{\theta} = (\widetilde{0}, \widetilde{0}, \widetilde{0}, \dots)$ and $\mathcal{F}$ is the space of finite sequences of soft numbers.

Definition 2.6 ([22]) A subspace of the $\mathfrak{psssf}$ is said to be a premodular $\mathfrak{psssf}$ if there is a function $\tau : \mathcal{E}^{\mathfrak{S}} \to [0, \infty)$ that verifies the following conditions:

(i) Suppose $\widetilde{h} \in \mathcal{E}^{\mathfrak{S}}$, $\widetilde{h} = \widetilde{\theta} \iff \tau(|\widetilde{h}|) = 0$, and $\tau(\widetilde{h}) \geq 0$;
(ii) assume $\widetilde{h} \in \mathcal{E}^{\mathfrak{S}}$ and $\varepsilon \in \mathbb{R}$, then there is $E_0 \geq 1$ with $\tau(\varepsilon\widetilde{h}) \leq |\varepsilon| E_0 \tau(\widetilde{h})$;
(iii) one has $G_0 \geq 1$ with $\tau(\widetilde{f} + \widetilde{g}) \leq G_0(\tau(\widetilde{f}) + \tau(\widetilde{g}))$ for all $\widetilde{f}, \widetilde{g} \in \mathcal{E}^{\mathfrak{S}}$;
(iv) if $|\widetilde{f_b}| \leq |\widetilde{g_b}|$ for all $b \in \mathbb{N}$, then $\tau(|\widetilde{f_b}|) \leq \tau(|\widetilde{g_b}|)$;
(v) there are $D_0 \geq 1$ with $\tau(|\widetilde{f}|) \leq \tau(|\widetilde{f_{[.]}}|) \leq D_0 \tau(|\widetilde{f}|)$;
(vi) the closure $\overline{\mathcal{F}}$ of $\mathcal{F} = \mathcal{E}_\tau^{\mathfrak{S}}$;
(vii) one obtains $\varepsilon > 0$ so that $\tau(\widetilde{\nu}, \widetilde{0}, \widetilde{0}, \widetilde{0}, \dots) \geq \varepsilon |\nu| \tau(\widetilde{1}, \widetilde{0}, \widetilde{0}, \widetilde{0}, \dots)$.

Definition 2.7 ([22]) The $\mathfrak{psssf}$ $\mathcal{E}_\tau^{\mathfrak{S}}$ is called a prequasinormed $\mathfrak{psssf}$ when τ satisfies parts (i)–(iii) of Definition 2.6. The space $\mathcal{E}_\tau^{\mathfrak{S}}$ is said to be a prequasi-Banach $\mathfrak{psssf}$ if $\mathcal{E}^{\mathfrak{S}}$ is complete equipped with τ.

Theorem 2.8 ([22]) *The space $\mathcal{E}_\tau^{\mathfrak{S}}$ is a prequasinormed $\mathfrak{psssf}$ if it is a premodular $\mathfrak{psssf}$.*

3 Properties of operators ideal

In this section, we examine some geometric and topological structures of the prequasiideal type of space of a weighted binomial matrix in Nakano sequence space of soft functions under the settings of Theorem 3.3.

Assume $\omega^{\mathfrak{S}}$ is the class of all sequence spaces of soft reals.

Definition 3.1 Suppose $(w_l) \in \mathbb{R}^{+\mathbb{N}}$, where $\mathbb{R}^{+\mathbb{N}}$ is the space of all sequences of positive reals. The sequence space $[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau$ with the function τ is defined by:

$$\left[E_{u,v}^{\mathfrak{S}}(q,w)\right]_\tau = \left\{\widetilde{h} = (\widetilde{h_m}) \in \omega^{\mathfrak{S}} : \tau(\delta\widetilde{h}) < \infty, \text{for some } \varepsilon > 0\right\},$$

where $\tau(\widetilde{h}) = \sum_{m=0}^{\infty} \left(\frac{\widetilde{\rho}(\sum_{z=0}^{m} A(m,z) q_{m,z} \widetilde{h_z}, \widetilde{0})}{(u+v)^m}\right)^{w_m}$.

Theorem 3.2 *If $(w_l) \in \ell_\infty \cap \mathbb{R}^{+\mathbb{N}}$, then*

$$\left[E_{u,v}^{\mathfrak{S}}(q,w)\right]_\tau = \left\{\widetilde{h} = (\widetilde{h_b}) \in \omega^{\mathfrak{S}} : \tau(\delta\widetilde{h}) < \infty, \textit{for all } \delta > 0\right\}.$$

Proof It is clear since (w_l) is a bounded sequence. □

The spaces of all monotonic increasing and decreasing sequences of positive reals are indicated by $\uparrow$ and $\downarrow$, respectively.

Theorem 3.3 $[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau$ *is a prequasi-Banach* $\mathfrak{psssf}$ *whenever the following settings are satisfied*:

(b1) $u + v > 1$;
(b2) $(w_p)_{p \in \mathbb{N}} \in \ell_\infty \cap \uparrow$;
(b3) $(A(a,k) q_{a,k})_{k=0}^{\infty} \in \downarrow$ *or*, $(A(a,k) q_{a,k})_{k=0}^{\infty} \in \uparrow \cap \ell_\infty$ *and there exists* $C \geq 1$ *such that*

$$A(a, 2k+1) q_{a,2k+1} \leq C A(a,k) q_k;$$

(b4) $(A(a,k) q_{a,k})_{a=0}^{\infty} \in \downarrow$.

Proof First, to prove that $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ is a premodular $\mathfrak{psssf}$.

(i) Clearly, $\tau(|\widetilde{h}|) = 0 \Leftrightarrow \widetilde{h} = \widetilde{\theta}$ and $\tau(\widetilde{h}) \geq 0$.

(a1) and (iii) Assume $\widetilde{f}, \widetilde{g} \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$, then

$$\begin{aligned}
\tau(\widetilde{f}+\widetilde{g}) &= \sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z)q_{l,z}(\widetilde{f}_z+\widetilde{g}_z),\widetilde{0})}{(u+v)^l}\right)^{w_l} \\
&\leq 2^{\hbar-1}\left(\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z)q_{l,z}\widetilde{f}_z,\widetilde{0})}{(u+v)^l}\right)^{w_l} + \sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z)q_{l,z}\widetilde{g}_z,\widetilde{0})}{(u+v)^l}\right)^{w_l}\right) \\
&= 2^{\hbar-1}\big(\tau(\widetilde{f})+\tau(\widetilde{g})\big) < \infty
\end{aligned}$$

hence, $\widetilde{f}+\widetilde{g} \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$;

(ii) If $\lambda \in \mathbb{R}, \widetilde{f} \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$, and since $(w_l) \in \uparrow \cap \ell_\infty$ then,

$$\begin{aligned}
\tau(\lambda\widetilde{f}) &= \sum_{m=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{m} A(m,z)q_{m,z}\lambda\widetilde{f}_z,\widetilde{0})}{(u+v)^m}\right)^{w_m} \\
&\leq \sup_m |\lambda|^{w_m}\sum_{m=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{m} A(m,z)q_{m,z}\widetilde{f}_z,\widetilde{0})}{(u+v)^m}\right)^{w_m} \\
&\leq E_0|\lambda|\tau(\widetilde{f}) < \infty,
\end{aligned}$$

where $E_0 = \max\{1, \sup_l |\lambda|^{w_l-1}\} \geq 1$. Therefore, $\lambda\widetilde{f} \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$.

If $(w_l) \in \uparrow \cap \ell_\infty$, then

$$\begin{aligned}
\sum_{m=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{m} A(m,z)q_{m,z}(\widetilde{e_b})_z,\widetilde{0})}{(u+v)^m}\right)^{w_m} &= \sum_{m=b}^{\infty}\left(\frac{A(m,b)q_{m,b}}{(u+v)^m}\right)^{w_m} \\
&\leq \sup_{m=b}^{\infty}\big(A(m,b)q_{m,b}\big)^{w_m}\sum_{m=b}^{\infty}\frac{1}{(u+v)^{mw_m}} < \infty.
\end{aligned}$$

Hence, $\widetilde{e_b} \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ for all $b \in \mathbb{N}$;

(a2) and (iv) Assume $|\widetilde{f_m}| \leq |\widetilde{g_m}|$ for all $m \in \mathbb{N}$ and $|\widetilde{g}| \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ hence

$$\begin{aligned}
\tau(|\widetilde{f}|) &= \sum_{m=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{m} A(m,z)q_{m,z}|\widetilde{f_z}|,\widetilde{0})}{(u+v)^m}\right)^{w_m} \\
&\leq \sum_{m=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{m} A(m,z)q_{m,z}|\widetilde{g_z}|,\widetilde{0})}{(u+v)^m}\right)^{w_m} \\
&= \tau(|\widetilde{g}|) < \infty.
\end{aligned}$$

Hence, $|\widetilde{f}| \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$;

(a3) and (v) If $(|\widetilde{f_z}|) \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ so that $(w_l) \in \uparrow \cap \ell_\infty$ and $(A(l,z)q_{l,z})_{z=0}^{\infty} \in \downarrow$ then

$$\tau(|\widetilde{f_{[\frac{z}{2}]}}|) = \sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z)q_{l,z}|\widetilde{f_{[\frac{z}{2}]}}|,\widetilde{0})}{(u+v)^l}\right)^{w_l}$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{2l} A(l,z)q_{l,z}|\widetilde{f_{[\frac{z}{2}]}}|,\widetilde{0})}{(u+v)^{2l}}\right)^{w_{2l}} + \sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{2l+1} A(l,z)q_{l,z}|\widetilde{f_{[\frac{z}{2}]}}|,\widetilde{0})}{(u+v)^{2l+1}}\right)^{w_{2l+1}} \\
&\le \sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{2l} A(l,z)q_{l,z}|\widetilde{f_{[\frac{z}{2}]}}|,\widetilde{0})}{(u+v)^{l}}\right)^{w_{l}} + \sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{2l+1} A(l,z)q_{l,z}|\widetilde{f_{[\frac{z}{2}]}}|,\widetilde{0})}{(u+v)^{l}}\right)^{w_{l}} \\
&\le \sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(A(l,2l)q_{l,2l}|\widetilde{f_l}| + \sum_{z=0}^{l}(A(l,2z)q_{2z} + A(l,2z+1)q_{l,2z+1})|\widetilde{f_z}|,\widetilde{0})}{(u+v)^{l}}\right)^{w_{l}} \\
&\quad + \sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}(A(l,2z)q_{2z} + A(l,2z+1)q_{l,2z+1})|\widetilde{f_z}|,\widetilde{0})}{(u+v)^{l}}\right)^{w_{l}} \\
&\le 2^{\hbar-1}\left(\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z)q_{l,z}|\widetilde{f_z}|,\widetilde{0})}{(u+v)^{l}}\right)^{w_{l}}\right. \\
&\quad \left. + \sum_{l=0}^{\infty}\left(\frac{2\widetilde{\rho}(\sum_{z=0}^{l} A(l,z)q_{l,z}|\widetilde{f_z}|,\widetilde{0})}{(u+v)^{l}}\right)^{w_{l}}\right) \\
&\quad + \sum_{l=0}^{\infty}\left(\frac{2\widetilde{\rho}(\sum_{z=0}^{l} A(l,z)q_{l,z}|\widetilde{f_z}|,\widetilde{0})}{(u+v)^{l}}\right)^{w_{l}} \le D_0\tau\big(|\widetilde{f}|\big) < \infty,
\end{aligned}
$$

where $D_0 \ge (2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}) \ge 1$. Therefore, $(|\widetilde{f_{[\frac{z}{2}]}}|) \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$;

(vi) Obviously, the closure of $\mathcal{F} = E^{\mathfrak{S}}_{u,v}(q,w)$;

(vii) One has $0 < \delta \le \sup_l |\lambda|^{w_l - 1}$ with $\tau(\widetilde{\lambda},\widetilde{0},\widetilde{0},\widetilde{0},\dots) \ge \delta|\lambda|\tau(\widetilde{1},\widetilde{0},\widetilde{0},\widetilde{0},\dots)$ for all $\lambda \neq 0$ and $\delta > 0$ if $\lambda = 0$.

The space $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$, given by Theorem 2.8, is a prequasinormed $\mathfrak{psssf}$. Secondly, to show that $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ is a Banach space, assume $\widetilde{h^i} = (\widetilde{h^i_k})_{k=0}^{\infty}$ is a Cauchy sequence in $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$. Hence, for all $\gamma \in (0,1)$ there exists $i_0 \in \mathbb{N}$, one has for all $i,j \ge i_0$ that

$$
\tau\big(\widetilde{h^i} - \widetilde{h^j}\big) = \sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z)q_{l,z}(\widetilde{h^i_z} - \widetilde{f^j_z}),\widetilde{0})}{(u+v)^{l}}\right)^{w_l} < \gamma^{\hbar}.
$$

Then, $\widetilde{\rho}(\sum_{z=0}^{l} A(l,z)q_{l,z}(\widetilde{h^i_z} - \widetilde{h^j_z}),\widetilde{0}) < \gamma$. Since $(\mathbb{R}(\mathbb{A}),\widetilde{\rho})$ is a complete metric space. So $(\widetilde{h^j_k})$ is a Cauchy sequence in $\mathbb{R}(\mathbb{A})$ for fixed $k \in \mathbb{N}$. Then, it is convergent to $\widetilde{h^0_k} \in \mathbb{R}(\mathbb{A})$. Hence, $\tau(\widetilde{h^i} - \widetilde{h^0}) < \gamma^{\hbar}$ for every $i \ge i_0$. Obviously, by setting (iii) one has $\widetilde{h^0} \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$. □

By Theorems 2.5 and 3.3 one can obtain the following theorem:

Theorem 3.4 *The space $\widetilde{\mathbb{D}}^s_{E^{\mathfrak{S}}_{u,v}(q,w)}$ is an operator ideal whenever the settings of Theorem 3.3 are established.*

Theorem 3.5 ([22]) *Assume s-type $\mathcal{E}^{\mathfrak{S}}_\tau := \{\widetilde{h} = (\widetilde{s_j(H)}) \in \mathbb{R}^{\mathbb{N}} : H \in \mathbb{D}(\mathcal{G},\mathcal{V})$ and $\tau(\widetilde{h}) < \infty\}$. If $\widetilde{\mathbb{D}}^s_{\mathcal{E}_\tau}$ is an operator ideal then the next settings are established:*

a. *s-type $\mathcal{E}^{\mathfrak{S}}_\tau \supset \mathcal{F}$;*
b. *Assume $(\widetilde{s_j(H_1)})_{j=0}^{\infty} \in$ s-type $\mathcal{E}^{\mathfrak{S}}_\tau$ and $(\widetilde{s_j(H_2)})_{j=0}^{\infty} \in$ s-type $\mathcal{E}^{\mathfrak{S}}_\tau$, then $(\widetilde{s_j(H_1 + H_2)})_{j=0}^{\infty} \in$ s-type $\mathcal{E}^{\mathfrak{S}}_\tau$;*
c. *Suppose $\varepsilon \in \mathbb{R}$ and $(\widetilde{s_j(H)})_{j=0}^{\infty} \in$ s-type $\mathcal{E}^{\mathfrak{S}}_\tau$, then $|\varepsilon|(\widetilde{s_j(H)})_{j=0}^{\infty} \in$ s-type $\mathcal{E}^{\mathfrak{S}}_\tau$;*
d. *If $(\widetilde{s_j(U)})_{j=0}^{\infty} \in$ s-type $\mathcal{E}^{\mathfrak{S}}_\tau$ and $\widetilde{s_j(T)} \le \widetilde{s_j(U)}$ for all $j \in \mathbb{N}$ where $T,U \in \mathbb{D}(\mathcal{G},\mathcal{V})$, then $(\widetilde{s_j(T)})_{j=0}^{\infty} \in$ s-type $\mathcal{E}^{\mathfrak{S}}_\tau$, i.e., $\mathcal{E}^{\mathfrak{S}}_\tau$ is a solid space.*

Some properties of s-type $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ are examined in the next theorem in view of Theorem 3.5 and Theorem 3.4.

Theorem 3.6 *The following statements hold*:

a. *s-type $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau \supset \mathcal{F}$*;
b. *If $(\widetilde{s_n(X_1)})_{n=0}^{\infty}, (\widetilde{s_n(X_2)})_{n=0}^{\infty} \in s$-type $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$, then $(\widetilde{s_n(X_1+X_2)})_{n=0}^{\infty} \in s$-type $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$*;
c. *If $\lambda \in \mathbb{R}$ and $(\widetilde{s_n(X)})_{n=0}^{\infty} \in s$-type $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ then $|\lambda|(\widetilde{s_n(X)})_{n=0}^{\infty} \in s$-type $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$*;
d. *s-type $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ is a solid space.*

Definition 3.7 ([25]) A subclass $\mathcal{U}$ of $\mathbb{D}$ is called a mappings' ideal assume all $\mathcal{U}(\mathcal{G},\mathcal{V}) = \mathcal{U} \cap \mathbb{D}(\mathcal{G},\mathcal{V})$ verifies the next conditions:

(i) $I_\Gamma \in \mathcal{U}$, where Γ is a one-dimensional Banach space;
(ii) The space $\mathcal{U}(\mathcal{G},\mathcal{V})$ is linear over $\mathbb{R}$;
(iii) Assume $W \in \mathbb{D}(\mathcal{G}_0,\mathcal{G})$, $X \in \mathcal{U}(\mathcal{G},\mathcal{V})$, and $Y \in \mathbb{D}(\mathcal{V},\mathcal{V}_0)$, then $YXW \in \mathcal{U}(\mathcal{G}_0,\mathcal{V}_0)$.

Definition 3.8 ([26]) A function $H \in [0,\infty)^{\mathcal{U}}$ is called a prequasinorm on the ideal $\mathcal{U}$ if the next settings hold:

(1) Assume $V \in \mathcal{U}(\mathcal{G},\mathcal{V})$, $H(V) \geq 0$ and $H(V) = 0$, if and only if, $V = 0$;
(2) there are $Q \geq 1$ so that $H(\alpha V) \leq D|\alpha|H(V)$ for all $V \in \mathcal{U}(\mathcal{G},\mathcal{V})$ and $\alpha \in \mathbb{R}$;
(3) one has $P \geq 1$ with $H(V_1+V_2) \leq P[H(V_1)+H(V_2)]$ for all $V_1, V_2 \in \mathcal{U}(\mathcal{G},\mathcal{V})$;
(4) one obtains $\sigma \geq 1$, if $V \in \mathbb{D}(\mathcal{G}_0,\mathcal{G})$, $X \in \mathcal{U}(\mathcal{G},\mathcal{V})$ and $Y \in \mathbb{D}(\mathcal{V},\mathcal{V}_0)$, then $H(YXV) \leq \sigma \|Y\| H(X) \|V\|$.

Theorem 3.9 ([26]) *Every quasinorm on the ideal $\mathcal{U}$ is a prequasinorm.*

Some properties of the ideal generated by our soft space and extended s-numbers are offered if the settings of Theorem 3.3 are established.

By $\downarrow^{\mathfrak{S}}$, we denote the space of all monotonic decreasing sequences of soft functions.

Theorem 3.10 *The settings of Theorem 3.3 are sufficient only for $\widetilde{\mathbb{D}^\alpha}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V}) = \overline{\mathbb{F}(\mathcal{G},\mathcal{V})}$.*

Proof Obviously, $\overline{\mathbb{F}(\mathcal{G},\mathcal{V})} \subseteq \widetilde{\mathbb{D}^\alpha}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$ by the linearity of the space $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ and $\widetilde{e_m} \in (E^{\mathfrak{S}}_{u,v}(q,w))_\tau$ for every $m \in \mathbb{N}$. To prove that $\widetilde{\mathbb{D}^\alpha}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V}) \subseteq \overline{\mathbb{F}(\mathcal{G},\mathcal{V})}$. When $H \in \widetilde{\mathbb{D}^\alpha}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$ then $(\widetilde{\alpha_l(H)})_{m=0}^{\infty} \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$. Since $\tau(\widetilde{\alpha_m(H)})_{m=0}^{\infty} < \infty$ and assume $\gamma \in (0,1)$ then there exists $l_0 \in \mathbb{N} - \{0\}$ with $\tau((\widetilde{\alpha_m(H)})_{m=l_0}^{\infty}) < \frac{\gamma}{2^{\hbar+3}\delta j}$ for some $j \geq 1$ and $\delta = \max\{1, \sum_{l=l_0}^{\infty}(\frac{1}{(u+v)^l})^{w_l}\}$. As $\widetilde{\alpha_l(H)} \in \downarrow^{\mathfrak{S}}$ one has

$$\begin{aligned}\sum_{l=l_0+1}^{2l_0}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\widetilde{\alpha_{2l_0}(H)},\widetilde{0})}{(u+v)^l}\right)^{w_l} &\leq \sum_{l=l_0+1}^{2l_0}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\widetilde{\alpha_z(H)},\widetilde{0})}{(u+v)^l}\right)^{w_l}\\ &\leq \sum_{l=l_0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\widetilde{\alpha_z(H)},\widetilde{0})}{(u+v)^l}\right)^{w_l} \\ &< \frac{\gamma}{2^{\hbar+3}\delta j}.\end{aligned} \tag{2}$$

Therefore, $U \in \mathbb{F}_{2l_0}(\mathcal{G},\mathcal{V})$ so that $\operatorname{rank}(U) \le 2l_0$ and

$$\begin{aligned}
&\sum_{l=2l_0+1}^{3l_0}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\|\widetilde{H-U}\|,\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
&\quad\le \sum_{l=l_0+1}^{2l_0}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\|\widetilde{H-U}\|,\widetilde{0})}{(u+v)^l}\right)^{w_l} < \frac{\gamma}{2^{\hbar+3}\delta j}.
\end{aligned} \tag{3}$$

As $(w_l) \in \uparrow \cap \ell_\infty$ one has

$$\sup_{l=l_0}^{\infty}\widetilde{\rho}^{w_l}\left(\sum_{z=0}^{l_0}A(l,z)q_{l,z}\|\widetilde{H-U}\|,\widetilde{0}\right) < \frac{\gamma}{2^{2\hbar+2}\delta}. \tag{4}$$

Hence,

$$\sum_{l=0}^{l_0}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\|\widetilde{H-U}\|,\widetilde{0})}{(u+v)^l}\right)^{w_l} < \frac{\gamma}{2^{\hbar+3}\delta j}. \tag{5}$$

By inequalities (1)–(5), we have

$$\begin{aligned}
d(H,U) &= \tau\left(\alpha_l(\widetilde{H-U})\right)_{l=0}^{\infty}\\
&= \sum_{l=0}^{3l_0-1}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\alpha_z(\widetilde{H-U}),\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
&\quad+\sum_{l=3l_0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\alpha_z(\widetilde{H-U}),\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
&\le \sum_{l=0}^{3l_0}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\|\widetilde{H-U}\|,\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
&\quad+\sum_{l=l_0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l+2l_0}A(l+2l_0,z)q_{l+2l_0,z}\alpha_z(\widetilde{H-U}),\widetilde{0})}{(u+v)^{l+2l_0}}\right)^{w_{l+2l_0}}\\
&\le \sum_{l=0}^{3l_0}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\|\widetilde{H-U}\|,\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
&\quad+\sum_{l=l_0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l+2l_0}A(l+2l_0,z)q_{l+2l_0,z}\alpha_z(\widetilde{H-U}),\widetilde{0})}{(u+v)^{l}}\right)^{w_l}\\
&\le 3\sum_{l=0}^{l_0}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\|\widetilde{H-U}\|,\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
&\quad+\sum_{l=l_0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{2l_0-1}A(l+2l_0,z)q_{l+2l_0,z}\alpha_z(\widetilde{H-U})+\sum_{z=2l_0}^{l+2l_0}A(l+2l_0,z)q_{l+2l_0,z}\alpha_z(\widetilde{H-U}),\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
&\le 3\sum_{l=0}^{l_0}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\|\widetilde{H-U}\|,\widetilde{0})}{(u+v)^l}\right)^{w_l}
\end{aligned}$$

$$
\begin{aligned}
&+ 2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{2l_0-1}A(l+2l_0,z)q_{l+2l_0,z}\alpha_z\widetilde{(H-U)},\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
&+ 2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=2l_0}^{l+2l_0}A(l+2l_0,z)q_{l+2l_0,z}\alpha_z\widetilde{(H-U)},\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
\leq{}& 3\sum_{l=0}^{l_0}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\widetilde{\|H-U\|},\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
&+ 2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{2l_0-1}A(l+2l_0,z)q_{l+2l_0,z}\alpha_z\widetilde{(Z-U)},\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
&+ 2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l+2l_0,z+2l_0)q_{z+2l_0}\alpha_{z+2l_0}\widetilde{(H-U)},\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
\leq{}& 3\sum_{l=0}^{l_0}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\widetilde{\|H-U\|},\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
&+ 2^{2\hbar-1}\sup_{l=l_0}^{\infty}\widetilde{\rho}^{w_l}\left(\sum_{z=0}^{l_0}A(l,z)q_{l,z}\widetilde{\|H-U\|},\widetilde{0}\right)\sum_{l=l_0}^{\infty}\left(\frac{1}{(u+v)^l}\right)^{w_l}\\
&+ 2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\widetilde{\alpha_z(H)},\widetilde{0})}{(u+v)^l}\right)^{w_l} < \gamma.
\end{aligned}
$$

On the other hand, we have a negative example since $I_2 \in \widetilde{\mathbb{D}^{\alpha}}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$, where $A(l,z)q_{l,z} = 1$ for all $l,z \in \mathbb{N}$ and $v = (0,-1,2,2,2,\ldots)$. However, $(v_l) \notin \uparrow$, which implies a negative answer of Rhoades' [27] open problem about the linearity of s-type $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ spaces. □

Theorem 3.11 *The class* $(\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}, \Xi)$ *is a prequasi-Banach ideal, where* $\Xi(H) = \tau((\widetilde{s_b(H)})_{b=0}^{\infty})$.

Proof Clearly, Ξ is a prequasinorm on $\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}$ since τ is a prequasinorm on $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$. If $(X_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\widetilde{\mathbb{D}^s}_{(E^{\mathfrak{S}}_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$ and as $\mathbb{D}(\mathcal{G},\mathcal{V}) \supseteq \widetilde{\mathbb{D}^s}_{(E^{\mathfrak{S}}_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$, then

$$
\Xi(H_j-H_m) = \sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}s_z\widetilde{(H_j-H_m)},\widetilde{0})}{(u+v)^l}\right)^{w_l} \geq \left(q_{0,0}\|H_j-H_m\|\right)^{w_0}.
$$

Hence, $(H_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{D}(\mathcal{G},\mathcal{V})$. Since $\mathbb{D}(\mathcal{G},\mathcal{V})$ is a Banach space, then $H \in \mathbb{D}(\mathcal{G},\mathcal{V})$ with $\lim_{m\to\infty}\|H_m - H\| = 0$. Since $(\widetilde{s_l(H_m)})_{l=0}^{\infty} \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ for every $m \in \mathbb{N}$. By Parts (ii), (iii), and (v) of Definition 2.6, one can observe that

$$
\begin{aligned}
\Xi(H) ={}& \sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\widetilde{s_z(H)},\widetilde{0})}{(u+v)^l}\right)^{w_l}\\
\leq{}& 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}s_{[\frac{z}{2}]}\widetilde{(H-H_m)},\widetilde{0})}{(u+v)^l}\right)^{w_l}
\end{aligned}
$$

$$+ 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\widetilde{s_{[\frac{z}{2}]}(H_m)},\widetilde{0})}{(u+v)^l}\right)^{w_l}$$

$$\le 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\|\widetilde{H-H_m}\|,\widetilde{0})}{(u+v)^l}\right)^{w_l}$$

$$+ 2^{\hbar-1}D_0\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\widetilde{s_z(H_m)},\widetilde{0})}{(u+v)^l}\right)^{w_l} < \infty.$$

Hence, $(\widetilde{s_b(H)})_{b=0}^{\infty} \in [E_{u,v}^{\mathfrak{S}}(q,w)]_\tau$, and $H \in \widetilde{\mathbb{D}}^s_{[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$. □

Theorem 3.12 *Assume* $1 < w_b^{(1)} < w_b^{(2)}$ *and* $0 < q_{b,z}^{(2)} \le q_{b,z}^{(1)}$ *for every* $b,z \in \mathbb{N}$, *then*

$$\widetilde{\mathbb{D}}^s_{[E_{u,v}^{\mathfrak{S}}((q_{b,z}^{(1)}),(w_b^{(1)}))]_\tau}(\mathcal{G},\mathcal{V}) \subsetneqq \widetilde{\mathbb{D}}^s_{[E_{u,v}^{\mathfrak{S}}((q_{b,z}^{(2)}),(w_b^{(2)}))]_\tau}(\mathcal{G},\mathcal{V}) \subsetneqq \mathbb{D}(\mathcal{G},\mathcal{V}).$$

Proof Suppose $H \in \widetilde{\mathbb{D}}^s_{[E_{u,v}^{\mathfrak{S}}((q_{b,z}^{(1)}),(w_b^{(1)}))]_\tau}(\mathcal{G},\mathcal{V})$. Then, $(\widetilde{s_b(H)}) \in [E_{u,v}^{\mathfrak{S}}((q_{b,z}^{(1)}),(w_b^{(1)}))]_\tau$. We have

$$\sum_{b=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{b}A(b,z)q_{b,z}^{(2)}\widetilde{s_z(H)},\widetilde{0})}{(u+v)^b}\right)^{w_b^{(2)}} < \sum_{b=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{b}A(b,z)q_{b,z}^{(1)}\widetilde{s_z(H)},\widetilde{0})}{(u+v)^b}\right)^{w_b^{(1)}} < \infty.$$

Hence, $H \in \widetilde{\mathbb{D}}^s_{[E_{u,v}^{\mathfrak{S}}((q_{b,z}^{(2)}),(w_b^{(2)}))]_\tau}(\mathcal{G},\mathcal{V})$. Put $(\widetilde{s_b(H)})_{b=0}^{\infty}$ so that $\widetilde{\rho}(\sum_{z=0}^{b}A(b,z)q_{b,z}^{(1)}\widetilde{s_z(H)},\widetilde{0}) = \frac{(u+v)^b}{w_b^{(1)}\sqrt{b+1}}$ then $H \in \mathbb{D}(\mathcal{G},\mathcal{V})$ so that

$$\sum_{b=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{b}A(b,z)q_{b,z}^{(1)}\widetilde{s_z(H)},\widetilde{0})}{(u+v)^b}\right)^{w_b^{(1)}} = \sum_{b=0}^{\infty}\frac{1}{b+1} = \infty,$$

and

$$\sum_{b=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{b}A(b,z)q_{b,z}^{(2)}\widetilde{s_z(H)},\widetilde{0})}{(u+v)^b}\right)^{w_b^{(2)}} \le \sum_{b=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{b}A(b,z)q_{b,z}^{(1)}\widetilde{s_z(H)},\widetilde{0})}{(u+v)^b}\right)^{w_b^{(2)}}$$

$$= \sum_{b=0}^{\infty}\left(\frac{1}{b+1}\right)^{\frac{w_b^{(2)}}{w_b^{(1)}}} < \infty.$$

Therefore, $H \notin \widetilde{\mathbb{D}}^s_{[E_{u,v}^{\mathfrak{S}}((q_{b,z}^{(1)}),(w_b^{(1)}))]_\tau}(\mathcal{G},\mathcal{V})$ and $H \in \widetilde{\mathbb{D}}^s_{[E_{u,v}^{\mathfrak{S}}((q_{b,z}^{(2)}),(w_b^{(2)}))]_\tau}(\mathcal{G},\mathcal{V})$.

Evidently, $\widetilde{\mathbb{D}}^s_{[E_{u,v}^{\mathfrak{S}}((q_{b,z}^{(2)}),(w_b^{(2)}))]_\tau}(\mathcal{G},\mathcal{V}) \subset \mathbb{D}(\mathcal{G},\mathcal{V})$. Put $(\widetilde{s_b(H)})_{b=0}^{\infty}$ such that $\widetilde{\rho}(\sum_{z=0}^{b}A(b,z) \times q_{b,z}^{(2)}\widetilde{s_z(H)},\widetilde{0}) = \frac{(u+v)^b}{w_b^{(2)}\sqrt{b+1}}$. Hence, $H \in \mathbb{D}(\mathcal{G},\mathcal{V})$ and $H \notin \widetilde{\mathbb{D}}^s_{[E_{u,v}^{\mathfrak{S}}((q_{b,z}^{(2)}),(w_b^{(2)}))]_\tau}(\mathcal{G},\mathcal{V})$. □

Suppose $\mathcal{G}$ and $\mathcal{V}$ are infinite-dimensional, in view of Dvoretzky's theorem [28] one has $\mathcal{G}/Y_j$ and $M_j \subseteq \mathcal{V}$ operated onto ℓ_2^j through isomorphisms V_j and X_j with $\|V_j\|\|V_j^{-1}\| \le 2$ and $\|X_j\|\|X_j^{-1}\| \le 2$ for all $j \in \mathbb{N}$. Assume T_j is the quotient mapping from $\mathcal{G}$ onto $\mathcal{G}/Y_j$, I_j is the identity operator on ℓ_2^j, and J_j is the natural embedding operator from M_j into $\mathcal{V}$. Suppose m_j are the Bernstein numbers [11].

Theorem 3.13 *The class* $\widetilde{\mathbb{D}^{\alpha}}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_{\tau}}$ *is minimum if* $(\frac{\sum_{z=0}^{l} A(l,z)q_{l,z}}{(u+v)^{l}})_{l=0}^{\infty} \notin \ell_{(w)}$.

Proof Suppose $\widetilde{\mathbb{D}^{\alpha}}_{E^{\mathfrak{S}}_{u,v}(q,w)}(\mathcal{G},\mathcal{V}) = \mathbb{D}(\mathcal{G},\mathcal{V})$. Then, there exists $\gamma > 0$ with $\Xi(H) \leq \gamma \|H\|$ for all $H \in \mathbb{D}(\mathcal{G},\mathcal{V})$ and $\Xi(H) = \sum_{b=0}^{\infty}(\frac{\widetilde{\rho}(\sum_{z=0}^{b} A(b,z)q_{b,z}\widetilde{\alpha_z(H)},\widetilde{0})}{(u+v)^{b}})^{w_b}$. One can see that

$$
\begin{aligned}
1 &= m_z(I_j) = m_z\bigl(X_jX_j^{-1}I_jV_jV_j^{-1}\bigr) \\
&\leq \|X_j\| m_z\bigl(X_j^{-1}I_jV_j\bigr)\bigl\|V_j^{-1}\bigr\| \\
&= \|X_j\| m_z\bigl(J_jX_j^{-1}I_jV_j\bigr)\bigl\|V_j^{-1}\bigr\| \\
&\leq \|X_j\| d_z\bigl(J_jX_j^{-1}I_jV_j\bigr)\bigl\|V_j^{-1}\bigr\| \\
&= \|X_j\| d_z\bigl(J_jX_j^{-1}I_jV_jT_j\bigr)\bigl\|V_j^{-1}\bigr\| \\
&\leq \|X_j\| \alpha_z\bigl(J_jX_j^{-1}I_jV_jT_j\bigr)\bigl\|V_j^{-1}\bigr\|.
\end{aligned}
$$

Let $0 \leq m \leq j$. Then, we derive that

$$
\begin{aligned}
&\sum_{z=0}^{m} A(m,z)q_{m,z} \leq \widetilde{\rho}\Biggl(\sum_{z=0}^{m} \|X_j\| A(m,z)q_{m,z}\alpha_z\bigl(\widetilde{J_jX_j^{-1}I_jV_jT_j}\bigr)\bigl\|V_j^{-1}\bigr\|,\widetilde{0}\Biggr) \\
&\quad\Rightarrow\quad \biggl(\frac{\sum_{z=0}^{m} A(m,z)q_{m,z}}{(u+v)^{m}}\biggr)^{w_m} \\
&\qquad\qquad \leq \bigl(\|X_j\|\bigl\|V_j^{-1}\bigr\|\bigr)^{w_m}\biggl(\frac{\widetilde{\rho}(\sum_{z=0}^{m} A(m,z)q_{m,z}\alpha_z(\widetilde{J_jX_j^{-1}I_jV_jT_j}),\widetilde{0})}{(u+v)^{m}}\biggr)^{w_m}.
\end{aligned}
$$

Hence, for some $\lambda \geq 1$, one has

$$
\begin{aligned}
&\sum_{m=0}^{j}\biggl(\frac{\sum_{z=0}^{m} A(m,z)q_{m,z}}{(u+v)^{m}}\biggr)^{w_m} \\
&\quad \leq \lambda\|X_j\|\bigl\|V_j^{-1}\bigr\|\sum_{m=0}^{j}\biggl(\frac{\widetilde{\rho}(\sum_{z=0}^{m} A(m,z)q_{m,z}\alpha_z(\widetilde{J_jX_j^{-1}I_jV_jT_j}),\widetilde{0})}{(u+v)^{m}}\biggr)^{w_m} \\
&\quad\Rightarrow\quad \sum_{m=0}^{j}\biggl(\frac{\sum_{z=0}^{m} A(m,z)q_{m,z}}{(u+v)^{m}}\biggr)^{w_m} \\
&\qquad\qquad \leq \lambda\|X_j\|\bigl\|V_j^{-1}\bigr\|\Xi\bigl(J_jX_j^{-1}I_jV_jT_j\bigr) \leq \lambda\gamma\|X_j\|\bigl\|V_j^{-1}\bigr\|\bigl\|J_jX_j^{-1}I_jV_jT_j\bigr\| \leq 4\lambda\gamma,
\end{aligned}
$$

by letting $j \to \infty$. Then, there is a contradiction. Therefore, $\mathcal{G}$ and $\mathcal{V}$ both cannot be infinite-dimensional if $\widetilde{\mathbb{D}^{\alpha}}_{E^{\mathfrak{S}}_{u,v}(q,w)}(\mathcal{G},\mathcal{V}) = \mathbb{D}(\mathcal{G},\mathcal{V})$. □

Theorem 3.14 *The class* $\widetilde{\mathbb{D}^{d}}_{E^{\mathfrak{S}}_{u,v}(q,w)}$ *is minimum if* $(\frac{\sum_{z=0}^{l} A(l,z)q_{l,z}}{(u+v)^{l}})_{l=0}^{\infty} \notin \ell_{(w)}$.

Lemma 3.15 ([12]) *Assume* $W \in \mathbb{D}(\mathcal{G},\mathcal{V})$ *and* $W \notin \mathcal{A}(\mathcal{G},\mathcal{V})$. *Then, there are* $P \in \mathbb{D}(\mathcal{G})$ *and* $A \in \mathbb{D}(\mathcal{V})$ *with* $AWPe_j = e_j$ *for all* $j \in \mathbb{N}$.

Theorem 3.16 ([12]) *Suppose* $\mathcal{E}^{\mathfrak{S}}$ *is an infinite-dimensional Banach space. Then, we have*

$$
\mathbb{F}\bigl(\mathcal{E}^{\mathfrak{S}}\bigr) \subsetneqq \mathcal{A}\bigl(\mathcal{E}^{\mathfrak{S}}\bigr) \subsetneqq \mathcal{K}\bigl(\mathcal{E}^{\mathfrak{S}}\bigr) \subsetneqq \mathbb{D}\bigl(\mathcal{E}^{\mathfrak{S}}\bigr).
$$

Theorem 3.17 *Assume* $1 < w_l^{(1)} < w_l^{(2)}$ *and* $0 < q_{l,z}^{(2)} \leq q_{l,z}^{(1)}$ *for all* $l, z \in \mathbb{N}$. *Then,*

$$\begin{aligned}&\mathbb{D}\big(\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(2)}),(w_l^{(2)}))]_\tau}(\mathcal{G},\mathcal{V}),\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(1)}),(w_l^{(1)}))]_\tau}(\mathcal{G},\mathcal{V})\big)\\&\quad=\mathcal{A}\big(\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(2)}),(w_l^{(2)}))]_\tau}(\mathcal{G},\mathcal{V}),\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(1)}),(w_l^{(1)}))]_\tau}(\mathcal{G},\mathcal{V})\big).\end{aligned}$$

Proof Suppose

$$X \in \mathbb{D}\big(\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(2)}),(w_l^{(2)}))]_\tau}(\mathcal{G},\mathcal{V}),\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(1)}),(w_l^{(1)}))]_\tau}(\mathcal{G},\mathcal{V})\big),$$

and

$$X \notin \mathcal{A}\big(\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(2)}),(w_l^{(2)}))]_\tau}(\mathcal{G},\mathcal{V}),\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(1)}),(w_l^{(1)}))]_\tau}(\mathcal{G},\mathcal{V})\big).$$

From Lemma 3.15 one has

$$Y \in \mathbb{D}\big(\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(2)}),(w_l^{(2)}))]_\tau}(\mathcal{G},\mathcal{V})\big)$$

and

$$Z \in \mathbb{D}\big(\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(1)}),(w_l^{(1)}))]_\tau}(\mathcal{G},\mathcal{V})\big)$$

such that $ZXYI_b = I_b$. Hence, for all $b \in \mathbb{N}$ we have

$$\begin{aligned}\|I_b\|_{\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(1)}),(w_l^{(1)}))]_\tau}(\mathcal{G},\mathcal{V})}&=\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}^{(1)}\widetilde{s_z(I_b)},\widetilde{0})}{(u+v)^l}\right)^{w_l^{(1)}}\\&\leq\|ZXY\|\|I_b\|_{\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(2)}),(w_l^{(2)}))]_\tau}(\mathcal{G},\mathcal{V})}\\&\leq\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}^{(2)}\widetilde{s_z(I_b)},\widetilde{0})}{(u+v)^l}\right)^{w_l^{(2)}}.\end{aligned}$$

This contradicts Theorem 3.12. Therefore,

$$X \in \mathcal{A}\big(\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(2)}),(w_l^{(2)}))]_\tau}(\mathcal{G},\mathcal{V}),\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(1)}),(w_l^{(1)}))]_\tau}(\mathcal{G},\mathcal{V})\big).\qquad\square$$

Corollary 3.18 *Assume* $1 < w_l^{(1)} < w_l^{(2)}$ *and* $0 < q_{l,z}^{(2)} \leq q_{l,z}^{(1)}$ *for all* $l, z \in \mathbb{N}$. *Then,*

$$\begin{aligned}&\mathbb{D}\big(\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(2)}),(w_l^{(2)}))]_\tau}(\mathcal{G},\mathcal{V}),\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(1)}),(w_l^{(1)}))]_\tau}(\mathcal{G},\mathcal{V})\big)\\&\quad=\mathcal{K}\big(\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(2)}),(w_l^{(2)}))]_\tau}(\mathcal{G},\mathcal{V}),\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((q_{l,z}^{(1)}),(w_l^{(1)}))]_\tau}(\mathcal{G},\mathcal{V})\big).\end{aligned}$$

Proof Since $\mathcal{A} \subset \mathcal{K}$, this is obvious. $\square$

Definition 3.19 ([12]) A Banach space $\mathcal{E}^{\mathfrak{S}}$ is called simple if $\mathbb{D}(\mathcal{E}^{\mathfrak{S}})$ contains a unique nontrivial closed ideal.

Theorem 3.20 *The class* $\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}$ *is simple.*

Proof Assume the closed ideal $\mathcal{K}(\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V}))$ has a mapping $H \notin \mathcal{A}(\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V}))$. From Lemma 3.15, there are $P,A \in \mathbb{D}(\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V}))$ with $AHPI_j = I_j$. Hence, $I_{\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})} \in \mathcal{K}(\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V}))$. Then, $\mathbb{D}(\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})) = \mathcal{K}(\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V}))$. Hence, $\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}$ is a simple Banach space. □

Theorem 3.21 *If* $\inf_l(\frac{\sum_{z=0}^{l} A(l,z)q_z}{(u+v)^l})^{w_l} > 0$, *then* $(\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau})^{\gamma}(\mathcal{G},\mathcal{V}) = \widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$.

Proof Assume $H \in (\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau})^{\gamma}(\mathcal{G},\mathcal{V})$, hence $(\widetilde{\gamma_m(H)})_{m=0}^{\infty} \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ and $\|H - \widetilde{\rho} \times (\widetilde{\gamma_m(H)},\widetilde{0})I\| = 0$ for all $m \in \mathbb{N}$. We have $H = \widetilde{\rho}(\widetilde{\gamma_m(H)},\widetilde{0})I$ for every $m \in \mathbb{N}$, then

$$\widetilde{\rho}\big(\widetilde{s_m(H)},\widetilde{0}\big) = \widetilde{\rho}\big(s_m\big(\widetilde{\widetilde{\rho}\big(\widetilde{\gamma_m(H)},\widetilde{0}\big)I}\big),\widetilde{0}\big) = \widetilde{\rho}\big(\widetilde{\gamma_m(H)},\widetilde{0}\big)$$

for every $m \in \mathbb{N}$. Therefore, $(\widetilde{s_m(H)})_{m=0}^{\infty} \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$. Hence, $H \in \widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$.

Next, suppose $H \in \widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$. Hence, $(\widetilde{s_m(H)})_{m=0}^{\infty} \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$. Hence, we obtain

$$\sum_{m=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{m} A(m,z)q_{m,z}\widetilde{s_z(H)},\widetilde{0})}{(u+v)^m}\right)^{w_m} \geq \inf_m\left(\frac{\sum_{z=0}^{m} A(m,z)q_{m,z}}{(u+v)^m}\right)^{w_m}\sum_{m=0}^{\infty}\big(\widetilde{\rho}\big(\widetilde{s_m(H)},\widetilde{0}\big)\big)^{w_m}.$$

Hence, $\lim_{m\to\infty}\widetilde{s_m(H)} = \widetilde{0}$. Assume $\|H - \widetilde{\rho}(\widetilde{s_m(H)},\widetilde{0})I\|^{-1}$ exists for every $m \in \mathbb{N}$. Hence, $\|H - \widetilde{\rho}(\widetilde{s_m(H)},\widetilde{0})I\|^{-1}$ exists and bounded for all $m \in \mathbb{N}$. Therefore, $\lim_{m\to\infty}\|H - \widetilde{\rho}(\widetilde{s_m(H)},\widetilde{0})I\|^{-1} = \|H\|^{-1}$ exists and is bounded. As $(\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau},\Xi)$ is prequasiideal, then

$$I = HH^{-1} \in \widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V}) \quad\Rightarrow\quad \big(\widetilde{s_m(I)}\big)_{m=0}^{\infty} \in E^{\mathfrak{S}}_{u,v}(q,w) \quad\Rightarrow\quad \lim_{m\to\infty}\widetilde{s_m(I)} = \widetilde{0}.$$

This implies a contradiction since $\lim_{m\to\infty}\widetilde{s_m(I)} = \widetilde{1}$. Hence, $\|H - \widetilde{\rho}(\widetilde{s_m(H)},\widetilde{0})I\| = 0$ for every $m \in \mathbb{N}$. Therefore, $\|H - \widetilde{\rho}(\widetilde{\gamma_m(H)},\widetilde{0})I\| = 0$ for every $m \in \mathbb{N}$. Hence, $H \in (\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau})^{\gamma}(\mathcal{G},\mathcal{V})$. □

4 Multiplication mappings on $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$

Some properties of the multiplication mapping acting on $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ are discussed in this section, assuming that the conditions of Theorem 3.3 are established.

Assume $(\text{Range}(V))^c$ is the complement of $\text{Range}(V)$, $\mathfrak{I}$ is the space of every set with a finite number of elements, and $\ell_\infty^{\mathfrak{S}}$ is the space of bounded sequences of soft functions.

Definition 4.1 If $\mathcal{E}_\tau^{\mathfrak{S}}$ is a prequasinormed $\mathfrak{psssf}$ and $\lambda = (\lambda_k) \in \mathbb{R}^{\mathbb{N}}$. The mapping $H_\lambda : \mathcal{E}_\tau^{\mathfrak{S}} \to \mathcal{E}_\tau^{\mathfrak{S}}$ is called a multiplication mapping on $\mathcal{E}_\tau^{\mathfrak{S}}$ if $H_\lambda\widetilde{f} = (\lambda_b\widetilde{f_b}) \in \mathcal{E}_\tau^{\mathfrak{S}}$ for every $f \in \mathcal{E}_\tau^{\mathfrak{S}}$. The multiplication mapping is said to be generated by λ whenever $H_\lambda \in \mathbb{D}(\mathcal{E}_\tau^{\mathfrak{S}})$.

Definition 4.2 ([29]) A mapping $V \in \mathbb{D}(\mathcal{E})$ is called Fredholm whenever $\dim(\text{Range}(V))^c < \infty$, $\text{Range}(V)$ is closed and $\dim(\ker(V)) < \infty$.

Theorem 4.3 *The following statements hold*:

(1) $\lambda \in \ell_\infty \Longleftrightarrow H_\lambda \in \mathbb{D}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)$;

(2) $|\lambda_a| = 1$ *for all* $a \in \mathbb{N}$, *if and only if*, H_λ *is an isometry*;

(3) $H_\lambda \in \mathcal{A}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau) \Longleftrightarrow (\lambda_a)_{a=0}^\infty \in c_0$;

(4) $H_\lambda \in \mathcal{K}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau) \Longleftrightarrow (\lambda_b)_{b=0}^\infty \in c_0$;

(5) $\mathcal{K}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau) \subsetneqq \mathbb{D}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)$;

(6) $0 < \alpha < |\lambda_a| < \eta$ *for all* $a \in (\ker(\lambda))^c$, *if and only if*, $\mathrm{Range}(H_\lambda)$ *is closed*;

(7) $0 < \alpha < |\lambda_a| < \eta$ *for every* $a \in \mathbb{N}$, *if and only if*, $H_\lambda \in \mathbb{D}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)$ *is invertible*;

(8) H_λ *is a Fredholm operator, if and only if*, $(g1)$ $\ker(\lambda) \subsetneqq \mathbb{N} \cap \mathfrak{I}$ *and* $(g2)$ $|\lambda_a| \geq \varrho$, *for every* $a \in (\ker(\lambda))^c$.

Proof

(1) Let $\lambda \in \ell_\infty$. Then, there exists $\nu > 0$ so that $|\lambda_a| \leq \nu$ for every $a \in \mathbb{N}$. Suppose $\widetilde{f} \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$, hence

$$\begin{aligned}
\tau(H_\lambda \widetilde{f}) &= \tau(\lambda \widetilde{f}) \\
&= \sum_{l=0}^{\infty} \left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} \lambda_z A(l,z) q_{l,z} \widetilde{f}_z, \widetilde{0})}{(u+v)^l} \right)^{w_l} \\
&\leq \sup_l \nu^{w_l} \sum_{l=0}^{\infty} \left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z) q_{l,z} \widetilde{f}_z, \widetilde{0})}{(u+v)^l} \right)^{w_l} \\
&= \sup_l \nu^{w_l} \tau(\widetilde{f}).
\end{aligned}$$

Hence, $H_\lambda \in \mathbb{D}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)$.

When $H_\lambda \in \mathbb{D}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)$ and $\lambda \notin \ell_\infty$. We have $x_b \in \mathbb{N}$ for all $b \in \mathbb{N}$ so that $\lambda_{x_b} > b$. Hence,

$$\begin{aligned}
\tau(H_\lambda \widetilde{e_{x_b}}) = \tau(\lambda \widetilde{e_{x_b}}) &= \sum_{l=0}^{\infty} \left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} \lambda_z A(l,z) q_{l,z} \widetilde{(e_{x_b})_z}, \widetilde{0})}{(u+v)^l} \right)^{w_l} \\
&= \sum_{l=x_b}^{\infty} \left(\frac{\lambda_{(x_b)} A(l,x_b) q_{l,x_b}}{(u+v)^l} \right)^{w_l} \\
&> \sum_{l=x_b}^{\infty} \left(\frac{b A(l,x_b) q_{l,x_b}}{(u+v)^l} \right)^{w_l} > b^{w_0} \tau(\widetilde{e_{x_b}}).
\end{aligned}$$

Hence, $H_\lambda \notin \mathbb{D}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)$. Therefore, $\lambda \in \ell_\infty$.

(2) Assume $\widetilde{f} \in [E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$ and $|\lambda_b| = 1$ for all $b \in \mathbb{N}$. Then,

$$\begin{aligned}
\tau(H_\lambda \widetilde{f}) = \tau(\lambda \widetilde{f}) &= \sum_{l=0}^{\infty} \left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z) q_{l,z} \lambda_z \widetilde{f}_z, \widetilde{0})}{(u+v)^l} \right)^{w_l} \\
&= \sum_{l=0}^{\infty} \left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z) q_{l,z} \widetilde{f}_z, \widetilde{0})}{(u+v)^l} \right)^{w_l} = \tau(\widetilde{f}).
\end{aligned}$$

Hence, H_λ is an isometry.

Next, suppose for some $b = b_0$ that $|\lambda_b| < 1$ we obtain

$$\begin{aligned}
\tau(H_\lambda \widetilde{e_{b_0}}) &= \tau(\lambda \widetilde{e_{b_0}}) \\
&= \sum_{l=0}^{\infty} \left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z) q_{l,z} \lambda_z \widetilde{(e_{b_0})_z}, \widetilde{0})}{(u+v)^l} \right)^{w_l} \\
&= \sum_{l=b_0}^{\infty} \left(\frac{|\lambda_{b_0}| A(l,b_0) q_{l,b_0}}{(u+v)^l} \right)^{w_l} \\
&< \sum_{l=b_0}^{\infty} \left(\frac{A(l,b_0) q_{l,b_0}}{(u+v)^l} \right)^{w_l} = \tau(\widetilde{e_{b_0}}).
\end{aligned}$$

If $|\lambda_{b_0}| > 1$, then $\tau(H_\lambda \widetilde{e_{b_0}}) > \tau(\widetilde{e_{b_0}})$. Therefore, $|\lambda_a| = 1$ for all $a \in \mathbb{N}$.

(3) Suppose $H_\lambda \in \mathcal{A}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)$, then $H_\lambda \in \mathcal{K}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)$. Assume $\lim_{b\to\infty} \lambda_b \neq 0$. We have $\varrho > 0$ so that $K_\varrho = \{a \in \mathbb{N} : |\lambda_a| \geq \varrho\} \supsetneqq \mathfrak{I}$. Suppose $\{\alpha_a\}_{a\in\mathbb{N}} \subset K_\varrho$. Then, $\{\widetilde{e_{\alpha_a}} : \alpha_a \in K_\varrho\} \in \ell^{\mathfrak{S}}_\infty$ is an infinite set in $[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau$. For every $\alpha_a, \alpha_b \in K_\varrho$ we have

$$\begin{aligned}
\tau(H_\lambda \widetilde{e_{\alpha_a}} - H_\lambda \widetilde{e_{\alpha_b}}) &= \tau(\lambda \widetilde{e_{\alpha_a}} - \lambda \widetilde{e_{\alpha_b}}) \\
&= \sum_{l=0}^{\infty} \left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z) q_{l,z} \lambda_z (\widetilde{(e_{\alpha_a})_z} - \widetilde{(e_{\alpha_b})_z}), \widetilde{0})}{(u+v)^l} \right)^{w_l} \\
&\geq \sum_{l=0}^{\infty} \left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z) q_{l,z} \varrho (\widetilde{(e_{\alpha_a})_z} - \widetilde{(e_{\alpha_b})_z}), \widetilde{0})}{(u+v)^l} \right)^{w_l} \\
&\geq \inf_l \varrho^{w_l} \tau(\widetilde{e_{\alpha_a}} - \widetilde{e_{\alpha_b}}).
\end{aligned}$$

Therefore, $\{\widetilde{e_{\alpha_b}} : \alpha_b \in K_\varrho\} \in \ell^{\mathfrak{S}}_\infty$ has no convergent subsequence under H_λ. Hence, $H_\lambda \notin \mathcal{K}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)$. Therefore, $H_\lambda \notin \mathcal{A}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)$, which implies a contradiction. Hence, $\lim_{b\to\infty} \lambda_b = 0$. Next, take $\lim_{a\to\infty} \lambda_a = 0$. Hence, for all $\varrho > 0$ one has $K_\varrho = \{b \in \mathbb{N} : |\lambda_b| \geq \varrho\} \subset \mathfrak{I}$. Hence, for every $\varrho > 0$, we obtain $\dim(([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)_{K_\varrho}) = \dim(\mathbb{R}^{K_\varrho}) < \infty$. Therefore, $H_\lambda \in \mathbb{F}(([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)_{K_\varrho})$. Suppose $\lambda_a \in \mathbb{R}^{\mathbb{N}}$ for every $a \in \mathbb{N}$, where

$$(\lambda_a)_b = \begin{cases} \lambda_b, & b \in K_{\frac{1}{a+1}}, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, $H_{\lambda_a} \in \mathbb{F}(([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)_{K_{\frac{1}{a+1}}})$, as $\dim(([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)_{K_{\frac{1}{a+1}}}) < \infty$ for every $a \in \mathbb{N}$. By $(w_l) \in \uparrow \cap \ell_\infty$ one obtains

$$\begin{aligned}
\tau\big((H_\lambda - H_{\lambda_a})\widetilde{f}\big) &= \tau\big(\big((\lambda_b - (\lambda_a)_b)\widetilde{f_b}\big)_{b=0}^{\infty}\big) \\
&= \sum_{l=0}^{\infty} \left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z) q_{l,z} (\lambda_z - (\lambda_a)_z) \widetilde{f_z}, \widetilde{0})}{(u+v)^l} \right)^{w_l} \\
&= \sum_{l=0, l\in K_{\frac{1}{a+1}}}^{\infty} \left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z) q_{l,z} (\lambda_z - (\lambda_a)_z) \widetilde{f_z}, \widetilde{0})}{(u+v)^l} \right)^{w_l}
\end{aligned}$$

$$+\sum_{l=0,l\notin K_{\frac{1}{a+1}}}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}(\lambda_z-(\lambda_a)_z)\widetilde{f}_z,\widetilde{0})}{(u+v)^l}\right)^{w_l}$$
$$=\sum_{l=0,l\notin K_{\frac{1}{a+1}}}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\lambda_z\widetilde{f}_z,\widetilde{0})}{(u+v)^l}\right)^{w_l}$$
$$\leq\frac{1}{(a+1)^{w_0}}\sum_{l=0,l\notin K_{\frac{1}{a+1}}}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\widetilde{f}_z,\widetilde{0})}{(u+v)^l}\right)^{w_l}$$
$$<\frac{1}{(a+1)^{w_0}}\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\widetilde{f}_z,\widetilde{0})}{(u+v)^l}\right)^{w_l}=\frac{1}{(a+1)^{w_0}}\tau(\widetilde{f}).$$

Hence, $\|H_\lambda-H_{\lambda_a}\|\leq\frac{1}{(a+1)^{w_0}}$. Hence, H_λ is a limit of finite rank mappings.

(4) Since $\mathcal{A}([E_{u,v}^{\mathfrak{S}}(q,w)]_\tau)\subsetneqq\mathcal{K}([E_{u,v}^{\mathfrak{S}}(q,w)]_\tau)$, the proof is immediate.

(5) As $I=I_\lambda$, where $\lambda=(1,1,\ldots)$, we have $I\notin\mathcal{K}([E_{u,v}^{\mathfrak{S}}(q,w)]_\tau)$ and $I\in\mathbb{D}([E_{u,v}^{\mathfrak{S}}(q,w)]_\tau)$.

(6) Assume the sufficient conditions are established. Then, there exists $\varrho>0$ so that $|\lambda_a|\geq\varrho$ for all $a\in(\ker(\lambda))^c$. To prove that Range(H_λ) is closed, assume $\widetilde{g}$ is a limit point of Range(H_λ). We obtain $H_\lambda\widetilde{f_b}\in[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau$ for every $b\in\mathbb{N}$ so that $\lim_{b\to\infty}H_\lambda\widetilde{f_b}=\widetilde{g}$. Evidently, $H_\lambda\widetilde{f_b}$ is a Cauchy sequence. As $(v_l)\in\uparrow\cap\ell_\infty$, then

$$\tau(H_\lambda\widetilde{f_a}-H_\lambda\widetilde{f_b})=\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}(\lambda_z\widetilde{(f_a)_z}-\lambda_z\widetilde{(f_b)_z}),\widetilde{0})}{(u+v)^l}\right)^{w_l}$$
$$=\sum_{l=0,l\in(\ker(\lambda))^c}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}(\lambda_z\widetilde{(f_a)_z}-\lambda_z\widetilde{(f_b)_z}),\widetilde{0})}{(u+v)^l}\right)^{w_l}$$
$$+\sum_{l=0,l\notin(\ker(\lambda))^c}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}(\lambda_z\widetilde{(f_a)_z}-\lambda_z\widetilde{(f_b)_z}),\widetilde{0})}{(u+v)^l}\right)^{w_l}$$
$$\geq\sum_{l=0,l\in(\ker(\lambda))^c}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}(\lambda_z\widetilde{(f_a)_z}-\lambda_z\widetilde{(f_b)_z}),\widetilde{0})}{(u+v)^l}\right)^{w_l}$$
$$=\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}(\lambda_z\widetilde{(u_a)_z}-\lambda_z\widetilde{(u_b)_z}),\widetilde{0})}{(u+v)^l}\right)^{w_l}$$
$$>\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\varrho\sum_{z=0}^{l}A(l,z)q_{l,z}(\widetilde{(u_a)_z}-\widetilde{(u_b)_z}),\widetilde{0})}{(u+v)^l}\right)^{w_l}$$
$$\geq\inf_l\varrho^{w_l}\tau(\widetilde{u_a}-\widetilde{u_b}),$$

where

$$\widetilde{(u_a)_k}=\begin{cases}\widetilde{(f_a)_k}, & k\in(\ker(\lambda))^c,\\ 0, & k\notin(\ker(\lambda))^c.\end{cases}$$

Hence, $\{\widetilde{u_a}\}$ is a Cauchy sequence in $[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau$. As $[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau$ is complete. We obtain $\widetilde{f}\in[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau$ such that $\lim_{b\to\infty}\widetilde{u_b}=\widetilde{f}$. Since $H_\lambda\in\mathbb{D}([E_{u,v}^{\mathfrak{S}}(q,w)]_\tau)$ one obtains $\lim_{b\to\infty}H_\lambda\widetilde{u_b}=H_\lambda\widetilde{f}$. Since $\lim_{b\to\infty}H_\lambda\widetilde{u_b}=\lim_{b\to\infty}H_\lambda\widetilde{f_b}=\widetilde{g}$. Therefore,

$H_\lambda\widetilde{f} = \widetilde{g}$. Hence, $\widetilde{g} \in \mathrm{Range}(H_\lambda)$, i.e., $\mathrm{Range}(H_\lambda)$ is closed. Next, if the necessary setup is verified, we have $\varrho > 0$, so that $\tau(H_\lambda\widetilde{f}) \geq \varrho\tau(\widetilde{f})$ and $\widetilde{f} \in ([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)_{(\ker(\lambda))^c}$. For $K = \{b \in (\ker(\lambda))^c : |\lambda_b| < \varrho\} \neq \emptyset$ one has for $a_0 \in K$ that

$$\begin{aligned}\tau(H_\lambda\widetilde{e_{a_0}}) &= \tau\big(\big(\lambda_b\widetilde{(e_{a_0})_b}\big)_{b=0}^\infty\big)\\ &= \sum_{l=0}^\infty\left(\frac{\widetilde{\rho}(\sum_{z=0}^l A(l,z)q_{l,z}\lambda_z\widetilde{(e_{a_0})_z},\widetilde{0})}{(u+v)^l}\right)^{w_l}\\ &< \sum_{l=0}^\infty\left(\frac{\widetilde{\rho}(\varrho\sum_{z=0}^l A(l,z)q_{l,z}\widetilde{(e_{a_0})_z},\widetilde{0})}{(u+v)^l}\right)^{w_l} \leq \sup_l \varrho^{w_l}\tau(\widetilde{e_{a_0}}).\end{aligned}$$

This implies a contradiction. Hence, $K = \phi$, one obtains $|\lambda_a| \geq \varrho$ for every $a \in (\ker(\lambda))^c$.

(7) First, if $\kappa \in \mathbb{R}^{\mathbb{N}}$ with $\kappa_a = \frac{1}{\lambda_a}$. From Theorem 4.3 setting (1) then $H_\lambda, H_\kappa \in \mathbb{D}([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)$. We have $H_\lambda.H_\kappa = H_\kappa.H_\lambda = I$. Hence, $H_\kappa = H_\lambda^{-1}$. Secondly, when H_λ is invertible. Hence, $\mathrm{Range}(H_\lambda) = ([E^{\mathfrak{S}}_{u,v}(q,w)]_\tau)_{\mathbb{N}}$. Hence, $\mathrm{Range}(H_\lambda)$ is closed. By Theorem 4.3 setting (5) then there is $\alpha > 0$ such that $|\lambda_a| \geq \alpha$ for every $a \in (\ker(\lambda))^c$. Hence, $\ker(\lambda) = \emptyset$ if $\lambda_{a_0} = 0$, where $a_0 \in \mathbb{N}$. This implies $e_{a_0} \in \ker(H_\lambda)$, which is a contradiction, as $\ker(H_\lambda)$ is trivial. Hence, $|\lambda_a| \geq \alpha$ for every $a \in \mathbb{N}$. Since $H_\lambda \in \ell_\infty$. By Theorem 4.3 setting (1) then there exists $\eta > 0$ such that $|\lambda_a| \leq \eta$ for every $a \in \mathbb{N}$. Hence, $\alpha \leq |\lambda_a| \leq \eta$ for every $a \in \mathbb{N}$.

(8) Suppose H_λ is a Fredholm operator. If $\ker(\lambda) \subsetneqq \mathbb{N}$ and $\ker(\lambda) \notin \mathfrak{I}$ we have $\widetilde{e_a} \in \ker(H_\lambda)$ for all $a \in \ker(\lambda)$. Since $\widetilde{e_a}$s are linearly independent, then $\dim(\ker(H_\lambda)) = \infty$. This implies a contradiction. Hence, $\ker(\lambda) \subsetneqq \mathbb{N} \in \mathfrak{I}$ and the setting (g1) holds. The setting (g2) follows from Theorem 4.3 setting (6). In the opposite direction, if the settings (g1) and (g2) are established, by Theorem 4.3 setting (6), then the condition (g2) implies that $\mathrm{Range}(H_\lambda)$ is closed. The setting (g1) explains that $\dim((\mathrm{Range}(H_\lambda))^c) < \infty$ and $\dim(\ker(H_\lambda)) < \infty$. So H_λ is Fredholm. □

5 Fixed points of Kannan contraction type

The existence of a fixed point of a Kannan contraction mapping acting on the prequasiideal type of weighted binomial matrix in a Nakano sequence space of soft functions under the settings of Theorem 3.3 is investigated in this section. Several numerical examples are discussed to explain our results.

In this part, we will use $\Xi(V) = \tau((\widetilde{s_b(V)})_{b=0}^\infty) = (\sum_{l=0}^\infty(\frac{\widetilde{\rho}(\sum_{z=0}^l A(l,z)q_{l,z}\widetilde{s_z(V)},\widetilde{0})}{(u+v)^l})^{w_l})^{\frac{1}{\hbar}}$, for every $V \in \widetilde{\mathbb{D}}^s_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$.

Definition 5.1 ([22]) A function Ξ on $\widetilde{\mathbb{D}}^s{}_{\mathcal{E}\mathfrak{S}}$ confirms the Fatou property when for every $\{V_b\}_{b\in\mathbb{N}} \subseteq \widetilde{\mathbb{D}}^s{}_{\mathcal{E}\mathfrak{S}}(\mathcal{G},\mathcal{V})$ so that $\lim_{b\to\infty}\Xi(V_b - V) = 0$ and every $T \in \widetilde{\mathbb{D}}^s{}_{\mathcal{E}\mathfrak{S}}(\mathcal{G},\mathcal{V})$, one has

$$\Xi(T - V) \leq \sup_b \inf_{j\geq b}\Xi(T - V_j).$$

Theorem 5.2 *The function* Ξ *does not satisfy the Fatou property.*

Proof Assume $\{W_m\}_{m\in\mathbb{N}} \subseteq \widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$ with $\lim_{m\to\infty}\Xi(W_m - W) = 0$. Then, $W \in \widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$. Hence, for all $V \in \widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$ one has

$$\begin{aligned}
\Xi(V-W) &= \left(\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}s_z(\widetilde{V-W}),\widetilde{0})}{(u+v)^l}\right)^{w_l}\right)^{\frac{1}{\hbar}} \\
&\le \left(\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}s_{[\frac{z}{2}]}(\widetilde{V-W_i}),\widetilde{0})}{(u+v)^l}\right)^{w_l}\right)^{\frac{1}{\hbar}} \\
&\quad + \left(\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}s_{[\frac{z}{2}]}(\widetilde{W-W_i}),\widetilde{0})}{(u+v)^l}\right)^{w_l}\right)^{\frac{1}{\hbar}} \\
&\le \left(2^{2\hbar-1}+2^{\hbar-1}+2^{\hbar}\right)^{\frac{1}{\hbar}} \\
&\quad \times \sup_m \inf_{i\ge m}\left(\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}s_z(\widetilde{V-W_i}),\widetilde{0})}{(u+v)^l}\right)^{w_l}\right)^{\frac{1}{\hbar}}.
\end{aligned}$$

Hence, Ξ does not satisfy the Fatou property. □

Definition 5.3 ([24]) A mapping $W : \widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}}(\mathcal{G},\mathcal{V}) \to \widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}}(\mathcal{G},\mathcal{V})$ is called a Kannan Ξ-contraction if one has $\zeta \in [0,\frac{1}{2})$ so that $\Xi(WV - WT) \le \zeta(\Xi(WV - V) + \Xi(WT - T))$ for every $V, T \in \widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}}(\mathcal{G},\mathcal{V})$.

Definition 5.4 ([22]) If $G : \widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}}(\mathcal{G},\mathcal{V}) \to \widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}}(\mathcal{G},\mathcal{V})$. The mapping G is said to be Ξ-sequentially continuous at $B \in \widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}}(\mathcal{G},\mathcal{V})$, if and only if, for every $\{W_m\}_{m\in\mathbb{N}} \subseteq \widetilde{\mathbb{D}^s}_{\mathcal{E}^{\mathfrak{S}}}(\mathcal{G},\mathcal{V})$ so that $\lim_{m\to\infty}\Xi(W_m - B) = 0$ then $\lim_{m\to\infty}\Xi(GW_m - GB) = 0$.

Theorem 5.5 *Suppose* $G : \widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V}) \to \widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$. *The operator* $A \in \widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$ *is the only fixed point of G if the next settings are established*:

(i) *G is a Kannan* Ξ*-contraction*;

(ii) *G is* Ξ*-sequentially continuous at* $A \in \widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$;

(iii) *there is* $B \in \widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$, *so that* $\{G^m B\}$ *contains* $\{G^{m_i}B\}$ *converges to A*.

Proof If A is not a fixed point of G then $GA \ne A$. From the settings (ii) and (iii), then

$$\lim_{m_i\to\infty}\Xi\left(G^{m_i}B - A\right) = 0 \quad \text{and} \quad \lim_{m_i\to\infty}\Xi\left(G^{m_i+1}B - GA\right) = 0.$$

Since G is a Kannan Ξ-contraction operator, one has

$$\begin{aligned}
0 &< \Xi(GA - A) \\
&= \Xi\left(\left(GA - G^{m_i+1}B\right) + \left(G^{m_i}B - A\right) + \left(G^{m_i+1}B - G^{m_i}B\right)\right) \\
&\le \left(2^{2\hbar-1}+2^{\hbar-1}+2^{\hbar}\right)^{\frac{1}{\hbar}}\Xi\left(G^{m_i+1}B - GA\right) + \left(2^{2\hbar-1}+2^{\hbar-1}+2^{\hbar}\right)^{\frac{2}{\hbar}}\Xi\left(G^{m_i}B - A\right) \\
&\quad + \left(2^{2\hbar-1}+2^{\hbar-1}+2^{\hbar}\right)^{\frac{2}{\hbar}}\zeta\left(\frac{\zeta}{1-\zeta}\right)^{m_i-1}\Xi(GB - B).
\end{aligned}$$

Take $m_i \to \infty$. This implies a contradiction. Hence, A is a fixed point of G. To prove the uniqueness of the fixed point A, assume we have two different fixed points $A, D \in$

$\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$ of G. Then,

$$\Xi(A-D)\leq \Xi(GA-GD)\leq \zeta\big(\Xi(GA-A)+\Xi(GD-D)\big)=0.$$

Hence, $A=D$. □

Example 5.6 *Consider*

$$M:\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}((\frac{1}{(l+z+4)A(l,z)})^{\infty}_{l=0},(\frac{2l+3}{l+2})^{\infty}_{l=0})]_\tau}(\mathcal{G},\mathcal{V})\to \widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}((\frac{1}{(l+z+4)A(l,z)})^{\infty}_{l=0},(\frac{2l+3}{l+2})^{\infty}_{l=0})]_\tau}(\mathcal{G},\mathcal{V}) \quad \textit{and}$$
$$M(H)=\begin{cases}\frac{H}{6}, & \Xi(H)\in[0,1),\\ \frac{H}{7}, & \Xi(H)\in[1,\infty).\end{cases}$$

Assume $H_1,H_2\in\widetilde{\mathbb{D}^s}_{[E^{\mathfrak{S}}_{u,v}((\frac{1}{(l+z+4)A(l,z)})^{\infty}_{l=0},(\frac{2l+3}{l+2})^{\infty}_{l=0})]_\tau}(\mathcal{G},\mathcal{V})$. *For* $\Xi(H_1),\Xi(H_2)\in[0,1)$, *then we obtain that*

$$\begin{aligned}\Xi(MH_1-MH_2)&=\Xi\left(\frac{H_1}{6}-\frac{H_2}{6}\right)\\&\leq\frac{\sqrt{2}}{\sqrt[4]{125}}\left(\Xi\left(\frac{5H_1}{6}\right)+\Xi\left(\frac{5H_2}{6}\right)\right)\\&=\frac{\sqrt{2}}{\sqrt[4]{125}}\big(\Xi(MH_1-H_1)+\Xi(MH_2-H_2)\big).\end{aligned}$$

If $\Xi(H_1),\Xi(H_2)\in[1,\infty)$ *then,*

$$\begin{aligned}\Xi(MH_1-MH_2)&=\Xi\left(\frac{H_1}{7}-\frac{H_2}{7}\right)\\&\leq\frac{\sqrt{2}}{\sqrt[4]{216}}\left(\Xi\left(\frac{6H_1}{7}\right)+\Xi\left(\frac{6H_2}{7}\right)\right)\\&=\frac{\sqrt{2}}{\sqrt[4]{216}}\big(\Xi(MH_1-H_1)+\Xi(MH_2-H_2)\big).\end{aligned}$$

Suppose $\Xi(H_1)\in[0,1)$ *and* $\Xi(H_2)\in[1,\infty)$. *Then,*

$$\begin{aligned}\Xi(MH_1-MH_2)&=\Xi\left(\frac{H_1}{6}-\frac{H_2}{7}\right)\\&\leq\frac{\sqrt{2}}{\sqrt[4]{125}}\Xi\left(\frac{5H_1}{6}\right)+\frac{\sqrt{2}}{\sqrt[4]{216}}\Xi\left(\frac{6H_2}{7}\right)\\&\leq\frac{\sqrt{2}}{\sqrt[4]{125}}\big(\Xi(MH_1-H_1)+\Xi(MH_2-H_2)\big).\end{aligned}$$

Therefore, M *is a Kannan* Ξ*-contraction and*

$$M^m(H)=\begin{cases}\frac{H}{6^m}, & \Xi(H)\in[0,1),\\ \frac{H}{7^m}, & \Xi(H)\in[1,\infty).\end{cases}$$

Obviously, M is Ξ-sequentially continuous at the zero operator Θ and $\{M^m H\}$ contains a $\{M^{m_j} H\}$ that converges to Θ. By Theorem 5.5 the zero operator is the only fixed point of M.

Assume $\{H^{(a)}\} \subseteq \widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((\frac{1}{(l+z+4)A(l,z)})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty})]_\tau}(\mathcal{G},\mathcal{V})$ such that $\lim_{a\to\infty}\Xi(H^{(a)}-H^{(0)})=0$, where $H^{(0)} \in \widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((\frac{1}{(l+z+4)A(l,z)})_{l=0}^{\infty},(\frac{2l+3}{l+2})_{l=0}^{\infty})]_\tau}(\mathcal{G},\mathcal{V})$ with $\Xi(H^{(0)})=1$. Since Ξ is continuous, then

$$\lim_{a\to\infty}\Xi\big(MH^{(a)}-MH^{(0)}\big)=\lim_{a\to\infty}\Xi\left(\frac{H^{(0)}}{6}-\frac{H^{(0)}}{7}\right)=\Xi\left(\frac{H^{(0)}}{42}\right)>0.$$

Hence, M is not Ξ-sequentially continuous at $H^{(0)}$. This implies M is not continuous at $H^{(0)}$.

6 Applications on a stochastic nonlinear dynamical system

The solution of nonlinear matrix equations (6) at $D\in\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$ under the settings of theorem 3.3 are investigated in this part, where $\Xi(G)=(\sum_{l=0}^{\infty}(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}\widetilde{s_z(G)},\widetilde{0})}{(u+v)^l})^{w_l})^{\frac{1}{\hbar}}$ for every $G\in\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$. Assume the stochastic nonlinear dynamical system [22]:

$$\widetilde{s_z(G)}=\widetilde{s_z(P)}+\sum_{m=0}^{\infty}\Pi(z,m)f\big(m,\widetilde{s_m(G)}\big) \tag{6}$$

and suppose $W:\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau}(\mathcal{G},\mathcal{V})\to\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$ is defined by

$$W(G)=\left(\widetilde{s_z(P)}+\sum_{m=0}^{\infty}\Pi(z,m)f\big(m,\widetilde{s_m(G)}\big)\right)I. \tag{7}$$

Theorem 6.1 *The stochastic nonlinear dynamical system (6) has a unique solution $D\in\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$ if the next settings are confirmed:*

(1) *$\Pi:\mathbb{N}^2\to\mathbb{R}$, $f:\mathbb{N}\times\mathbb{R}(\mathbb{A})\to\mathbb{R}(\mathbb{A})$, $P\in\mathbb{D}(\mathcal{G},\mathcal{V})$, $T\in\mathbb{D}(\mathcal{G},\mathcal{V})$, and for every $z\in\mathbb{N}$ there is a positive real κ with $\sup_z\kappa^{\frac{w_z}{\hbar}}\in[0,0.5)$ so that*

$$\begin{aligned}&\left|\sum_{m\in\mathbb{N}}\Pi(z,m)\big(f\big(m,\widetilde{s_m(G)}\big)-f\big(m,\widetilde{s_m(T)}\big)\big)\right|\\&\quad\widetilde{\leq}\kappa\left(\left|\widetilde{s_z(P)}-\widetilde{s_z(G)}+\sum_{m\in\mathbb{N}}\Pi(z,m)f\big(m,\widetilde{s_m(G)}\big)\right|\right.\\&\qquad\left.+\left|\widetilde{s_z(P)}-\widetilde{s_z(T)}+\sum_{m\in\mathbb{N}}\Pi(z,m)f\big(m,\widetilde{s_m(T)}\big)\right|\right);\end{aligned}$$

(2) *W is Ξ-sequentially continuous at a point $D\in\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$;*

(3) *one has $B\in\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$ with $\{W^aB\}$ has a $\{W^{a_i}B\}$ converging to D.*

Proof Assume $W:\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau}(\mathcal{G},\mathcal{V})\to\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$ is defined by (7). Then,

$$\begin{aligned}&\Xi(WG-WT)\\&\quad=\left(\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}A(l,z)q_{l,z}(\widetilde{s_z(G)}-\widetilde{s_z(T)}),\widetilde{0})}{(u+v)^l}\right)^{w_l}\right)^{\frac{1}{\hbar}}\end{aligned}$$

$$= \left(\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z)q_{l,z}\sum_{m\in\mathbb{N}}\Pi(z,m)(f(m,\widetilde{s_m(G)}) - f(m,\widetilde{s_m(T)})),\widetilde{0})}{(u+v)^l}\right)^{w_l}\right)^{\frac{1}{\hbar}}$$

$$\leq \sup_z \kappa^{\frac{t_z}{\hbar}}$$

$$\times \left(\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z)q_{l,z}(\widetilde{s_z(P)} - \widetilde{s_z(G)} + \sum_{m\in\mathbb{N}}\Pi(z,m)f(m,\widetilde{s_m(G)})),\widetilde{0})}{(u+v)^l}\right)^{w_l}\right)^{\frac{1}{\hbar}}$$

$$+ \sup_z \kappa^{\frac{t_z}{\hbar}}$$

$$\times \left(\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l} A(l,z)q_{l,z}(\widetilde{s_z(P)} - \widetilde{s_z(T)} + \sum_{m\in\mathbb{N}}\Pi(z,m)f(m,\widetilde{s_m(T)})),\widetilde{0})}{(u+v)^l}\right)^{w_l}\right)^{\frac{1}{\hbar}}$$

$$= \sup_z \kappa^{\frac{t_z}{\hbar}}\big(\Xi(WG-G) + \Xi(WT-T)\big).$$

By Theorem 5.5 we have a unique solution of equation (6) at $D \in \widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}(q,w)]_\tau}(\mathcal{G},\mathcal{V})$. □

Example 6.2 *Consider* $\widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((\frac{1}{(l+z)!}),(\frac{2l+3}{l+2}))]_\tau}(\mathcal{G},\mathcal{V})$, *where*

$$\Xi(G) = \sqrt{\sum_{l=0}^{\infty}\left(\frac{\widetilde{\rho}(\sum_{z=0}^{l}\frac{A(l,z)}{(l+z)!}\widetilde{s_z(G)},\widetilde{0})}{(u+v)^l}\right)^{\frac{2l+3}{l+2}}} \quad \textit{for every } G \in \widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((\frac{1}{(l+z)!}),(\frac{2l+3}{l+2}))]_\tau}(\mathcal{G},\mathcal{V}).$$

Assume the stochastic nonlinear dynamical system:

$$\widetilde{s_z(G)} = \widetilde{e^{-(2z+3)}} + \sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-z)\cos^b|\widetilde{s_{z-2}(G)}|}{\sinh^d|\widetilde{s_{z-1}(G)}| + \widetilde{\sin mz} + \widetilde{1}}, \tag{8}$$

for all $z \geq 2$, $b,d > 0$, *and assume* $W : \widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((\frac{1}{(l+z)!}),(\frac{2l+3}{l+2}))]_\tau}(\mathcal{G},\mathcal{V}) \to \widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((\frac{1}{(l+z)!}),(\frac{2l+3}{l+2}))]_\tau}(\mathcal{G},\mathcal{V})$ *is defined by*

$$W(G) = \left(\widetilde{e^{-(2z+3)}} + \sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-z)\cos^b|\widetilde{s_{z-2}(G)}|}{\sinh^d|\widetilde{s_{z-1}(G)}| + \widetilde{\sin mz} + \widetilde{1}}\right)I. \tag{9}$$

Assume W *is* Ξ*-sequentially continuous at a point* $D \in \widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((\frac{1}{(l+z)!}),(\frac{2l+3}{l+2}))]_\tau}(\mathcal{G},\mathcal{V})$ *and one has* $B \in \widetilde{\mathbb{D}^s}_{[E_{u,v}^{\mathfrak{S}}((\frac{1}{(l+z)!}),(\frac{2l+3}{l+2}))]_\tau}(\mathcal{G},\mathcal{V})$ *with* $\{W^a B\}$ *has a* $\{W^{a_i}B\}$ *converging to* D. *Obviously,*

$$\left|\sum_{m=0}^{\infty}\frac{\cosh(3m-z)\cos^b|\widetilde{s_{z-2}(G)}|}{\sinh^d|\widetilde{s_{z-1}(G)}| + \widetilde{\sin mz} + \widetilde{1}}\big(\tan(2m+1) - \tan(2m+1)\big)\right|$$

$$\precsim \frac{1}{25}\left|\widetilde{e^{-(2z+3)}} - \widetilde{s_z(G)} + \sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-z)\cos^b|\widetilde{s_{z-2}(G)}|}{\sinh^d|\widetilde{s_{z-1}(G)}| + \widetilde{\sin mz} + \widetilde{1}}\right|$$

$$+ \frac{1}{25}\left|\widetilde{e^{-(2z+3)}} - \widetilde{s_z(T)} + \sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-z)\cos^b|\widetilde{s_{z-2}(T)}|}{\sinh^d|\widetilde{s_{z-1}(T)}| + \widetilde{\sin mz} + \widetilde{1}}\right|.$$

In view of Theorem 6.1 the stochastic nonlinear dynamical system (8) has a unique solution D.

7 Conclusion

Some geometric and topological structures of the multiplication operator acting on the weighted binomial matrices in the Nakano sequence space of soft functions and the operators ideal are presented. The existence of a fixed point of the Kannan contraction operator in this prequasioperator ideal is confirmed. Finally, we discussed many applications of solutions to nonlinear stochastic dynamical matrix systems and illustrative examples of our findings.

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AB, OM and MA carried out the conceptualization of the paper. AB, MA and OM carried out methodology, formal analysis, and investigation. Writing the initial draft preparation was carried out by AB, MA and OM. Writing, reviewing, and editing was carried out by MA, AM and AB. AB, OM, MA and AM carried out project administration. All authors read and approved the final manuscript.

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