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On the shape-preserving properties of λ-Bernstein operators

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Abstract

We investigate the shape-preserving properties of λ-Bernstein operators $B_{n,\lambda}(f;x)$ that were recently introduced Bernstein-type operators defined by a new Beziér basis with shape parameter $\lambda \in [-1,1]$. For this purpose, we express $B_{n,\lambda}(f;x)$ as a sum of a classical Bernstein operator and a sum of first order divided differences of f. Using this new representation, we prove that $B_{n,\lambda}(f;x)$ preserves monotonic functions for all $\lambda \in [-1,1]$. However, we show by a counter example that $B_{n,\lambda}(f;x)$ does not preserve convex functions for some $\lambda \in [-1,1]$. We present a weaker result for the case $\lambda \in [0,1]$ for a special class of functions. Finally, we analyze the monotonicity of λ-Bernstein operators with n and show that $B_{n,\lambda}(f;x)$ is not monotonic with n for some λ if $1/2 < \lambda \leq 1$.

MSC: Primary 41A36; 47A58; secondary 41A10

Keywords: λ-Bernstein operators; Shape-preserving properties; Computer-aided design; Computer graphics

1 Introduction

Bernstein [1] introduced the famous Bernstein operators that are defined by

$$B_n(f;x) = \sum_{j=0}^{n} b_{n,j}(x) f\left(\frac{j}{n}\right), \tag{1}$$

where $f:[0,1] \to \mathbb{R}$ is a function, $n \in \mathbb{N} := \{1,2,\ldots\}$, $x \in [0,1]$ and $b_{n,j}(x)$ is defined by

$$b_{n,j}(x) := \binom{n}{j} x^j (1-x)^{n-j}, \tag{2}$$

where $j \in \{0,1,2,\ldots,n\}$. Bernstein [1] proved that $B_n(f;x)$ converges to $f(x)$ uniformly on $[0,1]$ as $n \to \infty$ for any continuous function $f:[0,1] \to \mathbb{R}$.

Among all linear positive operators, Bernstein operators are the most studied ones (see the monograph [2] for a survey of studies). This is due to their numerous applications in science and engineering, and also their favorable shape-preserving properties.

Since Bernstein operators possess favorable properties and are widely used in applications, there have been numerous generalizations and variants [3–8]. In particular, Ye et al. [9] introduced a new Bézier basis that is dependent on a shape parameter $\lambda \in [-1,1]$. Using this Bézier basis, new Bernstein-type operators (called λ-Bernstein operators) were introduced [3]

$$B_{n,\lambda}(f;x) := \sum_{j=0}^{n} b_{n,j}(\lambda;x) f\left(\frac{j}{n}\right),$$

where $\lambda \in [-1,1]$ and the Bézier basis is defined by [9]

$$\begin{aligned} b_{n,0}(\lambda;x) &= b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x), \\ b_{n,j}(\lambda;x) &= b_{n,j}(x) + \lambda \frac{n-2j+1}{n^2-1} b_{n+1,j}(x) - \lambda \frac{n-2j-1}{n^2-1} b_{n+1,j+1}(x), \\ b_{n,n}(\lambda;x) &= b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x), \end{aligned}$$

where $b_{n,j}(x)$ is given by (2). Note that taking $\lambda = 0$, one has the well-known Bernstein operator given by (1). Moreover, introducing the shape parameter λ, one has more modeling flexibility. We refer to [10–14] for more details about λ-Bernstein operators and their variants.

Bernstein operators have favorable shape-preserving properties and studying them is crucial for applications in computer-aided design and computer graphics (see [3, 5, 15] for recent studies). It is well known that Bernstein operators have a convexity-preserving property [2]. Namely, $B_n(f)$ is convex for every n, whenever $f \in C[0,1]$ is convex. Moreover, Bernstein operators preserve monotonic functions i.e., $B_n(f)$ is a decreasing (increasing) function for all $n \in \mathbb{N}$ whenever $f : [0,1] \to \mathbb{R}$ is a decreasing (increasing) function, respectively [2]. Temple [16] investigated the monotonicity of Bernstein operators with n. Namely, if f is a convex function on $[0,1]$, then $B_n(f;x)$ are monotonic in n, meaning that for all $n \in \mathbb{N}$ and $x \in [0,1]$ the inequality $B_{n+1}(f;x) \le B_n(f;x)$ holds. The converse of this property also holds [2].

The main purpose of this paper is to investigate the shape-preserving properties of recently introduced λ-Bernstein operators. To this end, we introduce a new representation of $B_{n,\lambda}(f;x)$ as a sum of a Bernstein operator $B_n(f;x)$ and a sum of first order divided differences of f. With the help of this new expression, we show that λ-Bernstein operators preserve monotonic functions. On the other hand, we show by a counter example that the convexity-preserving property is not satisfied for some $\lambda \in [-1,1]$. However, a weaker result for the convexity-preserving property is proven. Finally, we show that the monotonicity of λ-Bernstein operators with n fails for some $\lambda > 1/2$.

2 Preliminaries

Recall that Bernstein basis functions satisfy the following properties [2]

$$b_{n,j}(x) = 0, \quad \text{if } j > n \text{ or } j < 0,$$

$$b_{n,j}(x) = \left(1 - \frac{j}{n+1}\right) b_{n+1,j}(x) + \frac{j+1}{n+1} b_{n+1,j+1}(x), \tag{3}$$

$$\frac{d}{dx} b_{n,j}(x) = n\left[b_{n-1,j-1}(x) - b_{n-1,j}(x)\right]. \tag{4}$$

Definition 2.1 ([2]) Let $x_1, x_2, \dots, x_r \in [0,1]$ be distinct points and f be a real-valued function on $[0,1]$. Then, the divided difference of f with order $(r-1)$ is defined as

$$[x_1, x_2, \dots, x_r : f] := \sum_{j=1}^{r} \frac{f(x_j)}{(x_j - x_1)(x_j - x_2)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_r)}.$$

Fix $r \in \mathbb{N}$ and $f : [0,1] \to \mathbb{R}$. We say that f is a convex (respectively, concave) function of order r, if all its divided differences with order $(r+1)$ are positive (respectively, negative).

Theorem 2.1 ([2]) *The identity*

$$\frac{d^r}{dx^r} B_n(f;x) = \frac{n!r!}{(n-r)!n^r} \sum_{j=0}^{n-r} b_{n-r,j}(x) \left[\frac{j}{n}, \frac{j+1}{n}, \dots, \frac{j+r}{n} : f\right], \tag{5}$$

holds for any $f : [0,1] \to \mathbb{R}$ *and* $r \in \{0, 1, \dots, n\}$.

Corollary 2.1 ([2]) *Bernstein operators preserve convexities of all orders. In particular,* $B_n(f)$ *is decreasing (increasing) for every* n *whenever* f *is a decreasing (increasing) function on* $[0,1]$, *respectively. Similarly,* $B_n(f)$ *is convex (concave) for every* n *whenever* f *is a convex (concave) function on* $[0,1]$, *respectively.*

Theorem 2.2 ([2]) *Bernstein operators satisfy the identity*

$$B_{n+1}(f;x) - B_n(f;x) = -\frac{x(1-x)}{n(n+1)} \sum_{j=0}^{n-1} b_{n-1,j}(x) \left[\frac{j}{n}, \frac{j+1}{n+1}, \frac{j+1}{n} : f\right]$$

for $f : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, $x \in [0,1]$.

Corollary 2.2 ([2]) *If* $f : [0,1] \to \mathbb{R}$ *is convex, then* $B_{n+1}(f;x) \le B_n(f;x)$ *for all* $n \in \mathbb{N}$, $x \in [0,1]$.

Lemma 2.1 ([3]) *λ-Bernstein operators satisfy*

$$B_{n,\lambda}(1;x) = 1,$$

$$B_{n,\lambda}(t;x) = x + \lambda \frac{1 - 2x + x^{n+1} - (1-x)^{n+1}}{n(n-1)},$$

$$B_{n,\lambda}\left(t^2;x\right) = x^2 + \frac{x(1-x)}{n} + \lambda \left[\frac{2x - 4x^2 + 2x^{n+1}}{n(n-1)} + \frac{x^{n+1} + (1-x)^{n+1} - 1}{n^2(n-1)}\right].$$

Theorem 2.3 ([3]) *If* $f \in C[0,1]$ *and* $\lambda \in [-1,1]$, *then* $B_{n,\lambda}(f;x)$ *converges to* $f(x)$ *uniformly on* $[0,1]$ *as* $n \to \infty$.

3 Main results

From now on, we will use the notation $f_j := f(\frac{j}{n})$ for $j = 0, 1, 2, \dots, n$.

Lemma 3.1 *We can write* $B_{n,\lambda}(f;x)$ *in the following form*

$$B_{n,\lambda}(f;x) = B_n(f;x) + \lambda \sum_{j=1}^{n} \frac{n - 2j + 1}{n^2 - 1} b_{n+1,j}(x)[f_j - f_{j-1}]. \tag{6}$$

Proof By definition of λ-Bernstein operators, we have

$$\begin{aligned} B_{n,\lambda}(f;x) &= \sum_{j=0}^{n} b_{n,j}(x)f_j + \lambda \sum_{j=1}^{n} \frac{n-2j+1}{n^2-1} b_{n+1,j}(x)f_j \\ &\quad - \lambda \sum_{j=0}^{n-1} \frac{n-2j-1}{n^2-1} b_{n+1,j+1}(x)f_j \\ &= B_n(f;x) + \lambda \sum_{j=1}^{n} \frac{n-2j+1}{n^2-1} b_{n+1,j}(x)f_j \\ &\quad - \lambda \sum_{j=1}^{n} \frac{n-2j+1}{n^2-1} b_{n+1,j}(x)f_{j-1}. \end{aligned}$$

The last equation easily implies (6) and the proof is completed. □

Lemma 3.2 *λ-Bernstein operators satisfy the identity*

$$\begin{aligned} \frac{d}{dx}B_{n,\lambda}(f;x) &= \sum_{j=0}^{n-1} [f_{j+1}-f_j] b_{n,j}(x)\left[n-j+\lambda\frac{n-2j-1}{n-1}\right] \\ &\quad + \sum_{j=0}^{n-1} [f_{j+1}-f_j] b_{n,j+1}(x)\left[j+1-\lambda\frac{n-2j-1}{n-1}\right], \end{aligned} \tag{7}$$

for all $f:[0,1]\to\mathbb{R}$ *and* $\lambda\in[-1,1]$.

Proof Differentiating the expression (6) and using (4) and (5), we obtain

$$\begin{aligned} \frac{d}{dx}B_{n,\lambda}(f;x) &= \sum_{j=0}^{n-1} b_{n-1,j}(x)\left[\frac{j}{n},\frac{j+1}{n}:f\right] \\ &\quad + \lambda \sum_{j=1}^{n} \frac{n-2j+1}{n-1}\big(b_{n,j-1}(x)-b_{n,j}(x)\big)(f_j-f_{j-1}). \end{aligned}$$

Using the property (3), we obtain

$$\begin{aligned} \frac{d}{dx}B_{n,\lambda}(f;x) &= n\sum_{j=0}^{n-1}\left[\left(1-\frac{j}{n}\right)b_{n,j}(x)+\frac{j+1}{n}b_{n,j+1}(x)\right](f_{j+1}-f_j) \\ &\quad + \lambda \sum_{j=0}^{n-1} \frac{n-2j-1}{n-1}\big(b_{n,j}(x)-b_{n,j+1}(x)\big)(f_{j+1}-f_j), \end{aligned}$$

which gives (7) and completes the proof. □

Remark 3.1 Taking $\lambda = 0$ in equation (7), we arrive at equation (5) for $r = 1$.

Theorem 3.1 *λ-Bernstein operators preserve monotonic functions for all* $\lambda\in[-1,1]$, *i.e.*, $B_{n,\lambda}(f)$ *is decreasing (increasing) for all* $n\in\mathbb{N}$ *and* $\lambda\in[-1,1]$ *whenever* $f:[0,1]\to\mathbb{R}$ *is decreasing (increasing), respectively.*

Proof Let $f:[0,1]\to\mathbb{R}$ be an increasing function. Then, for all distinct points $u,v\in[0,1]$, one has

$$[u,v:f]=\frac{f(v)-f(u)}{v-u}>0. \tag{8}$$

Since $0\le j\le n-1$, it easily follows that $-1\le 1-\frac{2j}{n-1}\le 1$. Using $-1\le\lambda\le 1$, one easily obtains $-1\le -\lambda(1-\frac{2j}{n-1})\le 1$. As a result, we can write

$$0\le n-j-1\le n-j+\lambda\left(1-\frac{2j}{n-1}\right), \tag{9}$$

and

$$0\le (j+1)-1\le (j+1)-\lambda\left(1-\frac{2j}{n-1}\right). \tag{10}$$

Using Lemma 3.2, it follows from (8), (9), and (10) that $\frac{d}{dx}B_{n,\lambda}(f;x)>0$ and thus $B_{n,\lambda}(f;x)$ is increasing. Analogously, one can prove that if f is decreasing, then so is $B_{n,\lambda}(f;x)$. □

Lemma 3.3 *λ-Bernstein operators satisfy*

$$\begin{aligned}\frac{d^2}{dx^2}B_{n,\lambda}(f;x)&=\lambda\frac{n(n+1)}{n-1}\left\{b_{n-1,0}(x)(f_0-f_1)+b_{n-1,n-1}(x)(f_n-f_{n-1})\right\}\\&\quad+n\sum_{j=0}^{n-2}(f_{j+2}-2f_{j+1}+f_j)b_{n-1,j}(x)\left(n-j-1+\lambda\frac{n-2j-3}{n-1}\right)\\&\quad+n\sum_{k=0}^{n-2}(f_{j+2}-2f_{j+1}+f_j)b_{n-1,j+1}(x)\left(j+1-\lambda\frac{n-2j-1}{n-1}\right),\end{aligned}$$

for all $f:[0,1]\to\mathbb{R}$ *and* $\lambda\in[-1,1]$.

Proof Differentiating (7) and using (4) one has

$$\begin{aligned}\frac{d^2}{dx^2}B_{n,\lambda}(f;x)&=n\sum_{j=0}^{n-1}(f_{j+1}-f_j)\big(b_{n-1,j-1}(x)-b_{n-1,j}(x)\big)\left(n-j+\lambda\frac{n-2j-1}{n-1}\right)\\&\quad+n\sum_{j=0}^{n-1}(f_{j+1}-f_j)\big(b_{n-1,j}(x)-b_{n-1,j+1}(x)\big)\left(j+1-\lambda\frac{n-2j-1}{n-1}\right).\end{aligned}$$

The last equation can be written as

$$\begin{aligned}\frac{d^2}{dx^2}B_{n,\lambda}(f;x)&=n\sum_{j=0}^{n-2}(f_{j+2}-f_{j+1})b_{n-1,j}(x)\left(n-j-1+\lambda\frac{n-2j-3}{n-1}\right)\\&\quad+n\sum_{j=0}^{n-1}(f_j-f_{j+1})b_{n-1,j}(x)\left(n-j-1+\lambda\frac{n-2j-3}{n-1}\right)\end{aligned}$$

$$+ n\sum_{j=0}^{n-1}(f_{j+1} - f_j)b_{n-1,j}(x)\left(j - \lambda\frac{n-2j+1}{n-1}\right)$$
$$+ n\sum_{j=0}^{n-2}(f_j - f_{j+1})b_{n-1,j+1}(x)\left(j + 1 - \lambda\frac{n-2j-1}{n-1}\right).$$

Similarly, shifting the index in the third sum completes the proof. □

Remark 3.2 Taking $\lambda = 0$ in Lemma 3.3, we obtain (5) for $r = 2$. Moreover, if $0 \le j \le n-2$, then it is obvious that

$$-1 \le 1 - \frac{2(j+1)}{n-1} \le 1.$$

Since $-1 \le \lambda \le 1$, it immediately follows that

$$-1 \le \lambda\left(1 - \frac{2(j+1)}{n-1}\right) \le 1,$$

and thus

$$0 \le n - j - 2 \le n - j - 1 + \lambda\frac{n-2j-3}{n-1}.$$

Similarly, one can show that

$$0 \le j + 1 - \lambda\frac{n-2j-1}{n-1}.$$

Let $f : [0,1] \to \mathbb{R}$ be a convex function. Then, all divided differences $[x_1, x_2, x_3 : f]$ are positive. It easily follows that

$$\frac{2}{n^2}\left[\frac{j}{n}, \frac{j+1}{n}, \frac{j+2}{n} : f\right] = f_{j+2} - 2f_{j+1} + f_j > 0.$$

As a result

$$n\sum_{j=0}^{n-2}(f_{j+2} - 2f_{j+1} + f_j)\left[b_{n-1,j}(x)\left(n - j - 1 + \lambda\frac{n-2j-3}{n-1}\right)\right.$$
$$\left. + b_{n-1,j+1}(x)\left(j + 1 - \lambda\frac{n-2j-1}{n-1}\right)\right] > 0.$$

However, the term

$$\lambda\frac{n(n+1)}{n-1}\left\{b_{n-1,0}(x)\left[f(0) - f\left(\frac{1}{n}\right)\right] + b_{n-1,n-1}(x)\left[f(1) - f\left(\frac{n-1}{n}\right)\right]\right\}$$

can be negative or positive (since $-1 \le \lambda \le 1$), which may cause $\frac{d^2}{dx^2}B_{n,\lambda}(f;x) < 0$. We demonstrate this with an example.

Table 1 The intervals where $B_{n,-1}(t^2;x)$ is convex for corresponding values of n

n	the interval where $B_{n,-1}(t^2;x)$ is convex
2	$[0, 0.58333]$
3	$[0, 0.75]$
4	$[0, 0.83264]$
5	$[0, 0.88232]$
6	$[0, 0.91385]$
7	$[0, 0.93466]$
8	$[0, 0.94892]$
9	$[0, 0.95907]$
10	$[0, 0.96650]$
15	$[0, 0.98475]$
20	$[0, 0.99135]$
25	$[0, 0.99444]$
30	$[0, 0.99613]$
35	$[0, 0.99715]$
40	$[0, 0.99782]$
45	$[0, 0.99828]$
50	$[0, 0.99860]$

Example 3.1 Consider the convex function $f(t) = t^2$ on $[0,1]$. From Lemma 2.1, we obtain

$$\frac{d^2}{dx^2}B_{n,\lambda}\left(t^2;x\right) = 2 - \frac{2}{n} + \lambda\frac{-8 + (n+1)(2n+1)x^{n-1} + (n+1)(1-x)^{n-1}}{n(n-1)}. \tag{11}$$

Taking $\lambda = 1$ and $n = 2$ in the last equation, we have

$$\frac{d^2}{dx^2}B_{2,1}\left(t^2;x\right) = 6x - \frac{3}{2},$$

and it is obvious that $B_{2,1}(t^2;x)$ is convex on the interval $(\frac{1}{4}, 1)$, whereas it is concave on the interval $(0, \frac{1}{4})$. Therefore, $B_{n,1}(f;x)$ does not preserve convexity. Similarly, taking $\lambda = -1$ in equation (11), we give the intervals where $B_{n,-1}(t^2;x)$ is convex in Table 1 for different values of n. From the table, it can be observed that $B_{n,-1}(t^2;x)$ is not convex on $[0,1]$ for $n \leq 50$ and thus $B_{n,-1}(f)$ does not preserve convexity. We can see this in Fig. 1 for $n = 2, 3, 4, 5,$ and 10.

Solving the inequality

$$\frac{d^2}{dx^2}B_{n,\lambda}\left(t^2;x\right) > 0$$

for different values of λ and n, we see more examples in which $B_{n,\lambda}(t^2;x)$ is not convex on $[0,1]$. We collect these examples in Table 2, which shows the intervals where $B_{n,\lambda}(t^2;x)$ is convex for different values of λ and n.

The numerical data shows us that for $\lambda = -\frac{1}{2}$, $B_{n,\lambda}(t^2;x)$ is convex on $[0,1]$ only for $n \geq 5$, for $\lambda = \frac{1}{2}$, $B_{n,\lambda}(t^2;x)$ is convex on $[0,1]$ only for $n \geq 3$. Similarly, we see that $B_{n,\lambda}(f)$ does not preserve convexity for $\lambda = -\frac{15}{16}$ and $\lambda = -\frac{35}{36}$ when we look at the simple example $f(t) = t^2$. Therefore, we conclude the next result.

Remark 3.3 λ-Bernstein operators do not preserve convexity of functions for certain $\lambda \in [-1,1]$.

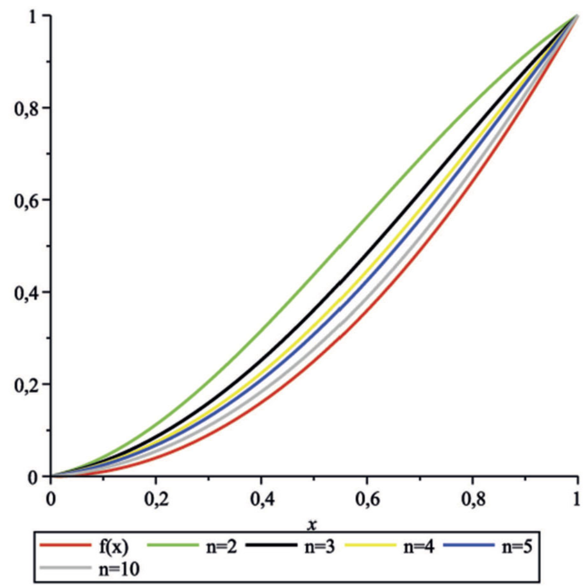

Figure 1 Graphs of $B_{n,-1}(t^2;x)$ for $n = 2,3,4,5$, and 10

Table 2 The intervals where $B_{n,\lambda}(t^2;x)$ is convex for the corresponding values of λ and n

n	$\lambda = 1$	$\lambda = 1/2$	$\lambda = -1/2$	$\lambda = -15/16$	$\lambda = -35/36$
2	[0, 0.25]	[0.08333]	[0, 0.75]	[0, 0.05944]	[0, 0.05881]
3	[0, 1]	[0, 1]	[0, 0.92539]	[0, 0.76319]	[0, 0.07557]
4	[0, 1]	[0, 1]	[0, 0.99254]	[0, 0.84532]	[0, 0.83812]
5	[0, 1]	[0, 1]	[0, 1]	[0, 0.89386]	[0, 0.88732]
10	[0, 1]	[0, 1]	[0, 1]	[0, 0.97314]	[0, 0.96939]
15	[0, 1]	[0, 1]	[0, 1]	[0, 0.98921]	[0, 0.98670]
20	[0, 1]	[0, 1]	[0, 1]	[0, 0.99469]	[0, 0.99281]
25	[0, 1]	[0, 1]	[0, 1]	[0, 0.99710]	[0, 0.99560]

We have seen that λ-Bernstein operators do not preserve convexity in general. However, we see that in some special cases, they preserve convexity as a result of the representation of $\frac{d^2}{dx^2}B_{n,\lambda}(f;x)$ given in Lemma 3.3.

Theorem 3.2 *If $f : [0,1] \to \mathbb{R}$ is a convex function that is nonincreasing on $(0,x_0)$ and nondecreasing on $(x_0,1)$ for an interior point $x_0 \in (0,1)$, then $B_{n,\lambda}(f)$ is also convex for all $\lambda \in [0,1]$ and $n > n_0$ where n_0 is dependent on x_0.*

Proof From Remark 3.2, it is enough to show that

$$\lambda \frac{n(n+1)}{n-1}\left\{b_{n-1,0}(x)\left[f(0)-f\left(\frac{1}{n}\right)\right]+b_{n-1,n-1}(x)\left[f(1)-f\left(\frac{n-1}{n}\right)\right]\right\} \geq 0. \tag{12}$$

If $x_0 < \frac{1}{2}$, then we can choose n such that $\frac{1}{n} < x_0$. Then, f is nonincreasing on the interval $(0, \frac{1}{n})$ and nondecreasing on the interval $(\frac{n-1}{n}, 1)$ for all such $n \in \mathbb{N}$ and thus

$$f(0)-f\left(\frac{1}{n}\right) \geq 0, \qquad f(1)-f\left(\frac{n-1}{n}\right) \geq 0. \tag{13}$$

Therefore, we have the inequality (12) for all n such that $\frac{1}{n} < x_0$ and for all $\lambda \in [0,1]$. Similarly if $x_0 > \frac{1}{2}$, then we can choose n such that $1 - \frac{1}{n} > x_0$. Then, f is nonincreasing on $(0, \frac{1}{n})$ and nondecreasing on $(\frac{n-1}{n}, 1)$ for all such $n \in \mathbb{N}$, which yields the inequality (12) for $\lambda \in [0,1]$. Finally, if $x_0 = \frac{1}{2}$, then the inequality (12) holds for all $n \geq 2$ and $\lambda \in [0,1]$. □

Definition 3.1 ([17]) Let $f : [0,1] \to \mathbb{R}$ be continuous. f is called quasiconvex on $[0,1]$ if

$$f\bigl(\eta u + (1-\eta)v\bigr) \leq \max\bigl\{f(u), f(v)\bigr\}, \quad \forall u, v, \eta \in [0,1].$$

f is quasiconvex on $[0,1]$ iff f is nonincreasing and nondecreasing on the intervals $[0,c]$ and $[c,1]$, respectively, where $c \in [0,1]$. Obviously, every nondecreasing, nonincreasing or convex function is quasiconvex on $[0,1]$.

Remark 3.4 Note that in the hypothesis of the last theorem, we have excluded the cases $x_0 = 0$ or $x_0 = 1$. This is because if f is nondecreasing or nonincreasing on $[0,1]$, we can not have both inequalities in (13). If f is a function that satisfies the hypothesis of the last theorem, then f is a quasiconvex function. Therefore, the assertion of the last theorem does not hold for all quasiconvex functions.

Now, we investigate the monotonicity of $B_{n,\lambda}(f;x)$ with n. Namely, we try to answer the question "is the inequality $B_{n+1,\lambda}(f;x) \leq B_{n,\lambda}(f;x)$ satisfied for every $n \in \mathbb{N}$ and $x \in [0,1]$ for fixed $\lambda \in [-1,1]$ if $f : [0,1] \to \mathbb{R}$ is an arbitrary convex function?" Again, we consider $f(t) = t^2$ and check if this property is satisfied. Basically, from Lemma 2.1 the problem reduces to checking when the inequality

$$\begin{aligned} &\frac{x(1-x)}{n+1} + \lambda\left[\frac{2x - 4x^2 + 2x^{n+2}}{n(n+1)} + \frac{x^{n+2} + (1-x)^{n+2} - 1}{(n+1)^2 n}\right] \\ &\quad\leq \frac{x(1-x)}{n} + \lambda\left[\frac{2x - 4x^2 + 2x^{n+1}}{n(n-1)} + \frac{x^{n+1} + (1-x)^{n+1} - 1}{n^2(n-1)}\right] \end{aligned} \tag{14}$$

is satisfied. We solve this inequality using computer algebra and obtain the data given in Table 3. We have observed that for different negative values of λ, the inequality (14) holds for all $n \in \mathbb{N}$ and $x \in [0,1]$. As for the positive values of λ, we have seen that the inequality (14) is satisfied for all $n \in \mathbb{N}$ and $x \in [0,1]$ if $\lambda \leq 1/2$. However, if $\lambda > 1/2$, then the inequality (14) does not hold for some x. In Table 3, we give the solutions of the inequality (14) for the corresponding values of n and λ.

From these observations, the next result easily follows.

Table 3 Solutions of the inequality (14) for the corresponding values of n and λ

n	$\lambda = 2/3$	$\lambda = 3/4$	$\lambda = 9/10$	$\lambda = 99/100$	$\lambda = 1$
2	[0, 1]	[0, 1]	[0, 0.68406]	[0, 0.60235]	[0, 0.59549]
3	[0, 1]	[0, 0.85964]	[0, 0.70958]	[0, 0.65802]	[0, 0.65314]
4	[0, 0.93051]	[0, 0.82643]	[0, 0.72916]	[0, 0.68982]	[0, 0.68598]
5	[0, 0.89504]	[0, 0.82452]	[0, 0.74676]	[0, 0.71358]	[0, 0.71029]
10	[0, 0.89650]	[0, 0.86049]	[0, 0.81344]	[0, 0.79140]	[0, 0.78916]
15	[0, 0.91618]	[0, 0.89009]	[0, 0.85422]	[0, 0.83676]	[0, 0.83496]
20	[0, 0.93072]	[0, 0.91000]	[0, 0.88073]	[0, 0.86616]	[0, 0.86464]
25	[0, 0.94122]	[0, 0.92397]	[0, 0.89918]	[0, 0.88666]	[0, 0.88535]

Remark 3.5 λ-Bernstein operators do not have monotonicity property with n for some $\lambda > 1/2$. Namely, the inequality

$$B_{n+1,\lambda}(f;x) \leq B_{n,\lambda}(f;x)$$

does not hold for all $n \in \mathbb{N}$ and $x \in [0,1]$ for some fixed $1/2 < \lambda \leq 1$ and arbitrary convex function $f \in C[0,1]$.

4 Conclusion

In this study, the shape-preserving properties of recently introduced λ-Bernstein operators $B_{n,\lambda}(f;x)$ have been revealed. These properties are fundamental for the applications in computer graphics and computer-aided design. It has been seen that the monotonicity-preserving property is satisfied for every $\lambda \in [-1,1]$. However, it has been demonstrated with a counter example that the convexity-preserving property fails for some $\lambda \in [-1,1]$. In this case, it has been proven that for a special class of convex functions, $B_{n,\lambda}(f;x)$ is convex for $\lambda \in [0,1]$ and $n > n_0$ (see Theorem 3.2). Furthermore, it has been shown with a counter example that the monotonicity of λ-Bernstein operators with n also fails for $\lambda > 1/2$. For further studies, a special class of functions for which λ-Bernstein operators preserve convexity for every $\lambda \in [-1,1]$ can be investigated. Moreover, as for the monotonicity property of λ-Bernstein operators with n, we have shown that this property is not satisfied, at least for some $\lambda > 1/2$. However, we were not able to obtain results for the other cases of λ. For this reason, it could be interesting to investigate whether this property is satisfied for $-1 \leq \lambda \leq 1/2$ or not.

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
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