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Weighted Lipschitz estimates for commutators of multilinear Calderón–Zygmund operators with Dini type kernels

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Abstract
In this paper, we discuss the properties for commutators and iterated commutators generated by the multilinear ω-CZO and weighted Lipschitz functions on Lebesgue space.

MSC: 42B20; 42B25

Keywords: Multilinear Calderón–Zygmund operator; Weight; Commutator; Weighted Lipschitz space

1 Introduction and results

The singular integral operator theory plays an important role in many aspects of harmonic analysis. The Calderón–Zygmund operator theory is one of the most important achievements of classical analysis in the last century, which has many important applications in Fourier analysis, complex analysis, operator theory, and so on. The multilinear Calderón–Zygmund theory was introduced by Coifman and Meyer in [1, 2]. This theory was then further studied by Grafakos and Torres [7, 8], who considered the multilinear Calderón–Zygmund operator with classical standard kernels. And this topic keeps attracting many researchers.

In 1985, Yabuta [23] firstly considered Calderón–Zygmund operators with kernels of type ω as the generalizations of Calderón–Zygmund operators when studied pseudodifferential operator. In 2009, Maldonado and Naibo [15] studied the bilinear Calderón–Zygmund operators of type ω. In 2014, Lu and Zhang [14] considered the multilinear case. Assume that $\omega(t) : [0,\infty) \to [0,\infty)$ is a nondecreasing function with $0 < \omega(1) < \infty$. For $a > 0$, we say $\omega \in \mathrm{Dini}(a)$ if

$$|\omega|_{\mathrm{Dini}(a)} = \int_0^1 \frac{\omega^a(t)}{t}\,\mathrm{d}t < \infty.$$

Definition 1.1 ([14]) A locally integrable function $K(x, y_1, \dots, y_m)$, defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, is called an m-linear Calderón–Zygmund kernel of type $\omega(t)$ if there exists a constant $A > 0$ such that

$$\left|K(x, y_1, \dots, y_m)\right| \le \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \tag{1.1}$$

or all $(x, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \neq y_j$ for some $j \in \{1, 2, \dots, m\}$, and

$$\begin{aligned} &\left|K(x, y_1, \dots, y_m) - K\left(x', y_1, \dots, y_m\right)\right| \\ &\quad \le \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \omega\left(\frac{|x-x'|}{|x-y_1| + \cdots + |x-y_m|}\right) \end{aligned} \tag{1.2}$$

whenever $|x - x'| \le \frac{1}{2} \max_{1 \le j \le m} |x - y_j|$, and

$$\begin{aligned} &\left|K(x, y_1, \dots, y_j, \dots, y_m) - K\left(x, y_1, \dots, y_j', \dots, y_m\right)\right| \\ &\quad \le \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \omega\left(\frac{|y_j-y_j'|}{|x-y_1| + \cdots + |x-y_m|}\right) \end{aligned} \tag{1.3}$$

whenever $|y_j - y_j'| \le \frac{1}{2} \max_{1 \le j \le m} |x - y_j|$.

We say $T : \mathscr{S}(\mathbb{R}^n) \times \cdots \times \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is an m-linear operator with an m-linear Calderón–Zygmund kernel $K(x, y_1, \dots, y_m)$ of type $\omega(t)$ if

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) \, \mathrm{d}y_1 \cdots \mathrm{d}y_m,$$

whenever $f_1, \dots, f_m \in C_c^\infty(\mathbb{R}^n)$ and $x \notin \cap_{j=1}^m \operatorname{supp} f_j$.

If T can be extended to a bounded multilinear operator from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ for some $1 < q, q_1, \dots, q_m < \infty$ with $1/q_1 + \cdots + 1/q_m = 1/q$ or from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ for some $1 < q_1, \dots, q_m < \infty$ with $1/q_1 + \cdots + 1/q_m = 1$, then T is called an m-linear Calderón–Zygmund operator of type ω, abbreviated to m-linear ω-CZO.

Obviously, when $\omega(t) = t^\varepsilon$ for some $\varepsilon > 0$, the m-linear ω-CZO is exactly the multilinear Calderón–Zygmund operator studied by Grafakos and Torres [7] and Lerner et al. [12].

To shorten the notation, we denote $\vec{f} = (f_1, \dots, f_m)$ and $\mathrm{d}\vec{y} = \mathrm{d}y_1 \cdots \mathrm{d}y_m$ in the following.

In 2014, Lu and Zhang [14] gave the endpoint estimate for the m-linear ω-CZO under some weaker assumptions of $\omega(t)$ and also got the following multiple weighted estimates.

Theorem A ([14]) *Let T be an m-linear ω-CZO with $\omega \in \mathrm{Dini}(1)$. Let $1/p = 1/p_1 + \cdots + 1/p_m$ and $\mu \in A_{\min\{p_1, p_2, \dots, p_m\}}(\mathbb{R}^n)$. If $1 < p_j < \infty$ for all $j = 1, \dots, m$, then*

$$\left\|T(\vec{f})\right\|_{L^p(\mu)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mu)}.$$

Theorem B ([14]) *Let T be an m-linear ω-CZO with $\omega \in \mathrm{Dini}(1)$. Then T can be extended to a bounded operator from $L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n)$ to $L^{1/m,\infty}(\mathbb{R}^n)$.*

Let $\vec{b} = (b_1, \dots, b_m)$ be a collection of locally integrable functions, the commutator generated by m-linear ω-CZO, and $\vec{b}$ is defined by

$$T_{\Sigma\vec{b}}(f_1, f_2, \dots, f_m)(x) = \sum_{j=1}^{m} T_{b_j}^{j}(f_1, f_2, \dots, f_m)(x),$$

where

$$T_{b_j}^{j}(\vec{f})(x) = [b_j, T]_j(\vec{f})(x) = b_j(x)T(f_1, \dots, f_j, \dots, f_m)(x) - T(f_1, \dots, f_j b_j, \dots, f_m)(x), \\ j = 1, \dots, m.$$

Let $\vec{b} = (b_1, \dots, b_m)$ be a collection of locally integrable functions, the commutator generated by m-linear ω-CZO, and $\vec{b}$ is defined by

$$T_{\Sigma\vec{b}}(f_1, f_2, \dots, f_m)(x) = \sum_{j=1}^{m} T_{b_j}^{j}(f_1, f_2, \dots, f_m)(x),$$

where

$$T_{b_j}^{j}(\vec{f})(x) = [b_j, T]_j(\vec{f})(x) = b_j(x)T(f_1, \dots, f_j, \dots, f_m) - T(f_1, \dots, f_j b_j, \dots, f_m)(x), \\ j = 1, \dots, m.$$

The iterated commutator $T_{\Pi\vec{b}}(\vec{f})$ is defined as follows:

$$T_{\Pi\vec{b}}(\vec{f})(x) = \bigl[b_1, \bigl[b_2, \dots \bigl[b_{m-1}, [b_m, T]_m\bigr]_{m-1}, \dots\bigr]_2\bigr]_1(\vec{f})(x),$$

which can also be given formally by

$$T_{\Pi\vec{b}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \prod_{j=1}^{m} \bigl(b_j(x) - b_j(y_j)\bigr) K(x, \vec{y}) f_1(y_1) \cdots f_m(y_m)\, d\vec{y}.$$

When $m = 1$, $T_{\Sigma\vec{b}}(\vec{f}) = T_{\Pi\vec{b}}(\vec{f}) = [b, T]f = bT(f) - T(bf)$, which is the well-known classical commutator studied in [3]. In 1995, Paluszynski [17] proved that the commutator $[b, T]$ generated by Calderón–Zygmund operators T with classical kernel and $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ whenever $0 < \beta < 1$, $1/q = 1/p - \beta/n$, and $1 < p < q < \infty$, and from L^p to homogenous Triebel–Lizorkin spaces $\dot{F}_p^{\beta,\infty}(\mathbb{R}^n)$ which is defined in [20].

For the weighted case, Hu and Gu [9] proved when $b \in \mathrm{Lip}_{\beta,\mu}(\mathbb{R}^n)$, the commutators $[b, T]$ is bounded from $L^p(\mu)$ to $L^q(\mu^{1-q})$. In 2011, Lian, Ma, and Wu [13] studied the m-linear commutators generated by the multilinear Calderón–Zygmund operators with nonsmooth kernels and weighted Lipschitz functions bounded from the product of weighted Lebesgue spaces to the weighted Lebesgue space. For more articles about multilinear operators, see [1, 2, 4, 6, 7, 10–12, 14, 16, 24, 25], and [26].

In this paper, we will discuss the mapping properties of multilinear commutators generated by m-linear Dini's type Calderón–Zygmund operators and weighted Lipschitz functions on some function spaces. We obtain the following results.

Theorem 1.1 *Let T be an m-linear ω-CZO satisfying*

$$\int_0^1 \frac{\omega(t)}{t}\left(1+\log\frac{1}{t}\right)\mathrm{d}t \le \infty. \tag{1.4}$$

Suppose $0<\beta<1$ and $1/r = 1/p-\beta/n$, $1<p<r<\infty$ for $1/p_1+\cdots+1/p_m = 1/p$ with $1<p_i<\infty$, $i=1,\dots,m$. If $\mu\in A_1(\mathbb{R}^n)$ and $b_j\in \mathrm{Lip}_{\beta,\mu}(\mathbb{R}^n)(1\le j\le m)$, then $T^j_{b_j}(\vec{f})$ is bounded from $L^{p_1}(\mu)\times\cdots\times L^{p_m}(\mu)$ to $L^r(\mu^{1-r})$.

Furthermore, $T_{\sum\vec{b}}(\vec{f})$ is bounded from $L^{p_1}(\mu)\times\cdots\times L^{p_m}(\mu)$ to $L^r(\mu^{1-r})$.

Theorem 1.2 *Let T be an m-linear ω-CZO satisfying*

$$\int_0^1 \frac{\omega(t)}{t}\left(1+\log\frac{1}{t}\right)^m\mathrm{d}t < \infty.$$

Suppose $0<\beta_i<1$, $i=1,\dots,m$, $1/r_i = 1/p_i-\beta_i/n$, $1<p_i<r_i<\infty$ with $1/p_1+\cdots+1/p_m=1/p$, $1/r_1+\cdots+1/r_m = 1/r$, $\beta_1+\cdots+\beta_m=\beta$, and $0<\beta<1$. If $\mu\in A_1(\mathbb{R}^n)$, $b_i\in\mathrm{Lip}_{\beta_i,\mu}(\mathbb{R}^n)(1\le i\le m)$, then $T_{\prod\vec{b}}(\vec{f})$ is bounded from $L^{p_1}(\mu)\times\cdots\times L^{p_m}(\mu)$ to $L^r(\mu^{1-mr})$.

Remark 1.1 Theorem 1.1 and 1.2 are also valid for commutators of multilinear Calderón–Zygmund operator with standard kernels.

Remark 1.2 Theorem 1.2 extends the corresponding result in [19] and [22].

The rest of this paper is organized as follows. After recalling some notations and lemmas in Sect. 2, we prove our results in Sect. 3.

Throughout this paper, we denote by p' the conjugate index of p, that is, $1/p+1/p'=1$. The letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same at each occurrence but independent of the main parameters.

2 Preliminaries and lemmas

A nonnegative locally integrable function is called a weight function.

Definition 2.1 ([5]) Let μ be a weight function, $1<p<\infty$. If there is a constant $C>0$ such that, for every ball $B\subseteq\mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B \mu(x)\,\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B \mu(x)^{-1/(p-1)}\,\mathrm{d}x\right)^{p-1}\le C,$$

then we say $\mu\in A_p$. We say $\mu\in A_1$ if there is a constant $C>0$ such that, for every ball $B\subseteq\mathbb{R}^n$,

$$\frac{1}{|B|}\int_B \mu(x)\,\mathrm{d}x \le C\operatorname*{essinf}_{x\in B}\mu(x).$$

A weight function $\mu\in A_\infty$ if it satisfies the A_p condition for some $1<p<\infty$. The smallest constant satisfying the formulas above is called A_p constant of w, we denote it by $[\mu]_{A_p}$.

For $1\le p<q<\infty$, we have $A_1\subset A_p\subset A_q$. And $A_\infty = \cup_{1\le p<\infty}A_p$.

If $\mu\in A_r$, $1<r<\infty$, then $\mu^{1-r'}\in A_{r'}$.

For a function $f \in L_{loc}(\mathbb{R}^n)$, the Hardy–Littlewood maximal and the sharp maximal functions are defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,\mathrm{d}y$$

and

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_B|\,\mathrm{d}y \approx \sup_{B \ni x} \inf_C \frac{1}{|Q|} \int_Q |f(y) - C|\,\mathrm{d}y,$$

where f_Q denotes the average of f over cube Q, that is, $f_Q = \frac{1}{|Q|} \int_Q f(x)\,\mathrm{d}x$.

For $\delta > 0$, we denote $M_\delta(f)$ and $M_\delta^\sharp(f)$ by $M_\delta(f) = M(|f|^\delta)^{1/\delta}$ and $M_\delta^\sharp(f) = [M^\sharp(|f|^\delta)]^{1/\delta}$. We denote the following fractional maximal operator:

$$M_{\alpha,\mu,s} f(x) = \sup_{Q \ni x} \left(\frac{1}{\mu(Q)^{1-s\alpha/n}} \int_Q |f(y)|^s \mu(y)\,\mathrm{d}y \right)^{1/s}.$$

Recall that $M_\alpha := M_{\alpha,1,1}$ is the fractional maximal operator

$$M_\alpha(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|\,\mathrm{d}y.$$

Definition 2.2 ([5]) Let $1 \le p \le \infty$, $0 < \beta < 1$, $\mu \in A_\infty$, the weighted Lipschitz space $\mathrm{Lip}^p_{\beta,\mu}$ contains all locally integrable functions f satisfying

$$\|f\|_{\mathrm{Lip}^p_{\beta,\mu}} = \sup_B \frac{1}{\mu(B)^{\beta/n}} \left[\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p \mu(x)^{1-p}\,\mathrm{d}x \right]^{1/p} \le C < \infty,$$

where $f_B = \frac{1}{|B|} \int_B f(y)\,\mathrm{d}y$, the supremum is taken over all balls B in $\mathbb{R}^n$.

The smallest number C satisfying the above inequality was denoted by $\|f\|_{\mathrm{Lip}^p_{\beta,\mu}}$, and we also denote by $\|f\|_{\mathrm{Lip}_{\beta,\mu}} = \|f\|_{\mathrm{Lip}^1_{\beta,\mu}}$. Obviously, when $\mu = 1$, $\mathrm{Lip}_{\beta,\mu} = \mathrm{Lip}_\beta$.

$A \sim B$ means there exist $C_1 > 0$, $C_2 > 0$ such that $C_1 A \le B \le C_2 A$. When $\mu \in A_1$, García–Cuerva in [5] proved that for $1 \le p, q \le \infty$, $\|f\|_{\mathrm{Lip}^p_{\beta,\mu}} \sim \|f\|_{\mathrm{Lip}^q_{\beta,\mu}}$.

As usual, we denote $\|f\|_{L^p(\mu)} = (\int_{\mathbb{R}^n} |f(x)|^p \mu(x)\,\mathrm{d}x)^{1/p}$ for $1 < p < \infty$ and $p = \infty$, $\|f\|_{L^\infty(\mu)} = \|f\|_{L^\infty}$.

We will use the following Kolmogorov inequality:

$$\|f\|_{L^p(Q, \frac{\mathrm{d}x}{|Q|})} \le C \|f\|_{L^{q,\infty}(Q, \frac{\mathrm{d}x}{|Q|})},$$

where $0 < p < q < \infty$. See [12, 21].

Lemma 2.1 ([18]) *Let $0 < p, \delta < \infty$, $\mu \in A_\infty$, then there exists a constant C such that*

$$\int_{\mathbb{R}^n} M_\delta f(x)^p \mu(x)\,\mathrm{d}x \le C \int_{\mathbb{R}^n} M_\delta^\sharp f(x)^p \mu(x)\,\mathrm{d}x \tag{2.1}$$

for any function f for which the left-hand side is finite.

Lemma 2.2 ([13]) *Suppose $\mu \in A_1$, $0 < \beta < 1$, $b \in \mathrm{Lip}_{\beta,\mu}(\mathbb{R}^n)$.*
(i) *For $k \geq 1$,*

$$|b_{B(x,2R)} - b_{B(x,2^{k+1}R)}| \leq Ck\mu(x)\mu\bigl(B\bigl(x,2^{k+1}R\bigr)\bigr)^{\beta/n}\|b\|_{\mathrm{Lip}_{\beta,\mu}}.$$

(ii) *For any $1 \leq s < \infty$ and any ball $B \ni x$, we have*

$$\frac{\mu(B)^{\beta/n}}{|B|}\int_B \bigl|f(y)\bigr|\,\mathrm{d}y \leq CM_{\beta,\mu,s}(f)(x).$$

(iii) *For any $1 < s < \infty$ and any ball $B \ni x$, we have*

$$\frac{1}{|B|}\int_B \bigl|\bigl(b(y) - b_B\bigr)f(y)\bigr|\,\mathrm{d}y \leq C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,s}(f)(x).$$

Lemma 2.3 ([13]) *Suppose that $0 < \alpha < n$, $0 < s < p < n/\alpha$, $1/q = 1/p - \alpha/n$. If $\mu \in A_\infty(\mathbb{R}^n)$, then*

$$\|M_{\alpha,\mu,s}f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\mu)}.$$

Lemma 2.4 ([13]) *Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$, and $1/q = 1/p - \alpha/n$. If $\mu \in A_{1+q/p'}(\mathbb{R}^n)$, then*

$$\|M_\alpha f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\mu^{p/q})}.$$

3 Proofs of theorems

For simplicity, we only prove for the case $m = 2$. The argument for the case $m > 2$ is similar. We first establish the following lemmas.

Lemma 3.1 *Let T be a 2-linear ω-CZO satisfying (1.4). Suppose $\mu \in A_1(\mathbb{R}^n)$ and $b_j \in \mathrm{Lip}_{\beta,\mu}(\mathbb{R}^n)$ with $0 < \beta < 1$, $j = 1,2$. Let $0 < \delta < 1/2 < 1 < s < n/\beta$. Then we have*

$$\begin{aligned} M_\delta^\sharp\bigl[T_{b_j}^j(f_1,f_2)\bigr](x) &\leq C\mu(x)\|b_j\|_{\mathrm{Lip}_{\beta,\mu}}\bigl[M_{\beta,\mu,s}\bigl(T(f_1,f_2)\bigr)(x) \\ &\quad + M_{\beta,\mu,s}(f_1)(x)M(f_2)(x) + M(f_1)(x)M_{\beta,\mu,s}(f_2)(x)\bigr] \end{aligned} \tag{3.1}$$

for $j = 1,2$.

Proof We only estimate $M_\delta^\sharp(T_{b_1}^1(f_1,f_2))$ and write $b_1 = b$ for simplicity. A similar discussion also works for $M_\delta^\sharp(T_{b_2}^2(f_1,f_2))$.

Fix $x \in \mathbb{R}^n$ for any cube $Q(x_Q,l_Q)$ containing x with side-length l_Q, set $Q^* = 8\sqrt{n}Q = Q(x_Q,8\sqrt{n}l_Q)$. We decompose $f_j = f_j^0 + f_j^\infty$, where $f_j^0 = f_j\chi_{Q^*}$ and $f_j^\infty = f\chi_{\mathbb{R}^n\setminus Q^*}$, $j = 1,2$.

Since $0 < \delta < 1/2$, then for any constant c, we have

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q \bigl|\bigl|T_b^1(f_1,f_2)(z)\bigr|^\delta - |c|^\delta\bigr|\,\mathrm{d}z\right)^{1/\delta} \\ &\quad\leq \left(\frac{1}{|Q|}\int_Q \bigl|T_b^1(f_1,f_2)(z) - c\bigr|^\delta\,\mathrm{d}z\right)^{1/\delta} \end{aligned}$$

$$
\begin{aligned}
&\le C\left(\frac{1}{|Q|}\int_Q\left|\big(b(z)-b_{Q^*}\big)T(f_1,f_2)(z)\right|^\delta \mathrm{d}z\right)^{1/\delta}\\
&\quad + C\left(\frac{1}{|Q|}\int_Q\left|T\big((b-b_{Q^*})f_1^0,f_2^0\big)(z)\right|^\delta \mathrm{d}z\right)^{1/\delta}\\
&\quad + C\left(\frac{1}{|Q|}\int_Q\left|T\big((b-b_{Q^*})f_1^\infty,f_2^0\big)(z)\right|^\delta \mathrm{d}z\right)^{1/\delta}\\
&\quad + C\left(\frac{1}{|Q|}\int_Q\left|T\big((b-b_{Q^*})f_1^0,f_2^\infty\big)(z)\right|^\delta \mathrm{d}z\right)^{1/\delta}\\
&\quad + C\left(\frac{1}{|Q|}\int_Q\left|T\big((b-b_{Q^*})f_1^\infty,f_2^\infty\big)(z)-c\right|^\delta \mathrm{d}z\right)^{1/\delta}\\
&:= I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

Since $0<\delta<1$, $\mu\in A_1$, and $b\in \mathrm{Lip}_{\beta,\mu}$, by Lemma 2.2(iii) and Hölder's inequality, we get

$$
I_1\le C\frac{1}{|Q|}\int_Q\left|\big(b(z)-b_{Q^*}\big)T(f_1,f_2)(z)\right|\mathrm{d}z\le C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,s}\big(T(f_1,f_2)\big)(x).
$$

For the second term I_2, since $0<\delta<1/2$, by Kolmogorov's inequality, Theorem B, and Lemma 2.2(iii), we obtain

$$
\begin{aligned}
I_2 &= C|Q|^{-\delta}\left\|T\big((b-b_{Q^*})f_1^0,f_2^0\big)\right\|_{L^\delta(Q)}\\
&\le C|Q|^{-2}\left\|T\big((b-b_{Q^*})f_1^0,f_2^0\big)\right\|_{L^{1/2,\infty}(Q)}\\
&\le C|Q|^{-2}\left\|T\big((b-b_{Q^*})f_1^0,f_2^0\big)\right\|_{L^{1/2,\infty}(\mathbb{R}^n)}\\
&\le C\left(\frac{1}{|Q^*|}\int_{Q^*}\left|\big(b(z)-b_{Q^*}\big)f_1(z)\right|\mathrm{d}z\right)\left(\frac{1}{|Q^*|}\int_{Q^*}\left|f_2(z)\right|\mathrm{d}z\right)\\
&\le C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,s}(f_1)(x)M(f_2)(x).
\end{aligned}
$$

For the term I_3, noting the fact that $|z-y_1|\sim|y_1-x_Q|$ for any $y_1\in(Q^*)^c$ and $z\in Q$, then by (1.1) and Lemma 2.2(i)(ii), we obtain

$$
\begin{aligned}
I_3 &\le \frac{C}{|Q|}\int_Q\left|T\big((b-b_{Q^*})f_1^\infty,f_2^0\big)(z)\right|\mathrm{d}z\\
&\le \frac{C}{|Q|}\int_Q\int_{Q^*}\int_{(Q^*)^c}\frac{A}{(|z-y_1|+|z-y_2|)^{2n}}\left|b(y_1)-b_{Q^*}\right|\left|f_1(y_1)\right|\left|f_2(y_2)\right|\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}z\\
&\le C\int_{(Q^*)^c}\frac{|b(y_1)-b_{Q^*}|}{|y_1-x_Q|^{2n}}\left|f_1(y_1)\right|\mathrm{d}y_1\int_{Q^*}\left|f_2(y_2)\right|\mathrm{d}y_2\\
&\le C\left|Q^*\right|\sum_{k=1}^\infty\int_{(2^{k+3}\sqrt{n}Q)^*\setminus(2^{k+2}\sqrt{n}Q)^*}\frac{|b(y_1)-b_{Q^*}|}{|y_1-x_Q|^{2n}}\left|f_1(y_1)\right|\mathrm{d}y_1M(f_2)(x)\\
&\le C\sum_{k=1}^\infty\frac{|Q^*|}{|2^kQ^*|^2}\int_{2^kQ^*}\left[\left|b(y_1)-b_{2^kQ^*}\right|+\left|b_{2^kQ^*}-b_{Q^*}\right|\right]\left|f_1(y_1)\right|\mathrm{d}y_1M(f_2)(x)\\
&\le CM(f_2)(x)\sum_{k=1}^\infty 2^{-k}\bigg[\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,s}(f_1)(x)
\end{aligned}
$$

$$
\begin{aligned}
&+ k\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}} \frac{\mu(2^kQ^*)^{\frac{\beta}{n}}}{|2^kQ^*|} \int_{2^kQ^*} |f_1(y_1)|\,\mathrm{d}y_1\Big]\\
&\leq C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}} M_{\beta,\mu,s}(f_1)(x)M(f_2)(x).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
I_4 &\leq \frac{C}{|Q|}\int_Q \big|T\big((b-b_{Q^*})f_1^0,f_2^\infty\big)(z)\big|\,\mathrm{d}z\\
&\leq \frac{C}{|Q|}\int_Q\int_{Q^*}\int_{(Q^*)^c} \frac{A}{(|z-y_1|+|z-y_2|)^{2n}}\big|b(y_1)-b_{Q^*}\big|\big|f_1(y_1)\big|\big|f_2(y_2)\big|\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}z\\
&\leq C\int_{Q^*}\big|b(y_1)-b_{Q^*}\big|\big|f_1(y_1)\big|\,\mathrm{d}y_1\int_{(Q^*)^c}\frac{|f_2(y_2)|}{|y_2-x_Q|^{2n}}\,\mathrm{d}y_2\\
&\leq C\mu(x)\|b\|_{Lip_{\beta,\mu}} M_{\beta,\mu,s}(f_1)(x)\sum_{k=1}^\infty \frac{|Q|}{|2^kQ^*|^2}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\big|f_2(y_2)\big|\,\mathrm{d}y_2\\
&\leq C\mu(x)\|b\|_{Lip_{\beta,\mu}} M_{\beta,\mu,s}(f_1)(x)M(f_2)(x).
\end{aligned}
$$

For the last term I_5, since $(\mathbb{R}^n\setminus Q^*)^2 \subseteq \mathbb{R}^{2n}\setminus(Q^*)^2 \subseteq \cup_{k=1}^\infty (2^{k+3}\sqrt{n}Q)^2\setminus(2^{k+2}\sqrt{n}Q)^2$, making use of assumption (1.2), we have

$$
\begin{aligned}
I_5 &\leq \frac{C}{|Q|}\int_Q \big|T\big((b-b_{Q^*})f_1^\infty,f_2^\infty\big)(z)-T\big((b-b_{Q^*})f_1^\infty,f_2^\infty\big)(x_Q)\big|\,\mathrm{d}z\\
&\leq \frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2}\big|K(z,y_1,y_2)-K(x_Q,y_1,y_2)\big|\big|\big(b(y_1)-b_{Q^*}\big)f_1^\infty(y_1)f_2^\infty(y_2)\big|\,\mathrm{d}y_1\,\mathrm{d}y_2 dz\\
&\leq \frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2}\frac{1}{(|x-y_1|+|x-y_2|)^{2n}}\omega\bigg(\frac{|z-x_Q|}{|x-y_1|+|x-y_2|}\bigg)\\
&\quad\times\big|\big(b(y_1)-b_{Q^*}\big)f_1(y_1)f_2(y_2)\big|\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}z\\
&\leq \frac{C}{|Q|}\int_Q\sum_{k=1}^\infty\int_{(2^{k+3}\sqrt{n}Q)^2\setminus(2^{k+2}\sqrt{n}Q)^2}\frac{\omega(\frac{|z-x_Q|}{|x-y_1|+|x-y_2|})}{(|x-y_1|+|x-y_2|)^{2n}}\\
&\quad\times\big|b(y_1)-b_{Q^*}\big|\big|f_1(y_1)f_2(y_2)\big|\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}z\\
&\leq \frac{C}{|Q|}\int_Q\sum_{k=1}^\infty\frac{\omega(2^{-k})}{|2^kQ^*|}\int_{2^kQ^*}\big|b(y_1)-b_{Q^*}\big|\big|f_1(y_1)\big|\,\mathrm{d}y_1\times\frac{1}{|2^kQ^*|}\int_{2^kQ^*}\big|f_2(y_2)\big|\,\mathrm{d}y_2\,\mathrm{d}z\\
&\leq CM(f_2)(x)\sum_{k=1}^\infty\frac{\omega(2^{-k})}{|2^kQ^*|}\int_{2^kQ^*}\big[\big|b(y_1)-b_{2^kQ^*}\big|+|b_{2^kQ^*}-b_{Q^*}|\big]\big|f_1(y_1)\big|\,\mathrm{d}y_1\\
&\leq C\sum_{k=1}^\infty k\omega\big(2^{-k}\big)\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}} M_{\beta,\mu,s}(f_1)(x)M(f_2)(x)\\
&\leq C\int_0^1\frac{\omega(t)}{t}\bigg(1+\log\frac{1}{t}\bigg)\,\mathrm{d}t\mu(x)\|b\|_{Lip_{\beta,\mu}} M_{\beta,\mu,s}(f_1)(x)M(f_2)(x)\\
&\leq C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}} M_{\beta,\mu,s}(f_1)(x)M(f_2)(x).
\end{aligned}
$$

□

Proof of Theorem 1.1 Since $\mu \in A_1 \subset A_{r'}$ and $\mu^{1-r} \in A_r \subset A_\infty$, by Lemma 2.1 and Lemma 3.1, for any $j = 1, 2$, we get

$$\begin{aligned}\left\|T^j_{b_j}(f_1,f_2)\right\|_{L^r(\mu^{1-r})} &\leq \left\|M_\delta\left[T^j_{b_j}(f_1,f_2)\right]\right\|_{L^r(\mu^{1-r})}\\ &\leq \left\|M^\sharp_\delta\left[T^j_{b_j}(f_1,f_2)\right]\right\|_{L^r(\mu^{1-r})}\\ &\leq C\left\|\mu M_{\beta,\mu,s}\big(T(f_1,f_2)\big)\right\|_{L^r(\mu^{1-r})}\\ &\quad + C\left\|\mu M_{\beta,\mu,s}(f_1)M(f_2)\right\|_{L^r(\mu^{1-r})} + C\left\|\mu M(f_1)M_{\beta,\mu,s}(f_2)\right\|_{L^r(\mu^{1-r})}\\ &:= U_1 + U_2 + U_3.\end{aligned}$$

For U_1, since $1/r = 1/p - \beta/n$ and select s satisfying $1 < s < p < n/\beta$, by Theorem A and Lemma 2.3, we have

$$\begin{aligned}\left\|\mu M_{\beta,\mu,s}\big(T(f_1,f_2)\big)\right\|_{L^r(\mu^{1-r})} &= \left\|M_{\beta,\mu,s}\big(T(f_1,f_2)\big)\right\|_{L^r(\mu)}\\ &\leq C\left\|T(f_1,f_2)\right\|_{L^p(\mu)}\\ &\leq C\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.\end{aligned}$$

For U_2, let $1/r = 1/p_2 + 1/l$, then we have $1/l = 1/p_1 - \beta/n$. Then, by Hölder's inequality and Lemma 2.3, we obtain

$$\begin{aligned}\left\|\mu M_{\beta,\mu,s}(f_1)M(f_2)\right\|_{L^r(\mu^{1-r})} &= \left\|M_{\beta,\mu,s}(f_1)M(f_2)\right\|_{L^r(\mu)}\\ &\leq C\left\|M_{\beta,\mu,s}(f_1)\right\|_{L^l(\mu)}\left\|M(f_2)\right\|_{L^{p_2}(\mu)}\\ &\leq C\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.\end{aligned}$$

Similarly as the estimate of U_2, we may get

$$U_3 \leq C\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.$$

Thus Theorem 1.1 is proved. □

Lemma 3.2 *Let T be a 2-linear ω-CZO satisfying $\int_0^1 \frac{\omega(t)}{t}(1+\log\frac{1}{t})^2 dt < \infty$. Suppose $\mu \in A_1(\mathbb{R}^n)$ and $b_j \in \mathrm{Lip}_{\beta_j,\mu}(\mathbb{R}^n)$, $j = 1, 2$. Let $\beta_1 + \beta_2 = \beta$, $0 < \beta < 1$, and $0 < \delta < 1/3 < 1 < s < n/\beta$. Then we have*

$$\begin{aligned}M^\sharp_\delta\left[T_{\Pi\vec{b}}(f_1,f_2)\right](x) &\leq C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}\\ &\quad\times\left[M_{\beta,\mu,s}\big(T(f_1,f_2)\big)(x) + M_{\beta_1,\mu,s}(f_1)(x)M_{\beta_2,\mu,s}(f_2)(x)\right]\\ &\quad+\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\mu(x)^{1+\beta_1/n}M_{\beta_1}\big(T^2_{b_2}(f_1,f_2)\big)(x)\\ &\quad+\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}\mu(x)^{1+\beta_2/n}M_{\beta_2}\big(T^1_{b_1}(f_1,f_2)\big)(x).\end{aligned}$$

Proof We fix $x \in \mathbb{R}^n$ for any cube $Q(x_Q, l_Q)$ containing x with side-length l_Q, $i = 1, 2$, $Q^* = 8\sqrt{n}Q$. Set $\lambda_i = (b_i)_{Q^*}$, let c be a constant to be fixed along the proof. Since $0 < \delta < 1/3$, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q\big||T_{\Pi\mathbf{b}}(f_1,f_2)(z)|^\delta-|c|^\delta\big|\,\mathrm{d}z\right)^{1/\delta}\\
&\le\left(\frac{1}{|Q|}\int_Q\big|T_{\Pi\mathbf{b}}(f_1,f_2)(z)-c\big|^\delta\,\mathrm{d}z\right)^{1/\delta}\\
&\le C\left(\frac{1}{|Q|}\int_Q\big|(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)\big|^\delta\,\mathrm{d}z\right)^{1/\delta}\\
&\quad+C\left(\frac{1}{|Q|}\int_Q\big|(b_1(z)-\lambda_1)T^2_{(b_2-\lambda_2)}(f_1,f_2)(z)\big|^\delta\,\mathrm{d}z\right)^{1/\delta}\\
&\quad+C\left(\frac{1}{|Q|}\int_Q\big|(b_2(z)-\lambda_2)T^1_{(b_1-\lambda_1)}(f_1,f_2)(z)\big|^\delta\,\mathrm{d}z\right)^{1/\delta}\\
&\quad+C\left(\frac{1}{|Q|}\int_Q\big|T((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(z)-c\big|^\delta\,\mathrm{d}z\right)^{1/\delta}\\
&=K_1+K_2+K_3+K_4.
\end{aligned}$$

Firstly we consider K_1. For $0<\delta<1/3$, it follows from Hölder's inequality and Lemma 2.2(ii) that

$$\begin{aligned}
K_1&\le C\left(\frac{1}{|Q|}\int_Q|b_1(z)-\lambda_1|^{3\delta}\,\mathrm{d}z\right)^{1/3\delta}\left(\frac{1}{|Q|}\int_Q|b_2(z)-\lambda_2|^{3\delta}\,\mathrm{d}z\right)^{1/3\delta}\\
&\quad\times\left(\frac{1}{|Q|}\int_Q|T(f_1,f_2)(z)|^{3\delta}\,\mathrm{d}z\right)^{1/3\delta}\\
&\le C\left(\frac{1}{|Q|}\int_Q|b_1(z)-\lambda_1|dz\right)\left(\frac{1}{|Q|}\int_Q|b_2(z)-\lambda_2|\,\mathrm{d}z\right)\\
&\quad\times\left(\frac{1}{|Q|}\int_Q|T(f_1,f_2)(z)|\,\mathrm{d}z\right)\\
&\le C\frac{\mu(Q^*)^{\beta_1/n+1}}{|Q|}\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\frac{\mu(Q^*)^{\beta_2/n+1}}{|Q|}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}\frac{1}{|Q|}\int_Q|T(f_1,f_2)(z)|\,\mathrm{d}z\\
&\le C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}\mu(Q^*)^{\beta/n}\frac{1}{|Q|}\int_Q|T(f_1,f_2)(z)|\,\mathrm{d}z\\
&\le C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}M_{\beta,\mu,s}\big(T(f_1,f_2)\big)(x).
\end{aligned}$$

For the terms K_2, K_3, notice that $0<\delta<1/3$, we use the facts $1/\delta=1+(1-\delta)/\delta$ and $0<\frac{\delta}{1-\delta}<1/2$. By Hölder's inequality, we get

$$\begin{aligned}
K_2&\le C\left(\frac{1}{|Q^*|}\int_{Q^*}|b_1(z)-(b_1)_{Q^*}|\,\mathrm{d}z\right)\left(\frac{1}{|Q^*|}\int_{Q^*}\big|T^2_{b_2-\lambda_2}(f_1,f_2)(z)\big|^{\delta/(1-\delta)}\,\mathrm{d}z\right)^{(1-\delta)/\delta}\\
&\le C\frac{\mu(Q^*)^{\beta_1/n+1}}{|Q^*|}\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\left(\frac{1}{|Q^*|}\int_{Q^*}\big|T^2_{b_2-\lambda_2}(f_1,f_2)(z)\big|\,\mathrm{d}z\right)\\
&\le C\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\mu(x)^{1+\beta_1/n}\left(\frac{1}{|Q^*|^{1-\beta_1/n}}\int_{Q^*}\big|T^2_{b_2-\lambda_2}(f_1,f_2)(z)\big|\,\mathrm{d}z\right)
\end{aligned}$$

$$\le C\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\mu(x)^{1+\beta_1/n}M_{\beta_1}\big(T^2_{b_2-\lambda_2}(f_1,f_2)\big)(x)$$
$$\le C\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\mu(x)^{1+\beta_1/n}M_{\beta_1}\big(T^2_{b_2}(f_1,f_2)\big)(x).$$

Similarly, we have

$$K_3 \le C\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}\mu(x)^{1+\beta_2/n}M_{\beta_2}\big(T^1_{b_1}(f_1,f_2)\big)(x).$$

Now, we consider the last term K_4. For each $i=1,2$, we decompose $f_i=f_i^0+f_i^\infty$, where $f_i^0=f_i\chi_{Q^*}$, then

$$\begin{aligned}
K_4 &\le C\bigg(\frac{1}{|Q|}\int_Q\big|T\big((b_1-\lambda_1)f_1^0\big),(b_2-\lambda_2)f_2^0\bigg)(z)\big|^\delta\,\mathrm{d}z)^{1/\delta}\\
&\quad + C\bigg(\frac{1}{|Q|}\int_Q\big|T\big((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty\big)(z)\big|^\delta\,\mathrm{d}z\bigg)^{1/\delta}\\
&\quad + C\bigg(\frac{1}{|Q|}\int_Q\big|T\big((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^0\big)(z)\big|^\delta\,\mathrm{d}z\bigg)^{1/\delta}\\
&\quad + C\bigg(\frac{1}{|Q|}\int_Q\big|T\big((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty\big)(z)-c\big|^\delta\,\mathrm{d}z\bigg)^{1/\delta}\\
&:= K_{41}+K_{42}+K_{43}+K_{44}.
\end{aligned}$$

We first estimate K_{41}. Applying Kolmgorov's inequality, Theorem B, and Lemma 2.2(iii), we have

$$\begin{aligned}
K_{41} &\le C|Q|^{-2}\|T\big((b_1-\lambda_1)f_1^0\big),(b_2-\lambda_2)f_2^0)\|_{L^{\frac{1}{2},\infty}(Q)}\\
&\le C|Q|^{-2}\|T\big((b_1-\lambda_1)f_1^0\big),(b_2-\lambda_2)f_2^0)\|_{L^{\frac{1}{2},\infty}(\mathbb{R}^n)}\\
&\le C\bigg(\frac{1}{|Q|}\int_{Q^*}\big|b_1(y_1)-(b_1)_{Q^*}\big|\big|f_1(y_1)\big|\,\mathrm{d}y_1\bigg)\bigg(\frac{1}{|Q|}\int_{Q^*}\big|b_2(y_2)-(b_2)_{Q^*}\big|\big|f_2(y_2)\big|\,\mathrm{d}y_2\bigg)\\
&\le C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}M_{\beta_1,\mu,s}(f_1)(x)M_{\beta_2,\mu,s}(f_2)(x).
\end{aligned}$$

Next, we consider the term K_{42}. Note that for any $z\in Q$, $y_2\in(Q^*)^c$, $|z-y_2|\sim|y_2-x_Q|$, by (1.1) and Lemma 2.2(iii), we have

$$\begin{aligned}
K_{42} &\le C\frac{1}{|Q|}\int_Q\big|T\big((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty\big)(z)\big|\,\mathrm{d}z\\
&\le C\frac{1}{|Q|}\int_Q\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}\frac{A}{(|z-y_1|+|z-y_2|)^{2n}}\\
&\quad\times\big|b_1(y_1)-\lambda_1\big|\big|f_1(y_1)\big|\big|b_2(y_2)-\lambda_2\big|\big|f_2(y_2)\big|\,\mathrm{d}y_2\,\mathrm{d}y_1\,dz\\
&\le C\int_{Q^*}\big|b_1(y_1)-(b_1)_{Q^*}\big|\big|f_1(y_1)\big|\,\mathrm{d}y_1\int_{\mathbb{R}^n\setminus Q^*}\frac{1}{|y_2-x_Q|^{2n}}\big|b_2(y_2)-(b_2)_{Q^*}\big|\big|f_2(y_2)\big|\,\mathrm{d}y_2\\
&\le C\mu(x)\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}M_{\beta_1,\mu,s}(f_1)(x)\big|Q^*\big|\\
&\quad\times\sum_{k=1}^{\infty}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\frac{1}{|y_2-x_Q|^{2n}}\big|b_2(y_2)-(b_2)_{Q^*}\big|\big|f_2(y_2)\big|\,\mathrm{d}y_2
\end{aligned}$$

$$
\begin{aligned}
&\le C\mu(x)\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}} M_{\beta_1,\mu,s}(f_1)(x)\sum_{k=1}^{\infty}\frac{|Q^*|}{|2^kQ|^2}\int_{2^kQ^*}\left|b_2(y_2)-(b_2)_{Q^*}\right|\left|f_2(y_2)\right|\,\mathrm{d}y_2\\
&\le C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}} M_{\beta_1,\mu,s}(f_1)(x)M_{\beta_2,\mu,s}(f_2)(x).
\end{aligned}
$$

Similarly,

$$
K_{43}\le C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}} M_{\beta_1,\mu,s}(f_1)(x)M_{\beta_2,\mu,s}(f_2)(x).
$$

Finally, we consider the term K_{44}. For any $z\in Q$ and $(y_1,y_2)\in(2^{k+3}\sqrt{n}Q)^2\setminus(2^{k+2}\sqrt{n}Q)^2$,

$$
\left|K(z,y_1,y_2)-K(x,y_1,y_2)\right|\le C\frac{\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|}. \tag{3.2}
$$

Set $c=T((b_1-\lambda_1)f_1^{\infty},(b_2-\lambda_2)f_2^{\infty})(x)$, then by (3.2) and Lemma 2.2(iii), we have

$$
\begin{aligned}
K_{44}&\le\frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2}\left|K(z,y_1,y_2)-K(x,y_1,y_2)\right|\left(\Pi_{i=1}^2\left|b_i(y_i)-\lambda_i\right|\left|f_i(y_i)\right|\right)\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}z\\
&\le C\sum_{k=1}^{\infty}\frac{\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|^{2n}}\int_{(2^{k+3}\sqrt{n}Q)^2}\left(\Pi_{i=1}^2\left|b_i(y_i)-\lambda_i\right|\left|f_i(y_i)\right|\right)\mathrm{d}y_1\,\mathrm{d}y_2\\
&\le C\mu(x)^2\sum_{k=1}^{\infty}k^2\omega\left(2^{-k}\right)\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}} M_{\beta_1,\mu,s}(f_1)(x)M_{\beta_2,\mu,s}(f_2)(x)\\
&\le C\mu(x)^2\int_0^1\frac{\omega(t)}{t}\left(1+\log\frac{1}{t}\right)^2dt\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}} M_{\beta_1,\mu,s}(f_1)(x)M_{\beta_2,\mu,s}(f_2)(x)\\
&\le C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}} M_{\beta_1,\mu,s}(f_1)(x)M_{\beta_2,\mu,s}(f_2)(x).
\end{aligned}
$$

This, together with the estimates for K_1, K_2, K_3, K_4 gives

$$
\begin{aligned}
M_\delta^{\sharp}\left[T_{\Pi\vec{b}}(f_1,f_2)\right](x)\le{}& C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}\\
&\times\left[M_{\beta,\mu,s}\left(T(f_1,f_2)\right)(x)+M_{\beta_1,\mu,s}(f_1)(x)M(f_2)(x)\right]\\
&+\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\mu(x)^{1+\beta_1/n}M_{\beta_1}\left(T_{b_2}^2(f_1,f_2)\right)(x)\\
&+\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}\mu(x)^{1+\beta_2/n}M_{\beta_2}\left(T_{b_1}^1(f_1,f_2)\right)(x).
\end{aligned}
$$

Thus we finish the proof of Lemma 3.2. □

Proof of Theorem 1.2 Similarly as the proof of Theorem 1.1, since $\mu\in A_1$, by Lemma 2.1 and Lemma 3.2, for $0<\delta<1/2$,

$$
\begin{aligned}
\left\|T_{\Pi\vec{b}}(f_1,f_2)\right\|_{L^r(\mu^{1-2r})}&\le C\|M_\delta(T_{\Pi\vec{b}}(f_1,f_2)\|_{L^r(\mu^{1-2r})}\\
&\le C\|M_\delta^{\sharp}(T_{\Pi\vec{b}}(f_1,f_2)\|_{L^r(\mu^{1-2r})}\\
&\le C\left\|\mu^2M_{\beta,\mu,s}\left(T(f_1,f_2)\right)\right\|_{L^r(\mu^{1-2r})}\\
&\quad+C\left\|\mu^2M_{\beta_1,\mu,s}(f_1)M_{\beta_2,\mu,s}(f_2)\right\|_{L^r(\mu^{1-2r})}\\
&\quad+C\left\|\mu^{1+\beta_1/n}M_{\beta_1}\left(T_{b_2}^2(f_1,f_2)\right)\right\|_{L^r(\mu^{1-2r})}
\end{aligned}
$$

$$+ C\big\| \mu^{1+\beta_2/n} M_{\beta_2}\big(T^1_{b_1}(f_1,f_2)\big)\big\|_{L^r(\mu^{1-2r})}$$
$$:= V_1 + V_2 + V_3 + V_4.$$

For V_1, by Lemma 2.3 and Theorem A, we have

$$\begin{aligned}\big\|\mu^2 M_{\beta,\mu,s}\big(T(f_1,f_2)\big)\big\|_{L^r(\mu^{1-2r})} &= \big\|M_{\beta,\mu,s}\big(T(f_1,f_2)\big)\big\|_{L^r(\mu)}\\ &\le C\big\|T(f_1,f_2)\big\|_{L^p(\mu)}\\ &\le C\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.\end{aligned}$$

For V_2, since $1/r_1 + 1/r_2 = 1/r$, by Hölder's inequality and Lemma 2.3, we get

$$\begin{aligned}\big\|\mu^2 M_{\beta_1,\mu,s}(f_1)M_{\beta_2,\mu,s}(f_2)\big\|_{L^r(\mu^{1-2r})} &= \big\|M_{\beta_1,\mu,s}(f_1)M_{\beta_2,\mu,s}(f_2)\big\|_{L^r(\mu)}\\ &\le C\big\|M_{\beta_1,\mu,s}(f_1)\big\|_{L^{r_1}(\mu)}\big\|M_{\beta_2,\mu,s}(f_2)\big\|_{L^{r_2}(\mu)}\\ &\le C\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.\end{aligned}$$

For the term V_3, by Theorem 1.1 and Lemma 2.4, let $1/r = 1/l_1 - \beta_1/n$, and then $1 + r/l_1{}' = r - r\beta_1/n > 1$, $1/l_1 = 1/p - \beta_2/n$. Since $\mu \in A_1$, then $\mu^{1-r+r\beta_1/n} \in A_{1+r/l_1{}'}$, then we have

$$\begin{aligned}\big\|\mu^{1+\beta_1/n} M_{\beta_1}\big(T^2_{b_2}(f_1,f_2)\big)\big\|_{L^r(\mu^{1-2r})} &= \big\|M_{\beta_1}\big(T^2_{b_2}(f_1,f_2)\big)\big\|_{L^r(\mu^{1-r+r\beta_1/n})}\\ &\le C\big\|T^2_{b_2}(f_1,f_2)\big\|_{L^{l_1}(\mu^{1-l_1})}\\ &\le C\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.\end{aligned}$$

Similarly as the discussion of V_3, we have

$$V_4 \le C\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.$$

By combining the estimates of V_1, V_2, V_3, V_4, we finish the proof of Theorem 1.2. □

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