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Further study on domains and quasihyperbolic distances

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Abstract

We establish constructive geometric tools for determining when a domain is L^s-averaging and obtain upper and lower bounds for the L^s-integrals of the quasihyperbolic distance. We also construct examples that are helpful to understand our geometric tools and the relationship between p-Poincaré domains and L^s-averaging domains. Finally, finite unions of $L^s(\mu)$-averaging domains are explored.

Keywords: Poincaré domain; L^s-averaging domain; Quasihyperbolic distance; Whitney subdivision

1 Introduction

Domains and mappings are fundamental objects thay have been well studied and applied in many fields of mathematics and engineering, including partial differential equations, potential analysis, and harmonic analysis. It is well known that domains affect the properties of objects defined on them such as functions, mappings, differential forms, integrals, and differential equations. There are a number of analytic criteria that can be used to classify various domains in $\mathbb{R}^n$, such as uniform domains, John domains, and L^s-averaging domains and a typical goal is to determine the relationships among these criteria. The quasihyperbolic distance provides a powerful tool that has been widely used in geometric analysis in recent years, for example, to characterize L^s-averaging domains and $L^s(\mu)$-averaging domains. In this paper, we provide constructive geometric tools for determining when this characterization is met.

This paper is organized as follows. After introducing notation and background information in Sect. 2, we then define essential tubes and provide basic examples in Sect. 3. In Sect. 4, the notion of a generalized Whitney subdivision is introduced and basic properties are established. In Sects. 5 and 6, essential tubes and the idea of generalized Whitney subdivision are used to prove necessary and sufficient conditions for cusps and domains built using particular families of cubical blocks to be L^s-averaging. Finally, in Sect. 7, finite unions of $L^s(\mu)$-averaging domains are explored.

2 Background

In this section, after introducing some notation, we review the analytical criteria of interest in this paper, along with some of the known relationships among these criteria.

Generally, we will use z to denote a point in a domain and we will reserve x and y for coordinates.

Throughout, we consider bounded and connected domains Ω of $\mathbb{R}^n$ with $n \geq 2$. For any set E in $\mathbb{R}^n$, we denote by $|E|$ the Lebesgue measure of E, and for the purposes of integration, we will use dz to denote the Lebesgue measure. We will also consider more general measures μ defined in terms of a weight function w so that $d\mu = w(z)\,dz$. In these instances, the measure of a set E will be denoted $\mu(E)$. For a function $u \in L^1(\Omega)$, we denote the mean by u_Ω.

We use a capital C to indicate a positive constant, with optional arguments, such as the dimension n, to indicate on what a constant may depend. This constant may be different in different instances. Subscripts may be used when distinctions are necessary.

The following definition of L^s-averaging domains was introduced by Staples in [8]. For $1 \leq s < \infty$, a domain Ω is called an *L^s-averaging domain* if for all $u \in L^1_{\mathrm{loc}}(\Omega, dz)$ it follows that

$$\left(\frac{1}{|\Omega|}\int_\Omega |u - u_\Omega|^s\,dz\right)^{1/s} \leq C(s,\Omega)\left(\sup_{B\subset\Omega}\frac{1}{|B|}\int_B |u - u_B|^s\,dz\right)^{1/s},$$

where B is any open ball in Ω.

Many results about differential forms and related operators were established in L^s-averaging domains, see for example [1, 3, 6]. In [4], L^s-averaging domains were extended to weighted averaging domains, $L^s(\mu)$-averaging domains, and a characterization in terms of Whitney cubes was provided. Generalizing further, in [2], $L^\varphi(\mu)$-averaging domains were considered, where φ is a convex function defined on $(0,\infty)$.

The following definition of the quasihyperbolic distance can be found in [5]. For any points z and z_0 in Ω, let $\Gamma = \Gamma_{z,z_0,\Omega}$ be the set of rectifiable curves in Ω connecting z to z_0. The quasihyperbolic distance between z and z_0 is given by

$$k(z,z_0;\Omega) = \inf_{\gamma\in\Gamma}\int_\gamma \frac{1}{d(\zeta,\partial\Omega)}\,d\sigma = \inf_{\gamma\in\Gamma}\int_I \frac{|\gamma'(t)|}{d(\gamma(t),\partial\Omega)}\,dt.$$

Gehring and Osgood [5] proved that for any two points in Ω there is a quasihyperbolic geodesic arc joining them. In [8], Staples showed that Ω is an L^s-averaging domain if and only if

$$\left(\frac{1}{|\Omega|}\int_\Omega k(z,z_0;\Omega)^s\,dz\right)^{1/s} \leq C,$$

where z_0 is any fixed point in Ω and C is a constant depending only on n, s, $|\Omega|$, the choice of $z_0 \in \Omega$, and the constant from the inequality in the definition of L^s-averaging domains. Using this characterization, it was also shown in [8] that John domains are L^s-averaging for all $1 \leq s < \infty$.

For $1 \leq p < \infty$ we say a domain Ω is a *p-Poincaré Domain* if for every function u in the Sobolev space $W^{1,p}(\Omega)$,

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C(p,\Omega)\|\nabla u\|_{L^p(\Omega)}.$$

In [8] it was shown that for $p \geq n$, if Ω is L^p-averaging, then it is p-Poincaré as well. Staples also showed, by an explicit example, that this relationship need not hold if $p < n$. Specifically, she constructed a "rooms-and-halls" domain that was L^s-averaging for all $s \geq 1$, but was not p-Poincaré for any $p < n$. Of course, by the previously mentioned result, this domain is necessarily p-Poincaré for $p \geq n$.

In somewhat of a contrast, in [7] it was shown that star-shaped domains are p-Poincaré for all $1 \leq p < \infty$, but as shown in [8] and in Sects. 5 and 6 below, there are star-shaped domains that are L^s-averaging if and only if $1 \leq s < k$ where k depends on the dimension and parameters defining the domain.

3 Essential tubes

In this section, we define essential tubes, use them to generate necessary conditions for domains to be L^s-averaging, and provide some examples of domains that are not L^s-averaging for any $1 \leq s < \infty$.

3.1 Essential tubes defined

Let $D_r^k \subset \mathbb{R}^k$ be the closed k-dimensional disk of radius r centered at the origin. Consider the cylinder $[0, l] \times D_r^{n-1}$ in $\mathbb{R}^n$. We define a tube $T_{l,r}$ to be the image under a Euclidean transformation of this cylinder. We say the images of $\{0\} \times D_r^{n-1}$ and $\{l\} \times D_r^{n-1}$ are the ends of the tube and we say the image of $[0, l] \times (\partial D_r^{n-1})$ is the wall of the tube.

Let Ω be a domain in $\mathbb{R}^n$. We define an essential tube $T = T_{l,r,c}$ for Ω to be a tube $T_{l,r}$ such that $T \cap \Omega$ has a connected component Ω_T satisfying the following properties:

- The intersection of Ω_T with the wall of T is empty;
- There exists $c > 0$ such that for all $t \in [0, l]$, the $(n-1)$-dimensional measure of the tth slice of Ω_T is at least c times the measure of D_r^{n-1}.

See Fig. 1.

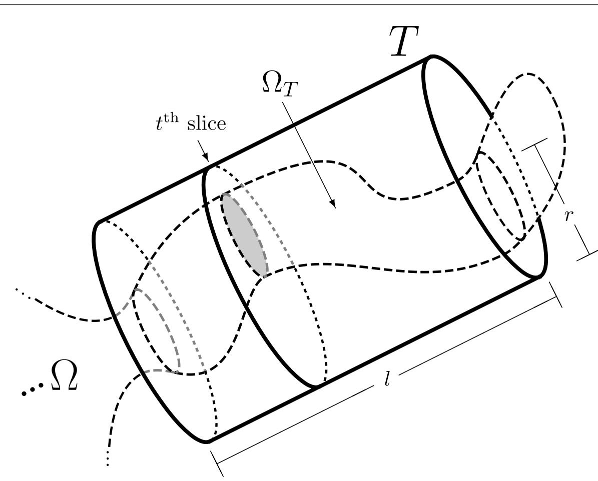


Figure 1 An essential tube $T = T_{l,r,c}$ enclosing a portion Ω_T of a domain Ω. The shaded region is the tth slice of Ω_T and its $(n-1)$-dimensional measure is at least c times the measure of the corresponding slice of T, which in turn is isometric to D_r^{n-1}

3.2 Quasihyperbolic distance calculations

First, we obtain the lower bound of the L^s-integral of the quasihyperbolic distance provided by Ω_T in the following theorem.

Theorem 3.1 *Let Ω be a domain, let $T = T_{l,r,c}$ be an essential tube for Ω with corresponding component Ω_T, and let $z_0 \in \Omega$ be any point not in Ω_T. Then,*

$$\int_{\Omega_T} k(z, z_0; \Omega_T)^s \, dz \geq C(s,n) c r^n \left(\frac{l}{r}\right)^{s+1}.$$

Proof First, choose coordinates so that the wall of the tube aligns with the first coordinate and, writing $z = (x_1, x_2, \dots, x_n)$, one end corresponds to $x_1 = 0$, and the other end corresponds to $x_1 = l$. For any point $z \in \Omega_T$, let $\gamma : [a,b] \to \Omega$ be a rectifiable curve connecting z to z_0. Let γ leave z for the last time at time α, and let γ leave Ω_T for the first time at time β. Then, for all $t \in [\alpha, \beta]$, $\gamma(t) \in \Omega_T$ and $d(\gamma(t), \partial\Omega_T) \leq r$. It is also always true that $|\gamma'(t)| \geq |\gamma_1'(t)|$, where γ_1 is the first component of γ. Using these estimates, and accounting for the fact that the curve may leave either end of the tube,

$$\begin{aligned} \int_\gamma \frac{1}{d(\zeta, \partial\Omega_T)} \, d\sigma &= \int_a^b \frac{|\gamma'(t)|}{d(\gamma(t), \partial\Omega_T)} \, dt \\ &\geq \int_\alpha^\beta \frac{|\gamma_1'(t)|}{r} \, dt \\ &\geq \min\left\{\frac{x_1}{r}, \frac{l - x_1}{r}\right\}. \end{aligned}$$

This is true for all rectifiable curves so for $z \in \Omega_T' = \Omega_T \cap \{z : x_1 \leq \frac{l}{2}\}$,

$$k(z, z_0; \Omega) \geq \frac{x_1}{r}.$$

Hence, letting V_k be the volume of the unit disk in $\mathbb{R}^k$,

$$\begin{aligned} \int_{\Omega_T} k(z, z_0; \Omega)^s \, dz &\geq \int_{\Omega_T'} k(z, z_0; \Omega)^s \, dz \\ &\geq \int_{\Omega_T'} \left(\frac{x_1}{r}\right)^s dz \\ &\geq \frac{1}{r^s} c\left(V_{n-1} r^{n-1}\right) \int_0^{\frac{l}{2}} x_1^s \, dx_1 \\ &= \frac{V_{n-1}}{(s+1)2^{s+1}} c r^n \left(\frac{l}{r}\right)^{s+1}. \end{aligned}$$

□

With this result in hand, essential tubes can be used to show when a given domain fails to be L^s-averaging. To help with this, we introduce the following notation: Given a family $\mathcal{T}$ of essential tubes T with parameters r_T, l_T, and c_T, define $E_{\mathcal{T}}$ to be the following sum:

$$E_{\mathcal{T}} = \sum_{T \in \mathcal{T}} c_T (r_T)^n \left(\frac{l_T}{r_T}\right)^{s+1}.$$

We now have the following:

Corollary 3.2 *Let $\mathcal{T}$ be a family of essential tubes for Ω such that the corresponding components Ω_T are pairwise disjoint. Let the parameters of $T \in \mathcal{T}$ be r_T, l_T, and c_T. Then, if $E_{\mathcal{T}}$ is infinite, Ω cannot be L^s-averaging.*

The proof of this result is left to the reader. We demonstrate how this can be used in the examples below and in later sections.

3.3 Examples

For the first example, we construct a "rooms-and-halls" domain $\Omega \subset \mathbb{R}^2$ that is not p-Poincaré for any $1 \leq p < \infty$. First, define two sequences $x_j = 1 - 1/2^j$ and $x'_j = x_j + 1/2^{j+2}$ for $j \in \mathbb{Z}^+$, and set $x'_0 = 0$. Next, define a sequence of "rooms" by

$$R_j = \left[x'_j, x_{j+1}\right] \times [0,1] \quad \text{for } j = 0,1,2,\ldots$$

and "halls" by

$$H_j = \left[x_j, x'_j\right] \times \left[0, \frac{1}{(j+1)!}\right] \quad \text{for } j = 1,2,\ldots.$$

Letting $f(x,y) = (-x,y)$, set $A = R_0 \cup [\bigcup_{j\in\mathbb{Z}^+}(R_j \cup H_j)]$, and define Ω to be the interior of $A \cup f(A)$. See Fig. 2.

Now, we construct a sequence of functions in $W^{1,p}(\Omega)$ that will demonstrate that Ω is not p-Poincaré. Let

$$v_j(x) = \begin{cases} 0 & \text{if } |x| < x_j, \\ 2^{j+2}(x - x_j) & \text{if } x_j \leq |x| \leq x'_j, \\ 1 & \text{if } |x| > x'_j \end{cases}$$

for $j \in \mathbb{Z}^+$ and $x > 0$, and let

$$u_j(x,y) = \begin{cases} v_j(x) & \text{if } x \geq 0, \\ -v_j(-x) & \text{if } x < 0 \end{cases}$$

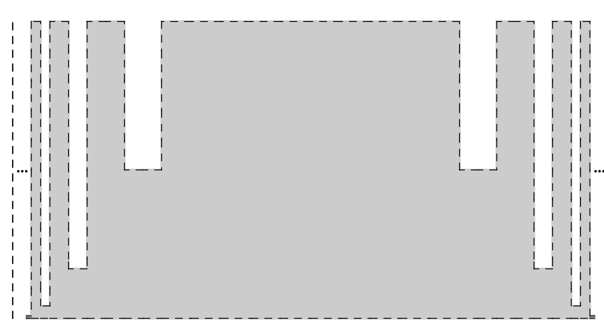


Figure 2 The rooms-and-halls domain

for $j \in \mathbb{Z}^+$. Then, $\{u_j\}_{j\in\mathbb{Z}^+}$ is the desired sequence of functions. To see this, note that each R_j and H_j has width $1/2^{j+2}$. Since $u_{j_\Omega} = 0$,

$$\|u_j - u_{j_\Omega}\|^p_{L^p(\Omega)} = \int_\Omega |u_j|^p\,dz \geq 2\int_{R_j} |1|^p\,dz = 2\cdot\frac{1}{2^{j+2}}$$

and

$$\|\nabla u_j\|^p_{L^p(\Omega)} = \int_\Omega |\nabla u_j|^p\,dz = 2\int_{H_j} \left|2^{j+2}\right|^p dz = 2\cdot\left(2^{j+2}\right)^p\left(\frac{1}{2^{j+2}}\cdot\frac{1}{(j+1)!}\right).$$

Therefore,

$$a_j = \frac{\|u_j - u_{j_\Omega}\|_{L^p(\Omega)}}{\|\nabla u_j\|_{L^p(\Omega)}} \geq \left(\frac{(j+1)!}{(2^{j+2})^p}\right)^{\frac{1}{p}}$$

and this sequence diverges as $j \to \infty$ regardless of the choice of p.

Next, we use essential tubes to show that this rooms-and-halls domain is not L^s-averaging for any $1 \leq s < \infty$. Note that the rectangles $T_j = [x'_j, x_{j+1}] \times [\frac{1}{2}, \frac{3}{4}]$ are essential tubes for $j \geq 1$, and for each tube we have $r_j = \frac{1}{2^{j+3}}$, $l = \frac{1}{4}$, and $c = 1$. Hence,

$$\begin{aligned} E_{\{T_j\}} &= \sum_{j=1}^{\infty} 1\cdot\left(\frac{1}{2^{j+3}}\right)^2\left(\frac{\frac{1}{4}}{\frac{1}{2^{j+3}}}\right)^{s+1} \\ &= \frac{1}{4^{s+1}}\sum_{j=2}^{\infty}\left(2^{s-1}\right)^{j+3} \\ &= \infty. \end{aligned}$$

Since the associated components Ω_{T_j} are pairwise disjoint, by Corollary 3.2, Ω cannot be L^s-averaging.

Of course, this is not much of a surprise. We already know that the rooms-and-halls domain is not p-Poincaré, so for $s \geq 2$ it cannot be L^s-averaging. The calculation above shows that it cannot be L^s-averaging for any $1 \leq s < \infty$.

The next, perhaps more interesting, example is a domain that is not L^s-averaging for any s, but is p-Poincaré for all p. For $j \in \mathbb{Z}^+$, let $\theta_j = [1 - (\frac{1}{2})^{j-1}]\pi$ and let $z_j = (\cos(\theta_j), \sin(\theta_j))$. Let R_j be the filled open rectangle with two vertices z_j and z_{j+1} and with the other two vertices lying on the circle of radius 3 centered at the origin. Let B be the open unit disk centered at the origin and define Ω to be the "disk-and-rooms" domain as follows:

$$\Omega = \bigcup_{j=1}^{\infty} R_j \cup B.$$

See Fig. 3. Note that Ω is star-shaped with respect to the origin. Hence, it is a p-Poincaré domain for all p.

Let T_j be the filled closed rectangle with two vertices z_j and z_{j+1} and with the other two vertices lying on the circle of radius 2 centered at the origin. Then, the T_j are essential tubes

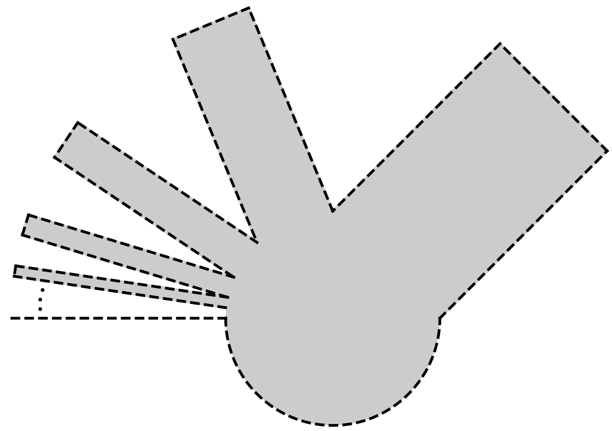


Figure 3 The disk-and-rooms domain

and the associated sets Ω_{T_j} are pairwise disjoint. The parameters of T_j can be estimated as follows: $r_j < \frac{\theta_{j+1}-\theta_j}{2} = \frac{\pi}{2^{j+1}}$, $l_j > 1$, and $c_j = 1$.

With this, we have

$$\begin{aligned} E_{\{T_j\}} &\geq \sum_{j=1}^{\infty} 1 \cdot \left(\frac{\pi}{2^{j+1}}\right)^2 \left(\frac{1}{\frac{\pi}{2^{j+1}}}\right)^{s+1} \\ &= \frac{1}{\pi^{s-1}} \sum_{j=1}^{\infty} \left(2^{s-1}\right)^{j+1} \\ &= \infty \end{aligned}$$

and therefore, by Corollary 3.2, Ω cannot be L^s-averaging.

4 Generalized Whitney subdivision

In this section, we discuss a general method that can be used to establish sufficient conditions for a domain to be L^s-averaging. In some ways this complements essential tubes, but this method is not as concrete.

Given a domain Ω, we say a collection $\mathcal{S}$ of sets is a *valid subdivision* if it has the following properties:

- Each element $S \in \mathcal{S}$ is a closed subset of Ω;
- Each element $S \in \mathcal{S}$ is star-shaped;
- For all distinct pairs $S, T \in \mathcal{S}$, $|S \cap T| = 0$;
- $|\Omega - \bigcup_{S\in\mathcal{S}} S| = 0$;
- For every pair of points $z_0, z \in \bigcup_{S\in\mathcal{S}} S$, there is a sequence $\{S_i : 0 \leq i \leq j\} \subset \mathcal{S}$ such that $z_0 \in S_0$, $z \in S_j$, and $\partial S_i \cap \partial S_{i+1} \neq \emptyset$.

For each $S \in \mathcal{S}$, define two parameters: let $d(S)$ be the diameter of S and let $\delta(S)$ be the distance between S and $\partial\Omega$. We then have the following:

Lemma 4.1 *Let $\mathcal{S}$ be a valid subdivision for Ω and let z_0 and z be two points in $\bigcup_{S\in\mathcal{S}} S$. Let $\{S_i : 0 \leq i \leq j\} \subset \mathcal{S}$ be a sequence of sets such that $z_0 \in S_0$, $z \in S_j$, and $\partial S_i \cap \partial S_{i+1} \neq \emptyset$.*

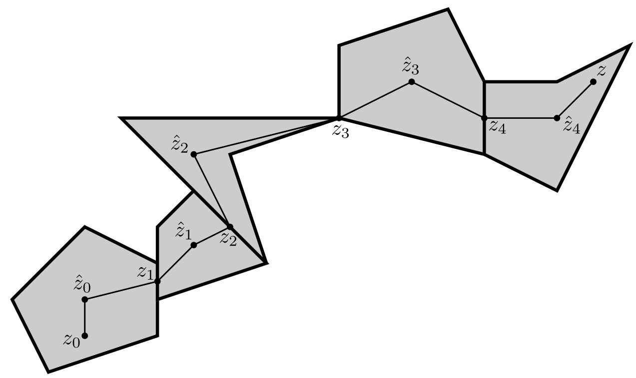


Figure 4 Sets $S_0,\dots,S_4$ in a valid subdivision, reference points, and path used for proof of Lemma 4.1

Then,

$$k(z,z_0;\Omega)\le 2\sum_{i=0}^{j}\frac{d(S_i)}{\delta(S_i)}.$$

Proof For $i\in\{0,\dots,j\}$ let $\hat{z}_i\in S_i$ be a point relative to which S_i is star-shaped. For $i\in\{1,\dots,j\}$, let $z_i\in S_{i-1}\cap S_i$, and let $z_{j+1}=z$. For each $i\in\{0,\dots,j\}$ let $\tilde{\gamma}_i:\tilde{I}_i\to S_i$ be a piecewise linear path connecting z_i to $\hat{z}_i$ and then to z_{i+1}. Note that in each S_i, $\tilde{\gamma}_i$ consists of two segments, both of which have length at most $d(S_i)$. Then, let $\tilde{\gamma}:\tilde{I}\to\Omega$ be the concatenation of these $\tilde{\gamma}_i$. See Fig. 4.

Since this path is rectifiable, it is an element of Γ, and so provides the following estimate for $k(z,z_0;\Omega)$:

$$\begin{aligned}
k(z,z_0;\Omega) &= \inf_{\gamma\in\Gamma}\int_{I_\gamma}\frac{|\gamma'(t)|}{d(\gamma(t),\partial D)}\,dt\\
&\le \int_{\tilde{I}}\frac{|\tilde{\gamma}'(t)|}{d(\tilde{\gamma}(t),\partial D)}\,dt\\
&= \sum_{i=0}^{j}\int_{\tilde{I}_i}\frac{|\tilde{\gamma}_i'(t)|}{d(\tilde{\gamma}_i(t),\partial D)}\,dt\\
&\le \sum_{i=0}^{j}\int_{\tilde{I}_i}\frac{|\tilde{\gamma}_i'(t)|}{\delta(S_i)}\,dt\\
&= \sum_{i=0}^{j}\frac{1}{\delta(S_i)}\int_{\tilde{I}_i}\left|\tilde{\gamma}_i'(t)\right|dt\\
&\le \sum_{i=0}^{j}\frac{1}{\delta(S_i)}2d(S_i).
\end{aligned}$$

□

The estimate in the above lemma is useful when the relationship between $d(S)$ and $\delta(S)$ is well behaved. With this in mind, we say a valid subdivision $\mathcal{S}$ is a *generalized Whitney subdivision* if there exists an M such that for each $S\in\mathcal{S}$, $d(S)\le M\delta(S)$. We call M the *distance factor*.

For a generalized Whitney subdivision $\mathcal{S}$ of Ω, with $z_0 \in S_0$, let L_j be the union of those sets S that are j sets away from S_0 (so $L_0 = S_0$). Then, we have the following upper bound for the L^s-integral of the quasihyperbolic distance:

Lemma 4.2 *Let $\mathcal{S}$ be a generalized Whitney subdivision for a domain Ω with distance factor M. Let $z_0 \in S_0$. Then, for $z \in L_j$, $k_\Omega(z, z_0) \le 2M(j+1)$ and*

$$\int_\Omega \bigl[k(z,z_0;\Omega)\bigr]^s\,dz \le (2M)^s \sum_{j=0}^{\infty} (j+1)^s |L_j|.$$

Proof Using Lemma 4.1,

$$\begin{aligned} k(z,z_0;\Omega) &\le 2\sum_{i=0}^{j} \frac{d(S_i)}{\delta(S_i)} \\ &\le 2\sum_{i=0}^{j} M \\ &= 2M(j+1) \end{aligned}$$

and so

$$\begin{aligned} \int_\Omega \bigl[k(z,z_0;\Omega)\bigr]^s\,dz &= \sum_{j=0}^{\infty} \int_{L_j} \bigl[k(z,z_0;\Omega)\bigr]^s\,dz \\ &\le \sum_{j=0}^{\infty} \int_{L_j} \bigl[2M(j+1)\bigr]^s\,dz \\ &= (2M)^s \sum_{j=0}^{\infty} (j+1)^s \int_{L_j} dz \\ &= (2M)^s \sum_{j=0}^{\infty} (j+1)^s |L_j|. \end{aligned}$$

□

To demonstrate how such an estimate can be used, we show that a cube is L^s-averaging for all s. This result is, of course, not new, but it illustrates how the analysis can be performed, and provides an upper bound to be used later.

Lemma 4.3 *Let Ω be the unit cube in $\mathbb{R}^n$ and let z_0 be the center point of Ω. Then,*

$$\int_\Omega \bigl[k(z,z_0;\Omega)\bigr]^s\,dz \le C(n,s) \sum_{j=0}^{\infty} (j+1)^s \left(\frac{1}{3}\right)^j, \tag{1}$$

which is finite for all $s \ge 1$.

Proof Let $L_0 = S_0$ be the closed cube of side length $\frac{1}{2}$ centered at z_0. After this, to produce the jth layer of cubes, subdivide each exposed $(n-1)$-dimensional face of the cubes in the $(j-1)$th layer into 3^{n-1} congruent square pieces and let these be the faces of a new set of

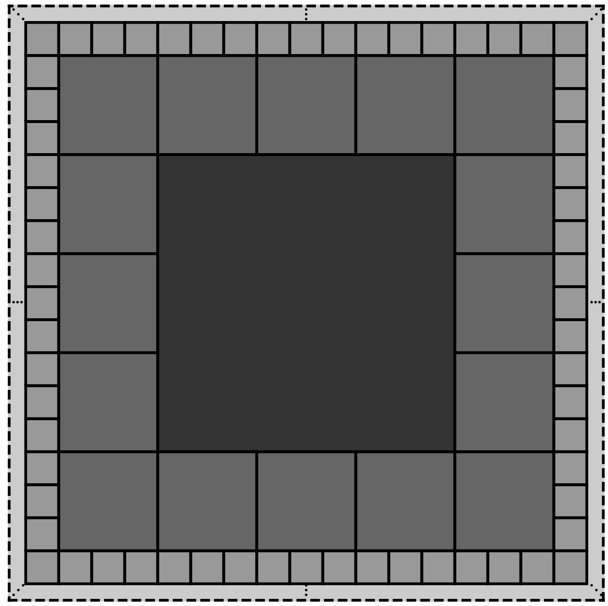


Figure 5 Generalized Whitney subdivision of a square. The first three layers L_0, L_1, and L_2 are shown in increasingly lighter shades of gray

cubes of side length $\frac{1}{2}\frac{1}{3^j}$. These cubes enclose most of the $(j-1)$th layer, but there are still lower-dimensional components that are accessible. With this in mind, complete the layer by adding more cubes of the same size so as to completely enclose the $(j-1)$th layer. See Fig. 5.

As a union of small cubes, these layers are hollow cubes. Let e_i be the number of cubes along a one-dimensional edge. Then, $e_0 = 1$ and $e_j = 3e_{j-1} + 2$, and from this we can conclude that $e_j = 2 \cdot 3^j - 1$.

Let ν_j be the number of cubes creating L_j. Then, $\nu_0 = 1$, and since an n-dimensional cube has $2n$ faces, for $j > 1$

$$\begin{aligned} \nu_j &< 2ne_j^{n-1} \\ &= 2n\left(2 \cdot 3^j - 1\right)^{n-1} \\ &< 2n\left(2 \cdot 3^j\right)^{n-1} \\ &= 2^n n 3^{j(n-1)}, \end{aligned}$$

where the first inequality comes from the fact that we are overcounting the cubes included to cover the lower-dimensional edges.

Since the side length of each cube S in L_j is $\frac{1}{2}(\frac{1}{3})^j$, the diameter is $d(S) = \frac{\sqrt{n}}{2}\frac{1}{3^j}$, and for $j \geq 1$

$$\begin{aligned} \delta(S) &= \frac{1}{4} - \sum_{i=1}^{j} \frac{1}{2}\left(\frac{1}{3}\right)^i \\ &= \frac{1}{4} - \frac{1}{4}\left(1 - \frac{1}{3^j}\right) \end{aligned}$$

$$= \frac{1}{4 \cdot 3^j}.$$

Hence, $d(S) = 2\sqrt{n}\delta(S)$ and so this is a generalized Whitney subdivision with a distance factor of $2\sqrt{n}$.

Estimating the measure of L_j, we have

$$\begin{aligned} |L_j| &= \nu_j \left(\frac{1}{2 \cdot 3^j}\right)^n \\ &\le 2^n n 3^{j(n-1)} \frac{1}{2^n} \frac{1}{3^{jn}} \\ &= n \frac{1}{3^j}. \end{aligned}$$

Applying Lemma 4.2 we have Equation (1), which converges for all s. □

For a given domain, cubes may not be an ideal object for subdivision, and other shapes, tailored to the domain, can be used. An example is provided in the next section.

5 Cusps

As a family, cusps demonstrate that a domain can be L^s-averaging for some s and not others. The cusps analyzed here were explored in [8]. We confirm those results using essential tubes and generalized Whitney subdivision.

Theorem 5.1 *For $\alpha \ge 0$, let $\Omega_\alpha \in \mathbb{R}^n$ be the following domain*:

$$\Omega_\alpha = \left\{(x, y_1, y_2, \dots, y_{n-1}) : 0 < x < 1, \left(y_1^2 + y_2^2 + \cdots + y_{n-1}^2\right)^{\frac{1}{2}} < x^\alpha\right\}.$$

Then, Ω_α is an L^s-averaging domain if and only if

$$(\alpha - 1)(s - n + 1) < n.$$

Note that if $0 \le \alpha \le 1$ then Ω_α is a John domain and hence L^s-averaging for all $s \ge 1$. At the same time, it is straightforward to check that the inequality is satisfied. With these observations in mind, we restrict to $\alpha > 1$ for the remainder of this section.

The proof is broken into parts. In the first part, essential tubes are used to show that a cusp cannot be L^s-averaging if the given inequality is not satisfied. In the second part, a generalized Whitney subdivision is used to show the converse.

Proof (*part one*) Let $0 < a < b < 1$ and consider the tube centered along the x-axis with ends at a and b, and its radius equal to b^α. This is an essential tube with $r = b^\alpha$, $l = b - a$, and $c = (\frac{a}{b})^{\alpha n}$.

Using the sequence $a_j = \frac{1}{2^j}$, consider the essential tubes T_j defined as above using $a = a_j$ and $b = a_{j-1}$ Then,

$$E_{T_j} = \sum_{j=3}^{\infty} \left(\frac{a_j}{a_{j-1}}\right)^{\alpha n} (a_{j-1})^{\alpha n} \left(\frac{a_{j-1} - a_j}{(a_{j-1})^\alpha}\right)^{s+1}$$

$$= 2^{-\alpha(s+1)} \sum_{j=3}^{\infty} \left(2^{(\alpha-1)(s-n+1)-n}\right)^j.$$

If $(\alpha - 1)(s - n + 1) \geq n$ then this series diverges, so by Corollary 3.2, Ω_α cannot be L^s-averaging. □

For the other direction, we first establish some structure and initial results. Because Ω_α is symmetric about the x-axis, it is beneficial to work in cylindrical coordinates (x, r, θ). In these coordinates,

$$\Omega_\alpha = \left\{(x, r, \theta) : 0 < x < 1, r < x^\alpha, \theta \in \mathbb{S}^{n-2}\right\},$$

and the volume element is

$$dx\, dy_1 \cdots dy_{n-1} = r^{n-2}\, dx\, dr\, d\theta,$$

where $d\theta$ is the volume element for the unit $(n-2)$-sphere $\mathbb{S}^{n-2}$.

Given $j, m \in \mathbb{Z}^+$, let

$$S_{j,m} = \left\{(x, r, \theta) : \frac{m}{2^{j+\ell}} \leq x \leq \frac{m+1}{2^{j+\ell}}, \left(1 - \frac{1}{2^{\ell-1}}\right)x^\alpha \leq r \leq \left(1 - \frac{1}{2^\ell}\right)x^\alpha\right\},$$

where $\ell = \lfloor \log_2(m) \rfloor + 1$. See Fig. 6.

These sets are created by first dividing the domain into disks indexed by j, then layers indexed by ℓ, and finally further subdividing into the sets described. Since, in the third step, each layer is subdivided into twice as many sets as the previous layer, the number of digits in the base-2 representation of m is the layer ℓ so that $m < 2^\ell \leq 2m$.

This subdivision is not quite a generalized Whitney subdivision for two different reasons. First, the union of all of the $S_{j,m}$ misses a significant portion of Ω. We define $S_0 = \Omega_\alpha \cap \{(x, r, \theta) : x > \frac{1}{4}\}$. This set is a John domain and we do not attempt to subdivide it.

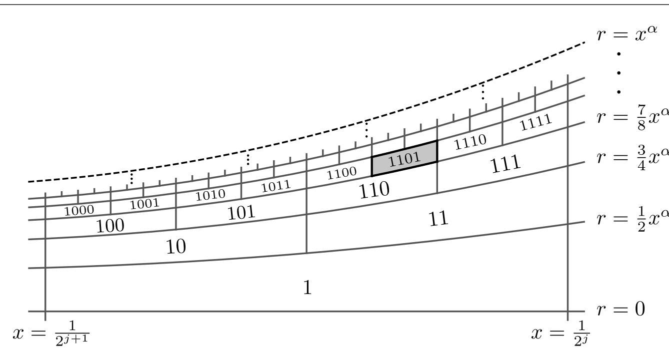


Figure 6 Part of the generalized Whitney subdivision of a Cusp Domain. Note that in order to show detail, the curves representing the layers are not to scale. The sets $S_{j,m}$ are labeled here with the binary representation of m. Moving from one such set to the set below it, the label is truncated by removing the right-most digit. For example, starting at $S_{j,13}$ (shaded), the sets below it have indices 6, 3, and 1

Secondly, when $m > 1$, the sets $S_{j,m}$ are not star-shaped. When $n = 2$, each $S_{j,m}$ is the disjoint union of two sets, one lying above the x-axis and one lying below. To resolve this, for $m > 1$ let $S^+_{j,m} = S_{j,m} \cap \{y_1 > 0\}$ and let $S^-_{j,m} = S_{j,m} \cap \{y_1 < 0\}$.

When $n > 2$, each $S_{j,m}$ has a solid ring shape, or disk-like shape if $m = 1$. To formally make use of the observations about diameter in the previous section, we could further subdivide each $S_{j,m}$ into star-shaped regions through some subdivision of $\mathbb{S}^{n-2}$. However, we will find that because of the choice of the path, each such region would contribute the same, so we keep $S_{j,m}$ as a single set.

To help with calculations later, we have the following:

Lemma 5.2 *For each j, m, denote the radial thickness and horizontal width of $S_{j,m}$ by d_r and d_x, respectively. Then,*

$$d_r < 2d_x.$$

Proof The horizontal width is

$$\begin{aligned} d_x &= \frac{m+1}{2^{j+\ell}} - \frac{m}{2^{j+\ell}} \\ &= \frac{1}{2^{j+\ell}}. \end{aligned}$$

The radial thickness for a given x is

$$\begin{aligned} d_r(x) &= \left(1 - \frac{1}{2^\ell}\right)x^\alpha - \left(1 - \frac{1}{2^{\ell-1}}\right)x^\alpha \\ &= \frac{x^\alpha}{2^\ell}, \end{aligned}$$

and this quantity is maximized at the right end of $S_{j,m}$ at $x = \frac{m+1}{2^{j+\ell}}$, hence

$$d_r = \frac{(\frac{m+1}{2^{j+\ell}})^\alpha}{2^\ell}.$$

Using the fact that $m < 2^\ell$, the ratio of these distances is

$$\begin{aligned} \frac{d_r}{d_x} &= \frac{\frac{(\frac{m+1}{2^{j+\ell}})^\alpha}{2^\ell}}{\frac{1}{2^{j+\ell}}} \\ &< \left(\frac{2^\ell + 1}{2^\ell}\right)^\alpha \frac{1}{2^{(\alpha-1)j}} \\ &< \frac{2^\alpha}{2^{(\alpha-1)j}} \\ &< \frac{2^\alpha}{2^{\alpha-1}} \\ &= 2. \end{aligned}$$

□

The curves used to estimate the quasihyperbolic distance for Ω_α will incorporate only horizontal and radial directions. Therefore, since in the previous section, the diameter of the set is used as a proxy for the length of a curve for a bound on k_Ω, we can restrict attention to d_r and d_x, and in light of the previous lemma, we may use $d_x(S_{j,m})$ in place of $d(S_{j,m})$ and we have

$$d_x(S_{j,m}) = \frac{1}{2^{j+\ell}}.$$

Next, we focus on distance to the boundary.

Lemma 5.3 *For each set $S_{j,m}$,*

$$\delta(S_{j,m}) \geq C(\alpha)\frac{1}{2^{\alpha j+\ell}}.$$

Proof We first consider the two-dimensional case. Let $f(x) = x^\alpha$ define the boundary. Let $z = (x,y) \in \Omega_\alpha \cap \{x \leq \frac{1}{2}\}$, and let $\delta(z)$ be its distance to the boundary. Then, since f is convex and increasing, $\delta(z)$ is at least the distance to the tangent line at $(x,f(x))$, and the distance to this tangent line is bounded below by a multiple of the vertical distance $x^\alpha - y$. This multiple, $C_1(\alpha)$ depends on α only, and is realized at $x = \frac{1}{2}$. The general case is similar, due to rotational symmetry.

Focusing now on $S_{j,m}$, the points closest to $\partial(\Omega_\alpha)$ are $(\frac{m}{2^{j+\ell}}, (1-\frac{1}{2^\ell})(\frac{m}{2^{j+\ell}})^\alpha, \theta)$. If we restrict our attention to just the radial distance we find

$$\begin{aligned}\delta(S_{j,m}) &\geq C_1(\alpha)\left[\left(\frac{m}{2^{j+\ell}}\right)^\alpha - \left(1-\frac{1}{2^\ell}\right)\left(\frac{m}{2^{j+\ell}}\right)^\alpha\right]\\ &= C_1(\alpha)\frac{1}{2^\ell}\left(\frac{m}{2^{j+\ell}}\right)^\alpha\\ &\geq C_1(\alpha)\frac{1}{2^\ell}\left(\frac{2^{\ell-1}}{2^{j+\ell}}\right)^\alpha\\ &= C(\alpha)\frac{1}{2^{\alpha j+\ell}}.\end{aligned}$$ □

With these estimates in hand, we now have the following lemma.

Lemma 5.4 *Using the basepoint $z_0 = (\frac{1}{3},0,0)$, for any point $z \in S_{j,m}$,*

$$k(z,z_0;\Omega_\alpha) \leq C(\alpha)(1+\ell)2^{(\alpha-1)j}.$$

Proof For any point $z = (x,r,\theta) \in S_{j,m}$, $j \geq 1$, define the L-shaped path $\gamma : [0,2] \longrightarrow \Omega_\alpha$ by

$$\gamma(t) = \begin{cases}(xt + \frac{1}{3}(1-t),0,0) & \text{if } t \in [0,1],\\ (x, r(t-1), \theta) & \text{if } t \in [1,2].\end{cases}$$

For the first part we only need the horizontal width of each set, and for the second part, we only need the radial thickness.

Now, we can estimate k_{Ω_α}. For the first step, we determine which sets intersect γ. Let $(x,r,\theta) \in S_{j,m}$. For the initial leg from z_0 to $(x,0,0)$, we use the sets $S_{i,1}$ for $i = 1,\ldots,j$. For the second leg from $(x,0,0)$ to (x,r,θ) we need to determine which sets lie between $S_{j,m}$ and

$S_{j,1}$ There is one at each layer out to the layer containing $S_{j,m}$, and the specific sets $S_{j,\lambda}$ are determined as follows. Express m in binary. Then, the λ values are represented in binary by truncating the binary representation of m by successively removing the rightmost digit. For example, if $m = 51$ then the λ to use are:

$$51 = 110011_2,$$

$$25 = 11001_2,$$

$$12 = 1100_2,$$

$$6 = 110_2,$$

$$3 = 11_2,$$

$$1 = 1_2.$$

Let $\Lambda(j,m)$ be the set of indices corresponding to these sets lying below $S_{j,m}$ and note that $|\Lambda(j,m)| = \ell$. With the specific sets through which γ passes known, Lemma 4.1, modified to account for the fact that only the radial or horizontal distances are needed, then Lemmas 5.2 and 5.3 are used to approximate k_{Ω_α} as follows:

$$\begin{aligned}
k(z,z_0;\Omega_\alpha) &\leq 2\sum_{i=1}^{j} \frac{d_x(S_{i,1})}{\delta(S_{i,1})} + 2\sum_{\lambda\in\Lambda(k,m)} \frac{d_r(S_{j,\lambda})}{\delta(S_{j,\lambda})} \\
&\leq 2\sum_{i=1}^{j} \frac{d_x(S_{i,1})}{\delta(S_{i,1})} + 4\sum_{\lambda\in\Lambda(k,m)} \frac{d_x(S_{j,\lambda})}{\delta(S_{j,\lambda})} \\
&\leq C_1(\alpha)\left(\sum_{i=1}^{j} \frac{2^{\alpha i+1}}{2^{i+1}} + \sum_{\lambda\in\Lambda(j,m)} \frac{2^{\alpha j+\ell}}{2^{j+\ell}}\right) \\
&= C_1(\alpha)\left(\sum_{i=1}^{j} 2^{(\alpha-1)i} + \sum_{\lambda\in\Lambda(j,m)} 2^{(\alpha-1)j}\right) \\
&\leq C(\alpha)\left(2^{(\alpha-1)j} + \ell 2^{(\alpha-1)j}\right) \\
&= C(\alpha)(1+\ell)2^{(\alpha-1)j},
\end{aligned}$$

where the first sum on the third-to-last line is approximated by a constant times the largest term. □

With $k(z,z_0;\Omega_\alpha)$ approximated, the next step is to estimate the measure of $S_{j,m}$.

Lemma 5.5 *For each $S_{j,m}$,*

$$|S_{j,m}| \leq C(\alpha,n)\frac{1}{2^{j[\alpha(n-1)+1]}}\frac{1}{2^{2\ell}}.$$

Proof We have

$$|S_{j,m}| = \int_{\mathbb{S}^{n-2}} \int_{\frac{m}{2^{j+\ell}}}^{\frac{m+1}{2^{j+\ell}}} \int_{(1-\frac{1}{2^{\ell-1}})x^\alpha}^{(1-\frac{1}{2^\ell})x^\alpha} r^{n-2}\,dr\,dx\,d\theta.$$

The integral over the sphere just produces a dimensional constant. For the other two integrals, the given functions are increasing, and so are approximated by $\int_a^b f(x)\,dx \le f(b)(b-a)$, resulting in

$$\begin{aligned} |S_{j,m}| &\le C_1(n)\int_{\frac{m}{2^{j+\ell}}}^{\frac{m+1}{2^{j+\ell}}}\left[\left(1-\frac{1}{2^\ell}\right)x^\alpha\right]^{n-2}x^\alpha\frac{1}{2^\ell}\,dx \\ &= C_1(n)\frac{1}{2^\ell}\left(1-\frac{1}{2^\ell}\right)^{n-2}\int_{\frac{m}{2^{j+\ell}}}^{\frac{m+1}{2^{j+\ell}}}x^{\alpha(n-1)}\,dx \\ &\le C_1(n)\frac{1}{2^\ell}\left(1-\frac{1}{2^\ell}\right)^{n-2}\left(\frac{m+1}{2^{j+\ell}}\right)^{\alpha(n-1)}\frac{1}{2^{j+\ell}}. \end{aligned}$$

Since $m+1\le 2m$ and $1-\frac{1}{2^\ell}<1$, this simplifies to

$$|S_{j,m}| \le C_1(n)\frac{1}{2^{j[\alpha(n-1)+1]}}\frac{(2m)^{\alpha(n-1)}}{2^{2\ell}2^{\ell\alpha(n-1)}}.$$

The final estimate then follows from the fact that $m<2^\ell$. □

We are now in position to complete the proof of Theorem 5.1.

Proof (*part two*) The domain Ω_α can be subdivided into a John domain and the family of $S_{j,m}$ as follows

$$\begin{aligned} \int_{\Omega_\alpha}\left[k(z,z_0;\Omega_\alpha)\right]^s dz &= \int_{\Omega_\alpha\cap\{x>\frac{1}{2}\}}\left[k(z,z_0,\Omega_\alpha)\right]^s dz + \sum_{j=1}^\infty\sum_{m=1}^\infty\int_{S_{j,m}}\left[k(z,z_0;\Omega_\alpha)\right]^s dz \\ &< \int_{S_0}\left[k(z,z_0;\Omega_\alpha)\right]^s dz + \sum_{j=1}^\infty\sum_{m=1}^\infty\int_{S_{j,m}}\left[k(z,z_0;\Omega_\alpha)\right]^s dz \\ &< \int_{S_0}\left[k(z,z_0;S_0)\right]^s dz + \sum_{j=1}^\infty\sum_{m=1}^\infty\int_{S_{j,m}}\left[k(z,z_0;\Omega_\alpha)\right]^s dz. \end{aligned}$$

The first integral on the right is finite since the domain is a John domain.

For the sum of integrals, the estimate for $k(z,z_0;\Omega_\alpha)$ and $|\Omega_\alpha|$ are combined to estimate $\int_{\Omega_\alpha}k(z,z_0;\Omega_\alpha)^s\,dz$ as follows:

$$\begin{aligned} \sum_{j=1}^\infty\sum_{m=1}^\infty\int_{S_{j,m}}\left[k(z,z_0;\Omega_\alpha)\right]^s dz &\le C(\alpha,n)\sum_{j=1}^\infty\sum_{m=1}^\infty\left[(1+\ell)2^{(\alpha-1)j}\right]^s\frac{1}{2^{j(\alpha(n-1)+1)}}\frac{1}{2^{2\ell}} \\ &= C(\alpha,n)\sum_{j=1}^\infty\left[2^{(\alpha-1)s-(\alpha(n-1)+1)}\right]^j\sum_{m=1}^\infty\frac{(1+\ell)^s}{2^{2\ell}}. \end{aligned}$$

For the sum over m, note that $2^{2\ell}>m^2$ and $(1+\ell)\le 2+\log_2(m)$, hence

$$\sum_{m=1}^\infty\frac{(1+\ell)^s}{2^{2\ell}} \le \sum_{m=1}^\infty\frac{[2+\log_2(m)]^s}{m^2},$$

which converges for all s.

For the sum over j, this converges if and only if

$$(\alpha-1)s-\big(\alpha(n-1)+1\big)<0,$$

which can be rearranged as $(\alpha-1)(s-n+1)<n$. □

We end with two comments. First, if $n=2$, the domain Ω_α is a finite intersection of John domains. Secondly, for all n, α, Ω_α is star-shaped and therefore p-Poincaré for all $1\le p<\infty$.

6 Block domains

In this section we build a domain using blocks, and show, by combining the techniques above, for which s it is L^s-averaging.

Consider the domain $\Omega\subset\mathbb{R}^n$ defined as follows: Starting with a closed unit cube Ω_1, perform a triadic subdivision of the top face and glue a closed cube Ω_2 onto the middle. Then, on the top face of Ω_2 glue on a cube Ω_3 that is the same size as Ω_2. Next, in a similar fashion, perform a triadic subdivision the top face of Ω_3, glue a cube Ω_4 onto the middle, and then extend with three more cubes, all the same size. Continue this process, doubling the number of same-sized cubes in each step so that the cubes $\Omega_{2^j},\dots\Omega_{2^{j+1}-1}$ have edge length equal to 3^{-j}. Finally, take the interior of the infinite union. See Fig. 7.

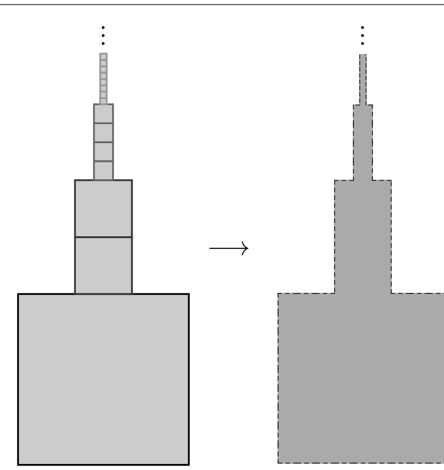
Theorem 6.1 *The set Ω is L^s-averaging if and only if $s<n\log_2(3)-1$.*

As before, we separate the proof into two parts, beginning with the proof of when Ω fails to be L^s-averaging.

Proof (*part one*) Essential tubes can be built for each set of cubes $\Omega_{2^j},\dots\Omega_{2^{j+1}-1}$. For the jth tube, $r_j=C_1(n)(\frac{1}{3})^j$, $l_j=(\frac{2}{3})^j$, and $c_j=C_2(n)$, hence

$$E_{\{T_j\}}=C(s,n)\sum_{j=1}^{\infty}\left[\left(\frac{1}{3}\right)^j\right]^n\left(\frac{\frac{2}{3}}{\frac{1}{3}}\right)^{j(s+1)}$$

Figure 7 Block domain construction



$$= \sum_{j=1}^{\infty} \left[\frac{2^{s+1}}{3^n} \right]^j.$$

If $2^{s+1} \geq 3^n$ this sum diverges and so by Corollary 3.2, Ω is not L^s-averaging, and this happens for $s \geq n \log_2(3) - 1$. □

Proof (*part two*) First, subdivide each Ω_m using the subdivision in the proof of Lemma 4.3. This subdivision has the problem of needing infinitely many elements of the subdivision for any path connecting points in Ω_m to points in Ω_{m+1}. This is resolved by noting that when Ω_m and Ω_{m+1} are the same size, the sets at the centers of Ω_m and Ω_{m+1} can be connected with a third set of the same size, and when Ω_m and Ω_{m+1} are not the same size, the center set in Ω_{m+1} is the same size as the sets in the layer L_1 of Ω_m and can be connected to this layer by a single set of the same size. See Fig. 8.

Thus, to build a path from $z_0 \in \Omega_1$ to z in the ith layer in Ω_m, first walk to the center of Ω_m, requiring at most $3m$ Whitney sets, and then to the ith layer, requiring at most $i + 1$ additional steps, maybe many fewer if z happens to be in or near one of the new big sets acting as a bridge into or out of Ω_m.

Note that this generalized Whitney subdivision does not cover all of Ω. Namely, it misses most of the points at $\Omega_i \cap \Omega_{i+1}$. This does not pose a difficulty though because it is a set of measure 0.

Combining this with the estimate in equation (1), and accounting for the sizes of the Ω_m,

$$\int_{\Omega} \left[k(z, z_0; \Omega) \right]^s dz \leq C(n, s) \sum_{m=1}^{\infty} \sum_{i=0}^{\infty} (i + 1 + 3m)^s \left(\frac{1}{3^n} \right)^{\lfloor \log_2(m) \rfloor} \left(\frac{1}{3} \right)^i,$$

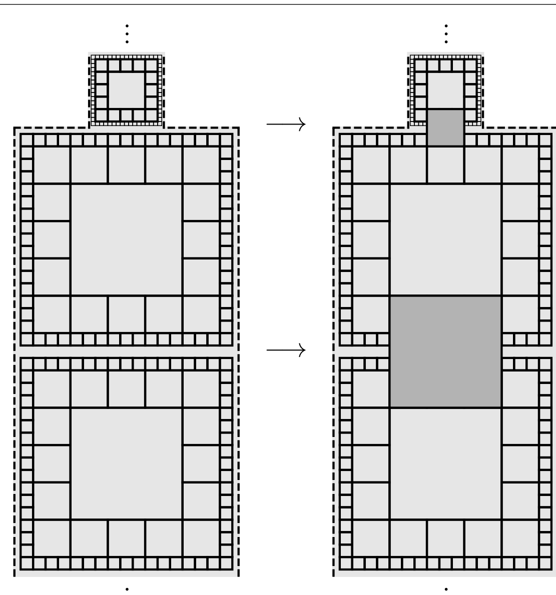


Figure 8 Modifying the subdivision. The dark gray sets replace the sets they cover so as to connect the blocks. Note that even with this modification, there are sets of measure zero at the boundary of each Ω_i that do not get covered

where $\lfloor \log_2(m) \rfloor$ accounts for the size of the Ω_m. Noting that $\lfloor \log_2(m) \rfloor \geq \log_2(m) - 1$, it follows that

$$\begin{aligned}\left(\frac{1}{3^n}\right)^{\lfloor \log_2(m) \rfloor} &\leq \left(\frac{1}{3^n}\right)^{\log_2(m)-1} \\ &= 3^n \left(\frac{1}{3^n}\right)^{\log_2(m)} \\ &= 3^n m^{-n\log_2(3)}.\end{aligned}$$

Inserting this into the estimate above yields

$$\int_\Omega \left[k(z, z_0; \Omega)\right]^s dz \leq C(n,s) \sum_{m=1}^{\infty} \sum_{i=0}^{\infty} (i+1+3m)^s m^{-n\log_2(3)} \left(\frac{1}{3}\right)^i.$$

Note that $i + 1 + 3m \geq m$ so if $s \geq n\log_2(3) - 1$ the double sum diverges and this estimate gives us no information. On the other hand,

$$\begin{aligned}(i+1+3m)^s &= m^s \left(\frac{i+1}{m} + 3\right)^s \\ &\leq m^s (i+4)^s,\end{aligned}$$

hence, if $s < n\log_2(3) - 1$ then the double sum above converges and so Ω is L^s-averaging. □

6.1 Variations

With this initial tower in hand, there are a number of modifications that can be made without significantly changing the analysis.

First, we could glue towers of cubes onto all faces of the initial cube, and more generally, we could add other smaller towers as well. As long as the number of additional towers is bounded, the estimates above will still hold.

We could consider more extreme ratios of side lengths of adjacent squares. This will only affect the contribution of the number of steps to go from one center to the next. As long as this stays bounded, the analysis above will hold.

We could glue the cubes together in different orientations to produce spirals, trees, or other interesting fractal shapes. For the above analysis to hold, the key thing that would need to be preserved is that the number of cubes of a given size stays comparable to the number introduced above. More exotic shapes could be considered with more careful analysis.

The "2" in the critical value comes from the growth in the number of cubes of a given size and the "3" comes from the ratio of one size to the next. Experimenting with these values would produce other relationships. In the current case, "3" was chosen because it was relatively easy to verify that the Whitney subdivision has the correct properties, and then "2" was the only available integer of any interest. For example, using "1" instead of "2", we obtain something like an Aztec pyramid, which is L^s-averaging for all $s \geq 1$, and in fact is John.

7 The union of $L^s(\mu)$-averaging domains

In 1999, Vaisala proved that, under appropriate conditions, the union of John domains is still a John domain in [9]. Since $L^s(\mu)$-averaging domains are extensions of John domains, a natural question is: Does the union of $L^s(\mu)$-averaging domains have the similar property? We will answer this question in this section.

We say a weight $w(z)$ satisfies the A_r condition in a domain Ω, and write $w \in A_r(\Omega)$, $r > 1$, if

$$\sup_{B\subset\Omega}\left(\frac{1}{|B|}\int_B w\,dz\right)\left(\frac{1}{|B|}\int_B w^{\frac{1}{1-r}}\,dz\right)^{r-1} < \infty.$$

Note that if $w \in A_r(\Omega)$ and $G \subset \Omega$ then $W \in A_r(G)$ as well. With this weight, the measure μ is defined by $d\mu = w(z)\,dz$

The following result, found in [6], gives a necessary and sufficient condition for a domain to be $L^s(\mu)$-averaging so long as the weight function defining μ satisfies the A_r condition.

Lemma 7.1 *Let $w \in A_r$ for $r > 1$ and μ be a measure defined by $d\mu = w(z)\,dz$. Then, Ω is an $L^s(\mu)$-averaging domain if and only if the inequality*

$$\left(\frac{1}{\mu(\Omega)}\int_\Omega k(z,z_0;\Omega)^s\,d\mu\right)^{\frac{1}{s}} \le C$$

holds for some fixed point z_0 in Ω and a constant C depending only on n, s, $\mu(\Omega)$, the choice of $z_0 \in \Omega$, and the constant from the inequality in the definition of $L^s(\mu)$-averaging domains.

Theorem 7.2 *Let G_1 and G_2 be bounded $L^s(\mu)$-averaging domains with $G_1 \cap G_2 \neq \emptyset$, where the measure μ is defined by $d\mu = w(z)\,dz$, and $w \in A_r(G_1 \cup G_2)$. Then, $G_1 \cup G_2$ is also an $L^s(\mu)$-averaging domain.*

Proof First, we show that for any two domains D and G with $D \subset G$, we have

$$k(z,z_0;G) \le k(z,z_0;D) \tag{2}$$

for any z, z_0 in D. We know that for any $z \in D$, it follows that

$$d(z,\partial G) \ge d(z,\partial D),$$

hence, for any rectifiable curve γ in D joining z to z_0, we have

$$\int_\gamma \frac{1}{d(\zeta,\partial G)}\,d\sigma \le \int_\gamma \frac{1}{d(\zeta,\partial D)}\,d\sigma.$$

Hence,

$$\inf_{\gamma\in\Gamma_D}\int_\gamma \frac{1}{d(\zeta,\partial G)}\,d\sigma \le \inf_{\gamma\in\Gamma_D}\int_\gamma \frac{1}{d(\zeta,\partial D)}\,d\sigma.$$

Thus,

$$
\begin{aligned}
k(z,z_0;G) &= \inf_{\gamma\in\Gamma_G}\int_\gamma \frac{1}{d(\zeta,\partial G)}\,d\sigma \\
&\le \inf_{\gamma\in\Gamma_D}\int_\gamma \frac{1}{d(\zeta,\partial G)}\,d\sigma \\
&\le \inf_{\gamma\in\Gamma_D}\int_\gamma \frac{1}{d(\zeta,\partial D)}\,d\sigma \\
&= k(z,z_0;D).
\end{aligned}
$$

Now, choose $z_0 \in G_1 \cap G_2$. For $i = 1,2$, extend the definitions of $k(z,z_0;G_i)$ to $G_1 \cup G_2$ by

$$
k_i^*(z,z_0) = \begin{cases} k(z,z_0;G_i), & z \in G_i, \\ 0, & z \notin G_i. \end{cases}
$$

Then, by Equation (2), we have

$$
k(z,z_0;G_1\cup G_2) \le k_1^*(z,z_0) + k_2^*(z,z_0). \tag{3}
$$

Since G_1 and G_2 are $L^s(\mu)$-averaging domains, by Lemma 7.1, for $i = 1,2$ and $z_0 \in G_1 \cap G_2$ we have

$$
\frac{1}{\mu(G_i)}\int_{G_i} k(z,z_0;G_i)^s\,d\mu \le C_i. \tag{4}
$$

Using Equations (3) and (4), and the elementary inequality

$$
\big(|a| + |b|\big)^s \le 2^s\big(|a|^s + |b|^s\big)
$$

for any $s > 0$, we obtain

$$
\begin{aligned}
&\frac{1}{\mu(G_1\cup G_2)}\int_{G_1\cup G_2}\big(k(z,z_0;G_1\cup G_2)\big)^s\,d\mu \\
&\quad\le \frac{1}{\mu(G_1\cup G_2)}\int_{G_1\cup G_2}\big(k_1^*(z,z_0) + k_2^*(z,z_0)\big)^s\,d\mu \\
&\quad\le \frac{1}{\mu(G_1\cup G_2)}\int_{G_1\cup G_2} 2^s\big(\big(\big(k_1^*(z,z_0)\big)^s + \big(k_2^*(z,z_0)\big)^s\big)\,d\mu \\
&\quad= \frac{2^s}{\mu(G_1\cup G_2)}\int_{G_1\cup G_2}\big(k_1^*(z,z_0)\big)^s\,d\mu \\
&\qquad+ \frac{2^s}{\mu(G_1\cup G_2)}\int_{G_1\cup G_2}\big(k_2^*(z,z_0)\big)^s\,d\mu \\
&\quad\le 2^s\left(\frac{1}{\mu(G_1)}\int_{G_1}\big(k(z,z_0;G_1)\big)^s\,d\mu + \frac{1}{\mu(G_2)}\int_{G_2}\big(k(z,z_0;G_2)\big)^s\,d\mu\right) \\
&\quad\le 2^s(C_1 + C_2) \\
&\quad= C_3,
\end{aligned}
$$

which means that

$$\left(\frac{1}{\mu(G_1 \cup G_2)} \int_{G_1 \cup G_2} \big(k(z, z_0; G_1 \cup G_2)\big)^s \, d\mu\right)^{\frac{1}{s}} \leq C_4 \tag{5}$$

and hence, by Lemma 7.1 and Equation (5), $G_1 \cup G_2$ is an $L^s(\mu)$-averaging domain. □

Using Theorem 7.2 and mathematical induction, we can prove the following theorem about the finite union of $L^s(\mu)$-averaging domains.

Theorem 7.3 *Let $w \in A_r(\bigcup_{i=1}^m G_i)$ and let G_i be $L^s(\mu)$-averaging domains, $i = 1, \dots, m$ such that $\bigcup_{i=1}^m G_i$ is connected. Then, $\bigcup_{i=1}^m G_i$ is also an $L^s(\mu)$-averaging domain.*

For any t with $0 < t < s < \infty$ and any z_0 in a domain G, by Hölder's inequality

$$\left(\int_G k(z, z_0; G)^t \, d\mu\right)^{\frac{1}{t}} \leq \left(\int_G k(z, z_0; G)^s \, d\mu\right)^{\frac{1}{s}} \left(\int_G d\mu\right)^{\frac{s-t}{st}},$$

that is,

$$\left(\frac{1}{\mu(G)} \int_G k(z, z_0; G)^t \, d\mu\right)^{\frac{1}{t}} \leq \left(\frac{1}{\mu(G)} \int_G k(z, z_0; G)^s \, d\mu\right)^{\frac{1}{s}}. \tag{6}$$

Applying Lemma 7.1 and Equation (6), we have the following corollary immediately, which also appeared in [6].

Corollary 7.4 *If G is an $L^s(\mu)$-averaging domain, then G is an $L^t(\mu)$-averaging domain for any t with $0 < t < s$.*

From Theorem 7.3 and Corollary 7.4, we have the following result:

Theorem 7.5 *Let $w \in A_r(\bigcup_{i=1}^m G_i)$ and let G_i be $L^{s_i}(\mu)$-averaging domains with $s_i > 0$, $i = 1, \dots, m$ such that $\bigcup_{i=1}^m G_i$ is connected. Then, $\bigcup_{i=1}^m G_i$ is also an $L^s(\mu)$-averaging domain, where $s = \min\{s_1, s_2, \dots, s_m\}$.*

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Declarations

Competing interests
One of the authors, Shusen Ding, is an Editor-In-Chief for the Journal of Inequalities and Applications. The other authors have no competing interests.

Author contributions
SD introduced the problems to the group, played a significant role in developing the results and writing the paper, and was primarily responsible for the last section about unions of domains. DH played a significant role in developing the results and writing the paper, and was primarily responsible for generating the final images. GP played a significant role in developing the results and writing the paper. AY played a significant role in developing the results and writing the paper.

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References

1. Agarwal, R.P., Ding, S., Nolder, C.: Inequalities for Differential Forms. Springer, New York (2009)
2. Ding, S.: $L^{\phi}(\mu)$-Averaging domains and the quasi-hyperbolic metric. Comput. Math. Appl. **47**(10–11), 1611–1618 (2004)
3. Ding, S., Liu, B.: Whitney covers and quasi-isometry of $L^s(\mu)$-averaging domains. J. Inequal. Appl. **6**(4), 435–449 (2001)
4. Ding, S., Nolder, C.A.: $L^S(\mu)$-Averaging domains. J. Math. Anal. Appl. **283**(1), 85–99 (2003)
5. Gehring, F.W., Osgood, B.G.: Uniform domains and the quasihyperbolic metric. J. Anal. Math. **36**, 50–74 (1979). (1980)
6. Liu, B., Ding, S.: The monotonic property of $L^s(\mu)$-averaging domains and weighted weak reverse Hölder inequality. J. Math. Anal. Appl. **237**(2), 730–739 (1999)
7. Smith, W., Stegenga, D.A.: Hölder domains and Poincaré domains. Trans. Am. Math. Soc. **319**(1), 67–100 (1990)
8. Staples, S.G.: L^p-Averaging domains and the Poincaré inequality. Ann. Acad. Sci. Fenn., Ser. A 1 Math. **14**(1), 103–127 (1989)
9. Väisälä, J.: Unions of John domains. Proc. Am. Math. Soc. **128**(4), 1135–1140 (2000)