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The method for solving the extension of general of the split feasibility problem and fixed point problem of the cutter

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Abstract

This paper first introduces a new iterative method for weak and strong convergence theorem to demonstrate the estimation potential for a fixed point of the cutter and the finite general split feasibility problem. Consequently, the set of fixed points of a quasi-nonexpansive mapping and the finite general split feasibility problem, the constrained minimization problem, and the general constrained minimization problems are proved using our main results. Finally, we give two numerical examples to advocate our main results.

MSC: Primary 47H10; secondary 47H09

Keywords: Split feasibility problem; Cutter; Nonexpansive mapping; Constrained minimization problem; Fixed point problem

1 Introduction

Throughout this article, let H, H_1, and H_2 be real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$, and norm $\|\cdot\|$. For each $i = 1,2,\dots,N$, for all $N \in \mathbb{N}$, let C, C_i be a nonempty closed convex subset of H_1 and Q, Q_i be a nonempty closed convex subset of H_2, and let $\Phi,\Psi : H_1 \to H_2$ be bounded linear operators.

Definition 1 Let $T : C \to C$ be a mapping. Then

(i) The *fixed point problem* for the mapping T is to find $x \in C$ such that

$$x = Tx.$$

We denote the fixed point set of a mapping T by $F(T)$.

(ii) A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \text{for all } x, y \in C.$$

The *split feasibility problem (SFP)*, is to find a point $x \in C$ and $\Phi(x) \in Q$. In 1994, the split feasibility problem was first introduced by Censor and Elfving [1]. The SFP can be applied and developed in various fields, such as radiation therapy treatment planning, sensor networks, resolution enhancement, etc. Many mathematicians have modified SFP; see previous works [2–5].

In 2016, Latif et al. [6] introduced the Generalized multiple-set split feasibility problem (GMSSFP), which is to find a point

$$x^* \in \bigcap_{i=1}^{p} C_i \quad \text{and} \quad A_k x^* \in \bigcap_{i=1}^{r} Q_i, \quad k = 1,2,\dots,m, \tag{1}$$

where $A_k : H_1 \to H_2$, $(k = 1,2,\dots,m)$ are family of bounded linear operators, $\{C_i\}_{i=1}^{p}$ and $\{Q_i\}_{i=1}^{r}$ are family of nonempty closed convex subsets in H_1 and H_2, respectively. The set of the problem (1) is denoted by $\Omega = \{x \in \bigcap_{i=1}^{p} C_i | A_k x \in \bigcap_{i=1}^{r} Q_i, k = 1,2,\dots,m\}$.

Applying the viscosity approximation method for solving the GMSSFP, Latif et al. [6] proved the best following result;

Theorem 1 *Let H and K be real Hilbert spaces, $A_k : H \to K$, $k = 1,2$ be two bounded linear operator, and let $\{C_i\}_{i=1}^{r}$ be a family of nonempty closed convex subsets in H and $\{Q_i\}_{i=1}^{r}$ be a family of nonempty closed convex subsets in K. Assume that GMSSFP has a nonempty solution set Ω. Suppose h is a contraction of H into itself with constant $b \in (0,1)$ and B is a strongly positive bounded linear self-adjoint operator on H with coefficient $\bar{\gamma}$ and $0 < \gamma < \frac{\bar{\gamma}}{b}$. Let $\{x_n\}$ be a sequence generated by $x_0 \in H$ and by*

$$\begin{cases} y_n = \alpha_n x_n + \sum_{i=1}^{r} \beta_{n,i} P_{C_i}(I - \lambda_{n,i} A_1^*(I - P_{Q_i})A_1)x_n \\ \qquad + \sum_{i=1}^{r} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A_2^*(I - P_{Q_i})A_2)x_n, \\ x_{n+1} = \theta_n \gamma h(x_n) + (I - \theta_n B)y_n, \quad \forall n \geq 0, \end{cases}$$

where $\alpha_n + \sum_{i=1}^{r} \beta_{n,i} + \sum_{i=1}^{r} \gamma_{n,i} = 1$ and the sequences $\{\alpha_n\}, \{\beta_{n,i}\}, \{\gamma_{n,i}\}, \{\theta_n\}$ and $\{\lambda_{n,i}\}$ satisfy the following conditions:

(i) $\liminf_n \alpha_n \beta_{n,i} > 0$ *and* $\liminf_n \alpha_n \gamma_{n,i} > 0$, *for each* $1 \leq i \leq r$,

(ii) $\lim_{n\to\infty} \theta_n = 0$ *and* $\sum_{n=0}^{\infty} \theta_n = \infty$,

(iii) *for each* $1 \leq i \leq r$, $0 < \lambda_{n,i} < \min\{\frac{2}{\|A_1\|^2}, \frac{2}{\|A_2\|^2}\}$ *and*

$$0 < \liminf_{n\to\infty} \lambda_{n,i} \leq \limsup_{n\to\infty} \lambda_{n,i} < \min\left\{\frac{2}{\|A_1\|^2}, \frac{2}{\|A_2\|^2}\right\}.$$

Then, the sequences $\{x_n\}$ converges strongly to $x^* \in \Omega$ which solves the variational inequality;

$$\langle (B - \gamma h)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega.$$

The *general split feasibility problem* introduced by Kangtunyakarn [5], which is to find a point

$$x^* \in C \quad \text{and} \quad \Phi(x^*), \Psi(x^*) \in Q, \tag{2}$$

where $\Phi, \Psi : H_1 \to H_2$ are bounded linear operators. The set of the problem (2) is denoted by $\Gamma = \{x \in C : \Phi(x), \Psi(x) \in Q\}$. Moreover, if we put $\Phi \equiv \Psi$, then the problem (2) is reduced to the SFP. Furthermore, under some control conditions, Kangtunyakarn [5] proved a strong convergence theorem for finding an element of the set of solutions to the variational inequality problem and the general split feasibility problem as follows:

Theorem 2 *Let $A, B : H_1 \to H_2$ be bounded linear operators with A^*, B^* that are adjoints of A and B, respectively, and $L = \max\{L_A, L_B\}$, where L_A and L_B are spectral radius of A^*A and B^*B, and let $D : C \to H_1$ be d-inverse strongly monotone. Assume that $\Gamma \cap VI(C, D) \neq \emptyset$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda D)x_n + \gamma_n P_C\left(I - a\left(\frac{A^*(I - P_Q)A}{2} + \frac{B^*(I - P_Q)B}{2}\right)\right)x_n,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subseteq (0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $f : C \to C$ is α-contractive mapping with $\alpha \in (0, 1)$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $c \le \beta_n, \gamma_n \le d$ *for some real number c, d with $c, d > 0$,*

(iii) $\lambda \in (0, 2d)$, $a \in (0, \frac{2}{L})$,

(iv) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}|, \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Gamma\cap VI(C,D)} f(x_0)$.

Inspired and motivated by Kangtunyakarn [5], we proposed problems more general than (2), i.e. find a point

$$x^* \in \bigcap_{i=1}^{N} C_i \quad \text{and} \quad \Phi\big(x^*\big), \Psi\big(x^*\big) \in \bigcap_{i=1}^{N} Q_i, \tag{3}$$

where $\Phi, \Psi : H_1 \to H_2$ are bounded linear operators. The set of solution of (3) is denoted by $\xi = \{x \in \bigcap_{i=1}^{N} C_i : \Phi(x), \Psi(x) \in \bigcap_{i=1}^{N} Q_i\}$. If we choose $N = 1$ in (3), then (3) is reduced to the general split feasibility problem. Obviously, problem (3) is more general than the problem (2) and the SFP, which can apply across many disciplines in mathematics and sciences, such as economics, finance, network analysis transportation, elasticity, and optimization. In the following sections, we construct a new process using techniques of solving the cutter and SFP to find solutions to problems (3). Moreover, if we put $\Phi \equiv \Psi$ in (3), then we have

$$x^* \in \bigcap_{i=1}^{N} C_i \quad \text{and} \quad \Phi\big(x^*\big) \in \bigcap_{i=1}^{N} Q_i. \tag{4}$$

If we put $p = r$ and $k = 1$ in (1), then problem (1) is reduced to the problem (4). It can be seen that both problems (1) and (3) can be reduced to problem (4). However, we have provided an example of the difference between problems (1) and (3) in Remark 1.

Example 1 Let $\mathbb{R}$ be the set of real numbers and Φ, Ψ be mappings from $\mathbb{R}$ to $\mathbb{R}$ defined by $\Phi(x) = \frac{x}{2}$ and $\Psi(x) = \frac{x}{3}$, respectively. For each $i = 1, 2, \dots, N$, let $C_i = [i, 100i]$ and $Q_i = [\frac{i}{3}, 50i]$. We choose $N = 10$. Then, we have $10 \in \xi$.

Remark 1 Let C_i and Q_i define as the same in Example 1. Let $k = 1, 2$, $A_1(x) = \frac{x}{2}$ and $A_2(x) = \frac{x}{3}$. If we choose $p = 10$ and $r = 20$ in (1), we have $10 \in \bigcap_{i=1}^{10} C_i$ but $A_1(10), A_2(10) \notin \bigcap_{i=1}^{20} Q_i$. This remark shows an example where problem (1) fails while problem (3) is applicable.

Given $x, y \in H$, let

$$H(x,y) := \{z \in H : \langle z - y, x - y\rangle \leq 0\},$$

be the half-space generated by (x, y). The boundary $\partial H(x,y)$ of $H(x,y)$ is

$$\partial H(x,y) = \{z \in H : \langle z - y, x - y\rangle = 0\}.$$

It is clear that $\partial H(x,y)$ is a closed and convex subset of H.

A mapping $T : H \to H$ is called a *cutter* if

$$\langle z - Tx, x - Tx\rangle \leq 0,$$

for all $x \in H$ and $z \in F(T)$. The cutter is fundamental to applied mathematics and optimization theory. For instance, the resolvent of a maximal monotone operator and the subgradient projectors is the cutter. Besides, the metric projection is the cutter with an essential tool for solving the variational inequality problem (VIP), the system of variational inequality problem, the subgradient extragradient method, etc.

Remark 2 Recently, Cegielski and Censor [7] proposed a new name "cutter" expresses the fact that for any $x \notin F(T)$, the hyperplane $H(x - Tx, \langle Tx, x - Tx\rangle)$ cuts the space into two half-spaces, one of which contains the point x while the other one contains the subset $F(T)$.

Over the past decade, many researchers introduced the new problem and iteration developed and modified the cutter; see more detail [8, 9].

The following Remark 3 includes important properties related to the cutter.

Remark 3

(i) If T is firmly nonexpansive, then T is a cutter.

(ii) T is a cutter if and only if $2T - I$ is quasi-nonexpansive (i.e. a mapping $G : H \to H$ is called to be *quasi nonexpansive* if $F(G)$ is nonempty and $\|Gx - y\| \leq \|x - y\|$, for all $x \in H$ and $y \in F(G)$). This property is important for proving Theorem 5 in the Application.

In 2013, Qiao-Li and Songnian [8] introduced a projection regularized Kranoselski-Mann iteration for a cutter $T : H \to H$ as follows:

$$\begin{cases} x_{n+1} = (1 - \beta_n)P_{H(x_n,Tx_n)\cap H(v_n,Tv_n)}v_n + \beta_n Tv_n, \\ v_n = (1 - \alpha_n)x_n, \end{cases} \tag{5}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and the sequence generated by (5) converges strongly to the least norm element of $F(T)$.

By concept Kangtunyakarn [5] and Qiao-Li and Songnian [8], we introduce the new iterative method for proving weak and strong convergence theorem of $\{x_n\}$ generated by the following algorithm:

Algorithm 1.1 Given $x_1 \in H_1$ and let the sequence $\{x_n\}$ be define by

$$\begin{aligned}x_{n+1} &= \eta_n x_n + \alpha_n P_{H(x_n,Tx_n)} x_n + \beta_n T x_n \\ &\quad + \gamma_n \sum_{i=1}^{N} a_i P_{C_i}\left(I - a\left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right)\right) x_n,\end{aligned}$$

where $\Phi, \Psi : H_1 \to H_2$ are bounded linear operators with Φ^*, Ψ^* that are adjoints of Φ and Ψ, respectively, $L = \max\{L_\Phi, L_\Psi\}$ where L_Φ, L_Ψ are spectral radius of $\Phi^*\Phi$ and $\Psi^*\Psi$, $a \in (0, \frac{2}{L})$, $T : H_1 \to H_1$ is a cutter with $I - T$ is demiclosed at 0, and the sequences $\{\eta_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are in $(0, 1)$ with $\eta_n + \alpha_n + \beta_n + \gamma_n = 1$.

Our paper is organized as follows. In Sect. 2, we first recall some basic definitions, and we give the lemma, which is crucial for proving our main results. In Sect. 3, we prove weak and strong convergence theorem for finding a common element of the set of fixed points of the cutter and the finite general split feasibility problem. In Sect. 4, we apply our main theorem to prove a weak and strong convergence theorem for finding solutions to the constrained minimization problem and the general constrained minimization problem. Moreover, we prove weak and strong convergence theorem for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of the finite general split feasibility problem. The last section gives two numerical examples to support our main result.

2 Preliminaries

This section provides a lemma that will be used for our main result in the next section.

We write $x_k \rightharpoonup x$ to indicate that the sequence $\{x_k\}_{k=0}^{\infty}$ converges weakly to x and $x_k \to x$ to indicate that the sequence $\{x_k\}_{k=0}^{\infty}$ converges strongly to x. For each point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$. That is,

$$\left\|x - P_C(x)\right\| \le \|x - y\|, \quad \forall y \in C.$$

The mapping $P_C : H \to C$ is called *the metric projection of H onto C*. It is well known that P_C is a firmly nonexpansive mapping of H onto C. From Remark 3, it obvious that P_C is a cutter. Moreover, if C is a hyperplane, then

$$\|x - y\|^2 \ge \left\|x - P_C(x)\right\|^2 + \left\|y - P_C(x)\right\|^2, \quad \forall x \in H, y \in C.$$

The following lemma is also well-known. It is important for solving VIP.

Lemma 1 ([10]) *If A is a mapping of C into H and $\lambda > 0$, then $F(P_C(I - \lambda A)) = VI(C, A)$.*

Lemma 2 (See [11]) *Let $A, B : C \to H$ be α and β-inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$ and $VI(C, A) \cap VI(C, B) \neq \emptyset$. Then*

$$VI\big(C, aA + (1 - a)B\big) = VI(C, A) \cap VI(C, B), \quad \forall a \in (0, 1).$$

Furthermore, if $0 < \gamma < \min\{2\alpha, 2\beta\}$, we have $I - \gamma(aA + (1 - a)B)$ is a nonexpansive mapping.

Lemma 3 (See [12]) *Let D be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on D. Suppose that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then a mapping S on D defined by*

$$S(x) = \sum_{n=1}^{\infty} \lambda_n T_n x,$$

for $x \in D$ is well defined, nonexpansive, and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ hold.

Lemma 4 (See [13]) *Let $\{\sigma_n\}$ and $\{\gamma_n\}$ be nonnegative sequences satisfying $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\gamma_{n+1} \le \gamma_n + \sigma_n$, for all $n = 1, 2, \dots$. Then $\{\gamma_n\}$ is a convergent sequence.*

Lemma 5 (Demiclosedness principle) *Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C that converges weakly to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in F(T)$.*

Lemma 6 (See [14]) *Let the sequence $\{x_k\}_{k=0}^{\infty} \subset H$ be Fejér-monotone with respect to C, i.e., for every $u \in C$,*

$$\|x_{k+1} - u\| \le \|x_k - u\|, \quad \forall k \ge 0.$$

Then $\{P_C(x_k)\}_{k=0}^{\infty}$ converges strongly to some $z \in C$.

The following lemmas are crucial for proving our main theorem.

Lemma 7 *For each $i = 1, 2, \dots, N$, for all $N \in \mathbb{N}$, let C_i, Q_i be a nonempty closed convex subset of H_1 and H_2, respectively, and let $\Phi, \Psi : H_1 \to H_2$ be bounded linear operators with Φ^*, Ψ^* that are adjoints of Φ and Ψ, respectively, with $\xi \neq \emptyset$. Assume that L_Φ and L_Ψ are spectral radius of $\Phi^*\Phi$ and $\Psi^*\Psi$ with $L = \max\{L_\Phi, L_\Psi\}$, and $\sum_{i=1}^{N} a_i = 1$. Then*

$$\xi = \bigcap_{i=1}^{N} \Gamma_i = F\left(P_{\bigcap_{i=1}^{N} C_i}\left(I - \sum_{i=1}^{N} a_i\left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right)\right)\right).$$

Proof *Firstly*, show that $\xi \subseteq \bigcap_{i=1}^{N} \Gamma_i$.

Let $x^* \in \xi$, we have $x^* \in \bigcap_{i=1}^{N} C_i$ and $\Phi(x^*), \Psi(x^*) \in \bigcap_{i=1}^{N} Q_i$, it follows that

$$x^* \in C_i, \quad \text{for all } i = 1, 2, \dots, N, \tag{6}$$

and

$$\Phi\left(x^*\right), \Psi\left(x^*\right) \in Q_i, \quad \text{for all } i = 1, 2, \dots, N. \tag{7}$$

From (6) and (7), we have

$$x^* \in \Gamma_i, \quad \text{for all } i = 1, 2, \dots, N.$$

Hence, $x^* \in \bigcap_{i=1}^{N} \Gamma_i$.

Secondly, show that $\bigcap_{i=1}^{N}\Gamma_i \subseteq \xi$.

Let $x^* \in \bigcap_{i=1}^{N}\Gamma_i$, then $x^* \in \Gamma_i$, for all $i = 1,2,\dots,N$, we have

$$x^* \in C_i \quad \text{and} \quad \Phi(x^*), \Psi(x^*) \in Q_i, \quad \text{for all } i = 1,2,\dots,N,$$

then $x^* \in \bigcap_{i=1}^{N} C_i$ and $\Phi(x^*), \Psi(x^*) \in \bigcap_{i=1}^{N} Q_i$.

Hence, $x^* \in \xi$.

Thirdly, show that $\xi \subseteq F(P_{\bigcap_{i=1}^{N} C_i}(I - \sum_{i=1}^{N} a_i(\frac{\Phi^*(I-P_{Q_i})\Phi}{2} + \frac{\Psi^*(I-P_{Q_i})\Psi}{2})))$.

Let $x^* \in \xi$, we have $x^* \in \bigcap_{i=1}^{N} C_i$ and $\Phi(x^*), \Psi(x^*) \in \bigcap_{i=1}^{N} Q_i$.

It implies that

$$(I - P_{Q_i})\Phi(x^*) = 0 = (I - P_{Q_i})\Psi(x^*), \quad \text{for all } i = 1,2,\dots,N.$$

Then

$$\frac{(I - P_{Q_i})\Phi(x^*)}{2} = \frac{(I - P_{Q_i})\Psi(x^*)}{2} = 0, \quad \text{for all } i = 1,2,\dots,N.$$

It follows that

$$x^* = P_{\bigcap_{i=1}^{N} C_i}\left(I - \sum_{i=1}^{N} a_i\left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right)\right)x^*.$$

Hence, $x^* \in F(P_{\bigcap_{i=1}^{N} C_i}(I - \sum_{i=1}^{N} a_i(\frac{\Phi^*(I-P_{Q_i})\Phi}{2} + \frac{\Psi^*(I-P_{Q_i})\Psi}{2})))$.

Finally, show that $F(P_{\bigcap_{i=1}^{N} C_i}(I - \sum_{i=1}^{N} a_i(\frac{\Phi^*(I-P_{Q_i})\Phi}{2} + \frac{\Psi^*(I-P_{Q_i})\Psi}{2}))) \subseteq \xi$.

Let $x^* = P_{\bigcap_{i=1}^{N} C_i}(I - \sum_{i=1}^{N} a_i(\frac{\Phi^*(I-P_{Q_i})\Phi}{2} + \frac{\Psi^*(I-P_{Q_i})\Psi}{2}))x^*$, where $\sum_{i=1}^{N} a_i = 1$ and let $w \in \xi$, we have $w \in \bigcap_{i=1}^{N} C_i$ and $\Phi(w), \Psi(w) \in \bigcap_{i=1}^{N} Q_i$.

Then, we have

$$\begin{aligned}
&\|x^* - w\|^2 \\
&\quad = \left\| P_{\bigcap_{i=1}^{N} C_i}\left(I - \sum_{i=1}^{N} a_i\left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right)\right)x^* - w\right\|^2 \\
&\quad \le \left\| \left(I - \sum_{i=1}^{N} a_i\left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right)\right)x^* - w\right\|^2 \\
&\quad = \|x^* - w\|^2 - 2\left\langle x^* - w, \sum_{i=1}^{N} a_i\left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right)x^*\right\rangle \\
&\qquad + \left\| \sum_{i=1}^{N} a_i\left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right)x^*\right\|^2 \\
&\quad \le \|x^* - w\|^2 - \sum_{i=1}^{N} a_i\langle x^* - w, \left(\Phi^*(I - P_{Q_i})\Phi + \Psi^*(I - P_{Q_i})\Psi\right)x^*\rangle \\
&\qquad + \sum_{i=1}^{N} a_i\left\| \left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right)x^*\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \|x^* - w\|^2 \\
&\quad - \sum_{i=1}^{N} a_i \big(\langle \Phi(x^*) - \Phi(w), (I - P_{Q_i})\Phi(x^*) \rangle + \langle \Psi(x^*) - \Psi(w), (I - P_{Q_i})\Psi(x^*) \rangle \big) \\
&\quad + \frac{\sum_{i=1}^{N} a_i}{2} \big(\|\Phi^*(I - P_{Q_i})\Phi(x^*)\|^2 + \|\Psi^*(I - P_{Q_i})\Psi(x^*)\|^2 \big) \\
&= \|x^* - w\|^2 \\
&\quad - \sum_{i=1}^{N} a_i \big(\langle \Phi(x^*) - P_{Q_i}\Phi(x^*), (I - P_{Q_i})\Phi(x^*) \rangle + \langle P_{Q_i}\Phi(x^*) - \Phi(w), (I - P_{Q_i})\Phi(x^*) \rangle \\
&\quad + \langle \Psi(x^*) - P_{Q_i}\Psi(x^*), (I - P_{Q_i})\Psi(x^*) \rangle + \langle P_{Q_i}\Psi(x^*) - \Psi(w), (I - P_{Q_i})\Psi(x^*) \rangle \big) \\
&\quad + \frac{\sum_{i=1}^{N} a_i}{2} \big(\|\Phi^*(I - P_{Q_i})\Phi(x^*)\|^2 + \|\Psi^*(I - P_{Q_i})\Psi(x^*)\|^2 \big) \\
&\leq \|x^* - w\|^2 - \sum_{i=1}^{N} a_i \big(\|(I - P_{Q_i})\Phi(x^*)\|^2 + \|(I - P_{Q_i})\Psi(x^*)\|^2 \big) \\
&\quad + \frac{L\sum_{i=1}^{N} a_i}{2} \big(\|(I - P_{Q_i})\Phi(x^*)\|^2 + \|(I - P_{Q_i})\Psi(x^*)\|^2 \big) \\
&= \|x^* - w\|^2 - \left(1 - \frac{L}{2}\right) \sum_{i=1}^{N} a_i \big(\|(I - P_{Q_i})\Phi(x^*)\|^2 + \|(I - P_{Q_i})\Psi(x^*)\|^2 \big).
\end{aligned}
$$

It implies that $\Phi(x^*) = P_{Q_i}\Phi(x^*)$ and $\Psi(x^*) = P_{Q_i}\Psi(x^*) \in \bigcap_{i=1}^{N} Q_i$.

It follows that

$$
x^* = P_{\bigcap_{i=1}^{N} C_i} \left(I - \sum_{i=1}^{N} a_i \left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2} \right) \right) x^* = P_{\bigcap_{i=1}^{N} C_i} x^* \in \bigcap_{i=1}^{N} C_i.
$$

Hence, $x^* \in \xi$. □

3 Main results

In this section, we prove weak and strong convergence theorem for finding a common element of the set of fixed points of the cutter and the finite general split feasibility problem.

Theorem 3 *For every $i = 1, 2, \dots, N$, let C_i, Q_i, Φ, Ψ, Φ^*, and Ψ^* define as the same in Lemma 7. Let $T : H_1 \to H_1$ be a cutter with $\varphi = F(T) \cap \xi \neq \emptyset$ and $I - T$ is demiclosed at 0. For given $x_1 \in H_1$ and let the sequence $\{x_n\}$ be generated by*

$$
\begin{aligned}
x_{n+1} &= \eta_n x_n + \alpha_n P_{H(x_n, Tx_n)} x_n + \beta_n T x_n \\
&\quad + \gamma_n \sum_{i=1}^{N} a_i P_{C_i} \left(I - a \left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2} \right) \right) x_n,
\end{aligned} \tag{8}
$$

for all $n, N \in \mathbb{N}$, where $\{\eta_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0, 1)$ with $\eta_n + \alpha_n + \beta_n + \gamma_n = 1$, $a \in (0, \frac{2}{L})$, and parameters L, L_Φ, L_Ψ define as the same in Lemma 7. Suppose that the following conditions hold:

(i) *$c \leq \eta_n, \alpha_n, \beta_n, \gamma_n \leq d$ for some real number c, d with $c, d > 0$,*

(ii) $\sum_{i=1}^{N} a_i = 1$, *where* $a_i > 0$ *for all* $N \in \mathbb{N}$.

Then, the sequence $\{x_n\}$ *converges weakly to* $z^* \in \varphi$ *and furthermore,*

$$z^* = \lim_{n\to\infty} P_\varphi(x_n).$$

Proof Putting $\nabla g_i = \frac{\Phi^*(I-P_{Q_i})\Phi}{2} + \frac{\Psi^*(I-P_{Q_i})\Psi}{2}$, for all $i = 1,2,\dots,N$. First, we show that ∇g_i are $\frac{1}{L}$-inverse strongly monotone.

Let $x, y \in C_i$. Since $\nabla g_i = \frac{\Phi^*(I-P_{Q_i})\Phi}{2} + \frac{\Psi^*(I-P_{Q_i})\Psi}{2}$, for all $i = 1,2,\dots,N$, we have

$$\begin{aligned}
&\bigl\|\nabla g_i(x) - \nabla g_i(y)\bigr\|^2 \\
&\quad = \left\|\frac{\Phi^*(I-P_{Q_i})\Phi(x)}{2} + \frac{\Psi^*(I-P_{Q_i})\Psi(x)}{2} - \frac{\Phi^*(I-P_{Q_i})\Phi(y)}{2} - \frac{\Psi^*(I-P_{Q_i})\Psi(y)}{2}\right\|^2 \\
&\quad \le \frac{1}{2}\bigl\|\Phi^*(I-P_{Q_i})\Phi(x) - \Phi^*(I-P_{Q_i})\Phi(y)\bigr\|^2 \\
&\qquad + \frac{1}{2}\bigl\|\Psi^*(I-P_{Q_i})\Psi(x) - \Psi^*(I-P_{Q_i})\Psi(y)\bigr\|^2 \\
&\quad \le \frac{L}{2}\bigl\|(I-P_{Q_i})\Phi(x) - (I-P_{Q_i})\Phi(y)\bigr\|^2 + \frac{L}{2}\bigl\|(I-P_{Q_i})\Psi(x) - (I-P_{Q_i})\Psi(y)\bigr\|^2.
\end{aligned} \tag{9}$$

For each $i = 1,2,\dots,N$. From property of P_{Q_i}, we have

$$\begin{aligned}
&\bigl\|(I-P_{Q_i})\Phi(x) - (I-P_{Q_i})\Phi(y)\bigr\|^2 \\
&\quad = \bigl\langle (I-P_{Q_i})\Phi(x) - (I-P_{Q_i})\Phi(y), \Phi(x) - \Phi(y) - \bigl(P_{Q_i}\Phi(x) - P_{Q_i}\Phi(y)\bigr)\bigr\rangle \\
&\quad = \bigl\langle \Phi^*(I-P_{Q_i})\Phi(x) - \Phi^*(I-P_{Q_i})\Phi(y), x-y\bigr\rangle \\
&\qquad - \bigl\langle (I-P_{Q_i})\Phi(x), P_{Q_i}\Phi(x) - P_{Q_i}\Phi(y)\bigr\rangle + \bigl\langle (I-P_{Q_i})\Phi(y), P_{Q_i}\Phi(x) - P_{Q_i}\Phi(y)\bigr\rangle \\
&\quad \le \bigl\langle \Phi^*(I-P_{Q_i})\Phi(x) - \Phi^*(I-P_{Q_i})\Phi(y), x-y\bigr\rangle.
\end{aligned} \tag{10}$$

Using the same method as (10), we have

$$\bigl\|(I-P_{Q_i})\Psi(x) - (I-P_{Q_i})\Psi(y)\bigr\|^2 \le \bigl\langle \Psi^*(I-P_{Q_i})\Psi(x) - \Psi^*(I-P_{Q_i})\Psi(y), x-y\bigr\rangle. \tag{11}$$

Substituting (10) and (11) into (9), we have

$$\begin{aligned}
&\bigl\|\nabla g_i(x) - \nabla g_i(y)\bigr\|^2 \\
&\quad \le \frac{L}{2}\bigl\|(I-P_{Q_i})\Phi(x) - (I-P_{Q_i})\Phi(y)\bigr\|^2 + \frac{L}{2}\bigl\|(I-P_{Q_i})\Psi(x) - (I-P_{Q_i})\Psi(y)\bigr\|^2 \\
&\quad \le \frac{L}{2}\bigl\langle \Phi^*(I-P_{Q_i})\Phi(x) - \Phi^*(I-P_{Q_i})\Phi(y), x-y\bigr\rangle \\
&\qquad + \frac{L}{2}\bigl\langle \Psi^*(I-P_{Q_i})\Psi(x) - \Psi^*(I-P_{Q_i})\Psi(y), x-y\bigr\rangle \\
&\quad = L\biggl\langle \frac{\Phi^*(I-P_{Q_i})\Phi(x)}{2} + \frac{\Psi^*(I-P_{Q_i})\Psi(x)}{2} \\
&\qquad - \biggl(\frac{\Phi^*(I-P_{Q_i})\Phi(y)}{2} + \frac{\Psi^*(I-P_{Q_i})\Psi(y)}{2}\biggr), x-y\biggr\rangle
\end{aligned}$$

$$= L\langle \nabla g_i(x) - \nabla g_i(y), x - y\rangle.$$

So, we have ∇g_i is $\frac{1}{L}$-inverse strongly monotone.

For each $i = 1, 2, \ldots, N$. From the definition of ∇g_i, we have

$$\begin{aligned}
&\left\|P_{C_i}(I - a\nabla g_i)x - P_{C_i}(I - a\nabla g_i)y\right\|^2 \\
&\quad\leq \left\|x - y - a\big(\nabla g_i(x) - \nabla g_i(y)\big)\right\|^2 \\
&\quad= \|x - y\|^2 - 2a\langle x - y, \nabla g_i(x) - \nabla g_i(y)\rangle + a^2\left\|\nabla g_i(x) - \nabla g_i(y)\right\|^2 \\
&\quad\leq \|x - y\|^2 - \frac{2a}{L}\left\|\nabla g_i(x) - \nabla g_i(y)\right\|^2 + a^2\left\|\nabla g_i(x) - \nabla g_i(y)\right\|^2 \\
&\quad= \|x - y\|^2 - a\left(\frac{2}{L} - a\right)\left\|\nabla g_i(x) - \nabla g_i(y)\right\|^2 \\
&\quad\leq \|x - y\|^2,
\end{aligned} \tag{12}$$

for all $x, y \in C_i$.

Let $z \in F(T) \cap \xi$.

Step 1. We show that $\{x_n\}$ is bounded.

From (8), (12), and Lemma 3, we have

$$\begin{aligned}
&\|x_{n+1} - z\| \\
&\quad= \left\|\eta_n x_n + \alpha_n P_{H(x_n,Tx_n)}x_n + \beta_n Tx_n + \gamma_n \sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)x_n - z\right\| \\
&\quad\leq \eta_n\|x_n - z\| + \alpha_n\|P_{H(x_n,Tx_n)}x_n - z\| + \beta_n\|Tx_n - z\| \\
&\qquad+ \gamma_n\left\|\sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)x_n - z\right\| \\
&\quad\leq \eta_n\|x_n - z\| + \alpha_n\|x_n - z\| + \beta_n\|x_n - z\| + \gamma_n\left\|\sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)x_n - z\right\| \\
&\quad= (\eta_n + \alpha_n + \beta_n)\|x_n - z\| + \gamma_n\left\|\sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)x_n - \sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)z\right\| \\
&\quad= (\eta_n + \alpha_n + \beta_n)\|x_n - z\| + \gamma_n\sum_{i=1}^{N} a_i\left\|P_{C_i}(I - a\nabla g_i)x_n - P_{C_i}(I - a\nabla g_i)z\right\| \\
&\quad\leq (\eta_n + \alpha_n + \beta_n)\|x_n - z\| + \gamma_n\|x_n - z\| \\
&\quad= \|x_n - z\|,
\end{aligned} \tag{13}$$

then $\{x_n\}$ is Fejér monotone with respect to φ, for all $z \in \varphi$.

Applying Lemma 4, we have that $\lim_{n\to\infty}\|x_n - z\|$ exists. In particular, this implies that $\{x_n\}$ is bounded.

Step 2. We show that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - \sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)x_n\| = 0$.

From (8) and (12), we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= \left\| \eta_n x_n + \alpha_n P_{H(x_n,Tx_n)} x_n + \beta_n T x_n + \gamma_n \sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)x_n - z \right\|^2 \\
&= \left\| \eta_n (x_n - z) + \alpha_n (P_{H(x_n,Tx_n)} x_n - z) + \beta_n (T x_n - z) \right. \\
&\quad \left. + \gamma_n \left(\sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)x_n - z \right) \right\|^2 \\
&\leq \|x_n - z\|^2 - \eta_n \beta_n \|x_n - Tx_n\|^2 - \eta_n \gamma_n \left\| x_n - \sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)x_n \right\|^2,
\end{aligned}$$

which yields that

$$\eta_n \beta_n \|x_n - Tx_n\|^2 + \eta_n \gamma_n \left\| x_n - \sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)x_n \right\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2. \quad (14)$$

From (14) and $\lim_{n\to\infty}(\|x_n - z\|^2 - \|x_{n+1} - z\|^2) = 0$, then

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \left\| x_n - \sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)x_n \right\| = 0. \quad (15)$$

Step 3. We show that the sequences $\{x_n\}$ converge weakly to $z^* \in \varphi$.

Since $\{x_n\}$ is bounded by Step 1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some element of $\bar{z}$.

By Lemma 5 and (15), we obtain

$$\bar{z} = T\bar{z},$$

then

$$\bar{z} \in F(T). \quad (16)$$

Assume that $\bar{z} \notin \xi$.

By Lemma 7 and Lemma 3, we also have $\bar{z} \neq \sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)\bar{z}$.

From (12), (15), and Opial's property, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \bar{z}\| &< \liminf_{k\to\infty} \left\| x_{n_k} - \sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)\bar{z} \right\| \\
&\leq \liminf_{k\to\infty} \left(\left\| x_{n_k} - \sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)x_{n_k} \right\| \right. \\
&\quad \left. + \left\| \sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)x_{n_k} - \sum_{i=1}^{N} a_i P_{C_i}(I - a\nabla g_i)\bar{z} \right\| \right)
\end{aligned}$$

$$\leq \liminf_{k\to\infty} \|x_{n_k} - \bar{z}\|.$$

This is a contradiction, then we have

$$\bar{z} \in \xi. \tag{17}$$

From (16) and (17), thus

$$\bar{z} \in \varphi.$$

Next, we will show that the entire sequence $\{x_n\}$ weakly converges to $\bar{z}$.

Since $\{x_n\}$ is bounded by Step 1, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to some element of $\bar{z}'$. Assume that $x_{n_j} \rightharpoonup \bar{z}'$ as $j \to \infty$, with $\bar{z}' \neq \bar{z}$ and $\bar{z}' \in \varphi$.

By the Opial condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \bar{z}\| &= \liminf_{k\to\infty} \|x_{n_k} - \bar{z}\| \\
&< \liminf_{k\to\infty} \left\|x_{n_k} - \bar{z}'\right\| \\
&= \lim_{n\to\infty} \left\|x_n - \bar{z}'\right\| \\
&< \liminf_{j\to\infty} \|x_{n_j} - \bar{z}\| \\
&= \lim_{n\to\infty} \|x_n - \bar{z}\|,
\end{aligned}$$

and this is a contradiction, thus $\bar{z}' = \bar{z}$. This implies that the sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to the same point $\bar{z} \in \varphi$.

Finally, if we take

$$u_n = P_\varphi x_n,$$

then by (13) and Lemma 6, we see that $\{P_\varphi x_n\}_{n=0}^{\infty}$ converges strongly to some $z^* \in \varphi$.

Since P_φ is a cutter and the convergences of $\{x_n\}$ and $\{u_n\}$, we have

$$\left\langle \bar{z} - z^*, z^* - \bar{z}\right\rangle \geq 0,$$

and hence $z^* = \bar{z}$, this completes the proof. □

Corollary 1 *For every $i = 1, 2, \dots, N$, let C_i, Q_i, Φ, Φ^*, and T define as the same in Theorem 3. Assume that $\varphi = F(T) \cap \Gamma^\Phi \neq \emptyset$, where $\Gamma^\Phi = \{x \in \bigcap_{i=1}^N C_i | \Phi(x) \in \bigcap_{i=1}^N Q_i, \forall i = 1, 2, \dots, N\}$. For given $x_1 \in H_1$ and let the sequence $\{x_n\}$ be generated by*

$$x_{n+1} = \eta_n x_n + \alpha_n P_{H(x_n, Tx_n)} x_n + \beta_n T x_n + \gamma_n \sum_{i=1}^{N} a_i P_{C_i}\bigl(I - a\bigl(\Phi^*(I - P_{Q_i})\Phi\bigr)\bigr)x_n, \tag{18}$$

for all $n, N \in \mathbb{N}$, where $\{\eta_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0,1)$ with $\eta_n + \alpha_n + \beta_n + \gamma_n = 1$, $a \in (0, \frac{2}{L})$, and L is spectral radius of $\Phi^*\Phi$. Suppose that the following conditions hold:

(i) *$c \leq \eta_n, \alpha_n, \beta_n, \gamma_n \leq d$ for some real number c, d with $c, d > 0$,*
(ii) *$\sum_{i=1}^N a_i = 1$, where $a_i > 0$ for all $N \in \mathbb{N}$.*

Then, the sequence $\{x_n\}$ *converges weakly to* $z^* \in \varphi$ *and furthermore,*

$$z^* = \lim_{n\to\infty} P_\varphi(x_n).$$

Proof Putting $\Phi \equiv \Psi$, in Theorem 3, we obtain the desired conclusion. □

The following corollary is a modification in terms of the iterative process of Theorem 1.

Corollary 2 *For every* $i = 1,2,\dots,N$, *let* C_i, Q_i, Φ, Ψ, Φ^*, Ψ^*, T, *all parameters, and the conditions* (*i*) *and* (*ii*) *define as the same in Theorem* 3. *Assume that* $\varphi = F(T) \cap \xi \neq \emptyset$, *and* $I - T$ *is demiclosed at* 0. *For given* $x_1 \in H_1$ *and let the sequence* $\{x_n\}$ *be generated by*

$$\begin{aligned} x_{n+1} &= \eta_n x_n + \alpha_n P_{H(x_n,Tx_n)} x_n + \beta_n T x_n + \gamma_n \left(\sum_{i=1}^{N} a_i t P_{C_i}\big(I - a\big(\Phi^*(I - P_{Q_i})\Phi\big)\big) \right. \\ &\quad \left. + \sum_{i=1}^{N} a_i (1-t) P_{C_i}\big(I - a\big(\Psi^*(I - P_{Q_i})\Psi\big)\big) \right) x_n, \end{aligned}$$

for all $n, N \in \mathbb{N}$. *where* $\{\eta_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0,1)$ *with* $\eta_n + \alpha_n + \beta_n + \gamma_n = 1$, $t \in (0,1)$, $a \in (0, \frac{2}{L})$, *and parameters* L, L_Φ, L_Ψ *define as the same in Lemma* 7. *Suppose the following conditions hold*:

(i) $c \leq \eta_n, \alpha_n, \beta_n, \gamma_n \leq d$ *for some real number* c, d *with* $c, d > 0$,

(ii) $\sum_{i=1}^{N} a_i = 1$, *where* $a_i > 0$ *for all* $N \in \mathbb{N}$.

Then, the sequence $\{x_n\}$ *converges weakly to* $z^* \in \varphi$ *and furthermore,*

$$z^* = \lim_{n\to\infty} P_\varphi(x_n).$$

Proof For each $i = 1,2,\dots,N$. From Lemma 2, then we get

$$\begin{aligned} \xi &\equiv F\left(P_{C_i}\left(I - a\left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right)\right)\right) \\ &= VI\left(C_i, \frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right) \\ &= VI\big(C_i, \Phi^*(I - P_{Q_i})\Phi\big) \cap VI\big(C_i, \Psi^*(I - P_{Q_i})\Psi\big) \\ &= F\big(P_{C_i}\big(I - \lambda_i\big(\Phi^*(I - P_{Q_i})\Phi\big)\big)\big) \cap F\big(P_{C_i}\big(I - \lambda_i\big(\Psi^*(I - P_{Q_i})\Psi\big)\big)\big), \end{aligned}$$

for all $N \in \mathbb{N}$ and $\lambda_i > 0$. Applying the above and Theorem 3, we obtain the desired conclusion. □

4 Application

4.1 The general constrained minimization problem

Let $\Phi : H_1 \to H_2$ be bounded linear operator, and let $g : H_1 \to \mathbb{R}$ be a continuous differentiable function. The minimization problem:

$$\min_{x\in C} g(x) := \frac{1}{2}\big\|(I - P_Q)\Phi(x)\big\|^2,$$

is to find a point $x^* \in C$ such that $g(x^*) \leq g(x)$, for all $x \in C$.

In 2019, Kangtunyakarn [5] introduced *the general constrained minimization problem* as follows:

$$\min_{x\in C} g(x) := \frac{\|(I-P_Q)\Phi(x)\|^2}{4} + \frac{\|(I-P_Q)\Psi(x)\|^2}{4}. \tag{19}$$

The set of all solution of (19) is denoted by $\Gamma_g = \{x^* \in C : g(x^*) \le g(x), \forall x \in C\}$.

Lemma 8, Kangtunyakarn [5] shows the relationship between the general split feasibility problem and the general constrained minimization problem.

Lemma 8 (See [5]) *Let Φ, Ψ, Φ^* and Ψ^* define as the same in Lemma 7. Let $g : H_1 \to \mathbb{R}$ be a continuous differentiable function defined by $g(x) = \frac{\|(I-P_Q)\Phi(x)\|^2}{4} + \frac{\|(I-P_Q)\Psi(x)\|^2}{4}$, for all $x \in H_1$. If $\Gamma \neq \emptyset$, then $\Gamma = \Gamma_g$.*

Remark 4 From Lemma 7 and Lemma 8, we have $\xi = \bigcap_{i=1}^N \Gamma_{g_i}$.

Theorem 4 *For each $i = 1,2,\dots,N$, let C_i, Q_i, Φ, Ψ, Φ^*, and Ψ^* define as the same in Lemma 7. Let the function $g_i : H_1 \to \mathbb{R}$ be differentiable continuous function defined by $g_i(x) = \frac{\|(I-P_{Q_i})\Phi(x)\|^2}{4} + \frac{\|(I-P_{Q_i})\Psi(x)\|^2}{4}$, for all $i = 1,2,\dots,N$. Let $T : H_1 \to H_1$ be a cutter with $\varphi = F(T) \cap \bigcap_{i=1}^N \Gamma g_i \neq \emptyset$ and $I - T$ is demiclosed at 0. For given $x_1 \in H_1$ and let the sequence $\{x_n\}$ be generated by*

$$x_{n+1} = \eta_n x_n + \alpha_n P_{H(x_n,Tx_n)} x_n + \beta_n T x_n + \gamma_n \sum_{i=1}^N a_i P_{C_i}(I - a\nabla g_i)x_n, \tag{20}$$

for all $n, N \in \mathbb{N}$, where $\{\eta_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0,1)$ with $\eta_n + \alpha_n + \beta_n + \gamma_n = 1$, $a \in (0, \frac{2}{L})$, and parameters L, L_Φ, L_Ψ define as the same in Lemma 7. Suppose the following conditions hold:

(i) *$c \le \eta_n, \alpha_n, \beta_n, \gamma_n \le d$ for some real number c, d with $c, d > 0$,*

(ii) *$\sum_{i=1}^N a_i = 1$, where $a_i > 0$ for all $N \in \mathbb{N}$.*

Then, the sequence $\{x_n\}$ converges weakly to $z^* \in \varphi$ and furthermore,

$$z^* = \lim_{n\to\infty} P_\varphi(x_n).$$

Proof We observe that $\nabla g_i = \frac{\|(I-P_{Q_i})\Phi\|^2}{2} + \frac{\|(I-P_{Q_i})\Psi\|^2}{2}$, where Φ^* and Ψ^* are adjoint of Φ and Ψ, respectively, and ∇g_i is a gradient of g_i. From Remark 4.2 [5], we have $\bigcap_{i=1}^N \Gamma g_i = \bigcap_{i=1}^N VI(C, \nabla g_i)$. By Theorem 3, Lemma 4, and Remark 4.2 [5], we can conclude Theorem 4. □

Corollary 3 *For each $i = 1,2,\dots,N$, let C_i, Q_i, Φ, and Φ^* define as the same in Lemma 7. Let the function $g_i : H_1 \to \mathbb{R}$ be differentiable continuous function defined by $g_i(x) = \frac{\|(I-P_{Q_i})\Phi(x)\|^2}{2}$, for all $i = 1,2,\dots,N$. Let $T : H_1 \to H_1$ is a cutter with $\varphi = F(T) \cap \bigcap_{i=1}^N \Gamma g_i \neq \emptyset$, and $I - T$ is demiclosed at 0. For given $x_1 \in H_1$ and let the sequence $\{x_n\}$ be generated by*

$$x_{n+1} = \eta_n x_n + \alpha_n P_{H(x_n,Tx_n)} x_n + \beta_n T x_n + \gamma_n \sum_{i=1}^N a_i P_{C_i}(I - a\nabla g_i)x_n, \tag{21}$$

for all $n, N \in \mathbb{N}$, where $\{\eta_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0,1)$ with $\eta_n + \alpha_n + \beta_n + \gamma_n = 1$, $a \in (0, \frac{2}{L_\Phi})$, and L_Φ is spectral radius of $\Phi^*\Phi$. Suppose the following conditions hold:

(i) $c \leq \eta_n, \alpha_n, \beta_n, \gamma_n \leq d$ *for some real number c, d with $c, d > 0$,*

(ii) $\sum_{i=1}^{N} a_i = 1$, *where $a_i > 0$ for all $N \in \mathbb{N}$.*

Then, the sequence $\{x_n\}$ converges weakly to $z^* \in \varphi$ and furthermore,

$$z^* = \lim_{n\to\infty} P_\varphi(x_n).$$

Proof Putting $\Phi \equiv \Psi$, in Theorem 4, we obtain the desired conclusion. □

Theorem 5 *For each $i = 1, 2, \ldots, N$, let C_i, Q_i, Φ, Ψ, Φ^*, and Ψ^* define as the same in Lemma 7. Let $U : H_1 \to H_1$ be a quasi nonexpansive mapping with $\varphi = F(U) \cap \xi \neq \emptyset$, and $I - U$ is demiclosed at 0. For given $x_1 \in H_1$ and let the sequence $\{x_n\}$ be generated by*

$$\begin{aligned} x_{n+1} &= \eta_n x_n + \alpha_n P_{H(x_n, \frac{1}{2}(U+I)x_n)} x_n + \beta_n \left(\frac{1}{2}(U+I)\right) x_n \\ &\quad + \gamma_n \sum_{i=1}^{N} a_i P_{C_i}\left(I - a\left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right)\right) x_n, \end{aligned} \tag{22}$$

for all $n, N \in \mathbb{N}$, where $\{\eta_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0,1)$ with $\eta_n + \alpha_n + \beta_n + \gamma_n = 1$, $a \in (0, \frac{2}{L})$, and parameters L, L_Φ, L_Ψ define as the same in Lemma 7. Suppose the following conditions hold:

(i) $c \leq \eta_n, \alpha_n, \beta_n, \gamma_n \leq d$ *for some real number c, d with $c, d > 0$,*

(ii) $\sum_{i=1}^{N} a_i = 1$, *where $a_i > 0$ for all $N \in \mathbb{N}$.*

Then, the sequence $\{x_n\}$ converges weakly to $z^* \in \varphi$ and furthermore,

$$z^* = \lim_{n\to\infty} P_\varphi(x_n).$$

Proof From Remark 3 (ii) and Theorem 3, we can conclude Theorem 5. □

4.2 The constrained minimization problem

Since the proximity operator is related to the minimization problem, that is $F(Prox_f) = \arg\min f$, and proximal mapping of f is a special case of the cutter, with such a relationship, we will prove Theorem 6, but let us first recall the definition and the critical lemma of the proximity operator as follows:

Definition 2 Let $f : H \to (-\infty, +\infty)$ and let $x \in H$. Then $Prox_f x$ is the unique point in H that satisfies

$$f(x) = \min_{y\in H}\left(f(y) + \frac{1}{2}\|x - y\|^2\right) = f(Prox_f x) + \frac{1}{2}\|x - Prox_f x\|^2.$$

The operator $Prox_f : H \to H$ is the *proximity operator* or *proximal mapping* of f.

Lemma 9 (See [15]) *Let $f : H \to (-\infty, +\infty)$ be the set of proper lower semicontinuous convex functions. Then, $Prox_f$ and $I - Prox_f$ are firmly nonexpansive.*

Lemma 10 (See [15]) *Let $f : H \to (-\infty, +\infty)$ be the set of proper lower semicontinuous convex functions. Then, $F(Prox_f) = \arg\min f$.*

Using the relationship of proximal mapping and cutter, we prove weak and strong convergence theorem for finding solutions to the proximal problem and common element of the set of the finite general split feasibility problem.

Theorem 6 *For each $i = 1, 2, \dots, N$, let C_i, Q_i, Φ, Ψ, Φ^*, and Ψ^* define as the same in Lemma 7. Let $f : H \to (-\infty, +\infty)$ be the set of proper lower semicontinuous convex functions and assume that $Prox_f : H \to H$ is the proximal mapping of f with $\varphi = \arg\min f \cap \xi \neq \emptyset$, and $I - Prox_f$ is demiclosed at 0. For given $x_1 \in H_1$ and let the sequence $\{x_n\}$ be generated by*

$$\begin{aligned} x_{n+1} &= \eta_n x_n + \alpha_n P_{H(x_n, Prox_f(x_n))} x_n + \beta_n Prox_f(x_n) \\ &\quad + \gamma_n \sum_{i=1}^{N} a_i P_{C_i}\left(I - a\left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right)\right)x_n, \end{aligned} \tag{23}$$

for all $n, N \in \mathbb{N}$, where $\{\eta_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0, 1)$ with $\eta_n + \alpha_n + \beta_n + \gamma_n = 1$, $a \in (0, \frac{2}{L})$, and parameters L, L_Φ, L_Ψ define as the same in Lemma 7. Suppose the following conditions hold:

(i) *$c \leq \eta_n, \alpha_n, \beta_n, \gamma_n \leq d$ for some real number c, d with $c, d > 0$,*
(ii) *$\sum_{i=1}^{N} a_i = 1$, where $a_i > 0$ for all $N \in \mathbb{N}$.*

Then, the sequence $\{x_n\}$ converges weakly to $z^* \in \varphi$ and furthermore,

$$z^* = \lim_{n \to \infty} P_\varphi(x_n).$$

Proof From Lemma 9, Lemma 10, and Remark 3, we can conclude Theorem 6. □

5 Numerical examples

In this section, we give the following examples to support our main theorem.

Example 2 Let $\mathbb{R}$ be the set of real numbers. Let $H_1 = H_2 = \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\}$, and let $\langle \cdot, \cdot \rangle : \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ be an inner product defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2$, where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\}$ and $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\}$ and a usual norm $\|\cdot\| : \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ be defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$ where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\}$. For each $i = 1, 2, \dots, N$, let $C_i = [-i - 1, i + 1] \times [-i - 2, i + 2]$ and $Q_i = [1 - i, i + 2] \times [1 - i, i + 3]$. Let the mapping $\Phi, \Psi : \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\}$ defined by

$$\Phi(\mathbf{x}) = (x_1 + 2x_2, 2x_1 + x_2),$$

and

$$\Psi(\mathbf{x}) = (x_1 + x_2, x_1 + x_2),$$

for all $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\}$ and $\Phi^*, \Psi^* : \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\}$ defined by

$$\Phi^*(\mathbf{z}) = (z_1 + 2z_2, 2z_1 + z_2),$$

and

$$\Psi^*(\mathbf{z}) = (z_1 + z_2, z_1 + z_2),$$

for all $\mathbf{z} = (z_1, z_2) \in \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\}$. Let $T\mathbf{x} = (|x_1|, |x_2|)$ where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^+ \cup \{0\} \times \mathbb{R}^+ \cup \{0\}$. The sequence $\{x_n\}$ is generated by (8), where $\eta_n = \frac{2n+1}{10n}$, $\alpha_n = \frac{5n-4}{10n}$, $\beta_n = \frac{1}{10n}$ and $\gamma_n = \frac{3n+2}{10n}$ for all $n \in \mathbb{N}$. Since $L_\Phi = 9$ and $L_\Psi = 4$, we have $L = 9$. So, we choose $a = \frac{1}{5}$. From Theorem 3, we can conclude that the sequence $\{x_n\}$ converges to $(0,0)$. We can rewrite (8) as follows:

$$\begin{aligned} x_{n+1} &= \left(\frac{2n+1}{10n}\right)(x_n) + \left(\frac{5n-4}{10n}\right) P_{H(x_n, Tx_n)} x_n + \left(\frac{1}{10n}\right) T x_n \\ &\quad + \left(\frac{3n+2}{10n}\right) \sum_{i=1}^{N} \left(\frac{1}{2^i} + \frac{1}{N2^N}\right) P_{C_i} \left(I - \frac{1}{5}\left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right)\right) x_n, \end{aligned}$$

for all $n \geq 1$, where $x_n = (x_n^1, x_n^2)$, $P_{C_i}(x_1, x_2) = (\max\{\min\{x_1, i+1\}, -i-1\}, \max\{\min\{x_2, i+2\}, -i-2\})$ and $P_{Q_i}(x_1, x_2) = (\max\{\min\{x_1, i+2\}, 1-i\}, \max\{\min\{x_2, i+3\}, 1-i\})$.

The Table 1 and Fig. 1 show the values of $\{x_n\}$ with $x_1 = (-100, -100)$ and $n = N = 15$.

Table 1 The values of $\{x_n\}$ with $x_1 = (-100, -100)$ and $n = N = 15$

n	$x_n = (x_n^1, x_n^2)$
1	(−100.0000, −100.0000)
2	(−20.0000, −20.0000)
3	(−3.6000, −3.4000)
⋮	⋮
7	(−0.0162, −0.0053)
⋮	⋮
13	(−0.0001, 0.0006)
14	(−0.0001, 0.0006)
15	(0.0000, 0.0006)

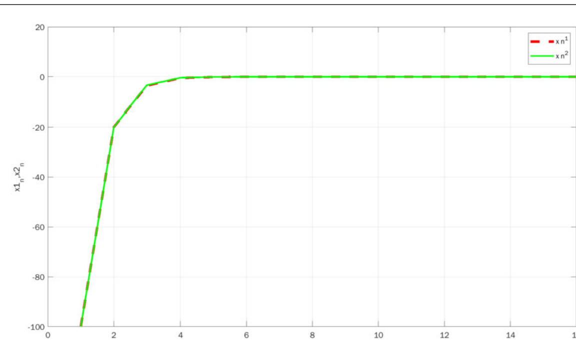

Figure 1 The convergence of $\{x_n\}$ with $x_1 = (-100, -100)$ and $n = N = 15$

Example 3 Let $\mathbb{R}$ be the set of real numbers. Let $H_1 = H_2 = \mathbb{R}^2$ and let $\langle\cdot,\cdot\rangle : \mathbb{R}^2 \to \mathbb{R}$ be an inner product defined by $\langle \mathbf{x},\mathbf{y}\rangle = \mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2$ where $\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2$ and $\mathbf{y} = (y_1,y_2) \in \mathbb{R}^2$ and a usual norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$ where $\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2$. For each $i = 1,2$, let $C_i = H(\alpha_i,\beta) = \{\mathbf{r} \in H_1 : \langle (2^i,3^i),\mathbf{r}\rangle = 0\}$ where $\mathbf{r} = (r_1,r_2) \in \mathbb{R}^2$ and $Q_i = \overline{H}(\bar{\alpha}_i,\bar{\beta}) = \{\bar{\mathbf{r}} \in H_2 : \langle ((-2)^i,2^i),\bar{\mathbf{r}}\rangle = 0\}$ where $\bar{\mathbf{r}} = (\bar{r}_1,\bar{r}_2) \in \mathbb{R}^2$. Let the mapping $\Phi,\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\Phi(\mathbf{x}) = (x_1 + 2x_2, 2x_1 + x_2),$$

and

$$\Psi(\mathbf{x}) = (x_1 + x_2, x_1 + x_2),$$

for all $\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2$ and $\Phi^*,\Psi^* : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\Phi^*(\mathbf{z}) = (z_1 + 2z_2, 2z_1 + z_2),$$

and

$$\Psi^*(\mathbf{z}) = (z_1 + z_2, z_1 + z_2),$$

for all $\mathbf{z} = (z_1,z_2) \in \mathbb{R}^2$. Let $T\mathbf{x} = (x_1,x_2)$ where $\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2$. The sequence $\{x_n\}$ is generated by (8), where $\eta_n = \frac{1}{3n}$, $\alpha_n = \frac{2n-1}{6n}$, $\beta_n = \frac{3n-2}{6n}$ and $\gamma_n = \frac{n+1}{6n}$. From the definition of Φ, Ψ, Φ^*, Ψ^* and T, we have $F(T) \cap \xi = (0,0)$. From Theorem 3, we can conclude that the sequence $\{x_n\}$ converges to $(0,0)$. We can rewrite (8) as follows:

$$\begin{aligned} x_{n+1} = {} & \frac{1}{3n}(x_n) + \left(\frac{2n-1}{6n}\right) P_{H(x_n,Tx_n)}x_n + \left(\frac{3n-2}{6n}\right) Tx_n \\ & + \left(\frac{n+1}{6n}\right) \sum_{i=1}^{N} \left(\frac{1}{2^i} + \frac{1}{N2^N}\right) P_{C_i}\left(I - \frac{1}{5}\left(\frac{\Phi^*(I - P_{Q_i})\Phi}{2} + \frac{\Psi^*(I - P_{Q_i})\Psi}{2}\right)\right)x_n, \end{aligned}$$

for all $n \geq 1$, where $x_n = (x_n^1, x_n^2)$, $P_{C_i}(x_1,x_2) = (x_1,x_2) - \frac{\langle (2^i,3^i),(x_1,x_2)\rangle}{\|(2^i,3^i)\|^2} \cdot (2^i,3^i)$ and $P_{Q_i}(x_1,x_2) = (x_1,x_2) - \frac{\langle ((-2)^i,3^i),(x_1,x_2)\rangle}{\|((-2)^i,2^i)\|^2} \cdot ((-2)^i,2^i)$.

The Table 2 and Fig. 2 show the values of $\{x_n\}$ with $x_1 = (-0.5,-0.5)$ and $n = N = 50$.

Table 2 The values of $\{x_n\}$ with $x_1 = (-0.5,-0.5)$ and $n = N = 50$

n	$x_n = (x_n^1, x_n^2)$
1	(−0.5000, −0.5000)
2	(−0.3333, −0.3333)
3	(−0.2570, −0.2454)
⋮	⋮
25	(0.0046, −0.0058)
⋮	⋮
48	(0.0018, −0.0011)
49	(0.0017, −0.0010)
50	(0.0016, −0.0010)

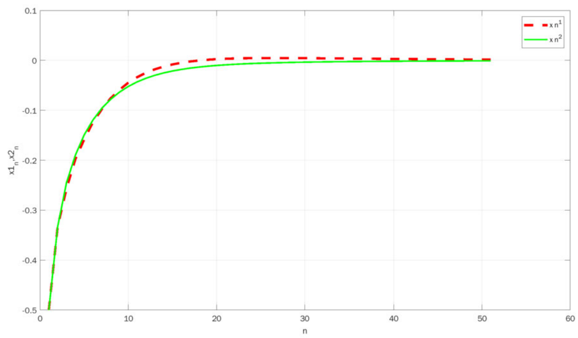

Figure 2 The convergence of $\{x_n\}$ with $x_1 = (-0.5, -0.5)$ and $n = N = 50$

6 Conclusion

1. Theorem 3 guarantees the convergence of $\{x_n\}$ in Example 2 and Example 3.
2. The convergence of $\{x_n\}$ in Example 2 is faster than the convergence of $\{x_n\}$ in Example 3.
3. Theorem 3 is an approximate solution to the fixed point problem of the cutter, which the cutter can be applied to various theorems in optimization theory (see more detail in [16, 17]) and can be reduced to firmly quasi-nonexpansive mapping.
4. Theorem 3 can be reduced to Corollary 2, which is a modification in terms of the iterative process of Theorem 1.
5. Theorem 1 makes it difficult to give an example in a real problem, especially condition (ii) in Theorem 1 must be calculated through the norm, which is quite complex. While calculating the conditions of Theorem 3, it is easier to provide examples that we give in Sect. 5.

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Competing interests
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Author contribution
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