Das and Swaminathan *Journal of Inequalities and Applications* (2022) 2022:147
https://doi.org/10.1186/s13660-022-02880-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

New asymptotic expansions and Padé approximants related to the triple gamma function

Sourav Das[1*] and A. Swaminathan[2]

*Correspondence:
souravdasmath@gmail.com;
souravdas.math@nitjsr.ac.in
[1]Department of Mathematics, National Institute of Technology Jamshedpur, Jamshedpur, 831014, Jharkhand, India
Full list of author information is available at the end of the article

Abstract
In this work, our main focus is to establish asymptotic expansions for the triple gamma function in terms of the triple Bernoulli polynomials. As application, an asymptotic expansion for hyperfactorial function is also obtained. Furthermore, using these asymptotic expansions, Padé approximants related to the triple gamma function are derived as a consequence. The results obtained are new, and their importance is demonstrated by deducing several interesting remarks and corollaries.

MSC: 33B15; 41A60; 41A21

Keywords: Multiple gamma function; Asymptotic expansion; Triple Bernoulli polynomial; Approximation; Padé approximant

1 Introduction

E.W. Barnes introduced the multiple gamma function Γ_n and expressed it in terms of the multiple Hurwitz zeta-functions [1–4]. After this invention, several researchers and mathematicians have studied this function. Γ_n was studied by G.H. Hardy [11, 12] in view of the theory of elliptic functions. Γ_n can be used to compute summation of series and infinite products. It is also applied in the theory of elliptic functions and theta functions [1–4]. Moreover, multiple gamma functions are also useful to study the determinant of Laplacians on the n-dimensional unit sphere S^n [7]. Also, Γ_n appears in functional equations for the Selberg zeta functions [18] associated with higher rank symmetric spaces, so obtaining their asymptotic expansion is of interest. It can be observed from the literature that the Γ_n plays a vital role in analytic number theory, approximation theory, mathematical physics, and several branches of science and engineering [7]. Γ_n satisfies the following recurrence relations [17]:

$$\Gamma_{n+1}(z+1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)}, \qquad \Gamma_1(z) = \Gamma(z), \qquad \Gamma_n(1) = 1, \quad n \in \mathbb{N}, \tag{1.1}$$

where $\Gamma(z)$ is Euler's gamma function. The reciprocal of $\Gamma_2(z)$ is the well-known Barnes G-function and is denoted by $G(z)$. For further information on Γ_n, we refer to [1–4, 8, 9, 15–17] and the references therein.

One of the most recent research interests in the theory of multiple gamma functions is to find their asymptotic expansions. In [6], asymptotic expansions for $G(z)$ are obtained by C.-P. Chen. Recently, Z. Xu and W. Wang [20] generalized the result. In [10], C. Ferreira and J.L. López derived an asymptotic expansion of $G(z)$. Recently, X. Li and C.-P. Chen [13] established Padé approximant involving asymptotics for the gamma function.

The above results motivate us to find asymptotic expansion for the triple gamma function Γ_3. In this work, we derive the asymptotic expansion for Γ_3. Moreover, Padé approximants related to these asymptotic expansions are also obtained. It can be noted that Γ_3 has the Weierstrass canonical product [7] form:

$$\begin{aligned}\Gamma_3(z+1) &= \exp\bigl[Dz^3+Ez^2+Fz\bigr] \\ &\quad\cdot\prod_{k=1}^{\infty}\left(\left(1+\frac{z}{k}\right)^{-\frac{k(k+1)}{2}}\exp\left[\frac{k+1}{2}z-\frac{1}{4}\left(1+\frac{1}{k}\right)z^2+\frac{1}{6k}\left(1+\frac{1}{k}\right)z^3\right]\right),\end{aligned} \tag{1.2}$$

where

$$\begin{aligned}&D=-\frac{1}{6}\left(\gamma+\frac{\pi^2}{6}+\frac{3}{2}\right),\qquad E=\frac{1}{4}\left(\gamma+\log(2\pi)+\frac{1}{2}\right),\quad\text{and}\\ &F=\frac{3}{8}-\frac{\log(2\pi)}{4}-\log A.\end{aligned}$$

Here A denotes the Glaisher–Kinkelin constant and is defined as [7]

$$\log A=\frac{1}{12}-\zeta'(-1)$$

with ζ being the Riemann zeta function. Γ_3 can also be expressed in terms of triple Bernoulli polynomials [15] as follows:

$$\begin{aligned}\log\Gamma_3(z) &= \frac{B_{3,3}(z)}{6}\log z-\frac{11}{36}B_{3,0}z^3-\frac{3}{4}B_{3,1}z^2-\frac{1}{2}B_{3,2}z \\ &\quad+\sum_{k=1}^{n}\frac{(-1)^{k-1}B_{3,k+3}}{k(k+1)(k+2)(k+3)z^k}+\mathcal{R}_{3,n}(z),\end{aligned} \tag{1.3}$$

where $B_{3,k}(x)$ are triple Bernoulli polynomials defined as

$$\frac{t^3e^{xt}}{(e^t-1)^3}=\sum_{k=0}^{\infty}B_{3,k}(x)\frac{t^k}{k!},\quad |t|<2\pi,$$

and $B_{3,k}=B_{3,k}(0)$ are triple Bernoulli numbers [14] with $\mathcal{R}_{3,n}(z)$, which is the remainder of order n and has the following integral representation:

$$\mathcal{R}_{3,n}(z)=\int_0^{\infty}\frac{e^{-zt}}{t^4}\left(\frac{t^3}{(1-e^{-t})^3}-\sum_{k=0}^{n}\frac{(-1)^k}{k!}B_{3,k}t^k\right)dt\quad\text{for }\Re z>0\text{ and }n\geq 3.$$

The following relation [14] between triple Bernoulli polynomials and Bernoulli polynomials will be helpful for computing triple Bernoulli polynomials to prove the main results:

$$B_{3,k}(x)=\frac{k(k-1)(k-2)}{2}\left((x-1)(x-2)\frac{B_{k-2}(x)}{k-2}-(2x-3)\frac{B_{k-1}(x)}{k-1}+\frac{B_k(x)}{k}\right), \tag{1.4}$$

where $B_k(x)$ are Bernoulli polynomials defined as

$$\frac{te^{tx}}{e^t-1}=\sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!},\quad |t|<2\pi,$$

and $B_k=B_k(0)$ are known as Bernoulli numbers.

It can be noted that putting $x=0$ in (1.4), we get the following recurrence relation, which will be useful to compute triple Bernoulli numbers:

$$B_{3,k}=\frac{k(k-1)(k-2)}{2}\left(\frac{2B_{k-2}}{k-2}+\frac{3B_{k-1}}{k-1}+\frac{B_k}{k}\right)\quad\text{for } k\geq 3. \tag{1.5}$$

The rest of the paper is organized as follows. In Sect. 2, two different classes of asymptotic expansions of triple gamma functions with the formulas for determining the coefficients of each class are found. An asymptotic expansion for hyperfactorial function is also obtained as a particular case. Moreover, Padé approximants related to these asymptotic expansions are obtained in Sect. 3.

2 Asymptotic expansions for the triple gamma function

Let

$$g(z)=\frac{\exp(\frac{11}{36}B_{3,0}z^3+\frac{3}{4}B_{3,1}z^2+\frac{1}{2}B_{3,2}z)\Gamma_3(z)}{z^{\frac{1}{6}B_{3,3}(z)}}. \tag{2.1}$$

Then, using the expansion for Γ_3 given in (1.3), we obtain

$$g(z)\sim\exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}B_{3,k+3}}{k(k+1)(k+2)(k+3)z^k}\right) \tag{2.2}$$

for $z\to\infty$ and $\Re z>0$. It follows from (1.5) [14, p. 187] that for $k\geq 1$,

$$\begin{aligned}B_{3,2k+1}&=\frac{3(2k+1)(2k-1)}{2}B_{2k},\\B_{3,2k+2}&=k(2k+1)(2k+2)\left(\frac{B_{2k}}{k}+\frac{B_{2k+2}}{2(k+1)}\right).\end{aligned}$$

Using the above expressions together with (1.5) in (2.2) gives

$$g(z)\sim\exp\left(\frac{19}{240z}+\frac{1}{160z^2}-\frac{2}{945z^3}-\frac{1}{672z^4}+\frac{19}{50{,}400z^5}\cdots\right) \tag{2.3}$$

for $z\to\infty$ and $\Re z>0$.

Using the expansion $e^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}$ in the exponential part of (2.3) containing Bernoulli numbers, a simple computation yields the following:

$$\begin{aligned}g(z)\sim\Bigg(1&+\frac{19}{240z}+\frac{1081}{115{,}200z^2}-\frac{893{,}507}{580{,}608{,}000z^3}-\frac{557{,}383{,}680{,}000z^4}{}^{-1}\cdot 900{,}113{,}513\\&+\frac{161{,}466{,}866{,}293}{668{,}860{,}416{,}000{,}000z^5}+\cdots\Bigg).\end{aligned} \tag{2.4}$$

Theorem 2.1 *Let $p \geq 0$ be any integer and $q \neq 0$ be any real number. Then $g(z)$ defined in* (2.1) *has the following asymptotic expansion*:

$$g(z) \sim \left(1 + \sum_{i=1}^{\infty} \frac{a_i}{z^i}\right)^{z^p/q} \tag{2.5}$$

for $z \to \infty$ and $\Re z > 0$, where the coefficients $a_i = a_i(p,q)$ $(i \in \mathbb{N})$ are given by

$$\begin{aligned} a_i = &\sum_{(1+p)k_1+(2+p)k_2+\cdots+(i+p)k_i=i} \frac{q^{k_1+k_2+\cdots+k_i}}{k_1!k_2!\cdots k_i!} \\ &\cdot \left(\frac{B_{3,4}}{1\cdot 2\cdot 3\cdot 4}\right)^{k_1} \left(\frac{-B_{3,5}}{2\cdot 3\cdot 4\cdot 5}\right)^{k_2} \cdots \left(\frac{(-1)^{i-1}B_{3,3+i}}{i(i+1)(i+2)(i+3)}\right)^{k_i}, \end{aligned} \tag{2.6}$$

where the summation is considered over all k_i, with k_is as nonnegative integers satisfying the following relation:

$$(1+p)k_1 + (2+p)k_2 + \cdots + (i+p)k_i = i.$$

Proof Our main objective is to determine a_i $(i \in \mathbb{N})$. For this we first express (2.5) as follows:

$$\bigl(g(z)\bigr)^{q/z^p} = 1 + \sum_{i=1}^{n} \frac{a_i}{z^i} + O\bigl(z^{-n-1}\bigr). \tag{2.7}$$

Note that even in the absence of any information on convergence, (2.7) is still valid as we are interested only in the asymptotic expansion. From (2.2), we have

$$g(z) = \exp\left(\sum_{k=1}^{n} \frac{(-1)^{k-1}B_{3,k+3}}{k(k+1)(k+2)(k+3)z^k} + \mathcal{R}(z)\right),$$

where $\mathcal{R}(z) = O(z^{-n-1})$, which implies

$$\begin{aligned} &\bigl(g(z)\bigr)^{q/z^p} \\ &= e^{q\mathcal{R}(z)/z^p} \exp\left(\sum_{k=1}^{n} \frac{(-1)^{k-1}qB_{3,k+3}}{k(k+1)(k+2)(k+3)z^{k+p}}\right) \\ &= e^{q\mathcal{R}(z)/z^p} \prod_{k=1}^{n}\left(1 + \left(\frac{(-1)^{k-1}qB_{3,k+3}}{k(k+1)(k+2)(k+3)z^{k+p}}\right)\right. \\ &\quad \left. + \frac{1}{2!}\left(\frac{(-1)^{k-1}qB_{3,k+3}}{k(k+1)(k+2)(k+3)z^{k+p}}\right)^2 + \cdots\right) \\ &= e^{q\mathcal{R}(z)/z^p} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} \frac{1}{k_1!k_2!\cdots k_n!}\left(\frac{qB_{3,4}}{1\cdot 2\cdot 3\cdot 4}\right)^{k_1}\left(\frac{-qB_{3,5}}{2\cdot 3\cdot 4\cdot 5}\right)^{k_2} \\ &\quad \cdots \left(\frac{(-1)^{n-1}qB_{3,n+3}}{n(n+1)(n+2)(n+3)}\right)^{k_n} \cdot \frac{1}{z^{(1+p)k_1+(2+p)k_2+\cdots+(n+p)k_n}}. \end{aligned} \tag{2.8}$$

Comparing the powers of z in (2.7) and (2.8), we obtain

$$a_i = \sum_{(1+p)k_1+(2+p)k_2+\cdots+(i+p)k_i=i} \frac{q^{k_1+k_2+\cdots+k_i}}{k_1!k_2!\cdots k_i!} \cdot \left(\frac{B_{3,4}}{1\cdot 2\cdot 3\cdot 4}\right)^{k_1}\left(\frac{-B_{3,5}}{2\cdot 3\cdot 4\cdot 5}\right)^{k_2}\cdots\left(\frac{(-1)^{i-1}B_{3,3+i}}{i(i+1)(i+2)(i+3)}\right)^{k_i},$$

which completes the proof. □

Remark 2.1 $p = 0$ and $q = 1$ in (2.5) yield (2.4). Setting $p = 0$, $q = 2$ and $p = 1$, $q = 1$ in (2.5), respectively, we have the following two asymptotic expansions:

$$\Gamma_3(z) \sim z^{\frac{B_{3,3}(z)}{6}} e^{-\frac{11}{36}B_{3,0}z^3-\frac{3}{4}B_{3,1}z^2-\frac{1}{2}B_{3,2}z}\left(1+\frac{19}{120z}+\frac{721}{28{,}800z^2}-\frac{115{,}547}{72{,}576{,}000z^3} - \frac{117{,}935{,}033}{34{,}836{,}480{,}000z^4}+\frac{3{,}885{,}027{,}493}{20{,}901{,}888{,}000{,}000z^5}+\cdots\right)^{\frac{1}{2}} \tag{2.9}$$

and

$$\Gamma_3(z) \sim z^{\frac{B_{3,3}(z)}{6}} e^{-\frac{11}{36}B_{3,0}z^3-\frac{3}{4}B_{3,1}z^2-\frac{1}{2}B_{3,2}z}\left(1+\frac{19}{240z^2}+\frac{1}{160z^3} + \frac{2461}{2{,}419{,}200z^4}-\frac{89}{89{,}600z^5}+\cdots\right)^{z}. \tag{2.10}$$

Remark 2.2 Let $H(n)$ be the hyperfactorial function defined as $H(n) := 1^1\cdot 2^2\cdot 3^3\cdots n^n$.

Then, using the relations $\Gamma_3(n) = G(n-1)\Gamma_3(n-1)$ and $G(n+1) = \frac{(n!)^n}{H(n)}$, we have

$$\Gamma_3(n) = G(3)\cdot G(4)\cdots G(n-1) = \frac{\prod_{k=1}^{n-2}(k!)^k}{\prod_{k=1}^{n-2}H(k)} = \frac{\prod_{k=1}^{n}(k!)^k}{((n-1)!)^{n-1}(n!)^n H_{n-2}}, \quad \text{where } \prod_{k=1}^{n} H(k) = H_n.$$

With the help of Remark 2.1, the following asymptotic expansions can be obtained:

$$(1!)^1(2!)^2\cdots(n!)^n \sim (n!)^n\big((n-1)!\big)^{n-1}H_{n-2}n^{\frac{B_{3,3}(n)}{6}}e^{-\frac{11}{36}B_{3,0}n^3-\frac{3}{4}B_{3,1}n^2-\frac{1}{2}B_{3,2}n} \times\left(1+\frac{19}{240n}+\frac{1081}{115{,}200n^2}-\frac{893{,}507}{580{,}608{,}000n^3}-\frac{900{,}113{,}513}{557{,}383{,}680{,}000n^4} + \frac{161{,}466{,}866{,}293}{668{,}860{,}416{,}000{,}000n^5}+\cdots\right) \tag{2.11}$$

and

$$(1!)^1(2!)^2\cdots(n!)^n \sim (n!)^n\big((n-1)!\big)^{n-1}H_{n-2}n^{\frac{B_{3,3}(n)}{6}}e^{-\frac{11}{36}B_{3,0}n^3-\frac{3}{4}B_{3,1}n^2-\frac{1}{2}B_{3,2}n} \times\left(1+\frac{19}{240n^2}+\frac{1}{160n^3}+\frac{2461}{2{,}419{,}200n^4}-\frac{89}{89{,}600n^5}+\cdots\right)^n. \tag{2.12}$$

Theorem 2.2 *Let $p \geq 0$ be any integer and $q \neq 0$ be any real number. Then $g(z)$ defined in* (2.1) *satisfies the following asymptotic expansion*:

$$g(z) \sim \left(1 + \ln\left(1 + \sum_{i=1}^{\infty} \frac{b_i}{z^i}\right)\right)^{z^p/q} \tag{2.13}$$

for $z \to \infty$ and $\Re z > 0$, where the coefficients $b_i = b_i(p,q)$ $(i \in \mathbb{N})$ are given by

$$b_i = \sum_{k_1+2k_2+\cdots+ik_i=i} \frac{1}{k_1!k_2!\cdots k_i!} a_1^{k_1} a_2^{k_2} \cdots a_i^{k_i},$$

a_i $(i \in \mathbb{N})$ are determined in (2.6).

Proof To prove this theorem, we will follow similar techniques as Theorem 2.1. To do so, first we express (2.13) as follows:

$$\exp\left(\left(g(z)\right)^{q/z^p} - 1\right) = 1 + \sum_{i=1}^{n} \frac{b_i}{z^i} + O\left(z^{-n-1}\right). \tag{2.14}$$

Note that, similar to (2.7), the discussion of asymptotic expansion does not require any information on the convergence of the series in (2.14). Writing (2.5) as

$$\left(g(z)\right)^{q/z^p} - 1 = \mathcal{R}(z) = O\left(z^{-m-1}\right),$$

we have

$$\begin{aligned}
&\exp\left(\left(g(z)\right)^{q/z^p} - 1\right) \\
&\quad = e^{\mathcal{R}(z)} \exp\left(\sum_{k=1}^{n} \frac{a_k}{z^k}\right) \\
&\quad = e^{\mathcal{R}(z)} \prod_{k=1}^{n} \left(1 + \left(\frac{a_k}{z^k}\right) + \frac{1}{2!}\left(\frac{a_k}{z^k}\right)^2 + \cdots\right) \\
&\quad = e^{\mathcal{R}(z)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \frac{1}{k_1!k_2!\cdots k_n!} a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n} \frac{1}{z^{k_1+2k_2+\cdots+nk_n}}.
\end{aligned} \tag{2.15}$$

Equating the coefficients of powers of z in (2.14) and (2.15), we have

$$b_i = \sum_{k_1+2k_2+\cdots+ik_i=i} \frac{1}{k_1!k_2!\cdots k_i!} a_1^{k_1} a_2^{k_2} \cdots a_i^{k_i},$$

where a_i $(i \in \mathbb{N})$ are determined in (2.6), which completes the proof. □

Remark 2.3 Putting $p = 0$, $q = 1$; $p = 0$, $q = 2$; and $p = 1$, $q = 1$ respectively in (2.13), the following asymptotic expansions can be obtained:

$$\begin{aligned}
\Gamma_3(z) \sim{}& z^{\frac{B_{3,3}(z)}{6}} e^{-\frac{11}{36}B_{3,0}z^3 - \frac{3}{4}B_{3,1}z^2 - \frac{1}{2}B_{3,2}z} \\
&\times \left(1 + \ln\left(1 + \frac{19}{240z} + \frac{721}{57{,}600z^2} - \frac{16{,}567}{23{,}224{,}320z^3} - \frac{245{,}178{,}299}{185{,}794{,}560{,}000z^4}\right.\right.
\end{aligned}$$

$$+\frac{5{,}299{,}227{,}679}{55{,}738{,}368{,}000{,}000z^5}+\cdots\biggr)\biggr), \tag{2.16}$$

$$\begin{aligned}\Gamma_3(z)\sim{}& z^{\frac{B_{3,3}(z)}{6}}e^{-\frac{11}{36}B_{3,0}z^3-\frac{3}{4}B_{3,1}z^2-\frac{1}{2}B_{3,2}z}\\ &\times\biggl(1+\ln\biggl(1+\frac{19}{240z^2}+\frac{1}{160z^3}-\frac{15{,}281{,}579}{11{,}612{,}160{,}000z^4}\\ &+\frac{1{,}613{,}002{,}879}{1{,}741{,}824{,}000{,}000z^5}+\cdots\biggr)\biggr)^{1/2},\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\Gamma_3(z)\sim{}& z^{\frac{B_{3,3}(z)}{6}}e^{-\frac{11}{36}B_{3,0}z^3-\frac{3}{4}B_{3,1}z^2-\frac{1}{2}B_{3,2}z}\\ &\times\biggl(1+\ln\biggl(1+\frac{19}{240z^2}+\frac{1}{160z^3}+\frac{5021}{1{,}209{,}600z^4}-\frac{67}{134{,}400z^5}+\cdots\biggr)\biggr)^z.\end{aligned} \tag{2.18}$$

3 Padé approximants related to the triple gamma function

Let $h(t)=\sum_{k=0}^{\infty}c_kt^k$ be a formal power series and $h_n(t)=\sum_{k=0}^{n}c_kt^k$ be its nth partial sum (g_n is identically zero for $n<0$). Then the Padé approximation [19] of order (m,n) of the function g is defined as the rational function

$$[m/n]_h(t):=\frac{\sum_{k=0}^{m}a_kt^k}{1+\sum_{k=1}^{n}b_kt^k}, \tag{3.1}$$

where $m\geq 0$ and $n\geq 1$ are two given integers, and the coefficients a_k and b_k are given by [13]

$$\begin{aligned}&a_0=c_0,\\ &a_1=c_0b_1+c_1,\\ &a_2=c_0b_2+c_1b_1+c_2,\\ &\quad\vdots\\ &a_m=c_0b_m+\cdots+c_{m-1}b_1+c_m,\\ &0=c_{m+1}+c_mb_1+\cdots+c_{m-n+1}b_n,\\ &\quad\vdots\\ &0=c_{m+n}+c_{m+n-1}b_1+\cdots+c_mb_n,\end{aligned} \tag{3.2}$$

with the following property:

$$[m/n]_h(t)-h(t)=O\bigl(t^{m+n+1}\bigr). \tag{3.3}$$

Clearly, the first $m+n+1$ coefficients of the series expansion of $[m/n]_g$ are identical to those of g. Further, we have

$$[m/n]_h(t) = \frac{\begin{vmatrix} t^n h_{m-n}(t) & t^{n-1} h_{m-n+1}(t) & \cdots & h_m(t) \\ c_{m-n+1} & c_{m-n+2} & \cdots & c_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & c_{m+1} & \cdots & c_{m+n} \end{vmatrix}}{\begin{vmatrix} t^n & t^{n-1} & \cdots & 1 \\ c_{m-n+1} & c_{m-n+2} & \cdots & c_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & c_{m+1} & \cdots & c_{m+n} \end{vmatrix}}. \tag{3.4}$$

Padé approximants related to the gamma function are discussed in [13]. In this section, we are interested in finding the Padé approximants for the function $g(x)$, which is defined in (2.1).

From (2.4) and (2.1), it can be noted that, as $x \to \infty$,

$$\begin{aligned} g(x) &\sim \sum_{k=0}^{\infty} \frac{c_k}{x^k} \\ &= 1 + \frac{19}{240x} + \frac{1081}{115{,}200x^2} - \frac{893{,}507}{580{,}608{,}000x^3} - \frac{900{,}113{,}513}{557{,}383{,}680{,}000x^4} + \cdots, \end{aligned} \tag{3.5}$$

where the coefficients c_k satisfy (2.6) for $p = 0$ and $q = 1$.

Now we will proceed for the Padé approximation of $g(x)$. Let us consider

$$[1/1]_g(x) = \frac{\sum_{k=0}^{1} \frac{a_k}{x^k}}{1 + \sum_{k=1}^{1} \frac{b_k}{x^k}}.$$

It follows that

$$\begin{aligned} &c_0 = 1, \qquad c_1 = \frac{19}{240}, \qquad c_2 = \frac{1081}{115{,}200}, \\ &c_3 = -\frac{893{,}507}{580{,}608{,}000}, \qquad c_4 = -\frac{900{,}113{,}513}{557{,}383{,}680{,}000}. \end{aligned} \tag{3.6}$$

From (3.2), we obtain

$$\begin{aligned} a_0 &= 1, \\ a_1 &= b_1 + \frac{19}{240}, \\ 0 &= \frac{19}{240} b_1 + \frac{1081}{115{,}200}, \end{aligned}$$

which gives $a_0 = 1$, $a_1 = -\frac{359}{9120}$, and $b_1 = -\frac{1081}{9120}$. Hence, we have

$$[1/1]_g(x) = \frac{1 - \frac{359}{9120x}}{1 - \frac{1081}{9120x}} = \frac{x - \frac{359}{9120}}{x - \frac{1081}{9120}}, \tag{3.7}$$

and using (3.3), it follows that

$$g(x) = \frac{x - \frac{359}{9120}}{x - \frac{1081}{9120}} + O\left(\frac{1}{x^3}\right).$$

Now we will derive a Padé approximant of order $(2, 2)$. For this, let us consider

$$[2/2]_g(x) = \frac{\sum_{k=0}^{2} \frac{a_k}{x^k}}{1 + \sum_{k=1}^{2} \frac{b_k}{x^k}}.$$

Using (3.2), we obtain

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= b_1 + \frac{19}{240}, \\
a_2 &= b_2 + \frac{19}{240} b_1 + \frac{1081}{115{,}200}, \\
0 &= \frac{1081}{115{,}200} b_1 + \frac{19}{240} b_2 - \frac{893{,}507}{580{,}608{,}000}, \\
0 &= \frac{893{,}507}{580{,}608{,}000} b_1 - \frac{1081}{115{,}200} b_2 + \frac{900{,}113{,}513}{557{,}383{,}680{,}000},
\end{aligned}$$

which implies that

$$a_1 = -\frac{12{,}947{,}658{,}827}{28{,}076{,}662{,}560}, \qquad a_2 = \frac{14{,}176{,}622{,}313{,}529}{283{,}012{,}758{,}604{,}800},$$
$$b_1 = -\frac{15{,}170{,}394{,}613}{28{,}076{,}662{,}560}, \qquad b_2 = \frac{23{,}626{,}895{,}894{,}809}{283{,}012{,}758{,}604{,}800},$$

and

$$\begin{aligned}
[2/2]_g(x) &= \frac{1 - \frac{12{,}947{,}658{,}827}{28{,}076{,}662{,}560x} + \frac{14{,}176{,}622{,}313{,}529}{283{,}012{,}758{,}604{,}800x^2}}{1 - \frac{15{,}170{,}394{,}613}{28{,}076{,}662{,}560x} + \frac{23{,}626{,}895{,}894{,}809}{283{,}012{,}758{,}604{,}800x^2}} \\
&= \frac{x^2 - \frac{12{,}947{,}658{,}827}{28{,}076{,}662{,}560} x + \frac{14{,}176{,}622{,}313{,}529}{283{,}012{,}758{,}604{,}800}}{x^2 - \frac{15{,}170{,}394{,}613}{28{,}076{,}662{,}560} x + \frac{23{,}626{,}895{,}894{,}809}{283{,}012{,}758{,}604{,}800}}.
\end{aligned} \tag{3.8}$$

Therefore, using (3.3), it follows that

$$g(x) = \frac{x^2 - \frac{12{,}947{,}658{,}827}{28{,}076{,}662{,}560} x + \frac{14{,}176{,}622{,}313{,}529}{283{,}012{,}758{,}604{,}800}}{x^2 - \frac{15{,}170{,}394{,}613}{28{,}076{,}662{,}560} x + \frac{23{,}626{,}895{,}894{,}809}{283{,}012{,}758{,}604{,}800}} + O\left(\frac{1}{x^5}\right).$$

Using the Padé approximation method and expansion (3.5), the following theorem can be derived.

Theorem 3.1 *The Padé approximation of order* (m, n) *of the asymptotic formula of the function* $g(x)$ (*at the point* $x = \infty$) *is given by the rational function*:

$$[m/n]_g(x) = \frac{1 + \sum_{k=1}^{m} \frac{a_k}{x^k}}{1 + \sum_{k=1}^{n} \frac{b_k}{x^k}} = x^{n-m} \left(\frac{x^m + a_1 x^{m-1} + \cdots + a_m}{x^n + b_1 x^{n-1} + \cdots + b_n} \right),$$

where $m \geq 1$ and $n \geq 1$ are given integers and the coefficients a_k and b_k satisfy the conditions (3.2) with $a_0 = c_0 = 1$ and c_k satisfies (2.6) for $p = 0$ and $q = 1$, and the following relation holds:

$$g(x) - [m/n]_g(x) = O\left(\frac{1}{x^{m+n+1}}\right), \quad x \to \infty.$$

Further, we have (see [5])

$$[m/n]_g(x) = \frac{\begin{vmatrix} x^n g_{m-n}(x) & x^{n-1} g_{m-n+1}(x) & \cdots & g_m(x) \\ c_{m-n+1} & c_{m-n+2} & \cdots & c_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & c_{m+1} & \cdots & c_{m+n} \end{vmatrix}}{\begin{vmatrix} x^n & x^{n-1} & \cdots & 1 \\ c_{m-n+1} & c_{m-n+2} & \cdots & c_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & c_{m+1} & \cdots & c_{m+n} \end{vmatrix}}, \tag{3.9}$$

where $g_n(x) = \sum_{k=0}^{n} \frac{c_k}{x^k}$ is the nth partial sum of the asymptotic series (3.5).

Remark 3.1 Using (3.9), it is also possible to derive (3.7) and (3.8). Let us verify these results.

$$[1/1]_g(x) = \frac{\begin{vmatrix} \frac{1}{x} g_0(x) & g_1(x) \\ c_1 & c_2 \end{vmatrix}}{\begin{vmatrix} \frac{1}{x} & 1 \\ c_1 & c_2 \end{vmatrix}} = \frac{\begin{vmatrix} \frac{1}{x} & 1 + \frac{19}{240x} \\ \frac{19}{240} & \frac{1081}{115{,}200} \end{vmatrix}}{\begin{vmatrix} \frac{1}{x} & 1 \\ \frac{19}{240} & \frac{1081}{115{,}200} \end{vmatrix}} = \frac{x - \frac{359}{9120}}{x - \frac{1081}{9120}}$$

and

$$[2/2]_g(x) = \frac{\begin{vmatrix} \frac{1}{x^2} g_0(x) & \frac{1}{x} g_1(x) & g_2(x) \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}}{\begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} & 1 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}} = \frac{\begin{vmatrix} \frac{1}{x^2} & \frac{1}{x}(1 + \frac{19}{240x}) & 1 + \frac{19}{240x} + \frac{1081}{115{,}200x^2} \\ \frac{19}{240} & \frac{1081}{115{,}200} & -\frac{893{,}507}{580{,}608{,}000} \\ \frac{1081}{115{,}200} & -\frac{893{,}507}{580{,}608{,}000} & -\frac{900{,}113{,}513}{557{,}383{,}680{,}000} \end{vmatrix}}{\begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} & 1 \\ \frac{19}{240} & \frac{1081}{115{,}200} & -\frac{893{,}507}{580{,}608{,}000} \\ \frac{1081}{115{,}200} & -\frac{893{,}507}{580{,}608{,}000} & -\frac{900{,}113{,}513}{557{,}383{,}680{,}000} \end{vmatrix}}$$

$$= \frac{x^2 - \frac{12{,}947{,}658{,}827}{28{,}076{,}662{,}560}x + \frac{14{,}176{,}622{,}313{,}529}{283{,}012{,}758{,}604{,}800}}{x^2 - \frac{15{,}170{,}394{,}613}{28{,}076{,}662{,}560}x + \frac{23{,}626{,}895{,}894{,}809}{283{,}012{,}758{,}604{,}800}}.$$

Setting $m = n = r$ in (3.3), the following result is immediate.

Corollary 3.1

$$g(x) = \frac{x^r + a_1 x^{r-1} + \cdots + a_r}{x^r + b_1 x^{r-1} + \cdots + b_r} + O\left(\frac{1}{x^{2r+1}}\right) \tag{3.10}$$

for $x \to \infty$ and $r \geq 1$ is any given integer, the coefficients a_i and b_i $(1 \leq i \leq r)$ satisfy (3.2) *with $a_0 = c_0 = 1$, and c_r is given in* (2.6) (*for $p = 0$ and $q = 1$*).

Remark 3.2 Setting $r = 3$ in (3.10), we obtain

$$g(x) = \frac{x^3 - a_1x^2 + a_2x - a_3}{x^3 - b_1x^2 + b_2x - b_3} + O\left(\frac{1}{x^7}\right),$$

where

$$\begin{aligned}
a_1 &= \frac{564{,}264{,}707{,}394{,}045{,}441{,}291{,}137}{777{,}288{,}995{,}553{,}534{,}212{,}638{,}560},\\
a_2 &= \frac{507{,}337{,}383{,}020{,}418{,}800{,}355{,}971{,}917}{932{,}746{,}794{,}664{,}241{,}055{,}166{,}272{,}000},\\
a_3 &= \frac{524{,}560{,}286{,}160{,}441{,}450{,}378{,}984{,}935{,}759}{18{,}804{,}175{,}380{,}431{,}099{,}672{,}152{,}043{,}520{,}000},\\
b_1 &= \frac{625{,}800{,}086{,}208{,}700{,}233{,}125{,}023}{777{,}288{,}995{,}553{,}534{,}212{,}638{,}560},\\
b_2 &= \frac{50{,}730{,}526{,}659{,}081{,}610{,}299{,}631{,}847}{84{,}795{,}163{,}151{,}294{,}641{,}378{,}752{,}000},\\
b_3 &= \frac{1{,}244{,}184{,}486{,}344{,}122{,}340{,}305{,}441{,}760{,}881}{18{,}804{,}175{,}380{,}431{,}099{,}672{,}152{,}043{,}520{,}000}.
\end{aligned}$$

Remark 3.3 Setting $r = 4$ in (3.10), we obtain

$$g(x) = \frac{x^4 - a_1x^3 + a_2x^2 - a_3x + a_4}{x^4 - b_1x^3 + b_2x^2 - b_3x + b^4} + O\left(\frac{1}{x^9}\right),$$

where

$$\begin{aligned}
a_1 &= \frac{215{,}703{,}766{,}781{,}168{,}876{,}220{,}885{,}513{,}532{,}207{,}573{,}037{,}853{,}391}{156{,}833{,}412{,}819{,}388{,}306{,}354{,}787{,}238{,}937{,}752{,}295{,}790{,}827{,}680},\\
a_2 &= \frac{439{,}021{,}821{,}607{,}824{,}678{,}358{,}124{,}666{,}374{,}361{,}977{,}917{,}184{,}670{,}503}{276{,}026{,}806{,}562{,}123{,}419{,}184{,}425{,}540{,}530{,}444{,}040{,}591{,}856{,}716{,}800},\\
a_3 &= \frac{2{,}697{,}600{,}344{,}247{,}344{,}205{,}104{,}432{,}196{,}561{,}229{,}100{,}202{,}237{,}946{,}568{,}861}{4{,}173{,}525{,}315{,}219{,}306{,}098{,}068{,}514{,}172{,}820{,}313{,}893{,}748{,}873{,}558{,}016{,}000},\\
a_4 &= \frac{940{,}070{,}303{,}236{,}823{,}651{,}814{,}961{,}986{,}110{,}532{,}058{,}170{,}155{,}544{,}925{,}558{,}921}{20{,}032{,}921{,}513{,}052{,}669{,}270{,}728{,}868{,}029{,}537{,}506{,}689{,}994{,}593{,}078{,}476{,}800{,}000},\\
b_1 &= \frac{228{,}119{,}745{,}296{,}037{,}117{,}140{,}639{,}503{,}281{,}446{,}296{,}454{,}627{,}249}{156{,}833{,}412{,}819{,}388{,}306{,}354{,}787{,}238{,}937{,}752{,}295{,}790{,}827{,}680},\\
b_2 &= \frac{468{,}216{,}358{,}741{,}523{,}424{,}525{,}741{,}582{,}917{,}397{,}922{,}238{,}128{,}383{,}943}{276{,}026{,}806{,}562{,}123{,}419{,}184{,}425{,}540{,}530{,}444{,}040{,}591{,}856{,}716{,}800},\\
b_3 &= \frac{290{,}424{,}419{,}148{,}610{,}158{,}964{,}741{,}537{,}112{,}783{,}630{,}996{,}531{,}879{,}384{,}489}{379{,}411{,}392{,}292{,}664{,}190{,}733{,}501{,}288{,}438{,}210{,}353{,}977{,}170{,}323{,}456{,}000},\\
b_4 &= \frac{1{,}822{,}684{,}110{,}010{,}671{,}047{,}977{,}349{,}196{,}992{,}991{,}024{,}335{,}965{,}841{,}858{,}437{,}321}{20{,}032{,}921{,}513{,}052{,}669{,}270{,}728{,}868{,}029{,}537{,}506{,}689{,}994{,}593{,}078{,}476{,}800{,}000}.
\end{aligned}$$

Acknowledgements
The work of the second author is supported by the NBHM(DAE) Project No. NBHM/RP-1/2019. The authors wish to thank the anonymous referee for the careful reading of the manuscript and for constructive comments, observations, and suggestions that have helped in significantly improving the manuscript.

Funding
Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare that there are no competing interest.

Author contribution
The manuscript was written by SD and AS. SD and AS worked together to accomplish the formal analysis, validation, and editing. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, National Institute of Technology Jamshedpur, Jamshedpur, 831014, Jharkhand, India.
[2]Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, 247667, Uttarakhand, India.

Publisher's Note

Received: 24 September 2021 Accepted: 28 October 2022 Published online: 22 November 2022

References

1. Barnes, E.W.: The theory of the G-function. Q. J. Math. **31**, 264–314 (1899)
2. Barnes, E.W.: Genesis of the double gamma function. Proc. Lond. Math. Soc. **31**, 358–381 (1900)
3. Barnes, E.W.: The theory of the double gamma function. Philos. Trans. R. Soc. Lond. Ser. A **196**, 265–388 (1901)
4. Barnes, E.W.: On the theory of the multiple gamma functions. Trans. Camb. Philos. Soc. **19**, 374–439 (1904)
5. Brezinski, C., Redivo-Zaglia, M.: New representations of Padé, Padé-type, and partial Padé approximants. J. Comput. Appl. Math. **284**, 69–77 (2015)
6. Chen, C.-P.: Asymptotic expansions for Barnes *G*-function. J. Number Theory **135**, 36–42 (2014)
7. Choi, J.: Multiple gamma functions and their applications. In: Analytic Number Theory, Approximation Theory, and Special Functions, pp. 93–129. Springer, New York (2014)
8. Das, S.: Inequalities involving the multiple psi function. C. R. Math. Acad. Sci. Paris **356**(3), 288–292 (2018)
9. Das, S., Swaminathan, A.: Bounds for triple gamma functions and their ratios. J. Inequal. Appl. **2016**, Paper No. 210 (2016)
10. Ferreira, C., López, J.L.: An asymptotic expansion of the double gamma function. J. Approx. Theory **111**(2), 298–314 (2001)
11. Hardy, G.H.: On the expression of the double zeta-function and double gamma function in terms of elliptic functions. Trans. Camb. Philos. Soc. **20**, 395–427 (1905)
12. Hardy, G.H.: On double Fourier series and especially these which represent the double zeta-function and incommensurable parameters. Q. J. Math. **37**, 53–79 (1906)
13. Li, X., Chen, C.-P.: Padé approximant related to asymptotics for the gamma function. J. Inequal. Appl. **2017**, Paper No. 53 (2017)
14. Nörlund, N.E.: Mémoire sur les polynomes de Bernoulli. Acta Math. **43**, 121–196 (1922)
15. Ruijsenaars, S.N.M.: On Barnes' multiple zeta and gamma functions. Adv. Math. **156**, 107–132 (2000)
16. Srinivasan, G.K.: The gamma function: an eclectic tour. Am. Math. Mon. **114**(4), 297–315 (2007)
17. Ueno, K., Nishizawa, M.: Multiple gamma functions and multiple *q*-gamma functions. Publ. Res. Inst. Math. Sci. **33**(5), 813–838 (1997)
18. Vignéras, M.-F.: L'équation fonctionnelle de la fonction zêta de Selberg du groupe modulaire $\mathrm{PSL}(2, \mathbf{Z})$. In: Journées Arithmétiques de Luminy (Colloq. Internat. CNRS, Centre Univ. Luminy, Luminy, 1978). Astérisque, vol. 61, pp. 235–249. Soc. Math. France, Paris
19. Wall, H.S.: Analytic Theory of Continued Fractions. Van Nostrand, New York (1948)
20. Xu, Z., Wang, W.: More asymptotic expansions for the Barnes *G*-function. J. Number Theory **174**, 505–517 (2017)