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Dynamics of a fractional plankton–fish model under the influence of toxicity, refuge, and combine-harvesting efforts

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Abstract

In this work, we analyze plankton–fish dynamics in the presence of toxicity, refuge, and combine-harvesting efforts by a considering Holling type-II functional response. We have considered phytoplankton, zooplankton, and fish populations, and the interdependent evolution is presented with the help of the Caputo fractional derivative. Since toxicity in phytoplankton spreads to zooplankton and hence to fish, we have introduced toxic terms in all the populations. On the other hand, to save the population from extinction harvesting is an essential tool. Theoretical aspects are studied in terms of nonnegativity, boundedness, existence, and uniqueness of the solution. Sufficient conditions are derived for the stability of various points of equilibrium. The composite behavior is studied by varying the values of different parameters and fractional derivatives. Numerical simulations are carried out to strengthen the theoretical findings.

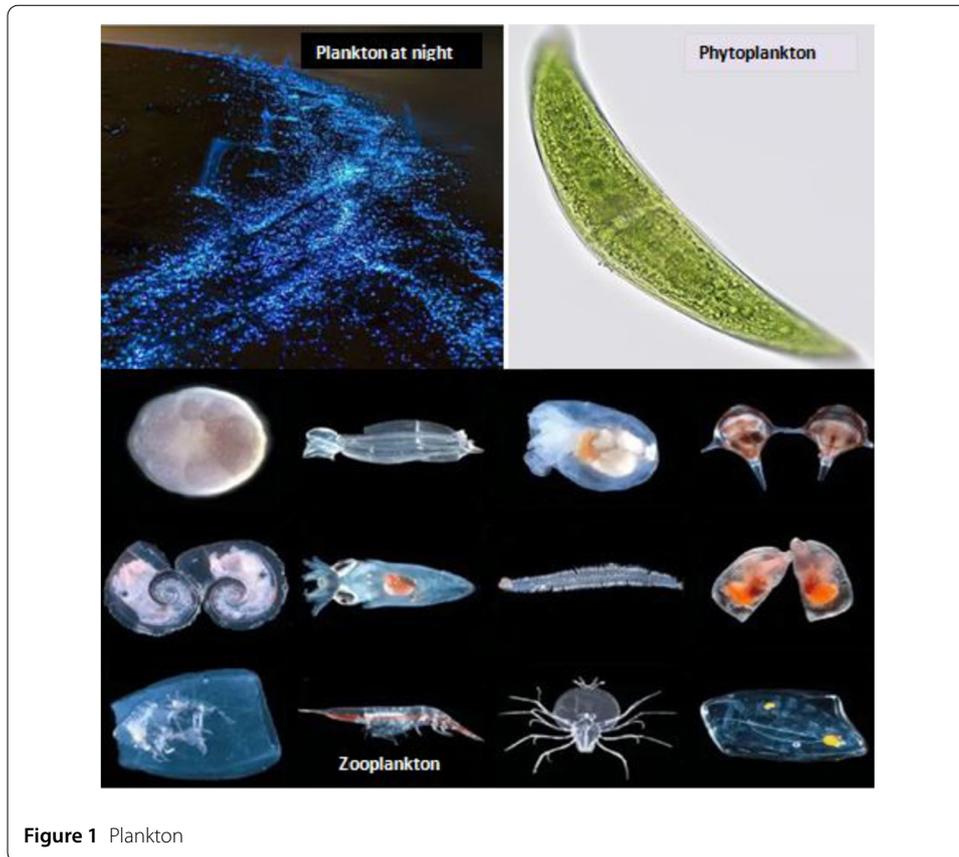
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1 Introduction

The marine ecosystem is comprised of a huge number of complex networks of energy providers and consumers, both large and tiny. One such organism is plankton as seen in Fig. 1 that consists mainly of two types: phytoplankton, also known as microalgae, which are microscopic marine plants, and zooplankton, which are microscopic marine animals. Zooplankton and other small marine creatures eat phytoplankton and then become food for fish, crustaceans, and other larger species. This forms the foundation of the aquatic food web. More sunlight, over supply of nutrients, and slow-moving water cause phytoplankton to grow out of control and this results in the formation of harmful algal blooms (HABs). These blooms produce extremely dangerous toxic compounds that cause a health hazard for animals and humans, damage industries that depend on clean water, elevate treatment costs for drinking water, and create dead zones in the water [1, 2]. Plankton dynamics have attracted the attention of mathematical biologists due to its key role in balancing the marine ecosystem [3–12]. Chattopadaya et al. found that toxin-generated

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phytoplankton influences the growth of the zooplankton population and it also affected phytoplankton–zooplankton association [3, 4]. Das and Roy investigated the phytoplankton and zooplankton association with delayed nutrient cycling from maturation and mortality of phytoplankton in the estuarine system [5]. Roy [7] established a mathematical model to analyze the influence between the toxic and nontoxic phytoplankton concerning a single nutrient. The dynamical behavior of the nutrient–phytoplankton system with a toxic chemical produced by phytoplankton and delay in toxic production is studied in [13].

In their exploratory work, Bertolo et al. [14] discovered that zooplankton refuge minimizes fish impact mortality and phytoplankton development in lakes. Furthermore, it lowers phytoplankton density. To protect themselves from predators, phytoplankton create poisonous chemicals (zooplankton). Thakur and Ojha [15], analyzed the phytoplankton–zooplankton interaction with phytoplankton defense and dormancy in zooplankton and they observed that dormancy controls the fluctuation in population density. The authors explored the study with three-stage plankton–fish interaction by using two different functional responses along with discrete time delay and mutual inference and noticed that the time delay with the defense exhibits chaotic behavior in the plankton system [16]. Saha and Bandopadhyaya suggested a toxin-producing phytoplankton–zooplankton model in which the toxin released by phytoplankton species distinctly varied over time. The impact of ingesting toxic and nontoxic phytoplankton on zooplankton density is examined in the interaction between plankton systems [17]. The influence of harvesting on cohabitation and competitive exclusion of competing predators was studied by Pei et al. [18]. They de-

vised and tested a harvesting paradigm with two zooplankton and one phytoplankton. A phytoplankton–zooplankton model with harvesting was proposed by Yunfei et al. [19]. They concluded that overexploitation may lead to population extinction, but proper harvesting would assure population sustainability. Rehim and Imran [20] looked at a toxic phytoplankton–zooplankton system and looked at how it behaved dynamically. In [21], plankton interaction with time delay in phytoplankton growth and dormancy in zooplankton density is analyzed, also the stability and direction of a bifurcating periodic solution are analyzed. Furthermore, numerous research studies have looked at phytoplankton–zooplankton systems with a nutrient supply, plankton coexistence, the plankton system's harmful effect, or the effect of harvesting [12, 22–28]. The effect of toxicity on the phytoplankton–zooplankton model with harvesting is studied by Chakraborty and Das [29]. Kaur investigates the importance of zooplankton's defense mechanism against fish predation for the plankton–fish ecosystem's coexistence [30]. However, little attention has been dedicated to a model that combines the effects of harvest on phytoplankton and zooplankton populations, as well as zooplankton survival instincts. The expanding human need for more food and energy has resulted in the increased exploitation of numerous biological resources; the world's fish population has decreased dramatically over the last half-century. Moreover, the effect of toxicity is another aspect of concern. Therefore, there is a need for investigating refuge, harvesting, and toxic effects in the plankton–fish dynamics.

Even though fractional differentiation has a 323-year history, the concept has only been widely recognized in the last 55 years. The year 1967, when it was initially updated as part of the Caputo exploration [31], was a breakpoint moment. In recent years, using fractional derivatives (FDs) such as Riemann–Liouville, Caputo, Weyl, Riesz, Grunwald–Letnikov, Marchand and Hifler, Caputo–Fabrizio, and Atangana–Baleanu operators, researchers could derive remarkable results in the fields of signal and image processing, mechanics, chemistry, biology, economics, electricity, and control theory by modeling numerous real-world phenomena [32, 33]. The most vital topics include anomalous diffusion, continuous time random walk, Levy statistics, vibration and control, power law, Riesz potential, fractional Brownian motion, computational fractional differential equations, FD and fractals, history-dependent processes, nonlocal phenomena, porous media, biomedical engineering, fractional variational principles, fractional wavelet, fractional transforms, soft-matter mechanics, fractional predator–prey system, singularities analysis and integral representations for fractional differential systems, fractional signal and image processing, non-Fourier heat conduction, special functions related to fractional calculus, geophysics, acoustic dissipation, relaxation, rheology, fluid dynamics, viscoelasticity, chaos, and groundwater problems. Significant literature of these studies can be found in [34–36]. The research community has paid significant attention to Riemann–Liouville and Caputo FD over the years. The Caputo FD has the advantage of allowing the model formulation to include typical initial and boundary conditions [31]. Additionally, the derivative of a constant is zero under the Caputo FD and the Caputo derivative is defined only for differentiable functions. The fact that the Riemann–Liouville derivative of a constant is not zero causes several changes in modeling assumptions, and the initial conditions must be generalized with fractional order. In addition, the fractional derivation produces singularity at the origin whenever an arbitrary function is a constant at the origin, for instance, exponential and Mittag–Leffler functions. Due to the singularity of the power-law-based FDs,

the commonly used fractional-differential operators have some limitations for simulating certain problems. To deal with this, Caputo and Fabrizio have nurtured a new operator with fractional order. They stressed that many physical occurrences are nonsingular and that using singular operators to simulate nonsingular events could lead to erroneous results. To tackle the problem, they devised the Caputo–Fabrizio FD, a fractional-differential operator using the exponential function as the kernel [37].

In this work, a fractional-order Caputo derivative is employed to study the dynamics of a plankton–fish model in the presence of the toxic compounds produced by harmful algal blooms, which in turn effects all the populations. A combine-harvesting effort is used to harvest the population from extinction. Many times zooplankton will hide to protect themselves from fish. To examine the influence of this phenomenon, we have incorporated refuge of zooplankton in the projected mode. Theoretical aspects are analytically examined in terms of the existence of the solution, positiveness, and boundedness, the existence of equilibrium points, and their stability. The influence of the components such as toxic chemicals, harvesting, and zooplankton refuge are numerically investigated in the presence of various values of the fractional derivative. The numerical results are computed by using the generalized fractional Adams–Bashforth–Moultan technique [38–40].

2 Preliminaries

In this section, we present some results related to the Caputo fractional derivatives that are applied in this work to prove the theoretical aspects.

Definition 2.1 ([35] (Caputo Fractional Derivative)) Suppose $g(t)$ is a k -times continuously differentiable function and $g^{(k)}(t)$ is integrable in $[t_0, T]$. Then, the Caputo fractional derivative of order α for a function $g(t)$ is defined as

$${}^C D_t^\alpha g(t) = \frac{1}{\Gamma(k - \alpha)} \int_{t_0}^t \frac{g^{(k)}(\tau)}{(t - \tau)^{\alpha+1-k}} d\tau,$$

where $\Gamma(\cdot)$ refers to the Gamma function, $t > a$ and k is a positive integer such that $k - 1 < \alpha < k$.

Definition 2.2 ([35]) The Riemann–Liouville fractional-order integral operator is defined by

$$J_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x - t)^{\alpha-1}} dt, \quad \alpha > 0.$$

Lemma 2.3 ([41]) Consider the system

$${}^C D_t^\alpha x(t) = g(t, x), \quad t > t_0, \tag{1}$$

with the initial condition $x(t_0)$, where $0 < \mu \leq 1$ and $g : [t_0, \infty] \times \Omega \rightarrow \mathbb{R}^n$, $\Omega \in \mathbb{R}^n$. If $g(t, x)$ satisfies the locally Lipchitz condition with respect to x , then there exists a unique solution of Eq. (1) on $[t_0, \infty) \times \Omega$.

Lemma 2.4 ([41]) Let $0 < \alpha \leq 1$. Suppose that $g(t) \in C[a, b]$ and ${}^C D_t^\alpha g(t) \in C[a, b]$. If ${}^C D_t^\alpha g(t) \geq 0 \forall t \in (a, b)$, then $g(t)$ is a nondecreasing function for each $t \in [a, b]$. If ${}^C D_t^\alpha g(t) \leq 0 \forall t \in (a, b)$, then $g(t)$ is a nonincreasing function for each $t \in [a, b]$.

Lemma 2.5 ([41]) *We assume that $g(t)$ is a continuous function on $[t_0, +\infty)$ that satisfies*

$$D_{t_0}^\alpha g(t) \leq -\lambda g(t) + \xi, \quad g(t_0) = f_{t_0},$$

here $0 < \alpha \leq 1, (\lambda, \xi) \in \mathbb{R}^2$ and $\lambda \neq 0$ and consider $t_0 \geq 0$ as the initial time. Now,

$$g(t) \leq \left(g(t_0) - \frac{\xi}{\lambda} \right) E_\alpha [-\lambda(t - t_0)^\alpha] + \frac{\xi}{\lambda}.$$

Lemma 2.6 ([41]) *Let $x(t) \in \mathbb{R}_+$ be a continuous and derivable function. Then, for any instant of time $t > t_0$,*

$${}^C D_t^\alpha \left(x(t) - x^* - x^* \ln \frac{x}{x^*} \right) \leq {}^C D_t^\alpha x(t) \left(1 - \frac{x^*}{x} \right), \quad x^* \in \mathbb{R}_+, \forall \alpha \in (0, 1).$$

3 Mathematical model

In the marine ecosystem, the dynamics between plankton and fish is a significant phenomenon. Fish depend on zooplankton, which feed on phytoplankton, for their survival. The poisonous phytoplankton produced by damaging algal blooms affects zooplankton, which in turn affects fish. Since phytoplankton is directly exposed to toxicity, its effect is represented by a cubic term. On the other hand, indirect effects of toxicity on zooplankton and fish are represented by a quadratic term. Zooplankton frequently hide from their attackers and as a result a portion of the zooplankton only are available for prey. To project this phenomenon, we have introduced a refuge term in zooplankton. Due to toxicity, refuge and predation impacts populations that may head towards extinction. For this, we have introduced combine-harvesting terms in all the population. More significantly, it has been found in the literature that the fractional derivative can be used to depict coexistence in population dynamics. We have included the Caputo fractional derivative to explore additional information regarding plankton–fish dynamics because coexistence is crucial for a balanced ecosystem. In this work, we have incorporated the Caputo fractional-order derivative to analyze plankton–fish dynamics by considering phytoplankton, zooplankton, and fish populations in the marine ecosystem. The nature of the interaction between phytoplankton and zooplankton, and zooplankton and fish is represented by a Holling type-II functional response [29, 30]. To establish the biological significance of the system we have made some assumptions. The size of the phytoplankton population, zooplankton population, and fish population, at time t is denoted by $P, Z,$ and $F,$ respectively.

We assume that the phytoplankton population grows logistically with an intrinsic growth rate r and carrying capacity K . $\frac{\beta_1 PZ}{1+P}$ represents the density of the phytoplankton population consumed by zooplankton and $\frac{\beta_2 PZ}{1+P}$ represents the growth that happens in zooplankton due to this. Considering m as the refuge coefficient, we assume that $\frac{\nu m ZF}{1+(1-m)z}$ zooplankton population can hide from the fish population. As a result, a $\frac{\nu(1-m)ZF}{1+(1-m)z}$ amount of the zooplankton population will be consumed by the fish population and hence contribute to the growth of the fish population by $\frac{c\nu(1-m)ZF}{1+(1-m)Z}$, where ν is the attack rate and c is the conversion factor. We considered that the phytoplankton is the first-hand infector by an external toxic substance and zooplankton is affected indirectly by this toxicity after consuming the phytoplankton. Finally, fish consume the infected zooplankton and become

infected. The functional form of harvesting is considered using the catch-per-unit-effort hypothesis [42] and the combined effort (E).

The plankton–fish model in terms of a Caputo fractional derivative is projected as:

$$\begin{aligned} {}^C D_t^\alpha P(t) &= rP \left(1 - \frac{P}{K}\right) - \frac{\beta_1 PZ}{1+P} - \sigma_1 P^3 - q_1 EP, \\ {}^C D_t^\alpha Z(t) &= \frac{\beta_2 PZ}{1+P} - \frac{\nu(1-m)ZF}{1+(1-m)Z} - \delta_1 Z - \sigma_2 Z^2 - q_2 EZ, \\ {}^C D_t^\alpha F(t) &= \frac{c\nu(1-m)ZF}{1+(1-m)Z} - \delta_2 F - wF^2 - q_3 EF, \end{aligned} \quad (2)$$

with initial conditions $P(t_0) > 0$, $Z(t_0) > 0$, $F(t_0) > 0$, where t_0 is the initial time. All the parameters are assumed to be nonnegative, where β_1 and ν are the highest consumption rate of zooplankton and fish, respectively. Accordingly, β_2 and c are the conversion rates of phytoplankton to zooplankton and zooplankton to fish, respectively. δ_1 and δ_2 are the mortality rates of the zooplankton and fish populations, respectively, and the constants q_1 , q_2 , and q_3 are the catchability coefficients of the three species.

The term $\sigma_1 P^3$ represents the loss of the phytoplankton population due to the infection caused by an external toxic substance. $\sigma_2 Z^2$ depicts the loss in zooplankton due to toxicity acquired after consuming phytoplankton and wF^2 states the loss in fish population due to the effect of toxicity acquired on consumption of infected zooplankton. Here, σ_1 , σ_2 , and w are the measure of toxicity with $0 < \sigma_2 < \sigma_1$ and $0 < w < \sigma_1$ [43].

4 Nonnegativity and boundedness

In this section, we illustrate that the solutions of the system (2) are nonnegative and bounded.

Theorem 4.1 *All the solutions of the system (2) that initiate in Θ_+ are nonnegative and uniformly bounded.*

Proof First, we shall show that the solution $P(t)$ that initiates in Θ_+ is nonnegative. That is, $P(t) \geq 0$ for all $t \geq t_0$. Suppose this is not true. Then, there exists a constant time t_2 such that $P(t_2) < 0$. Then, there also exists a constant time t_1 with $t_2 > t_1 > t_0$, such that

$$P(t_1) = 0. \quad (3)$$

Based on the relation in equation (3) and system (2), we have

$${}^C D_{t_1}^\alpha P(t) = 0.$$

From Lemma 2.4, we have $P(t_2) = 0$, which contradicts the fact that $P(t_2) < 0$. Hence, $P(t) \geq 0$ for all $t \geq t_0$. Similar justification is valid for the solutions $Z(t)$ and $F(t)$.

Now, we shall prove that the solutions of the system (2) that bring in Θ_+ are uniformly bounded.

Let us define the function $\chi(t) = P(t) + Z(t) + \frac{1}{c}F(t)$.

On applying the Caputo fractional derivative, we have

$${}^C D_t^\alpha \chi(t) = {}^C D_t^\alpha P(t) + {}^C D_t^\alpha Z(t) + \left(\frac{1}{c}\right) {}^C D_t^\alpha F(t)$$

$$\begin{aligned}
 &= rP\left(1 - \frac{P}{K}\right) - \frac{\beta_1 PZ}{1+P} - \sigma_1 P^3 \\
 &\quad - q_1 EP + \frac{\beta_2 PZ}{1+P} - \frac{v(1-m)ZF}{1+(1-m)Z} - \delta_1 Z - \sigma_2 Z^2 - q_2 EZ \\
 &\quad + \frac{1}{c}\left(\frac{cv(1-m)ZF}{1+(1-m)Z} - \delta_2 F - wF^2 - q_3 EF\right) \\
 &= rP\left(1 - \frac{P}{K}\right) - \frac{\beta_1 PZ}{1+P} - \sigma_1 P^3 - q_1 EP + \frac{\beta_2 PZ}{1+P} - \delta_1 Z - \sigma_2 Z^2 - q_2 EZ \\
 &\quad - \frac{1}{c}(\delta_2 F + wF^2 + q_3 EF). \tag{4}
 \end{aligned}$$

Consider,

$$\begin{aligned}
 {}^C D_t^\alpha \chi(t) + L\chi(t) &= rP\left(1 - \frac{P}{K}\right) - \frac{\beta_1 PZ}{1+P} - \sigma_1 P^3 - q_1 EP + \frac{\beta_2 PZ}{1+P} - \delta_1 Z - \sigma_2 Z^2 - q_2 EZ \\
 &\quad - \frac{1}{c}(\delta_2 F + wF^2 + q_3 EF) + L\left(P(t) + Z(t) + \frac{1}{c}F(t)\right) \\
 &= P\left(\left(r\left(1 - \frac{P}{K}\right) + L\right) - \frac{\beta_1 Z}{1+P} - \sigma_1 P^2 - q_1 E + \frac{\beta_2 Z}{1+P}\right) \\
 &\quad - Z((\delta_1 + q_2 E) - L) - \frac{1}{c}(\delta_2 - L) - \frac{1}{c}(wF^2 + q_3 EF + \sigma_2 cz^2), \tag{5}
 \end{aligned}$$

where, L is a positive constant. If $L = \min\{(\delta_1 + q_2 E), \delta_2\}$, then

$$\begin{aligned}
 {}^C D_t^\alpha \chi(t) + L\chi(t) &\leq \left(-\frac{P^2 r}{K} + Pr + PL + \frac{\beta_2 PZ}{1+P}\right) \\
 &\leq -\frac{r}{K}(P - Q)^2 + \frac{K}{r}Q^2, \quad \text{where } Q = \frac{K}{2r}\left(r + L + \frac{\beta_2 Z}{1+P}\right) \tag{6}
 \end{aligned}$$

which implies that,

$${}^C D_t^\alpha \chi(t) + L\chi(t) \leq \frac{K}{r}Q^2. \tag{7}$$

On applying Lemma 2.5, we obtain $0 < \chi(t) \leq \frac{K}{rL}Q^2 + (\chi(0) - \frac{K}{rL}Q^2)E_\alpha(-Lt^\alpha)$, $t \geq 0$.

By Lemma 2.5, this implies that, $E_\alpha(-Lt^\alpha) \rightarrow 0$ as $t \rightarrow \infty$.

Therefore, $0 < \lim_{t \rightarrow \infty} \chi(t) \leq \frac{K}{rL}Q^2$.

Hence, all the solutions of (2) initiating in \mathbb{R}_+^3 are limited in $\Theta = \{(P, Z, F) \in \mathbb{R}_+^3 : 0 < P(t) + Z(t) + \frac{1}{c}F(t) < \frac{K}{rL}Q^2 + \phi, \phi > 0\}$.

This complete the proof. □

5 Existence and uniqueness of the solution

In this section, we establish the existence and uniqueness of the solution of the model (2) by applying the Banach Fixed-Point Theorem. As the model is complex and nonlocal, there are no accurate methods or approaches for evaluating exact solutions. However, if certain conditions are satisfied, the existence of a solution is assured. System (2) can be rewritten as follows:

$${}^C D_t^\alpha [P(t)] = \zeta_1(t, P),$$

$$\begin{aligned} {}^C D_t^\alpha [Z(t)] &= \zeta_2(t, Z), \\ {}^C D_t^\alpha [F(t)] &= \zeta_3(t, F). \end{aligned} \quad (8)$$

The Volterra integral equation corresponding to the above equation is:

$$\begin{aligned} P(t) - P(t_0) &= \frac{1}{\Gamma(\alpha)} \int_0^t \zeta_1(\tau, P(\tau))(t - \tau)^{\alpha-1} d\tau, \\ Z(t) - Z(t_0) &= \frac{1}{\Gamma(\alpha)} \int_0^t \zeta_2(\tau, Z(\tau))(t - \tau)^{\alpha-1} d\tau, \\ F(t) - F(t_0) &= \frac{1}{\Gamma(\alpha)} \int_0^t \zeta_3(\tau, F(\tau))(t - \tau)^{\alpha-1} d\tau. \end{aligned} \quad (9)$$

Theorem 5.1 *In the region $\Theta \times [t_0, T]$, where $\Theta = \{(P, Z, F) \in \mathbb{R}_+^3 : \max\{|P|, |Z|, |F|\} \leq M\}$ and $T \leq +\infty$, the Lipschitz condition is satisfied and contraction occurs by the kernel ζ_1 if $0 \leq r + \frac{2Mr}{K} + \beta_1 M + 3\sigma_1 M^2 + q_1 E < 1$ is held true.*

Proof Consider the two functions P and \bar{P} such that

$$\begin{aligned} &\| \zeta_1(t, P) - \zeta_1(t, \bar{P}) \| \\ &= \left\| rP \left(1 - \frac{P}{K} \right) - \frac{\beta_1 PZ}{1+P} - \sigma_1 P^3 - q_3 EP - r\bar{P} \left(1 - \frac{\bar{P}}{K} \right) + \frac{\beta_1 \bar{P}Z}{1+\bar{P}} + \sigma_1 \bar{P}^3 + q_3 E\bar{P} \right\| \\ &\leq \left(r + q_3 E + \frac{2Mr}{K} \right) \| P - \bar{P} \| + \frac{\beta_1 Z}{(1+P)(1+\bar{P})} \| P - \bar{P} \| + \sigma_1 \| P^3 - \bar{P}^3 \| \\ &\leq \left(r + \frac{2Mr}{K} + \beta_1 M + 3\sigma_1 M^2 + q_3 E \right) \| P - \bar{P} \| \\ &= \eta_1 \| P - \bar{P} \|, \end{aligned} \quad (10)$$

where $\eta_1 = (r + \frac{2Mr}{K} + \beta_1 M + 3\sigma_1 M^2 + q_1 E)$. As a result, the Lipschitz condition is met for η_1 and if $0 \leq \eta_1 < 1$, then η_1 follows contraction. Similarly, it can be exhibited and demonstrated in the other equations as follows:

$$\begin{aligned} \| \zeta_2(t, Z) - \zeta_2(t, \bar{Z}) \| &\leq \eta_2 \| Z - \bar{Z} \|, \\ \| \zeta_3(t, F) - \zeta_3(t, \bar{F}) \| &\leq \eta_3 \| F - \bar{F} \|, \end{aligned} \quad (11)$$

where $\eta_2 = \delta_1 + 2M\sigma_2 + \beta_2 + M\nu(1-m) + q_2 E$ and $\eta_3 = \delta_2 + 2Mw + q_3 E + c\nu(1-m)$, η_i are the contractions if $0 < \eta_i < 1$, $i = 2, 3$ \square

Theorem 5.2 *The solution of the fractional model (2) exists and will be unique, if we acquire some t_α such that $\frac{1}{\Gamma(\alpha)} \eta_i t_\alpha < 1$, for $i = 1, 2, 3$.*

Proof The proof of this theorem is illustrated in three steps.

1. From system (9), the recurrent form can be written as follows:

$$G_{1,n}(t) = P_n(t) - P_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\zeta_1(\tau, P_{n-1}) - \zeta_1(\tau, P_{n-2}))(t - \tau)^{\alpha-1} d\tau,$$

$$\begin{aligned}
 G_{2,n}(t) &= Z_n(t) - Z_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\zeta_2(\tau, Z_{n-1}) - \zeta_2(\tau, Z_{n-2}))(t - \tau)^{\alpha-1} d\tau, \\
 G_{3,n}(t) &= F_n(t) - F_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\zeta_3(\tau, Z_{n-1}) - \zeta_3(\tau, Z_{n-2}))(t - \tau)^{\alpha-1} d\tau. \tag{12}
 \end{aligned}$$

The conditions are: $P_0(t) = P(0)$, $Z_0(t) = Z(0)$, $F_0(t) = F(0)$. By applying the norm to the equation (12), we obtain

$$\begin{aligned}
 \|G_{1,n}\| &= \|P_n(t) - P_{n-1}(t)\| \\
 &= \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (\zeta_1(\tau, P_{n-1}) - \zeta_1(\tau, P_{n-2}))(t - \tau)^{\alpha-1} d\tau \right\| \\
 &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \|(\zeta_1(\tau, P_{n-1}) - \zeta_1(\tau, P_{n-2}))(t - \tau)^{\alpha-1} d\tau\|. \tag{13}
 \end{aligned}$$

Using the Lipchitz condition in equation (13), we obtain:

$$\|G_{1,n}(t)\| \leq \frac{1}{\Gamma(\alpha)} \eta_1 \int_0^t \|G_{1,n-1}(\tau) d\tau\|. \tag{14}$$

Subsequently we have,

$$\begin{aligned}
 \|G_{2,n}(t)\| &\leq \frac{1}{\Gamma(\alpha)} \eta_2 \int_0^t \|G_{2,n-1}(\tau) d\tau\|, \\
 \|G_{3,n}(t)\| &\leq \frac{1}{\Gamma(\alpha)} \eta_3 \int_0^t \|G_{3,n-1}(\tau) d\tau\|, \tag{15}
 \end{aligned}$$

which implies that it can be written as: $P_n(t) = \sum_{i=1}^n G_{1,i}$, $Z_n(t) = \sum_{i=1}^n G_{2,i}$, $F_n(t) = \sum_{i=1}^n G_{3,i}$. Applying equations (14) and (15) recursively, we have:

$$\begin{aligned}
 \|G_{1,i}(t)\| &\leq \|P_n(t_0)\| \left[\frac{1}{\Gamma(\alpha)} \eta_1 t \right]^n, \\
 \|G_{2,i}(t)\| &\leq \|Z_n(t_0)\| \left[\frac{1}{\Gamma(\alpha)} \eta_2 t \right]^n, \\
 \|G_{3,i}(t)\| &\leq \|F_n(t_0)\| \left[\frac{1}{\Gamma(\alpha)} \eta_3 t \right]^n. \tag{16}
 \end{aligned}$$

As a consequence, the existence and continuity are demonstrated.

- To show that the association equation (16) works out the solutions for equation (2), we review the following:

$$\begin{aligned}
 P(t) - P(t_0) &= P_n(t) - \psi_{1n}(t), \\
 Z(t) - Z(t_0) &= Z_n(t) - \psi_{2n}(t), \\
 F(t) - F(t_0) &= F_n(t) - \psi_{3n}(t). \tag{17}
 \end{aligned}$$

In order to execute the desired results, we set

$$\|\psi_{1n}(t)\| = \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (\zeta_1(\tau, P) - \zeta_1(\tau, P_{n-1})) d\tau \right\|$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(\alpha)} \eta_1 \int_0^t \|\zeta_1(\tau, P) - \zeta_1(\tau, P_{n-1})\| d\tau \\ &\leq \frac{1}{\Gamma(\alpha)} \eta_1 \|P - P_{n-1}\| t. \end{aligned} \tag{18}$$

Continuing the same procedure recursively, we obtain

$$\|\psi_{1n}(t)\| \leq \left(\frac{1}{\Gamma(\alpha)} \eta_1 t\right)^{n+1} M. \tag{19}$$

At t_α , we have

$$\|\psi_{1n}(t)\| \leq \left(\frac{1}{\Gamma(\alpha)} \eta_1 t_\alpha\right)^{n+1} M. \tag{20}$$

From equation (20), as we can see n tends to ∞ , $\|\psi_{1n}\|$ approaches to 0. Similarly, it may be proven that $\|\psi_{2n}\|, \|\psi_{3n}\|$ tends to 0

- To establish the uniqueness for a solution of the system (2). Consider the different set of solutions for the system (2), say $\tilde{P}, \tilde{Z}, \tilde{F}$. Then, as an outcome of the first equation of (2) we write:

$$P(t) - \tilde{P}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\zeta_1(t, P) - \zeta_1(t, \tilde{P})) d\tau.$$

Using the norm, the above equation becomes:

$$\|P(t) - \tilde{P}(t)\| = \frac{1}{\Gamma(\alpha)} \int_0^t \|(\zeta_1(t, P) - \zeta_1(t, \tilde{P}))\| d\tau. \tag{21}$$

By applying the Lipchitz condition we obtain

$$\|P(t) - \tilde{P}(t)\| \leq \frac{1}{\Gamma(\alpha)} \eta_1 t \|P - \tilde{P}\|.$$

At some t_α this result yields,

$$\|P(t) - \tilde{P}(t)\| \left(1 - \frac{1}{\Gamma(\alpha)} \eta_1 t_\alpha\right) \leq 0.$$

Since, $(1 - \frac{1}{\Gamma(\alpha)} \eta_1 t_\alpha) > 0$, we have $\|P(t) - \tilde{P}(t)\| = 0$. Hence, $P(t) = \tilde{P}(t)$. □

6 Existence equilibrium points and their stability

In this section, we examine the existence and stability of equilibrium points of the system (2). To evaluate the stability of the equilibrium points we determine the Jacobian matrix as:

$$J(P, Z, F) = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

where

$$\begin{aligned}
 b_{11} &= -\frac{Pr}{K} + r\left(1 - \frac{P}{K}\right) - Eq_1 + \frac{\beta_1 PZ}{(1+P)^2} - \frac{\beta_1 Z}{1+P} - P^2 \sigma_1, \\
 b_{12} &= \frac{\beta_1 P}{1+P}, \quad b_{13} = 0, \quad b_{21} = -\frac{PZ\beta_2}{(1+P)^2} + \frac{Z\beta_2}{1+P}, \\
 b_{22} &= \frac{\nu F(1-m)^2 Z}{(1+(1-m)Z)^2} - \frac{\nu F(1-m)}{1+(1-m)Z} - Eq_2 + \frac{P\beta_2}{1+P} - \delta_1 - 2Z\sigma_2, \\
 b_{23} &= -\frac{\nu(1-m)Z}{1+(1-m)Z}, \quad b_{31} = 0, \quad b_{32} = -\frac{c\nu(1-m)^2 FZ}{(1+(1-m)Z)^2} + \frac{c\nu(1-m)ZF}{1+(1-m)Z}, \\
 b_{33} &= \frac{c\nu(1-m)Z}{1+(1-m)Z} - 2wF - \delta_2 - q_3E.
 \end{aligned}$$

1. The trivial equilibrium point $A_0(0, 0, 0)$ always exists.

Theorem 6.1 *Stability of the trivial equilibrium point $A_0(0, 0, 0)$ is biologically impossible.*

Proof The eigenvalues of the Jacobian matrix at A_0 are:

$$\begin{aligned}
 \lambda_{11} &= r - q_1E, \\
 \lambda_{12} &= -(\delta_1 + q_2E), \\
 \lambda_{13} &= -(\delta_2 + q_3E).
 \end{aligned}$$

Consequently, $A_0(0, 0, 0)$ is a stable node when the growth rate (r) is less than the combine-harvesting effort (Eq_1) but from the biological point of view it is not possible because overutilization $Eq_1 > r$ would result in the annihilation of the whole population. \square

2. Let $R_0 = \frac{r}{Eq_1}$.

The boundary equilibrium point is $A_1(P^*, 0, 0)$, where $P^* = \frac{1}{2\sigma_1} \left(\sqrt{\left(\frac{r}{K}\right)^2 - 4\sigma_1(r - Eq_1) - \frac{r}{K}} \right)$. A_1 exists if $R_0 > 1$.

Theorem 6.2 *Let $R_1 = \frac{\beta_2 P^*}{(1+P^*)(Eq_2 + \delta_1)}$. Then, A_1 is locally asymptotically stable if $R_1 < 1$.*

Proof The following are the set of eigenvalues of Jacobian matrix J at A_1 :

$$\begin{aligned}
 \lambda_{21} &= -(Eq_3 + \delta_2), \\
 \lambda_{22} &= -\left(Eq_2 + \delta_1 - \frac{\beta_2 P^*}{1+P^*}\right) = (R_1 - 1)(Eq_2 + \delta_1), \\
 \lambda_{23} &= -\left(Eq_1 + 3P^{*2} \sigma_1 + \frac{2P^* r}{K} - r\right).
 \end{aligned}$$

At A_1 , $|\arg(\lambda_{21})| = \pi > \frac{\alpha\pi}{2}$ and $|\arg(\lambda_{23})| = \pi > \frac{\alpha\pi}{2}$. Hence, A_1 is a stable node if $|\arg(\lambda_{22})| > \frac{\alpha\pi}{2}$. For this, the required condition is $R_1 < 1$. \square

3. The fish-free equilibrium point is $A_2(P_1, Z_1, 0)$, where P_1, Z_1 are solutions of the system:

$$r\left(1 - \frac{P_1}{K}\right) - \frac{\beta_1 Z_1}{1 + P_1} - \sigma_1 P_1^2 - q_1 E = 0, \tag{22}$$

$$\frac{\beta_2 P_1}{1 + P_1} - \delta_1 - \sigma_2 Z_1 - q_2 E = 0. \tag{23}$$

Sequentially, we can obtain Z_1 from equation (23) as follows:

$$Z_1 = \frac{1}{\sigma_2} \left(\frac{\beta_2 P_1}{1 + P_1} - \delta_1 - q_2 E \right). \tag{24}$$

Therefore, P_1 satisfies the fourth-degree equation,

$$H_1 P_1^4 + H_2 P_1^3 + H_3 P_1^2 + H_4 P_1 + H_5 = 0, \tag{25}$$

where,

$$H_1 = K\sigma_1\sigma_2,$$

$$H_2 = \sigma_2(2K\sigma_1 + r),$$

$$H_3 = K\sigma_1\sigma_2 + 2r\sigma_2 - K\sigma_2(r - Eq_1),$$

$$H_4 = r\sigma_2 - K\beta_1(\delta_1 - \beta_2) - 2K\sigma_2(r - Eq_1),$$

$$H_5 = -K\sigma_2(r - Eq_1) - K\beta_1(\delta_1 + Eq_2).$$

Since H_5 is negative, equation (25) has at least one positive root. The zooplankton biomass will be positive, if from (22) we obtain

$$\begin{aligned} r\left(1 - \frac{P_1}{K}\right) - \sigma_1 P_1^2 - q_1 E &> 0 \\ \implies r - \sigma_1\left(P_1^2 + \frac{r}{K\sigma_1}P_1\right) - q_1 E &> 0, \\ q_1 E < r + \frac{r^2}{4K^2\sigma_1} - \sigma_1\left(P_1 + \frac{r}{2K\sigma_1}\right)^2 &< r + \frac{r^2}{4K^2\sigma_1}. \end{aligned}$$

Therefore,

$$E < \frac{1}{q_1} \left(r + \frac{r^2}{4K^2\sigma_1} \right) = \Upsilon_1. \tag{26}$$

Also, from (24), we have:

$$Z_1 = \frac{1}{\sigma_1} \left(\frac{\beta_2 P_1}{1 + P_1} - (\delta_1 + q_2 E) \right) > 0.$$

This implies,

$$\frac{\beta_2 P_1}{1 + P_1} - (\delta_1 + q_2 E) > 0.$$

Thus,

$$\delta_1 + Eq_2 < \frac{\beta_2 P_1}{1 + P_1} < \beta_2.$$

Hence, we obtain the following inequality,

$$\beta_2 - \delta_1 > Eq_2 \implies E < \frac{\beta_2 - \delta_1}{q_2} = \Upsilon_2. \tag{27}$$

Finally, using inequalities (26) and (27), we can limit the harvesting effort as

$$0 < E < \min\{\Upsilon_1, \Upsilon_2\}.$$

Hence, the fish-free equilibrium point $A_2(P_1, Z_1, 0)$ exists.

- The interior equilibrium point $A_3(P_2, Z_2, F_2)$ is obtained by solving the following equations

$$r\left(1 - \frac{P_2}{K}\right) - \frac{\beta_1 Z_2}{1 + P_2} - \sigma_1 P_2^2 - q_1 E = 0, \tag{28}$$

$$\frac{\beta_2 P_2}{1 + P_2} - \frac{v(1 - m)F_2}{1 + (1 - m)Z_2} - \delta_1 - \sigma_2 Z_2 - q_2 E = 0, \tag{29}$$

$$\frac{cv(1 - m)Z_2}{1 + (1 - m)Z_2} - \delta_2 - wF_2 - q_3 E = 0. \tag{30}$$

Solving the equation (30) we obtain $F_2 = \frac{(1-m)Z_2(cv-\delta_2-Jq_3)-\delta_2-Jq_3}{w((1-m)Z_2+1)}$.

Solving the equation (29) we obtain $P_2 = \frac{F_2(1-m)vZ_2+((1-m)Z_2+1)(\delta_1+Jq_2+\sigma_2Z_2)}{((1-m)Z_2+1)(\beta_2-Jq_2-\sigma_2Z_2)-F(1-m)vZ_2}$.

Substituting the values of F_2 and Z_2 into equation (28) we obtain an equation in the variable Z_2 , solving which we obtain the interior equilibrium point A_3 . The dynamics of the interior equilibrium point is analyzed with the help of numerical simulation in the following section.

Theorem 6.3 *The interior equilibrium point A_3 of the system (2) is globally asymptotically stable if $(\beta_2 c Z_2 - \frac{\beta_2 c r}{K \beta_1} - \frac{\sigma_1 \beta_2 c}{\beta_1} (P + P_2)) < 0$, $cv(1 - m)^2 F_2 < \sigma_2 c$, $\Gamma_1 \phi_1 + P_2 Z_2 < (1 + \Gamma_2)(\Gamma_2 Z_2 + \phi_2 P_2)$, $\phi_1 \theta_1 + Z_2 F_2 > (1 + (1 - m)\phi_2)(\phi_2 F_2 + \theta_2 Z_2)$.*

Proof Consider the positive-definite function,

$$Y(P, Z, F) = \frac{\beta_2 c}{\beta_1} \left(P - P_2 - P_2 \ln \frac{P}{P_2} \right) + c \left(Z - Z_2 - Z_2 \ln \frac{Z}{Z_2} \right) + \left(F - F_2 - F_2 \ln \frac{F}{F_2} \right).$$

By employing the Caputo fractional-order derivative we have,

$$\begin{aligned} {}^c_0 D_t^\alpha Y(P, Z, F) &\leq \frac{\beta_2 c}{\beta_1} \left(1 - \frac{P_2}{P} \right) D_t^\alpha P + c \left(1 - \frac{Z_2}{Z} \right) D_t^\alpha Z + \left(1 - \frac{F_2}{F} \right) D_t^\alpha F \\ &= \frac{\beta_2 c}{\beta_1} \left(\frac{P - P_2}{P} \right) \left(rP \left(1 - \frac{P}{K} \right) - \frac{\beta_1 P Z}{1 + P} - \sigma_1 P^3 - q_1 E P \right) \\ &\quad + c \left(\frac{Z - Z_2}{Z} \right) \left(\frac{\beta_2 P Z}{1 + P} - \frac{v(1 - m) Z F}{1 + (1 - m) Z} - \sigma_2 Z^2 \right) \end{aligned}$$

$$\begin{aligned}
& - Z(\delta_1 + q_2 E) \Big) + \left(\frac{F - F_2}{F} \right) \left(\frac{cv(1-m)ZF}{1 + (1-m)Z} - wF^2 - F(\delta_2 + q_3 E) \right) \\
& = \frac{\beta_2 c}{\beta_1} (P - P_2) \left(\frac{-r}{k} (P - P_2) - \sigma_1 (P^2 - P_2^2) \right) \\
& \quad - \sigma_2 c (Z - Z_2)^2 - w(F - F_2)^2 - \frac{\beta_2 c Z P_2 (P - P_2)}{(1 + P)(1 + P_2)} \\
& \quad + \frac{\beta_2 c Z_2 P (P - P_2)}{(1 + P)(1 + P_2)} - \frac{cv(1-m)^2 (Z - Z_2) F Z_2}{(1 + (1-m)Z)((1 + (1-m)Z_2))} \\
& \quad + \frac{cv(1-m)^2 (Z - Z_2) F_2 Z}{(1 + (1-m)Z)((1 + (1-m)Z_2))} \\
& \leq \left(\frac{\beta_2 c Z_2}{(1 + P)(1 + P_2)} - \frac{\beta_2 cr}{K \beta_1} - \frac{\sigma_1 \beta_2 c}{\beta_1} (P + P_2) \right) (P - P_2)^2 \\
& \quad - \frac{\beta_2 P_2 c (P - P_2) (Z - Z_2)}{(1 + P)(1 + P_2)} \\
& \quad + \left(\frac{cv(1-m)^2 F_2}{(1 + (1-m)z)(1 + (1-m)Z_2)} - \sigma_2 c \right) (Z - Z_2)^2 \\
& \quad - \frac{cv Z_2 (1-m)^2 (Z - Z_2) (F - F_2)}{(1 + (1-m)Z)(1 + (1-m)Z_2)} \\
& \quad - w(F - F_2)^2 \\
& = \left(\frac{\beta_2 c Z_2}{(1 + P)(1 + P_2)} - \frac{\beta_2 cr}{K \beta_1} - \frac{\sigma_1 \beta_2 c}{\beta_1} (P + P_2) \right) (P - P_2)^2 \\
& \quad + \left(\frac{cv(1-m)^2 F_2}{(1 + (1-m)z)(1 + (1-m)Z_2)} - \sigma_2 c \right) (Z - Z_2)^2 - w(F - F_2)^2 \\
& \quad - \frac{1}{(1 + P)(1 + P_2)} (\beta_2 P_2 c (PZ - PZ_2 - P_2 Z + P_2 Z_2)) \\
& \quad - \frac{1}{(1 + (1-m)Z)(1 + (1-m)Z_2)} \\
& \quad \times (cv Z_2 (1-m)^2 ((ZF - ZF_2 - Z_2 F + Z_2 F_2)))
\end{aligned}$$

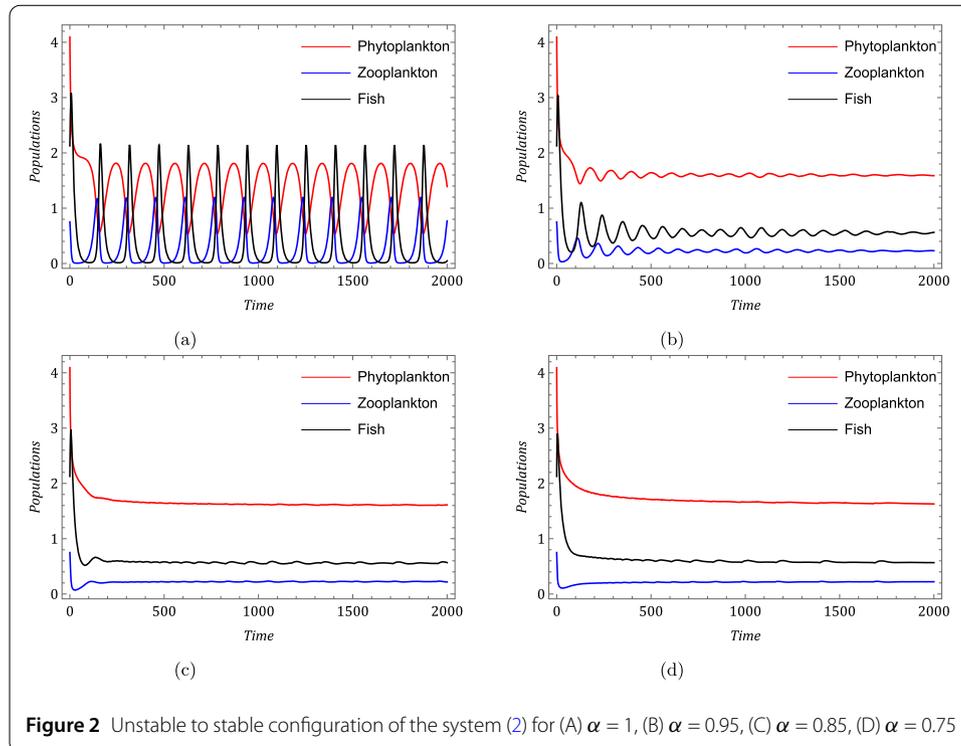
Let $\Gamma_1 < P < \Gamma_2$, $\phi_1 < Z < \phi_2$, $\theta_1 < F < \theta_2$.

$$\begin{aligned}
{}^C D_t^\alpha Y(P, Z, F) & \leq \left(\beta_2 c Z_2 - \frac{\beta_2 cr}{K \beta_1} - \frac{\sigma_1 \beta_2 c}{\beta_1} (P + P_2) \right) (P - P_2)^2 \\
& \quad + (cv(1-m)^2 F_2 - \sigma_2 c) (Z - Z_2)^2 - w(F - F_2)^2 \\
& \quad - \frac{c \beta_2 P_2}{1 + P_2} \left(\frac{\Gamma_1 \phi_1 + P_2 Z_2}{1 + \Gamma_2} - \Gamma_2 Z_2 - \phi_2 P_2 \right) \\
& \quad - \frac{cv Z_2 (1-m)^2}{1 + (1-m)Z_2} \left(\frac{\phi_1 \theta_1 + Z_2 F_2}{1 + (1-m)\phi_2} - \phi_2 F_2 - \theta_2 Z_2 \right)
\end{aligned}$$

${}^C D_t^\alpha Y(P, Z, F) \leq 0$ if $(\beta_2 c Z_2 - \frac{\beta_2 cr}{K \beta_1} - \frac{\sigma_1 \beta_2 c}{\beta_1} (P + P_2)) < 0$, $cv(1-m)^2 F_2 < \sigma_2 c$, $\Gamma_1 \phi_1 + P_2 Z_2 < (1 + \Gamma_2)(\Gamma_2 Z_2 + \phi_2 P_2)$, $\phi_1 \theta_1 + Z_2 F_2 > (1 + (1-m)\phi_2)(\phi_2 F_2 + \theta_2 Z_2)$. \square

7 Numerical simulation

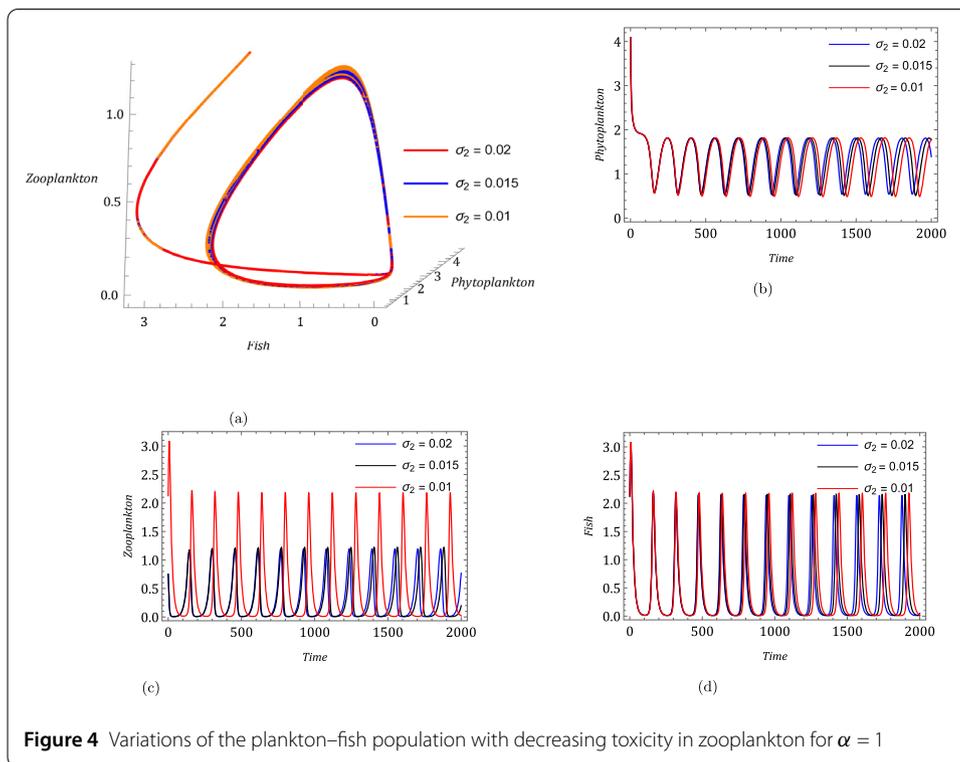
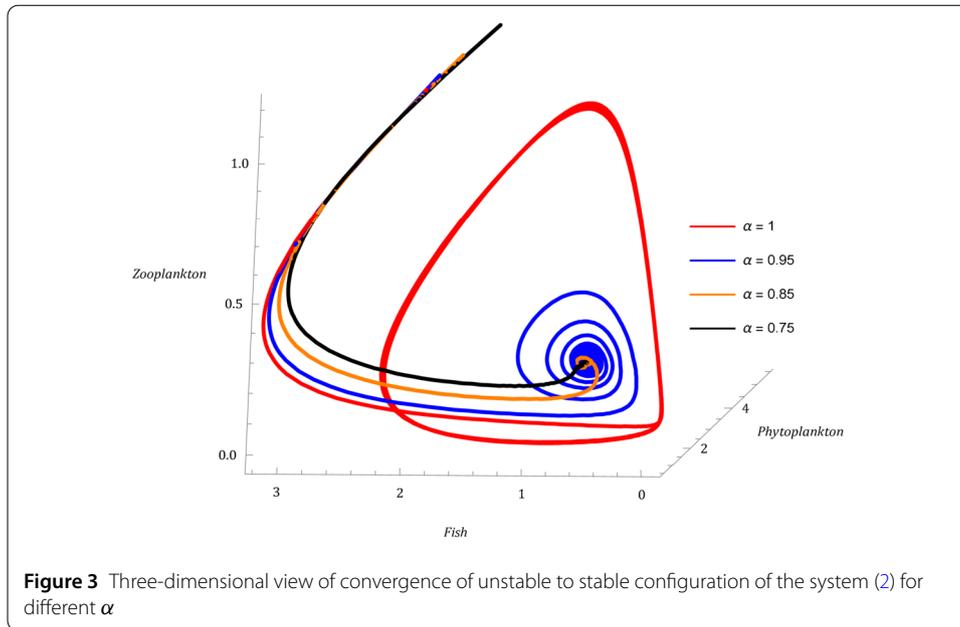
In this section, we analyze the dynamical behavior of the proposed fractional order system (2) numerically with a different set of parameters using the generalized Adams–



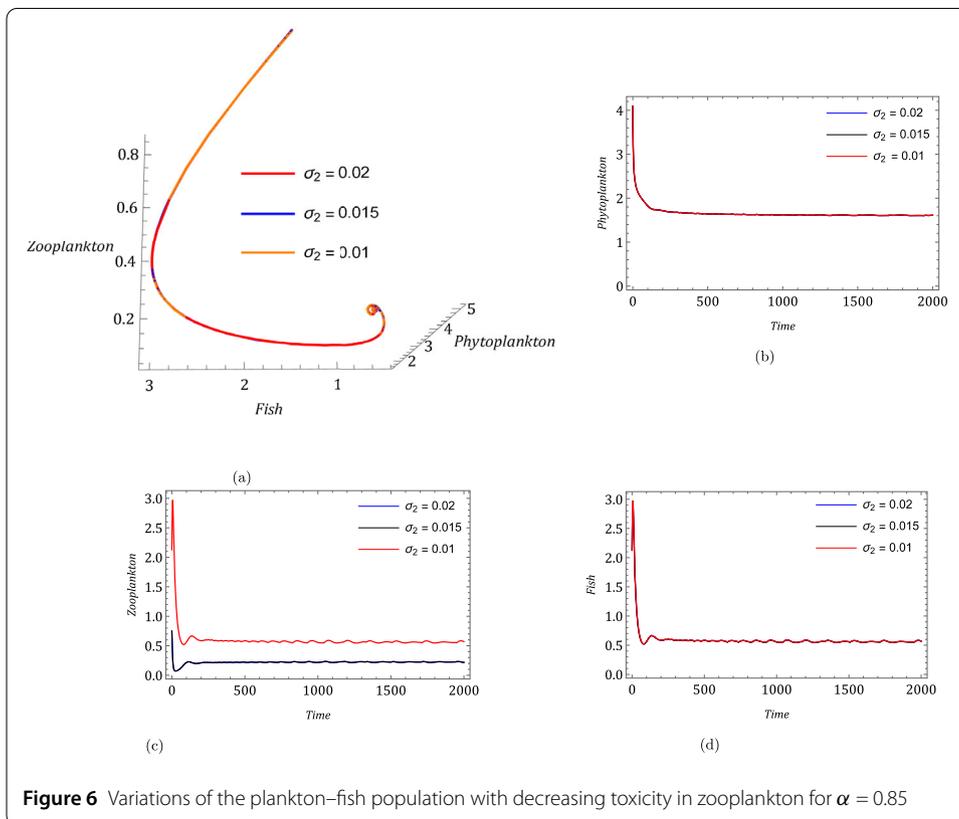
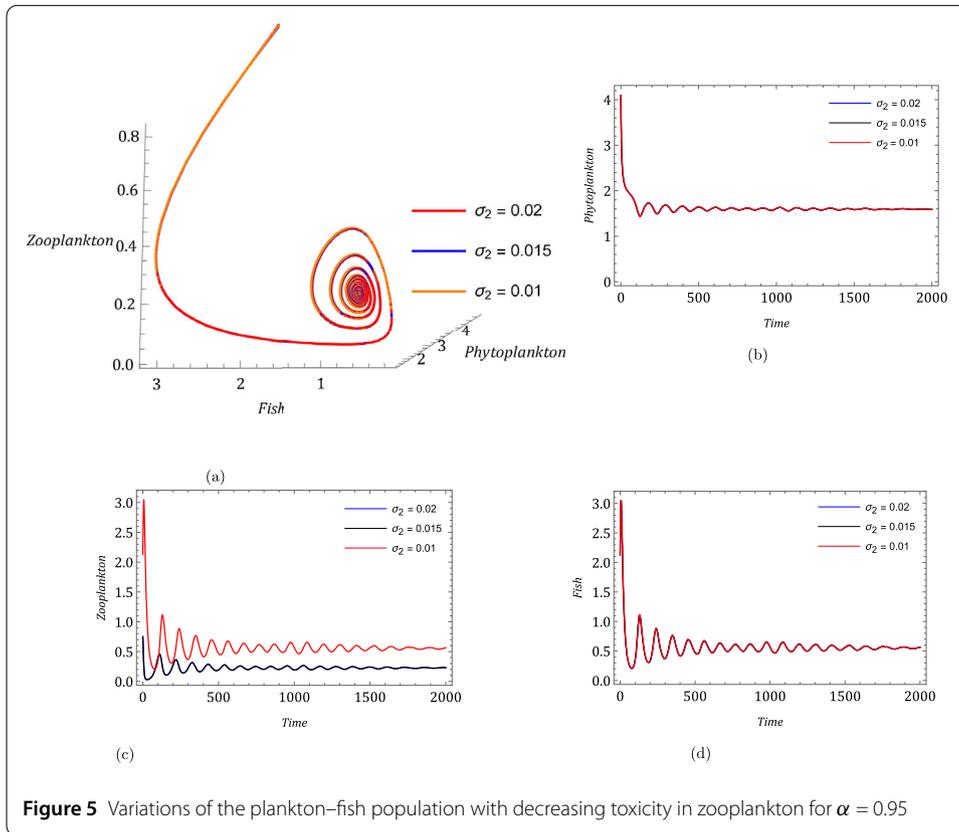
Bashforth–Moulton method [38–40]. By considering the parameters values as $r = 0.5$, $K = 5$, $\beta_1 = 1.1$, $\beta_2 = 2$, $\sigma_1 = 0.057$, $\sigma_2 = 0.02$, $w = 0.025$, $\delta_1 = 0.5$, $\delta_2 = 0.5$, $\nu = 1.42$, $c = 3.02$, $q_1 = 0.2$, $q_2 = 0.3$, $q_3 = 0.4$, and $E = 0.5$ we have presented the stability profile of the solutions of the system (2) for various values of α . Figure 2a shows the unstable behavior of the system (2) for the integer-order derivative around the coexistence equilibrium point $A_3(P_2, Z_2, F_2) = A_3(1.5922, 0.2269, 0.5509)$. However, as the value of α is considered to be fractional in Figs. 2b–2d we observe that the system (2) converges towards the stable point at a finite time. Figure 3 shows the clear result of this observation. From this, we can conclude that the dynamics of the system (2) can be changed from unstable to stable by the composition of the fractional-order derivative and hence it ensures the presence of a Hopf bifurcation.

Since zooplankton directly consumes toxin-producing phytoplankton, its density has a greater impact on the chemical toxicity of the water than do phytoplankton or fish populations. As toxicity rises, zooplankton density falls. From Figs. 4–6 we can observe the effect of toxicity present in zooplankton (σ_2) with the changing values of α . The influence of the fractional derivative in bringing the system to stability is visible in these figures. From Fig. 4, when $\alpha = 1$, decreased toxicity of zooplankton shows the fluctuation in zooplankton population, whereas the toxicity effect on phytoplankton and fish populations is minimum. When $\alpha = 0.95$, the toxicity effect on the zooplankton population is delayed (Fig. 5). A similar consequence is observed in Fig. 6 when $\alpha = 0.85$. From Figs. 5 and 6 we can conclude that if we have to depict a situation where oscillation in marine-population dynamics varies in the presence of a changing toxic effect, then the fractional derivative is an essential tool for this.

Fish can become infected by marine biotoxins produced by specific types of algae. These toxins have the potential to harm a lot of fish. The increased toxicity of fish affects the fish



population as well as the plankton population. For $\alpha = 1$ the oscillatory nature in plankton and fish populations is very prominent (Fig. 7). When $\alpha = 0.95$ and $\alpha = 0.8$ the solution profile converge from an unstable spiral to a stable focus (Figs. 8 and 9). An increase in toxicity in fish has a delayed effect on plankton because plankton directly do not depend on fish. However, in long run, this has an influence by varying the plankton population. In Figs. 10–12 we have projected the influence of the combine-harvesting effort on phytoplankton, zooplankton, and fish populations. Since equal harvesting effort is made for



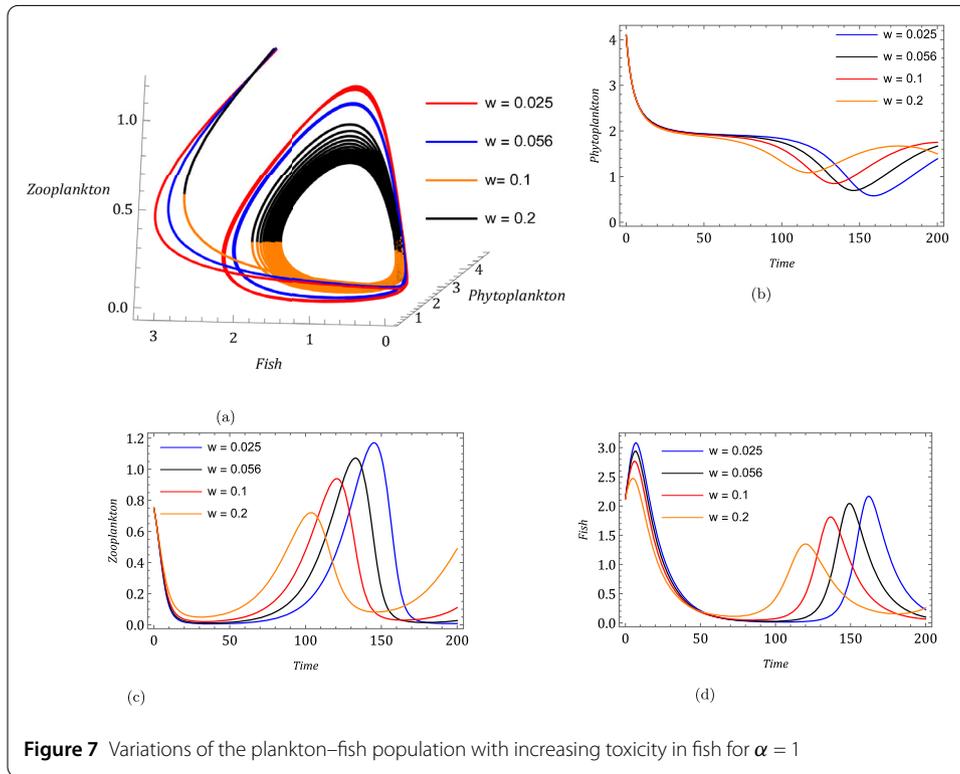


Figure 7 Variations of the plankton–fish population with increasing toxicity in fish for $\alpha = 1$

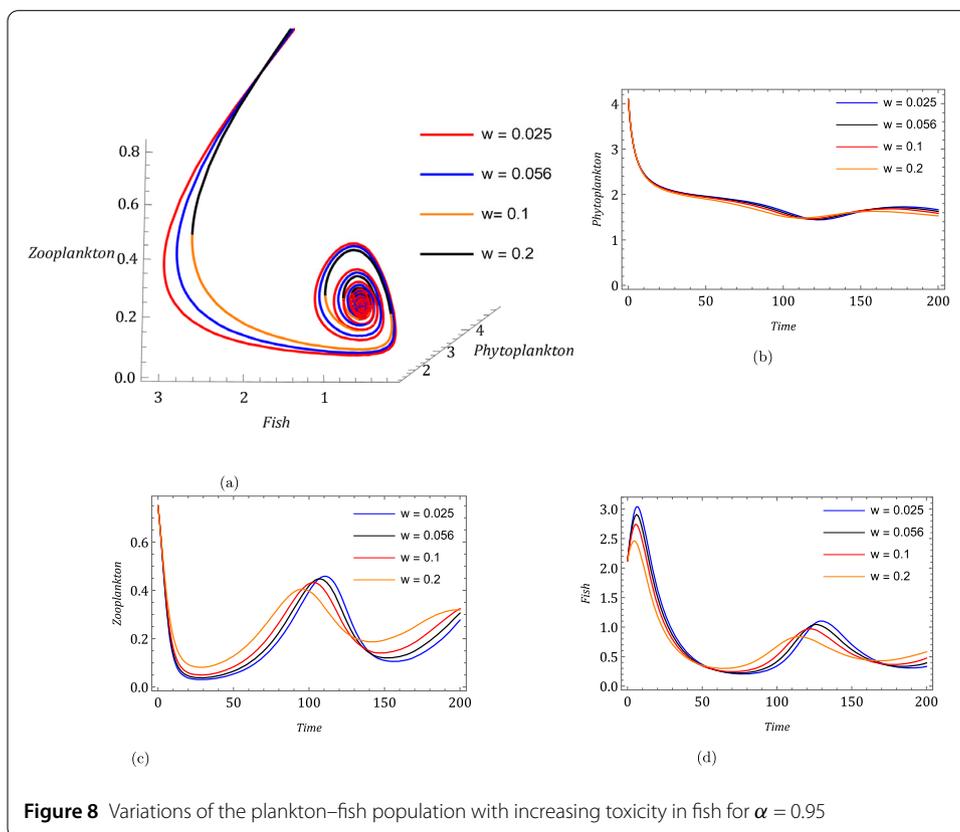
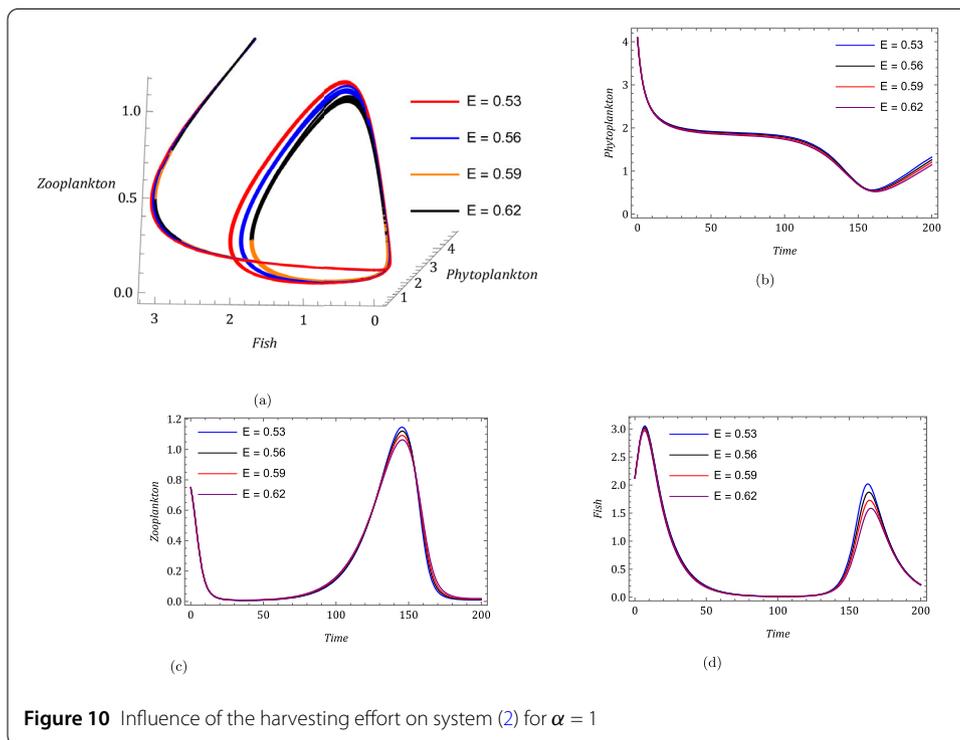
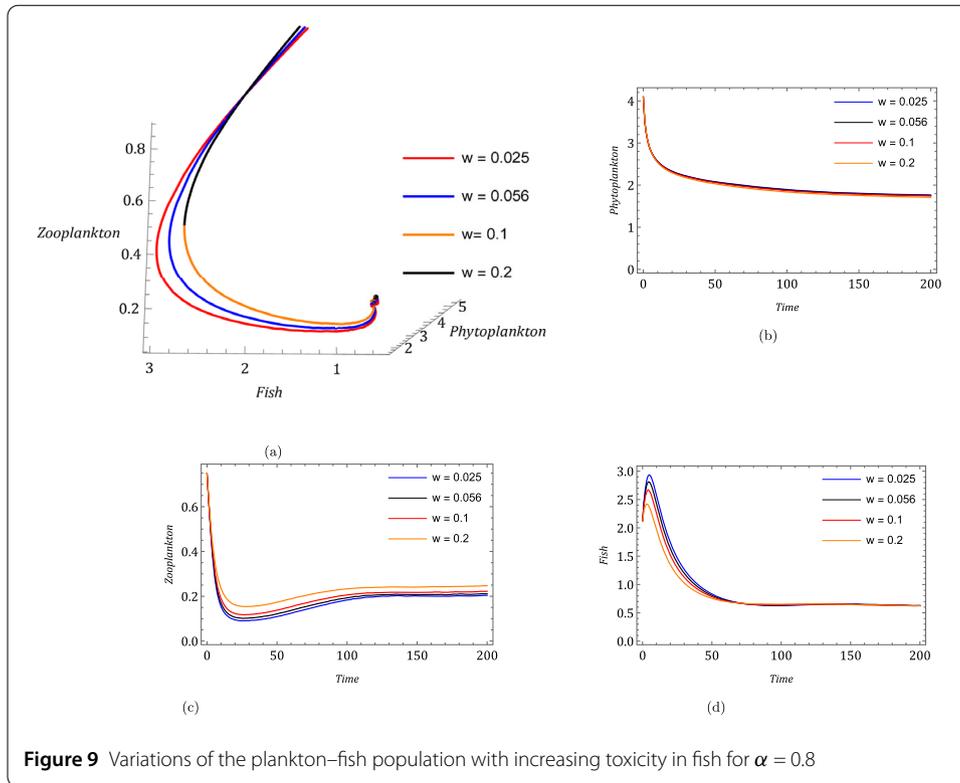
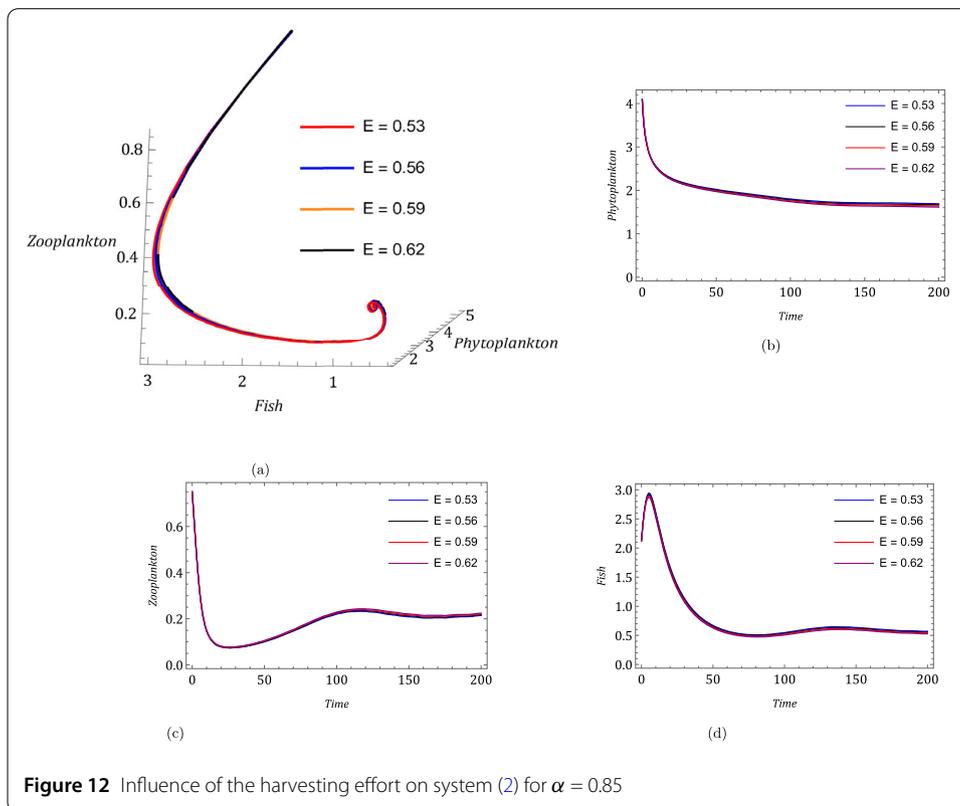
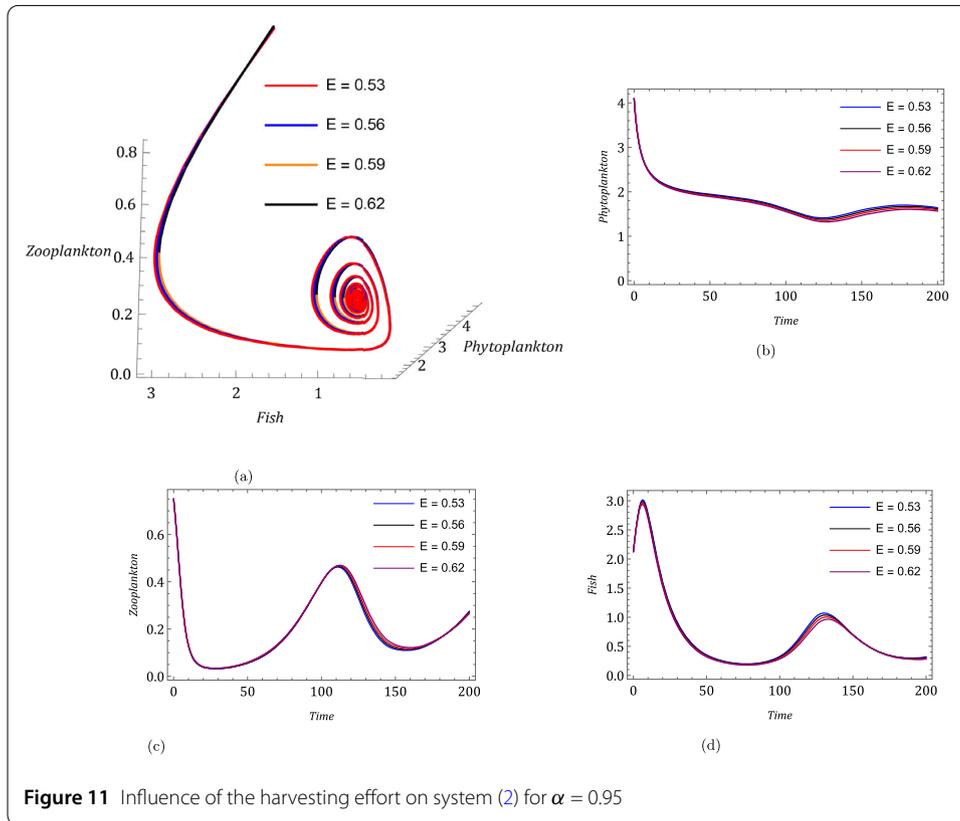


Figure 8 Variations of the plankton–fish population with increasing toxicity in fish for $\alpha = 0.95$



all the categories, we can observe the balancing effect in the population. However, it is visible that increased harvesting influences decrease the considered marine species at a considerably slower pace. Figures 11 and 12 show that impact of the combine-harvesting



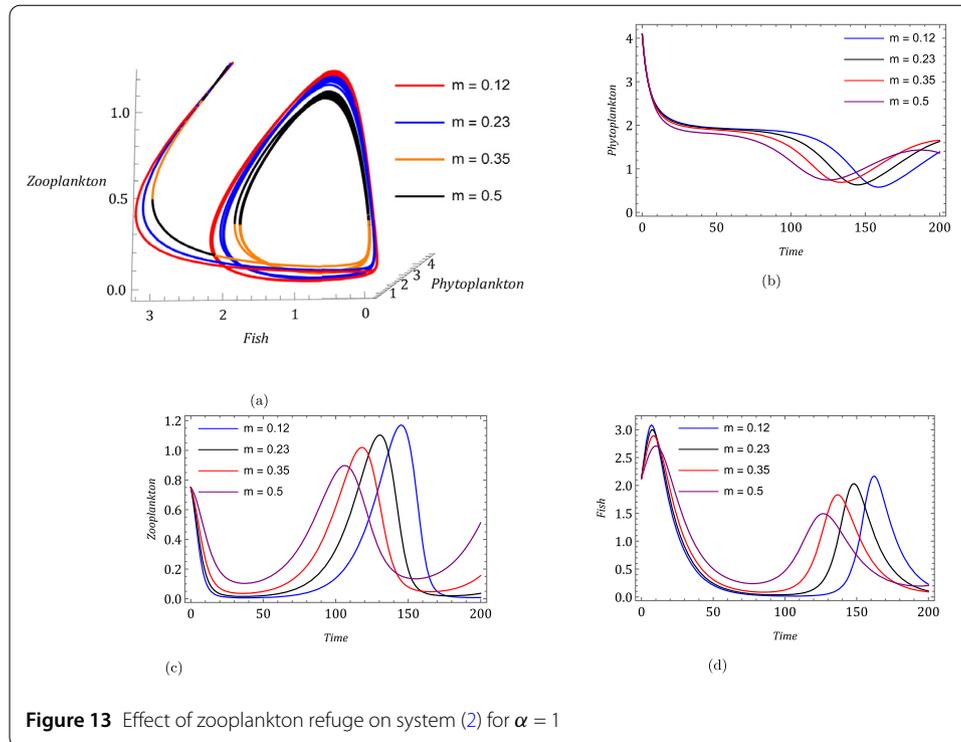


Figure 13 Effect of zooplankton refuge on system (2) for $\alpha = 1$

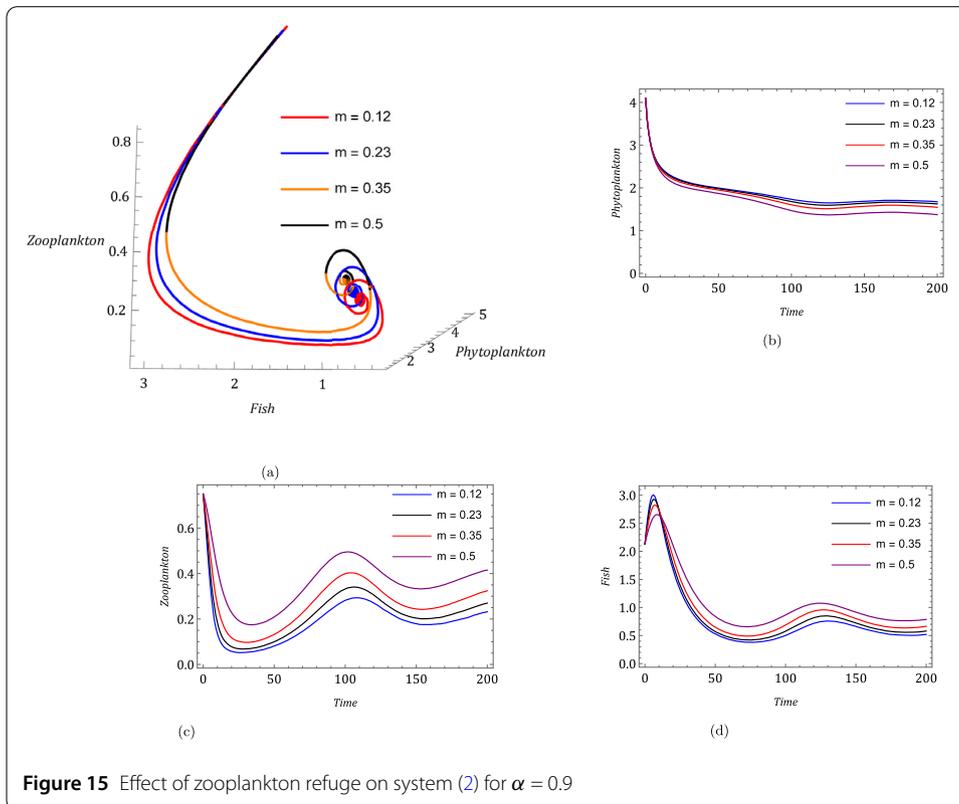
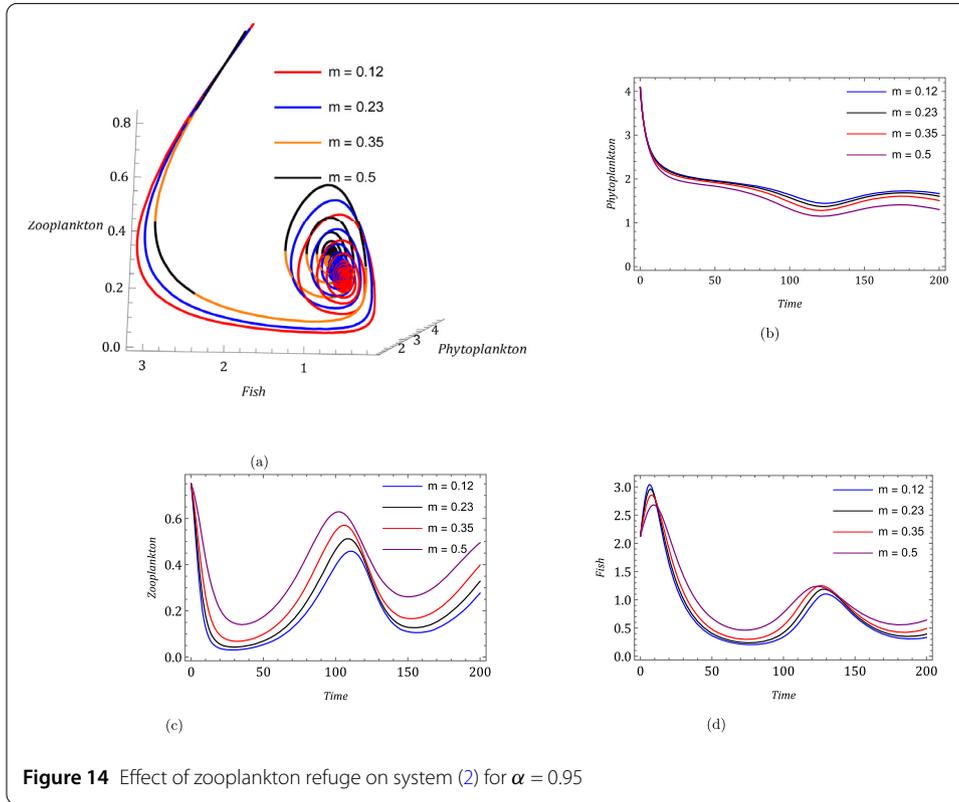
effort on phytoplankton, zooplankton, and fish populations is more in the presence of fractional-derivative values.

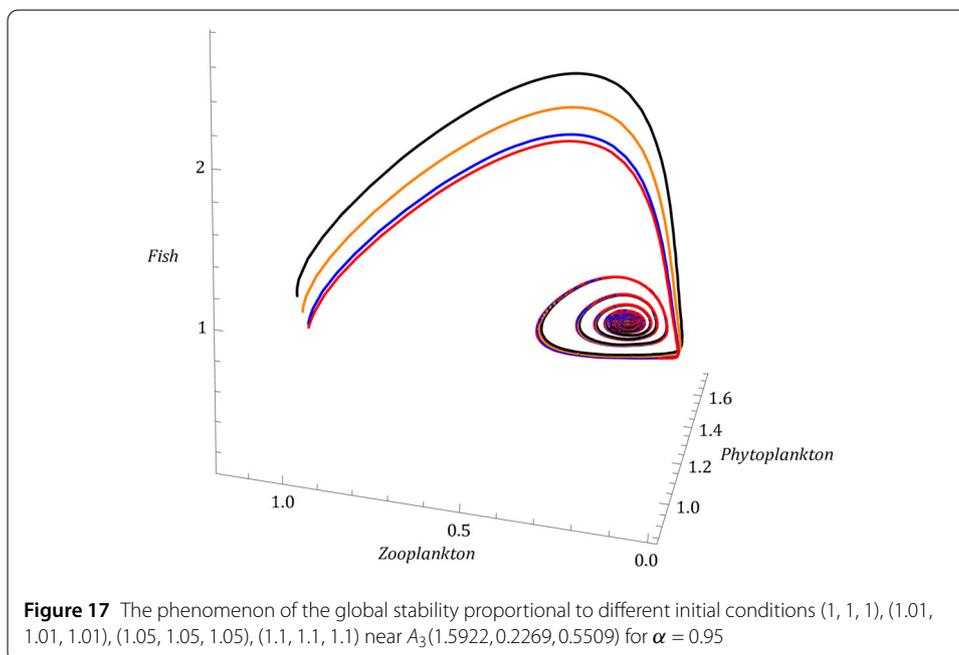
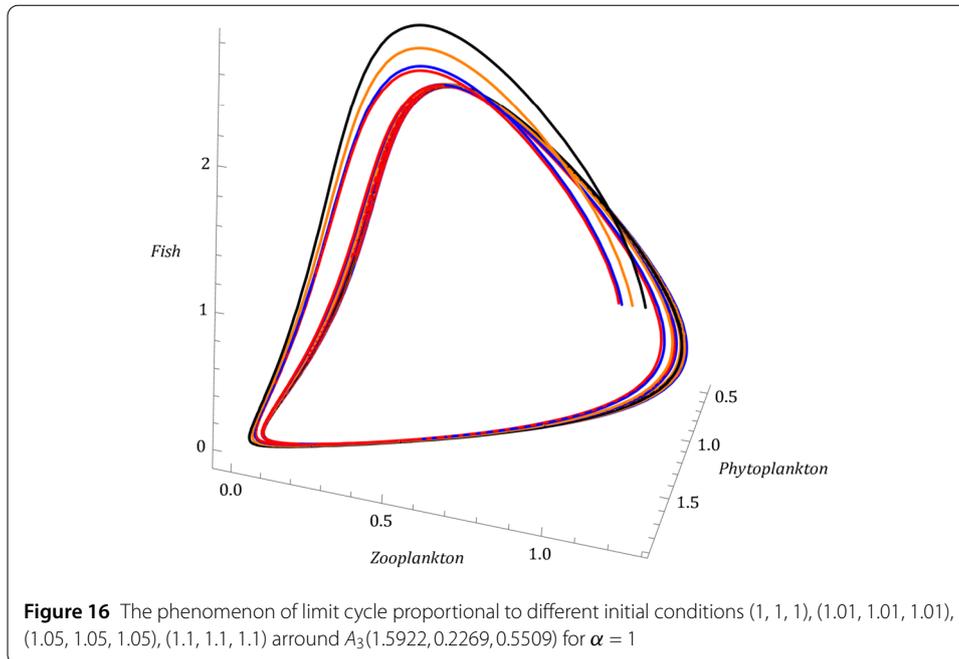
In Figs. 13–15, we have presented the effect of refuge in zooplankton under the influence of fractional derivatives. We observe an interesting biological behavior here. When refuge in zooplankton increases the zooplankton population decreases. This is because due to hiding they also suffer a lack of food and face elimination. Figure 13 shows the decrease in the fish population who are dependent on zooplankton only. From Figs. 10 and 13, it is visible that both the factors are inversely proportional to the growth of the population. Harvesting is the forced hiding of the population but refuge is the self-hiding of the population. Moreover, since we have introduced a combine-harvesting effort, it affected all the populations proportionally and hence oscillation is less compared to the oscillation visible due to the refuge effect. Figures 14 and 15 show the impact of the fractional derivative on this changing zooplankton refuge on the system (2).

In Figs. 16–18 we have observed the impact of the fractional derivative in transforming the system from instability to global stability around the coexistence equilibrium point $A_3(1.5922, 0.2269, 0.5509)$. A fractional derivative has a significant influence in projecting the coexistence of elements of the ecosystem in a stable atmosphere.

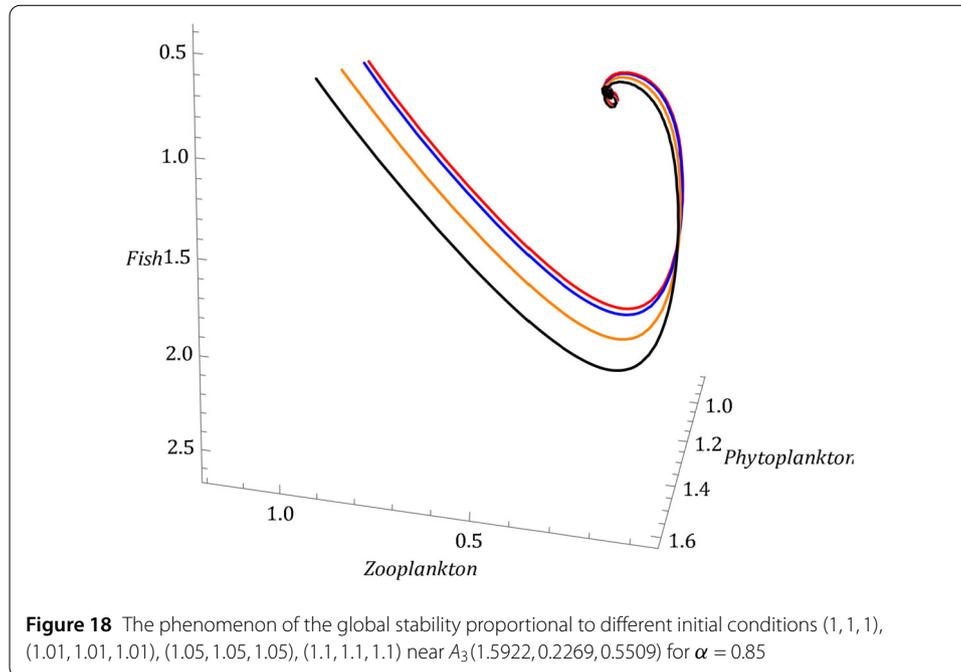
8 Conclusion

In this paper, we have proposed the three-species model connecting the phytoplankton, zooplankton, and fish populations under the effect of the Caputo fractional derivative. To establish the theoretical aspects of the solution, we have examined the nonnegativity, boundedness, and the existence and uniqueness of the solutions of the projected model. The requirements for the existence and local stability of the trivial, boundary, and fish-free equilibrium points are evaluated. Conditions for global stability of the interior equilibrium





point for the fractional-order derivative are derived. Since zooplankton depend only on phytoplankton for its livelihood, in the numerical simulation we have observed that their populations are inversely proportional to one another. The influence of toxins, combine-harvesting effort, and zooplankton refuge are studied for different fractional-derivative values. Toxicity in zooplankton has a prominent effect on the oscillatory behavior of their population dynamics, whereas its effect on phytoplankton and fish is less. However, increasing toxicity in fish influences the population significantly. On the other hand, changes in the combine-harvesting effort have a minimum influence on the oscillating behavior of



the dynamics. An increase in zooplankton refuge is inversely proportional to the population of the considered marine species. The produced results of the proposed mathematical model show that the fractional value of the derivative has a remarkable impact on the co-existence of the plankton–fish dynamics in the ecosystem. We have observed this in the form of numerical simulation where implementation of the fractional derivative changes the profile of the system from unstable to stable.

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Availability of data and materials

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Competing interests

The authors declare no competing interests.

Author contributions

PRN contributed to actualization, validated the proposed model, studied methodology, did formal analysis, investigated model, performed simulations using software, and prepared initial draft. CB contributed to actualization, validated the proposed model, studied methodology, did formal analysis, investigated model, performed simulations using software, and prepared initial draft. MKAK contributed to actualization, validated the proposed model, studied methodology, did formal analysis, investigated model, performed simulations using software, prepared initial and final draft, and supervised the whole project. All the authors have equal contribution in the article and there is no competing interest among them.

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