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Exponential stability for a class of set dynamic equations on time scales

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Abstract

We first present a new definition for some form of exponential stability of solutions, including H-exponential stability, H-exponentially asymptotic stability, H-uniformly exponential stability, and H-uniformly exponentially asymptotic stability for a class of set dynamic equations on time scales. Employing Lyapunov-like functions on time scales, we provide the sufficient conditions for the exponential stability of the trivial solution for such set dynamic equations.

Keywords: Time scales; Matrix-valued Lyapunov functions; Set dynamic equations; Exponential stability

1 Introduction

In this paper, we denote the n-dimensional real number set, natural number set, and (positive) integer number set by $\mathbb{R}^n$, $\mathbb{N}$ and $(\mathbb{Z}_+)$ $\mathbb{Z}$, respectively, and stipulate $\mathbb{R} = \mathbb{R}^1$, $\mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The set $K_c(\mathbb{R}^n)$ consists of all nonempty compact convex subsets of $\mathbb{R}^n$ and $K_c^m(\mathbb{R}^n) = K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^n) \times \cdots \times K_c(\mathbb{R}^n)$ (m-times). The purpose of this paper is to discuss the exponential stability of the set dynamic equations (SDE) on time scales

$$X^{\Delta}(t) = F(t,X), \qquad X(t_0) = X_0 \in K_c^m\left(\mathbb{R}^n\right), \tag{1}$$

where $X = (X_1, X_2, \ldots, X_m)^T$ with $m \in \mathbb{Z}_+$ and $X_i : \mathbb{T} \to K_c(\mathbb{R}^n)$ for $1 \leq i \leq m$, $t_0 \in \mathbb{T}$ is given, $X^{\Delta}(t)$ is the Δ-derivative of X at the moment $t \in \mathbb{T}$ and F is a set-valued function from $\mathbb{T} \times K_c^m(\mathbb{R}^n)$ into $K_c^m(\mathbb{R}^n)$.

The study of set differential and difference equations has been initiated as an independent subject and some results of interest can be found in [1–13]. More attention has been paid to the stability criteria for such equation's solutions in recent years. For instance, the comparison results and the stability considerations for hybrid dynamic systems were discussed in [14]. Since then, much progress has been made in studying various fundamental aspects of the stability of set differential or difference equations (see [3–6, 8, 15–20]). For instance, certain Lyapunov-like functions were used to study their stability criteria by Lakshmikantham in [15], Bhaskar and Devi [5] studied the Lyapunov stability for the solutions of set differential equations, using Lyapunov-like functions that are continuous. Moreover, in [5] the authors employed an important comparison result in the light of Lyapunov functions to characterize various stability behaviors of the solutions of the initial-value problem for a class of set differential equations, such as, the equistability, equiasymptotic stability, uniform, and uniformly asymptotic stability. In [18] the authors obtained the necessary and sufficient conditions of the globally asymptotic stability for a class of nonlinear neutral set-valued functional differential equations via the fixed-point method.

On the other hand, according to the references [21, 22], the theory of time scales was introduced by Stefan Hilger in his PhD thesis as a mean of unifying structure for the study of differential equations in the continuous case and the study of finite-difference equations in the discrete case. The stability theory of dynamic systems on time scales recently received much attention and is undergoing rapid development (see [23–28]). We mention that in [28] Martynyuk et al. researched the stability of a family of dynamic equations on time scales and proposed efficient sufficient conditions for several stability types of the sets of trajectories on time scales by means of scalar and vector Lyapunov-like functions constructed on the basis of matrix-valued functions. Very recently, since the derivative of set-valued functions on the time scale has been established (see Hong [29]), the qualitative problems of set differential equations have received extensive attention (see [11, 28, 30–34]). However, we observe that there are very few results for the stability to set differential equations on time scales. For example, Ahmad and Sivasundaram [35] discussed some basic problems of set differential equations on time scales and obtained some stability criteria. Wang and Sun [34] obtained a comparison principle by introducing a notion of upper quasimonotone nondecreasing provided the practical stability criteria for set differential equations in terms of two measures on time scales by using the vector Lyapunov function together with the comparison principle. In [36], notions of stability for the solutions of set dynamic equations on time scales are considered by using Lyapunov-like functions. Moreover, criteria for the equistability, equiasymptotic stability, uniform, and uniformly asymptotic stability are developed. In [33], the authors considered the exponential stability, exponentially asymptotic stability, uniformly exponential stability, and uniformly exponentially asymptotic stability for the trivial solution of set dynamic equations on time scales by using Lyapunov-like functions.

In this paper, inspired by the above-mentioned literature, we also consider the exponential stability for the solutions of set dynamic equations on time scales. More precisely, applying the method of matrix-valued functions in the theory of stability of classical dynamic equations on time scales described in [28], we similarly define appropriate matrix-valued Lyapunov-like functions and then formulate certain inequalities on these functions. Moreover, we employ these results to provide a generalized stability called the H-exponential stability in the paper, as well as H-exponentially asymptotic stability, H-uniformly exponential stability, and H-uniformly exponentially asymptotic stability of trivial solutions to a class of set dynamic equations on time scales. In addition, we present some sufficient conditions for the exponential stability for the trivial solution to SDE (1).

2 Preliminaries

In this section, some hypotheses and background materials are given that are necessary in this paper. We first recall the notion of the time scale built by Hilger and Bohner. For more details, we refer the reader to [21, 22].

Definition 2.1 For any $t \in \mathbb{T}$, the forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator is given by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

The distance from an arbitrary element $t \in \mathbb{T}$ to the nearest element on the right is called the graininess of the time scale $\mathbb{T}$ and denoted by $\mu(t)$, i.e.,

$$\mu(t) = \sigma(t) - t.$$

In Definition 2.1, it is assumed that $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ contains the largest element t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ contains the smallest element t).

Definition 2.2 Using the operators $\sigma : \mathbb{T} \to \mathbb{T}$ and $\rho : \mathbb{T} \to \mathbb{T}$, the points t on time scale $\mathbb{T}$ are classified as follows: if $\sigma(t) = t$, then t is said to be right-dense, while the point t is said to be right-scattered if $\sigma(t) > t$. Similarly, the point t is called left-dense if $\rho(t) = t$, while t is called left-scattered if $\rho(t) < t$.

Unless otherwise stated, we stipulate that $\mathbb{T}$ stands for $\mathbb{T}\backslash\{\hat{t}\}$ if $\mathbb{T}$ contains the left-scattered point maximum $\hat{t}$.

Definition 2.3 ([22]) A function f is right(left)-dense-continuous (rd(ld)-continuous, for short) if f is continuous at each right(left)-dense point in $\mathbb{T}$ and its left(right)-sided limits exist at each left(right)-dense point in $\mathbb{T}$. By $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ and $C_{\mathrm{ld}}(\mathbb{T}, \mathbb{R})$ we denote the set of all right- and left-dense continuous functions from $\mathbb{T}$ to $\mathbb{R}$, respectively.

We say that a function $p : \mathbb{T} \to \mathbb{R}$ (a $n \times n$-matrix-valued function $A : \mathbb{T} \to \mathbb{R}^{n\times n}$) is regressive provided

$$1 + \mu(t)p(t) \neq 0\bigl(I + \mu(t)A(t) \text{ is invertible, where } I \text{ denotes the } n \times n\text{-identity matrix}\bigr)$$

for all $t \in \mathbb{T}$. The set of all regressive rd-continuous functions $p : \mathbb{T} \to \mathbb{R}(\mathbb{R}_+)$ (a regressive rd-continuous matrix-valued function) is denoted by

$$\Re = \Re(\mathbb{T}, \mathbb{R})\bigl(\Re\bigl(\mathbb{T}, \mathbb{R}^{n\times n}\bigr)\bigr) \quad \text{and} \quad \Re_+ = \Re(\mathbb{T}, \mathbb{R}_+).$$

Let $p, q(A, B) \in \Re$. For all $t \in \mathbb{T}$, we define

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t), \qquad (p \ominus q)(t) = \bigl(p \oplus (\ominus q)\bigr)(t),$$
$$(\ominus p)(t) = -\bigl[1 + \mu(t)p(t)\bigr]^{-1}p(t)$$

and

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t), \qquad (A \ominus B)(t) = \bigl(A \oplus (\ominus B)\bigr)(t),$$
$$(\ominus A)(t) = -\bigl[I + \mu(t)A(t)\bigr]^{-1}A(t).$$

Definition 2.4 ([17], Definition 1.10) Assume that $g : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then, we define $g^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\delta > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\bigl|\bigl[g\bigl(\sigma(t)\bigr) - g(s)\bigr] - g^{\Delta}(t)\bigl[\sigma(t) - s\bigr]\bigr| \leq \varepsilon\bigl|\sigma(t) - s\bigr|$$

for all $s \in U$. We call $g^{\Delta}(t)$ the Δ-derivative of g at t. Moreover, we say that g is Δ-differentiable on $\mathbb{T}$ provided $g^{\Delta}(t)$ exists for all $t \in \mathbb{T}$.

If a single valued function g is Δ-differentiable and its Δ-derivative $g^{\Delta}(t)$ at $t \in \mathbb{T}$ equals $f(t)$, then we define the Cauchy integral by

$$\int_a^t f(s)\Delta s = g(t) - g(a).$$

In this case, we say f is Δ-integrable on interval $[a, t] \cap \mathbb{T}$.

Definition 2.5 ([22], Definition 2.30) If $p \in \Re$, then we define the exponential function by

$$e_p(t, s) = \exp\biggl(\int_s^t \xi_{\mu(t)}\bigl(p(\tau)\bigr)\Delta\tau\biggr) \quad \text{for } s, t \in \mathbb{T},$$

where $\xi_h(z)$ with $h > 0$ is the cylinder transformation from the set $\{z \in \mathbb{C} : z \neq \frac{1}{h}\}$ ($\mathbb{C}$ stands for the complex number set) into the strip $\{z \in \mathbb{C} : \frac{-\pi}{h} < \operatorname{Im}(z) \leq \frac{\pi}{h}\}$ defined by

$$\xi_h(z) = \frac{1}{h}\operatorname{Log}(1 + zh).$$

Lemma 2.1 ([22],Theorem 2.36) *Let $p, q \in \Re$. Then,*

(i) $e_p(t, t) \equiv 1$, $e_0(t, s) \equiv 1$, $e_p(t, s) = 1/e_p(s, t)$, *and* $e_p(t, s) = e_p(t, u)e_p(u, s)$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$, $e_p(s, \sigma(t)) = \frac{e_p(s,t)}{1+\mu(t)p(t)}$;
(iii) $e_p^{\Delta}(\cdot, s) = pe_p(\cdot, s)$, $e_p^{\Delta}(s, \cdot) = (\ominus p)e_p(s, \cdot)$;
(iv) $e_{p\oplus q} = e_p e_q$ *and* $e_{p\ominus q} = e_p/e_q$;
(v) *if* $p \in \Re_+$, *then* $e_p(t, s) > 0$ *for all* $t, s \in \mathbb{T}$;
(vi) $e_\alpha(t, s) \geq 1 + \alpha(t - s)$ *for* $\alpha \in \Re_+$, $\alpha \in \mathbb{R}_+$ *and any* $t, s \in \mathbb{T}$ *with* $t \geq s$.

We continue with a description of the basic known results for Hausdorff metrics, continuity, and differentiability for set-valued mappings on time scales and their corresponding properties within the framework of time scales. We refer readers to [7, 29] for details. The following operations can be naturally defined on $K_c(\mathbb{R}^n)$: for $X, Y \in K_c(\mathbb{R}^n)$,

$$X + Y = \{x + y : x \in X, y \in Y\}, \qquad \lambda \cdot X = \{\lambda \cdot x : x \in X\} \quad \text{for } \lambda \in \mathbb{R}_+,$$
$$XY = \{xy : x \in X, y \in Y\}.$$

In addition, the set $Z \in K_c(\mathbb{R}^n)$ satisfying $X = Y + Z$ is known as the geometric difference of the sets X and Y and is denoted by the symbol $X - Y$. It is worth noting that the geometric difference of two sets does not always exist, but if it does then it is unique.

We define the Hausdorff metric as

$$D[X,Y] = \max\Bigl\{\sup_{y\in Y} d(y,X), \sup_{x\in X} d(x,Y)\Bigr\},$$

where $d(x,Y) = \inf\{d(x,y) : y \in Y\}$ and X, Y are bounded subsets of $\mathbb{R}^n$. Denote $\|X\| = D[X,\Theta]$.

A set-valued function $F:\mathbb{T}\to K_c(\mathbb{R}^n)$ is said to be continuous at $t_0 \in \mathbb{T}$ if $D[F(t),F(t_0)] \to 0$ whenever $t \to t_0$.

Definition 2.6 ([31], Definition 3.1) Assume that $F:\mathbb{T}\to K_c(\mathbb{R}^n)$ is a set-valued function and $t\in\mathbb{T}$. Let $F^{\Delta}(t)$ be an element of $K_c(\mathbb{R}^n)$ with the property that for a given $\varepsilon > 0$, there exists a neighborhood $U_{\mathbb{T}}$ of t (i.e., $U_{\mathbb{T}} = (t-\delta, t+\delta)\cap\mathbb{T}$ for some $\delta > 0$) such that

$$D\bigl[F(t+h) - F\bigl(\sigma(t)\bigr), F^{\Delta}(t)\bigl(h-\mu(t)\bigr)\bigr] \le \varepsilon\bigl(h-\mu(t)\bigr),$$
$$D\bigl[F\bigl(\sigma(t)\bigr) - F(t-h), F^{\Delta}(t)\bigl(\mu(t)+h\bigr)\bigr] \le \varepsilon\bigl(\mu(t)+h\bigr)$$

for all $t-h, t+h \in U_{\mathbb{T}}$ with $0 \le h < \delta$. Then, F is called Δ-derivable or Δ-differentiable at t and $F^{\Delta}(t)$ is called the Δ-derivative of F at t.

A function F is called Δ-differentiable on $\mathbb{T}$ if its Δ-derivative exists at each $t\in\mathbb{T}$.

Lemma 2.2 ([31], Theorem 3.5) *Let $F:\mathbb{T}\to K_c(\mathbb{R}^n)$ and the following results hold*:

(1) *If the Δ-derivative of F exists, then it is unique. Hence, the Δ-derivative is well defined.*

(2) *If F is Δ-differentiable at $t\in\mathbb{T}$, then F is continuous at t.*

(3) *Let F be continuous at $t\in\mathbb{T}$. Then, we have the following.*

 (i) *If t is right-scattered, then F is Δ-differentiable at t with*

$$F^{\Delta}(t) = \frac{F(\sigma(t)) - F(t)}{\mu(t)}.$$

 (ii) *If t is right-dense, then F is Δ-differentiable at t if and only if the limits*

$$\lim_{h\to 0^+}\frac{F(t+h)-F(t)}{h}, \qquad \lim_{h\to 0^+}\frac{F(t)-F(t-h)}{h}$$

 exist and satisfy the equations

$$\lim_{h\to 0^+}\frac{F(t+h)-F(t)}{h} = \lim_{h\to 0^+}\frac{F(t)-F(t-h)}{h} = F^{\Delta}(t).$$

Definition 2.7 Let $\mathbb{I}\subset\mathbb{T}$ be a subset. The set-valued function $F:\mathbb{I}\to K_c(\mathbb{R}^n)$ is called rd-continuous on $\mathbb{I}$, if it is continuous at all right-dense points of $\mathbb{I}$ and left-hand limits exist and are finite numbers at all left-dense points of $\mathbb{I}$. By $C_{\mathrm{rd}}(\mathbb{I}, K_c(\mathbb{R}^n))$ we denote the set consisting of all rd-continuous set-valued functions on $\mathbb{I}$.

At the end of this section, for fixed $m\in\mathbb{Z}_+$, we define

$$K_c^m\bigl(\mathbb{R}^n\bigr) = K_c\bigl(\mathbb{R}^n\bigr)\times K_c\bigl(\mathbb{R}^n\bigr)\times\cdots\times K_c\bigl(\mathbb{R}^n\bigr).$$

Moreover, $K_c^m(\mathbb{R}^n)$ is endowed with the distance as follows

$$D_0[X,Y] = \sum_{i=1}^{m} D[X_i, Y_i]$$

for $X, Y \in K_c^m(\mathbb{R}^n)$. It is not difficult to check that $(K_c^m(\mathbb{R}^n), D_0)$ is a metric space.

3 Main results

Let $F \in C_{\mathrm{rd}}(\mathbb{T} \times K_c^m(\mathbb{R}^n), K_c^m(\mathbb{R}^n))$ and $\mathcal{X} = \{X : X \in C_{\mathrm{rd}}(\mathbb{T}, K_c^m(\mathbb{R}^n))$ with $X(t) = X(t, t_0, X_0)$ is a solution of SDE (1)$\}$ be nonempty. The generalized direct Lyapunov's method for families of equations is discussed in a number of articles on the basis of scalar and vector auxiliary functions $v(t,X) \in C(\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+)$ (respectively, $V(t,X) \in C(\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+^m)$), constructed on the basis of two-indexed systems of functions, as a suitable manner for constructing a Lyapunov-like matrix function. Next, together with SDE (1), we will consider the matrix-valued function $U : \mathbb{T} \times K_c^m(\mathbb{R}^n) \to \mathbb{R}^{m\times m}$ defined by

$$U(t,X) = \bigl[U_{ij}(t,X)\bigr], \quad 1 \le i,j \le m, \forall (t,X) \in \mathbb{T} \times K_c^m\bigl(\mathbb{R}^n\bigr) \tag{2}$$

with

$$\begin{aligned} &U_{ii} \in C_{\mathrm{rd}}\bigl(\mathbb{T} \times K_c^m\bigl(\mathbb{R}^n\bigr), \mathbb{R}_+\bigr), \quad i = 1,2,\dots,m \quad \text{and} \\ &U_{ij} \in C_{\mathrm{rd}}\bigl(\mathbb{T} \times K_c^m\bigl(\mathbb{R}^n\bigr), \mathbb{R}\bigr), \quad i,j = 1,2,\dots,m, i \neq j. \end{aligned}$$

In the following, we suppose that $\det U(t,\Theta_0) = 0$, where $\det U(t,X)$ stands for the determinant of $U(t,X)$ and Θ_0 stands for the zero element in $K_c^m(\mathbb{R}^n)$. Let $\mathcal{A}(t,X)$ denote the set consisting of all solutions of the homogeneous linear system

$$U(t,X)a = 0, \quad (t,X) \in \mathbb{T} \times K_c^m\bigl(\mathbb{R}^n\bigr), a \in \mathbb{R}_+^m.$$

It is clear that system $U(t,\Theta_0)a = 0$ has nontrivial solutions. More precisely, $\mathcal{A}(t,\Theta_0)$ contains at least a nonvanishing vector for $t \in \mathbb{T}$.

Based on the matrix function (2) we define a scalar function

$$v(t,X,a) = a^T U(t,X)a, \quad (t,X,a) \in \mathbb{T} \times K_c^m\bigl(\mathbb{R}^n\bigr) \times \mathbb{R}_+^m. \tag{3}$$

Clearly, $v \in C_{\mathrm{rd}}(\mathbb{T} \times K_c^m(\mathbb{R}^n) \times \mathbb{R}_+^m, \mathbb{R}_+)$ and $v(t,\Theta_0,a) = 0$ for each $t \in \mathbb{T}$ and each $a \in \mathcal{A}(t,\Theta_0)$.

For the function (3) we will consider Δ-derivatives with respect to $t \in \mathbb{T}$, that is,

$$v^{\Delta}(t,X,a) = a^T U^{\Delta}(t,X)a,$$

with

$$U^{\Delta}(t,A) = \begin{cases} \frac{U(\sigma(t),A(\sigma(t)))-U(t,A(t))}{\mu(t)}, & \sigma(t) > t, \\ \limsup_{s\to t^+} \frac{U(t+\mu(t),A+(s-t)F(t,A))-U(t,A)}{s-t}, & \sigma(t) = t \end{cases}$$

for any given $A \in C_{\mathrm{rd}}(\mathbb{T}, K_c^m(\mathbb{R}^n))$.

The following definition is due to [36].

Definition 3.1 Let $v \in C_{\mathrm{rd}}(\mathbb{T} \times K_c^m(\mathbb{R}^n) \times \mathbb{R}_+^m, \mathbb{R}_+)$. We call $\Delta^r v(t,A,a)$ and $\Delta_r v(t,A,a)$ the right upper(ru) and the right lower(rl) derivatives of v with respect to t at $(t,A(t),a)$ for $A \in C_{rd}(\mathbb{T}, K_c^m(\mathbb{R}^n))$, $a \in \mathbb{R}_+^m$, $t \in \mathbb{T}$, respectively, if

$$\Delta^r v(t,A(t),a) = \begin{cases} \frac{v(\sigma(t),A(\sigma(t)),a)-v(t,A(t),a)}{\mu(t)}, & \sigma(t) > t, \\ \limsup_{s\to t^+} \frac{v(s,A(t)+(s-t)F(t,A(t)),a)-v(t,A(t),a)}{s-t}, & \sigma(t) = t, \end{cases}$$

$$\Delta_r v(t,A(t),a) = \begin{cases} \frac{v(\sigma(t),A(\sigma(t)),a)-v(t,A(t),a)}{\mu(t)}, & \sigma(t) > t, \\ \liminf_{s\to t^+} \frac{v(s,A(t)+(s-t)F(t,A(t)),a)-v(t,A(t),a)}{s-t}, & \sigma(t) = t. \end{cases}$$

In this case, v is said to be a matrix-valued Lyapunov-like function on $\mathbb{T} \times K_c^m(\mathbb{R}^n) \times \mathbb{R}_+^m$.

Theorem 3.1 *Let $v(t,X,a) = a^T U(t,X)a$ be a matrix-valued Lyapunov-like function. Then, its ru and rl derivatives exist. Moreover, for any fixed $A \in C_{rd}(\mathbb{T}, K_c^m(\mathbb{R}^n))$, $a \in \mathbb{R}_+^m$, the Δ-derivative of v with respect to $t \in \mathbb{T}$ exists and*

$$v^\Delta(t,A(t),a) = \Delta^r v(t,A(t),a) = \Delta_r v(t,A(t),a).$$

The proof of this theorem is similar to that of Theorem 3.1 of [33], and therefore we omit it.

In [28] the authors introduced two sets of functions Q and Q_0 that characterize the current and initial states of the set of solutions of SDE (1), respectively, as follows:

$$Q = \Bigl\{H : \mathbb{T} \times K_c^m(\mathbb{R}^n) \to \mathbb{R}_+ \Bigm| \inf_{X \in K_c^m(\mathbb{R}^n)} H(t,X) = 0\Bigr\},$$

$$Q_0 = \Bigl\{H_0 \in Q \Bigm| \inf_X H_0(t,X) = 0 \text{ for } t \in \mathbb{R}_+\Bigr\}.$$

They established some stability conditions under two different measures based on a class of matrix-valued Lyapunov functions for SDE (1), such as (H_0, H)-stable, (H_0, H)-uniformly stable, and (H_0, H)-asymptotically stable, etc.

In this section, we will develop the H-exponential stability, H-exponentially asymptotic stability, H-uniformly exponential stability, and H-uniformly exponentially asymptotic stability for the trivial solution of SDE (1). To this end, we assume that the initial value of SDE (1) $t_0 \in \mathbb{T}$ is positive and $\mathbb{T}$ is not bounded above.

In what follows, we consider two sets $\mathcal{M}$ and $\mathcal{M}_0$ consisting of functions that characterize the current and initial states of the set $\mathcal{X}$ of solutions to SDE (1), respectively, that is,

$$\begin{aligned} \mathcal{M} = \bigl\{&H \in C_{\mathrm{rd}}\bigl(\mathbb{T} \times K_c^m(\mathbb{R}^n), \mathbb{R}_+\bigr) | H(t,\cdot) \text{ cannot be a constant relative to the} \\ &\text{second variable on } \mathcal{X}\bigr\}, \\ \mathcal{M}_0 = \bigl\{&h_0 | h_0 = H_0(t_0, X_0) \text{ for some given } H_0 \in \mathcal{M}\bigr\}. \end{aligned}$$

In addition, we need the following notations:

$$\Gamma = \{\gamma : [0,\infty) \to [0,\infty) | \gamma \text{ is strictly increasing continuous with } \gamma(0) = 0\},$$
$$\Lambda = \{\lambda : [0,\infty) \to [0,\infty) | \lambda \text{ is continuous}, \lambda(0) = 0 \text{ and } \lambda(s) > 0 \text{ for } s > 0\}.$$

Definition 3.2 Let $H, H_0 \in \mathcal{M}$, $X \in \mathcal{X}$ and a constant $p \in (0,+\infty)$. The trivial solution of SDE (1) is said to be

(I) H-exponentially stable on $\mathbb{T}$ if there exist $\alpha \in \Gamma$ and a function $\varrho : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}_+$ such that

$$\alpha\big(\big(H\big(t,X(t)\big)\big)^p\big) \le \varrho(h_0,t_0)\big(e_{\ominus M}(t,t_0)\big)^d, \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{4}$$

where $M \in \Re_+$, $d \in (0,+\infty)$ and $h_0 = H_0(t_0,X_0)$;

(II) H-uniformly exponentially stable if (I) holds with the function ϱ independent of t_0;

(III) H-exponentially asymptotically stable if (I) holds, as well as, for any $\varepsilon > 0$, there exists a positive real number T such that

$$\alpha\big(\big(H\big(t,X(t)\big)\big)^p\big) < \varepsilon \quad \text{for all } t \in [t_0 + T,\infty)_{\mathbb{T}};$$

(IV) H-uniformly exponentially asymptotically stable if there exists $\alpha \in \Gamma$ such that (II) and (III) hold simultaneously.

Theorem 3.2 *Assume that v is a matrix-valued Lyapunov-like function on $\mathbb{T} \times K_c^m(\mathbb{R}^n) \times \mathbb{R}_+^m$ and satisfies the following conditions: for $H \in \mathcal{M}$, $X \in \mathcal{X}$, a constant $p > 0$ and a vector $a \in \mathbb{R}_+^m$,*

(i) *there exist functions $\lambda_1, \lambda_2 \in \Lambda$ such that*

$$\lambda_1\big(\big(H\big(t,X(t)\big)\big)^p\big) \le v\big(t,X(t),a\big) \le \lambda_2\big(\big(H\big(t,X(t)\big)\big)^p\big) \quad \textit{for all } t \in \mathbb{T};$$

(ii) *there exist a nondecreasing continuous function $\lambda_3 : \mathbb{R}_+ \to \mathbb{R}$, functions $\gamma \in \Gamma$, $\delta \in \Re$, $M \in \Re_+$ with $\lambda_2(t) \le \gamma(t)$ for $t \in \mathbb{T}_+$ and constants L, r with $r > 0$ such that*

$$v^\Delta\big(t,X(t),a\big) \le \frac{-\lambda_3((H(t,X(t)))^r) - L(M \ominus \delta)(t)e_{\ominus\delta}(t,0)}{1 + M\mu(t)}, \tag{5}$$

$$\begin{aligned} &Mv\big(t,X(t),a\big) \\ &\quad \le \lambda_3\big(\big[\gamma^{-1}\big(v\big(t,X(t),a\big)\big)\big]^{r/p}\big) + L(M \ominus \delta)(t)e_{\ominus\delta}(t,0) \quad \textit{for all } t \in \mathbb{T}. \end{aligned} \tag{6}$$

Then, the trivial solution of SDE (1) is H-exponentially stable on $[t_0,\infty)_{\mathbb{T}}$.

Proof Let $\bar{\gamma} \in \Gamma$ satisfy $\bar{\gamma}((H(t,X(t)))^p) \le \lambda_1((H(t,X(t)))^p)$ and $\lambda_2((H(t,X(t)))^p) \le \gamma((H(t, X(t)))^p)$. Combined with the condition (i), we have

$$\begin{aligned} \bar{\gamma}\big(\big(H\big(t,X(t)\big)\big)^p\big) &\le \lambda_1\big(\big(H\big(t,X(t)\big)\big)^p\big) \le v\big(t,X(t),a\big) \\ &\le \lambda_2\big(\big(H\big(t,X(t)\big)\big)^p\big) \le \gamma\big(\big(H\big(t,X(t)\big)\big)^p\big) \end{aligned} \tag{7}$$

for all $t \in \mathbb{T}$. To fulfill our wish, it is sufficient to verify (4). The remainder of the proof is divided into three steps.

Step 1. We first verify that $v(t,X(t),a)e_M(t,t_0)$ is nonincreasing in $t \in [t_0,+\infty)_{\mathbb{T}}$. Indeed, by means of (5), together with Lemma 2.1(ii) and Theorem 3.1, we have

$$\begin{aligned}
&\bigl[v\bigl(t,X(t),a\bigr)e_M(t,t_0)\bigr]^{\Delta}\\
&\quad= v^{\Delta}\bigl(t,X(t),a\bigr)e_M\bigl(\sigma(t),t_0\bigr) + Mv\bigl(t,X(t),a\bigr)e_M(t,t_0)\\
&\quad= \bigl(v^{\Delta}\bigl(t,X(t),a\bigr)e_M\bigl(\sigma(t),t\bigr) + Mv\bigl(t,X(t),a\bigr)\bigr)e_M(t,t_0)\\
&\quad\le \biggl(\frac{-\lambda_3((H(t,X(t)))^r) - L(M\ominus\delta)(t)e_{\ominus\delta}(t,0)}{1+M\mu(t)}e_M\bigl(\sigma(t),t\bigr) + Mv\bigl(t,X(t),a\bigr)\biggr)e_M(t,t_0)\\
&\quad= \bigl(-\lambda_3\bigl(\bigl(H\bigl(t,X(t)\bigr)\bigr)^r\bigr) - L(M\ominus\delta)(t)e_{\ominus\delta}(t,0) + Mv\bigl(t,X(t),a\bigr)\bigr)e_M(t,t_0).
\end{aligned}$$

From (7) it follows that $[\gamma^{-1}(v(t,X(t),a))]^{r/p} \le (H(t,X(t)))^r$. By virtue of the monotonicity of λ_3, we have $-\lambda_3([\gamma^{-1}(v(t,X(t),a))]^{r/p}) \ge -\lambda_3((H(t,X(t)))^r)$. Thus, combining (6) and (7), implies that

$$\begin{aligned}
&\bigl[v\bigl(t,X(t),a\bigr)e_M(t,t_0)\bigr]^{\Delta}\\
&\quad\le \bigl(-\lambda_3\bigl(\bigl(H\bigl(t,X(t)\bigr)\bigr)^r\bigr) - L(M\ominus\delta)(t)e_{\ominus\delta}(t,0) + Mv\bigl(t,X(t),a\bigr)\bigr)e_M(t,t_0)\\
&\quad\le \bigl(-\lambda_3\bigl(\bigl[\gamma^{-1}\bigl(v\bigl(t,X(t),a\bigr)\bigr)\bigr]^{r/p}\bigr) - L(M\ominus\delta)(t)e_{\ominus\delta}(t,0) + Mv\bigl(t,X(t),a\bigr)\bigr)e_M(t,t_0)\\
&\quad\le 0.
\end{aligned}$$

Consequently, $v(t,X(t),a)e_M(t,t_0)$ is nonincreasing in t.

Step 2. For the sake of convenience, let $N > 1$ be a given constant and $u(t_0,X_0) = Nv(t_0,X_0,a)$. We claim that

$$v\bigl(t,X(t),a\bigr) \le u(t_0,X_0)e_{\ominus M}(t,t_0), \quad t \in [t_0,\infty)_{\mathbb{T}}. \tag{8}$$

Suppose that the inequality (8) does not hold. Then, there exists $t \in [t_0,\infty)_{\mathbb{T}}$ such that

$$v\bigl(t,X(t),a\bigr) > u(t_0,X_0)e_{\ominus M}(t,t_0).$$

Set $\bar{t} = \inf\{t \in [t_0,\infty)_{\mathbb{T}} | v(t,X(t),a) > u(t_0,X_0)e_{\ominus M}(t,t_0)\}$. From Step 1 it follows that $\bar{t} > t_0$ (otherwise, our claim is achieved). Without loss of generality, assume that

$$v\bigl(\bar{t},X(\bar{t}),a\bigr) \ge u(t_0,X_0)e_{\ominus M}(\bar{t},t_0) \quad \text{and} \tag{9}$$

$$v\bigl(t,X(t),a\bigr) \le u(t_0,X_0)e_{\ominus M}(t,t_0), \quad t \in [t_0,\bar{t})_{\mathbb{T}}. \tag{10}$$

Next, let us choose $\varphi \in \Gamma$ to satisfy $s < \varphi(s) \le Ns$ for any $s \ge 0$. Then, we have

$$\varphi\bigl(v\bigl(\bar{t},X(\bar{t}),a\bigr)\bigr) \ge \varphi\bigl(u(t_0,X_0)e_{\ominus M}(\bar{t},t_0)\bigr) > u(t_0,X_0)e_{\ominus M}(\bar{t},t_0)$$

and

$$\varphi\bigl(v(t_0,X_0,a)\bigr) = \varphi\bigl(N^{-1}u(t_0,X_0)\bigr) \le u(t_0,X_0).$$

Note that the set $\{t \in [t_0, \bar{t}]_{\mathbb{T}} | \varphi(v(t, X(t), a)) \leq u(t_0, X_0)e_{\ominus M}(t, t_0)\}$ is nonempty since it includes at least the element t_0, we can define

$$\tilde{t} = \sup\bigl\{t \in [t_0, \bar{t}]_{\mathbb{T}} | \varphi\bigl(v\bigl(t, X(t), a\bigr)\bigr) \leq u(t_0, X_0)e_{\ominus M}(t, t_0)\bigr\}.$$

Thus, we deduce that

$$\varphi\bigl(v\bigl(\tilde{t}, X(\tilde{t}), a\bigr)\bigr) \leq u(t_0, X_0)e_{\ominus M}(\tilde{t}, t_0), \quad \tilde{t} \in [t_0, \bar{t})_{\mathbb{T}} \quad \text{and} \tag{11}$$

$$\varphi\bigl(v\bigl(t, X(t), a\bigr)\bigr) > u(t_0, X_0)e_{\ominus M}(t, t_0), \quad t \in (\tilde{t}, \bar{t}]_{\mathbb{T}}. \tag{12}$$

Step 1 guarantees that $v(t, X(t), a)e_M(t, t_0)$ is nonincreasing in $t \in [\tilde{t}, \bar{t}]_{\mathbb{T}}$, which implies that

$$v\bigl(\bar{t}, X(\bar{t}), a\bigr)e_M(\bar{t}, t_0) \leq v\bigl(\tilde{t}, X(\tilde{t}), a\bigr)e_M(\tilde{t}, t_0).$$

On the other hand, from (9) and (11) it follows that

$$v\bigl(\bar{t}, X(\bar{t}), a\bigr)e_M(\bar{t}, t_0) \geq u(t_0, X_0) \geq \varphi\bigl(v\bigl(\tilde{t}, X(\tilde{t}), a\bigr)\bigr)e_M(\tilde{t}, t_0) > v\bigl(\tilde{t}, X(\tilde{t}), a\bigr)e_M(\tilde{t}, t_0).$$

This is a contradiction and hence (8) is true.

Step 3. Finally, according to the condition (i), we derive

$$\bar{\gamma}\bigl(\bigl(H\bigl(t, X(t)\bigr)\bigr)^p\bigr) \leq v\bigl(t, X(t), a\bigr) \leq u(t_0, X_0)e_{\ominus M}(t, t_0), \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{13}$$

Let $\alpha = \bar{\gamma} \in \Gamma$, $\varrho(h_0, t_0) = u(t_0, X_0) = Nv(t_0, X_0, a)$, as well as, $h_0 = H_0(t_0, X_0) = v(t_0, X_0, a)$ with $H_0 \in \mathcal{M}$. Consequently, inequality (13) guarantees that (4) is satisfied. Hence, the trivial solution of SDE (1) is H-exponentially stable on $[t_0, \infty)_{\mathbb{T}}$. This proof is complete. □

Corollary 3.1 *Let v be a matrix-valued Lyapunov-like function on $\mathbb{T} \times K_c^m(\mathbb{R}^n) \times \mathbb{R}_+^m$. For $H \in \mathcal{M}$, $X \in \mathcal{X}$, suppose that the following conditions are satisfied*:

(i) *there exist positive functions $\mu_1, \mu_2 \in \Lambda$ and positive constants p, q such that*

$$\mu_1(t)\bigl(H\bigl(t, X(t)\bigr)\bigr)^p \leq v\bigl(t, X(t), a\bigr) \leq \mu_2(t)\bigl(H\bigl(t, X(t)\bigr)\bigr)^q \quad \textit{for all } t \in \mathbb{T};$$

(ii) *there exist a positive function $\lambda_3 : \mathbb{T}_+ \to \mathbb{R}_+$ with $M =: \inf_{s \geq t_0} \lambda_3(s)/[\mu_2(s)]^{r/q} \in \mathfrak{R}_+$, function $\delta \in \mathfrak{R}$, and constants L, r with $r > 0$ such that*

$$v^{\Delta}\bigl(t, X(t), a\bigr) \leq \frac{-\lambda_3(t)(H(t, X(t)))^r - L(M \ominus \delta)(t)e_{\ominus \delta}(t, 0)}{1 + M\mu(t)},$$

$$M\bigl(v\bigl(t, X(t), a\bigr) - v^{r/q}\bigl(t, X(t), a\bigr)\bigr) \leq L(M \ominus \delta)(t)e_{\ominus \delta}(t, 0),$$

where q is given as in (i).

Then, the trivial solution of SDE (1) *is H-exponentially stable on* $[t_0, \infty)_{\mathbb{T}}$.

Proof In fact, let us take

$$\lambda_1\big(\big(H\big(t,X(t)\big)\big)^p\big)=\mu_1(t)\big(H\big(t,X(t)\big)\big)^p,$$
$$\lambda_2\big(\big(H\big(t,X(t)\big)\big)^p\big)=\mu_2(t)\big(H\big(t,X(t)\big)\big)^q.$$

Then, the condition (i) of Theorem 3.2 holds. It is easy to verify that the remaining conditions of Theorem 3.2 are satisfied. By virtue of Theorem 3.2, the trivial solution of SDE (1) is H-exponentially stable on $[t_0,\infty)_{\mathbb{T}}$. The proof is completed. □

Corollary 3.2 *Let $Y\in C^1_{\mathrm{rd}}(\mathbb{T}\times K_c(\mathbb{R}),\mathbb{R}_+)$ be a Lyapunov-like function satisfying the following conditions on $\mathbb{T}\times K_c(\mathbb{R})$ for $X\in\mathcal{X}$,*

(i) *there exist positive functions $\mu_1,\mu_2\in\Lambda$ and positive constants p, q such that*

$$\mu_1(t)\big\|X(t)\big\|^p\le Y\big(t,X(t)\big)\le\mu_2(t)\big\|X(t)\big\|^q\quad\textit{for all } t\in\mathbb{T};$$

(ii) *there exist a positive function λ_3 on $\mathbb{T}$ with $M=:\inf_{s\ge t_0}\lambda_3(s)/[\mu_2(s)]^{r/q}\in\Re_+$, function $\delta\in\Re$, and constants L, r with $r>0$ such that*

$$Y^{\Delta}\big(t,X(t)\big)\le\frac{-\lambda_3(t)\|X(t)\|^r-L(M\ominus\delta)(t)e_{\ominus\delta}(t,0)}{1+M\mu(t)},$$
$$M\big(Y\big(t,X(t)\big)-Y^{r/q}\big(t,X(t)\big)\big)\le L(M\ominus\delta)(t)e_{\ominus\delta}(t,0),$$

where q is given as in (i).

Then, the trivial solution of the following SDE

$$X^{\Delta}(t)=F(t,X),\qquad X(t_0)=X_0\in K_c(\mathbb{R})\tag{14}$$

is H-exponentially stable on $[t_0,\infty)_{\mathbb{T}}$.

Proof Set $v(t,X(t),a)=Y(t,X(t))$, $H(t,X(t))=\|X(t)\|$, $H_0(t_0,X_0)=\|X_0\|$ and $m=1$, $n=1$, we see from Corollary 3.1 that the trivial solution of (14) is H-exponentially stable on $[t_0,\infty)_{\mathbb{T}}$. □

Remark 3.1 Corollary 3.2 is essentially an extension and improvement of Theorem 4.2 in [33].

Theorem 3.3 *Let v be a matrix-valued Lyapunov-like function on $\mathbb{T}\times K_c^m(\mathbb{R}^n)\times\mathbb{R}^m_+$ and the function $\bar{v}:K_c^m(\mathbb{R}^n)\times\mathbb{R}^m_+\to\mathbb{R}_+$ satisfy $Nv(t,X_0,a)\le\bar{v}(X_0,a)$ for the constant $N>1$. Moreover, suppose that the following conditions hold: for $H\in\mathcal{M}$, $X\in\mathcal{X}$,*

(i) *there exist constants $k_1,p>0$ such that*

$$k_1\big(H\big(t,X(t)\big)\big)^p\le v\big(t,X(t),a\big)\quad\textit{for all } t\in\mathbb{T};$$

(ii) *there exist constants k_2, L, and functions $\varepsilon\in\Re_+$, $\delta\in\Re$ such that*

$$v^{\Delta}\big(t,X(t),a\big)\le\frac{-k_2v(t,X(t),a)-L(\varepsilon\ominus\delta)(t)e_{\ominus\delta}(t,0)}{1+\varepsilon\mu(t)},\tag{15}$$

$$(\varepsilon - k_2)v\big(t, X(t), a\big) \le L(\varepsilon \ominus \delta)(t)e_{\ominus\delta}(t, 0) \quad \textit{for } t \in \mathbb{T}. \tag{16}$$

Then, the trivial solution of SDE (1) *is H-uniformly exponentially stable on* $[t_0, \infty)_{\mathbb{T}}$.

Proof Let $\gamma \in \Gamma$ satisfy $\gamma(s) \le k_1 s$ for $s \ge 0$. By the assumption (i), we have

$$\gamma\big(\big(H\big(t, X(t)\big)\big)^p\big) \le k_1\big(H\big(t, X(t)\big)\big)^p \le v\big(t, X(t), a\big) \quad \text{for all } t \in \mathbb{T}. \tag{17}$$

In the light of Definition 3.2 and (17), it suffices to check that

$$v\big(t, X(t), a\big) \le N\bar{v}(X_0, a)e_{\ominus\varepsilon}(t, t_0) \tag{18}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$ and fixed $a \in \mathbb{R}^m_+$. If the above inequality is false, then there exists $t \in [t_0, \infty)_{\mathbb{T}}$ such that $v(t, X(t), a) > N\bar{v}(X_0, a)e_{\ominus\varepsilon}(t, t_0) > \bar{v}(X_0, a)e_{\ominus\varepsilon}(t, t_0)$ since $N > 1$. This implies that $\bar{t} = \inf\{t \in [t_0, \infty)_{\mathbb{T}} | v(t, X(t), a) > \bar{v}(X_0, a)e_{\ominus\varepsilon}(t, t_0)\}$ exists. Therefore, we obtain

$$v\big(\bar{t}, X(\bar{t}), a\big) \ge \bar{v}(X_0, a)e_{\ominus\varepsilon}(\bar{t}, t_0), \quad \bar{t} \in (t_0, \infty)_{\mathbb{T}} \quad \text{and} \tag{19}$$

$$v\big(t, X(t), a\big) \le \bar{v}(X_0, a)e_{\ominus\varepsilon}(t, t_0), \quad t \in [t_0, \bar{t})_{\mathbb{T}}. \tag{20}$$

Again, we consider the function $\varphi \in \Gamma$ with $s < \varphi(s) \le Ns$ for any $s \ge 0$. Then, by (19) we have

$$\varphi\big(v\big(\bar{t}, X(\bar{t}), a\big)\big) \ge \varphi\big(\bar{v}(X_0, a)e_{\ominus\varepsilon}(\bar{t}, t_0)\big) > \bar{v}(X_0, a)e_{\ominus\varepsilon}(\bar{t}, t_0).$$

However, we observe that $\varphi(v(t_0, X_0, a)) \le \varphi(N^{-1}\bar{v}(X_0, a)) \le \bar{v}(X_0, a)$. This implies that the set

$$\big\{t \in [t_0, \bar{t}]_{\mathbb{T}} | \varphi\big(v\big(t, X(t), a\big)\big) \le \bar{v}(X_0, a)e_{\ominus\varepsilon}(t, t_0)\big\}$$

is nonempty and hence its supremum, denoted by $\tilde{t}$, exists. Thus, we have

$$\varphi\big(v\big(\tilde{t}, X(\tilde{t}), a\big)\big) \le \bar{v}(X_0, a)e_{\ominus\varepsilon}(\tilde{t}, t_0), \quad \tilde{t} \in [t_0, \bar{t})_{\mathbb{T}} \quad \text{and} \tag{21}$$

$$\varphi\big(v\big(t, X(t), a\big)\big) > \bar{v}(X_0, a)e_{\ominus\varepsilon}(t, t_0), \quad t \in (\tilde{t}, \bar{t}]_{\mathbb{T}}. \tag{22}$$

On the other hand, similar to the proof of Step 1 in Theorem 3.2, we obtain that $v(t, X(t), a)e_{\varepsilon}(t, t_0)$ is nonincreasing in $t \in [\tilde{t}, \bar{t}]$, which infers that

$$\begin{aligned} v\big(\tilde{t}, X(\tilde{t}), a\big)e_{\varepsilon}(\tilde{t}, t_0) &\ge v\big(\bar{t}, X(\bar{t}), a\big)e_{\varepsilon}(\bar{t}, t_0) \ge \bar{v}(X_0, a) \\ &\ge \varphi\big(v\big(\tilde{t}, X(\tilde{t}), a\big)\big)e_{\varepsilon}(\tilde{t}, t_0) > v\big(\tilde{t}, X(\tilde{t}), a\big)e_{\varepsilon}(\tilde{t}, t_0), \end{aligned}$$

a contradiction. Consequently, (18) holds and hence the trivial solution of SDE (1) is H-uniformly exponentially stable on $[t_0, \infty)_{\mathbb{T}}$. The proof is completed. □

Theorem 3.4 *Let the assumptions of Theorem* 3.2 *hold, except that function M is changed into a constant and the estimate of* (5) *is strengthened to*

$$v^{\Delta}\big(t, X(t), a\big) \le -\lambda_3\big(\big(H\big(t, X(t)\big)\big)^r\big), \tag{23}$$

where $\lambda_3 : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive nondecreasing function with $\lim_{t\to+\infty} \lambda_3(t) = +\infty$ and constant $r > 0$. Then, the trivial solution of SDE (1) is H-exponentially asymptotically stable.

Proof Theorem 3.2 has proved the H-exponential stability of the trivial solution of SDE (1), that is, (4) holds. By Lemma 2.1(vi) and the Bernoulli inequality, we obtain

$$e_{\ominus M}(t,t_0) = \frac{1}{e_M(t,t_0)} \le \frac{1}{1+M(t-t_0)}.$$

This means that

$$\lim_{t\to\infty} \varrho(h_0,t_0)\big(e_{\ominus M}(t,t_0)\big)^d = 0.$$

Integrating both sides of (23) from t_0 to t, one has

$$v\big(t,X(t),a\big) \le v(t_0,X_0,a) - \int_{t_0}^{t} \lambda_3\big(\big(H\big(s,X(s)\big)\big)^r\big)\Delta s, \quad t \in [t_0,\infty)_{\mathbb{T}}. \tag{24}$$

We observe that (23) guarantees $v(t,X(t),a)$ to be nonincreasing in $t \in \mathbb{T}$. In addition, $v(t,X(t),a) \ge 0$. From the monotone convergence theorem, it follows that there exists β with $\lim_{t\to\infty} v(t,X(t),a) = \beta$.

Now, we will prove that $\beta = 0$. Suppose that this is false. Then, we have $\beta > 0$. By virtue of the monotonicity of $v(t,X(t),a)$, we have $v(t,X(t),a) \ge \beta > 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$. On the other hand, since the function λ_3 is positive nondecreasing and $\lim_{t\to+\infty} \lambda_3(t) = +\infty$, we see that $\int_{t_0}^{t} \lambda_3((H(s,X(s)))^r)\Delta s$ is larger than $v(t_0,X_0,a)$ when $t \in \mathbb{T}$ is sufficiently large. Combining with (24), we obtain that $v(t,X(t),a) < 0$, which contradicts $v(t,X(t),a) \ge 0$. Hence, $\beta = 0$, that is $\lim_{t\to\infty} v(t,X(t),a) = 0$.

Next, we will prove that $\lim_{t\to\infty} H(t,X(t)) = 0$. If this is not true, then there exists $\varepsilon_0 > 0$ such that $H(t_m,X(t_m)) > \varepsilon_0 > 0$ for any $m \in \mathbb{Z}_+$ and some $t_m \in \mathbb{T}$ with $t_m \ge m$. From this, combined with (7), we have

$$v\big(t_m,X(t_m),a\big) \ge \bar{\gamma}\big(\big(H\big(t_m,X(t_m)\big)\big)^p\big) \ge \bar{\gamma}\big(\varepsilon_0^p\big) > \bar{\gamma}(0) = 0.$$

This contradicts $\lim_{t\to\infty} v(t,X(t),a) = 0$. Hence, $\lim_{t\to\infty} H(t,X(t)) = 0$. By virtue of the definition of $\bar{\gamma} \in \Gamma$, we obtain $\lim_{t\to\infty} \bar{\gamma}((H(t,X(t)))^p) = 0$. This guarantees that for any $\varepsilon > 0$, there exists a positive real number T such that

$$\bar{\gamma}\big(\big(H\big(t,X(t)\big)\big)^p\big) < \varepsilon \quad \text{for all } t \in [t_0+T,\infty)_{\mathbb{T}}.$$

Thus, the trivial solution of SDE (1) is H-exponentially asymptotically stable and the proof is completed. □

Theorem 3.5 *Let the assumptions of Theorem 3.3 hold except that function ε changes into a constant and the estimate of (15) is strengthened to*

$$v^{\Delta}\big(t,X(t),a\big) \le -k_2 v\big(t,X(t),a\big). \tag{25}$$

In addition, assume that $v(t,\Theta_0,a) = 0$ for each $t \in \mathbb{T}$ and each $a \in \mathcal{A}(t,\Theta_0)$. Then, the trivial solution of SDE (1) is H-uniformly exponentially asymptotically stable.

Proof Theorem 3.3 has proved the H-uniformly exponential stability, so that (4) holds. As an analogy of the proof of Theorem 3.4, we have

$$\lim_{t\to\infty} \varrho(h_0,t_0)\big(e_{\ominus\varepsilon}(t,t_0)\big)^q = 0$$

and

$$\lim_{t\to\infty} \gamma\Big(\big(H\big(t,X(t)\big)\big)^p\Big) = 0.$$

Thus, the trivial solution of SDE (1) is H-uniformly exponentially asymptotically stable. The proof is completed. □

At the end of this section, we will start with some simple examples of SDEs to illustrate that our approach and results are applicable. For the sake of convenience, we only consider $\mathbb{T}_+$ instead of $\mathbb{T}$ as the working platform.

Example 3.1 The trivial solution of the following SDE

$$X^\Delta = \ominus\zeta(t)X, \qquad X(0) = X_0 \in K_c^m\big(\mathbb{R}^n\big), \quad t \in \mathbb{T}_+ \tag{26}$$

is H-exponentially stable, where the function $\zeta \in C(\mathbb{T}_+,\mathbb{R}_+) \cap \Re_+$ is nondecreasing and satisfies that $w \le \zeta(t)$ on $\mathbb{T}_+$ with the constant $w > 0$.

Proof From Lemma 12 in [31] it follows that SDE (26) has a unique solution $X : \mathbb{T}_+ \to K_c^m(\mathbb{R}^n)$ as follows

$$X(t) = X_0 e_{\ominus\zeta}(t,0).$$

Choose $v(t,X(t),a) = H(t,X(t)) = \|X(t)\|$ for $t \in \mathbb{T}_+$. Next, we prove that the conditions in the Theorem 3.2 are satisfied. It is easy to see that the condition (i) is satisfied with $\lambda_1(s) = s$, $\lambda_2(s) = s$ for $s \ge 0$, $p = 1$. In order to verify condition (ii), we first calculate the Δ-derivative of $v(t,X(t),a)$. By Lemma 2.1(iii), we obtain

$$\begin{aligned} v^\Delta\big(t,X(t),a\big) &= \|X_0\| \ominus \zeta(t)e_{\ominus\zeta}(t,0) \\ &= -\frac{\zeta(t)}{1+\mu(t)\zeta(t)}\|X_0\|e_{\ominus\zeta}(t,0) \\ &\le -\frac{w}{1+w\mu(t)}v\big(t,X(t),a\big). \end{aligned}$$

Next, we take

$$M(t) = \delta(t) = w, \qquad r = 1, \qquad L = 1, \qquad \lambda_3(s) = ws, \qquad \gamma(s) = s$$

for all $t \in \mathbb{T}_+$. Then, the condition (ii) is satisfied. Consequently, the trivial solution of SDE (26) is H-exponentially stable. □

Example 3.2 The trivial solution of the following SDE

$$X^{\Delta}(t) = -\eta(t)X\bigl(\sigma(t)\bigr) + F(t), \qquad X(0) = X_0 \in K_c^m\bigl(\mathbb{R}^n\bigr), \quad t \in \mathbb{T}_+ \tag{27}$$

is H-uniformly exponentially stable, where the functions $\eta \in C(\mathbb{T}_+, \mathbb{R}_+) \cap \Re_+$ is nondecreasing and satisfies that $w \le \eta(t)$ on $\mathbb{T}_+$ with the constant $w > 0$ and $F \in C_{\mathrm{rd}}(\mathbb{T}_+, K_c^m(\mathbb{R}^n))$ satisfies

$$\int_0^t e_{\ominus\eta}(t,s)\bigl\|F(s)\bigr\|\Delta s \le \|X_0\|e_{\ominus\eta}(t,0) \quad \text{for } t \in \mathbb{T}_+.$$

Proof From Lemma 13 in [31] it follows that SDE (27) has a unique solution $X : \mathbb{T}_+ \to K_c^m(\mathbb{R}^n)$ as follows

$$X(t) = X_0 e_{\ominus\eta}(t,0) + \int_0^t e_{\ominus\eta}(t,s)F(s)\Delta s.$$

Thus,

$$\begin{aligned}
\bigl\|X(t)\bigr\| &= \left\|X_0 e_{\ominus\eta}(t,0) + \int_0^t e_{\ominus\eta}(t,s)F(s)\Delta s\right\| \\
&\ge \bigl\|X_0 e_{\ominus\eta}(t,0)\bigr\| - \left\|\int_0^t e_{\ominus\eta}(t,s)F(s)\Delta s\right\| \\
&\ge \|X_0\|e_{\ominus\eta}(t,0) - \int_0^t e_{\ominus\eta}(t,s)\bigl\|F(s)\bigr\|\Delta s.
\end{aligned} \tag{28}$$

Let

$$v\bigl(t, X(t), a\bigr) = H\bigl(t, X(t)\bigr) = \|X_0\|e_{\ominus\eta}(t,0) - \int_0^t e_{\ominus\eta}(t,s)\bigl\|F(s)\bigr\|\Delta s \tag{29}$$

for $t \in \mathbb{T}_+$ and $a \in \mathbb{R}_+^m$. Our hypothesis guarantees that $v(t, X(t), a) \ge 0$. Next, we prove that the conditions in the Theorem 3.2 are satisfied. It is easy to see that the condition (i) holds with $\lambda_1(s) = s$, $\lambda_2(s) = s$ for $s \ge 0$, $p = 1$. In order to verify condition (ii), we first calculate the Δ-derivative of $v(t, X(t), a)$. By Lemma 2.1(iii) we obtain

$$\begin{aligned}
v^{\Delta}\bigl(t, X(t), a\bigr) &= \left[\|X_0\|e_{\ominus\eta}(t,0) - \int_0^t e_{\ominus\eta}(t,s)\bigl\|F(s)\bigr\|\Delta s\right]^{\Delta} \\
&= \|X_0\| \ominus \eta e_{\ominus\eta}(t,0) - e_{\ominus\eta}\bigl(\sigma(t), t\bigr)\bigl\|F\bigl(\sigma(t)\bigr)\bigr\| \\
&\quad - \int_0^t \ominus\eta e_{\ominus\eta}(t,s)\bigl\|F(s)\bigr\|\Delta s \\
&= -\frac{\eta(t)}{1+\mu(t)\eta(t)}\|X_0\|e_{\ominus\eta}(t,0) - \frac{1}{1+\mu(t)\eta(t)}\bigl\|F\bigl(\sigma(t)\bigr)\bigr\| \\
&\quad + \frac{\eta(t)}{1+\mu(t)\eta(t)}\int_0^t e_{\ominus\eta}(t,s)\bigl\|F(s)\bigr\|\Delta s \\
&= \frac{1}{1+\mu(t)\eta(t)}\Bigl[-\eta(t)\|X_0\|e_{\ominus\eta}(t,0) - \bigl\|F\bigl(\sigma(t)\bigr)\bigr\|
\end{aligned}$$

$$
\begin{aligned}
&+ \eta(t) \int_0^t e_{\ominus\eta}(t,s)\big\|F(s)\big\|\Delta s\Bigg] \\
&\leq \frac{-\eta(t)}{1+\mu(t)\eta(t)}\left[\|X_0\|e_{\ominus\eta}(t,0) - \int_0^t e_{\ominus\eta}(t,s)\big\|F(s)\big\|\Delta s\right] \\
&\leq \frac{-w}{1+w\mu(t)} v\big(t,X(t),a\big). \qquad (30)
\end{aligned}
$$

Next, we take

$$
M(t) = \delta(t) = w, \qquad r = 1, \qquad L = 1, \qquad \lambda_3(s) = ws, \qquad \gamma(s) = s.
$$

Then, the condition (ii) is satisfied. Consequently, the trivial solution of SDE (27) is H-exponentially stable.

Finally, it is not difficult to see that all conditions in Theorem 3.3 are satisfied if we choose $k_1 = \frac{1}{2}$ and $\varepsilon(t) = k_2 = w$. Therefore, Theorem 3.3 guarantees that the trivial solution of SDE (27) is H-uniformly exponentially stable. □

Example 3.3 SDE (27) is H-exponentially asymptotically stable and H-uniformly exponentially asymptotically stable under the hypotheses of Example 3.2.

Proof Let $v(t,X(t),a)$ be given as in (29). It has been shown that the assumptions of Theorem 3.2 are fulfilled with $M = \omega$ (a positive constant) in the proof of Example 3.2. Let $\lambda_3(s) = \frac{w}{1+w\mu(t)}s$. Then, λ_3 satisfies the hypothesis of Theorem 3.4 and the inequality (23) holds by (30). Now, Theorem 3.4 guarantees that the trivial solution of SDE (27) is H-exponentially asymptotically stable.

Let $\varsigma = \max_{t\in\mathbb{T}_+}\mu(t)$. To check the conditions of Theorem 3.5, let us take constants $k_1 = \frac{1}{2}$, k_2, ε, and the function δ as follows

$$
k_2 = \varepsilon = \delta(t) = \begin{cases} \frac{w}{2(1+w\varsigma)}, & \varsigma < +\infty; \\ 0, & \varsigma = +\infty, \end{cases}
$$

for all $t \in \mathbb{T}_+$. Thus, if $\varsigma < +\infty$, similar to the derivation in the proof of Example 3.2, we obtain

$$
\begin{aligned}
v^\Delta\big(t,X(t),a\big) &\leq \frac{-w}{1+w\mu(t)} v\big(t,X(t),a\big) \leq \frac{-w}{1+w\varsigma} v\big(t,X(t),a\big) \\
&< \frac{-w}{2(1+w\varsigma)} v\big(t,X(t),a\big) = -k_2 v\big(t,X(t),a\big)
\end{aligned}
$$

for all $t \in \mathbb{T}_+$. If $\varsigma = +\infty$, we have

$$
v^\Delta\big(t,X(t),a\big) \leq \frac{-w}{1+w\mu(t)} v\big(t,X(t),a\big) \leq 0 = -k_2 v\big(t,X(t),a\big)
$$

for all $t \in \mathbb{T}_+$. Consequently, (25) holds.

From (28) and (29) it follows that $v(t,\Theta_0,a) = 0$ for each $t \in \mathbb{T}_+$ and each $a \in \mathcal{A}(t,\Theta_0)$. In addition, all the conditions in Theorem 3.3 are clearly satisfied under our assumptions. Now, Theorem 3.5 guarantees that the trivial solution of SDE (27) is H-uniformly exponentially asymptotically stable. □

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