https://doi.org/10.1186/s13660-022-02872-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

# Study of weak solutions for degenerate parabolic inequalities with nonstandard conditions

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**Abstract**

In this paper, we study the degenerate parabolic variational inequalities in a bounded domain. By solving a series of penalty problems, the existence and uniqueness of the solutions in the weak sense are proved by the energy method and a limit process.

**MSC:** 35K99; 97M30

**Keywords:** Parabolic variational inequality; Weak solution; Penalty problem; Existence; Uniqueness

## 1 Introduction

Let $0 < T < \infty$ and $\Omega \subset \mathrm{R}_N (N \geq 2)$ be a bounded simple domain with appropriately smooth boundary $\partial\Omega$. In this article, we consider the following quasilinear degenerate parabolic inequalities:

$$\begin{cases} \min\{Lu, u(x,0) - u_0\} = 0, & (x,t) \in Q_T, \\ u(x,t) = 0, & (x,t) \in \Gamma_T, \\ u(x,0) = u_0, & x \in \Omega, \end{cases} \tag{1.1}$$

with

$$Lu = u_t - u\operatorname{div}\big(a(u)|\nabla u|^{p(x,t)-2}\nabla u\big) - \gamma|\nabla u|^{p(x,t)} - f(x,t), \tag{1.2}$$

where $Q_T = \Omega \times (0, T]$, $\mathrm{a}(u) = u^\sigma + d_0$, and $\Gamma_T$ is the lateral boundary of cylinder $Q_T$.

In applications, Problem (1.1) arises in the model of American option pricing in the Black–Scholes models. We refer to [1–4] for the financial background of parabolic inequalities. Among them, the most interesting research topic is to construct different types of variational parabolic inequalities and analyze the existence and uniqueness for their solutions (see, for example, [3–10] and the references therein). In 2014, the authors in [5] discussed the problem

$$\begin{cases} u_t - Lu - F(u,x,t) \geq 0 & \text{in } Q_T, \\ u(x,t) \geq u_0(x) & \text{in } \Omega, \\ (u_t - Lu - F(u,x,t)) \cdot (u(x,t) - u_0(x)) = 0 & \text{in } Q_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma_T, \end{cases}$$

with second-order elliptic operator

$$Lu = -u\operatorname{div}\big(a(u)|\nabla u|^{p(x,t)-2}\nabla u\big) - \gamma|\nabla u|^{p(x,t)} - f(x,t).$$

They proved the existence and uniqueness of a solution to this problem with some restrictions on $u_0$, $F$, and $L$. Later, the authors in [6, 7] extended the relative conclusions with the assumption that $a(u)$ is a constant, and $p(x) = 2$. The authors discussed the existence and numerical algorithm of the solution.

To the best of our knowledge, the existence and uniqueness of this problem with the assumption that $p(x,t)$ are variables have been less studied. We cannot easily apply the method in [6, 7] to the case that $p(x,t)$ and $a(u)$ are not constants.

The aim of this paper is to study the existence and uniqueness of solutions for a degenerate parabolic variational inequality problem. Throughout the paper, we assume that the exponent $p(x,t)$ is continuous in $Q_T$ with a logarithmic module of a country:

$$1 < p^- < p(x,t) < p^+ < \infty, \tag{1.3}$$

where $p^- = \inf_{(x,t)\in Q_T} p(x,t)$ and $p^+ = \sup_{(x,t)\in Q_T} p(x,t)$.

The outline of this paper is as follows: In Sect. 2, we introduce the function spaces of Orlicz–Sobolev type, give the definition of the weak solution to the problem, and state our main theorems. In Sect. 3, we give some estimates of the penalty problem (approximating problem). Section 4 proves the existence and uniqueness of the solution obtained in Sect. 2.

## 2 The main results of weak solutions

In this section, we recall some useful definitions and known results, which can be found in [11–14]. Set

$$\begin{aligned}
&L^{p(x,t)}(Q_T) = \Big\{u(x,t)|u \text{ is measurable in } Q_T, A_{p(\cdot)}(u) = \int\int_{Q_T} |u|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t\Big\}, \\
&\|u\|_{p(\cdot)} = \inf\big\{\lambda > 0, A_{p(\cdot)}(u/\lambda) \leq 1\big\}, \\
&V_t(\Omega) = \big\{u|u \in L^2(\Omega) \cap W_0^{1,1}, |\nabla u| \in L^{p(x,t)}(\Omega)\big\}, \\
&\|u\|_{V_t(\Omega)} = \|u\|_{2,\Omega} + \|\nabla u\|_{p(\cdot,t),\Omega}, \\
&W(Q_T) = \big\{u : [0,T] \to V_t(\Omega)|u \in L^2(Q_T) \cap W_0^{1,1}, \\
&|\nabla u| \in L^{p(x,t)}(Q_T), u = 0 \text{ on } \Gamma_T\big\},
\end{aligned}$$

$$\|u\|_{W_t(Q_T)} = \|u\|_{2,Q_T} + \|\nabla u\|_{p(\cdot,t),Q_T}$$

and denote by $W'(Q_T)$ the dual of $W(Q_T)$ with respect to the inner product in $L^2(Q_T)$.

In the spirit of [3] and [4], we introduce the following maximal monotone graph

$$G(x) = \begin{cases} 0, & x > 0, \\ \theta, & x = 0, \end{cases} \tag{2.1}$$

where $\theta \in [0, M)$ and $M$ depends only on $|u_0|_\infty$.

The purpose of the paper is to obtain the existence and uniqueness of weak solutions of (1.1). Let $B = W(Q_T) \cap L^\infty(0,T;L^\infty(\Omega))$, and the weak solution is defined as:

**Definition 2.1** A pair is called a weak solution of problem (1.1), if (a) $u(x,t) \geq u_0(x)$, (b) $u(x,0) = u_0(x)$, (c) $\xi \in G(u - u_0)$, (d) for every test function $\phi \in Z \equiv \{\eta(z) : \eta \in W(Q_T) \cap L^\infty(0,T;L^2(\Omega)), \eta_t \in W(Q_T)\}$ and every $t_1, t_2 \in [0,T]$ the following identity holds:

$$\begin{aligned} &\int_{t_1}^{t_2} \int_\Omega u \cdot \phi_t - a(u)|\nabla u|^{p(x,t)-2}\nabla u \nabla \phi - \big(a(u) - \gamma\big)|\nabla u|^{p(x,t)} \phi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{t_1}^{t_2} \int_\Omega f(x,t)\phi + \xi\phi \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega u\phi \,\mathrm{d}x \int_{t_1}^{t_2}. \end{aligned} \tag{2.2}$$

Our main results are the following two theorems.

**Theorem 2.1** *Let us satisfy conditions* (1.3). *If the following conditions hold*:

(H1) $\max\{1, \frac{2N}{N+1}\} < p^- < N$, $2 \leq \sigma < \frac{2p^+}{p^+-1}$, $0 < \gamma < d_0$, *and*

(H2) $u_0(x) \geq 0, f \geq 0, \|u_0\|_{\infty,\Omega} + \int_0^T \|f(x,t)\|_{\infty,\Omega}\,\mathrm{d}t + |\Omega| \cdot T = K(T) < \infty$,

*then Problem* (1.1) *has at least one weak solution in the sense of Definition* 2.1.

**Theorem 2.2** *Suppose that the conditions in Theorem* 2.1 *are fulfilled and* $p^+ \geq 2$. *Then, Problem* (1.1) *admits a unique solution in the sense of Definition* 2.1.

## 3 Penalty problems

In this section, we consider a family of auxiliary parabolic problems

$$\begin{cases} L_\varepsilon u_\varepsilon + \beta(u_\varepsilon - u_0) = 0, & (x,t) \in Q_T, \\ u_\varepsilon(x,t) = \varepsilon, & (x,t) \in \Gamma_T, \\ u_\varepsilon(x,0) = u_0 + \varepsilon, & x \in \Omega, \end{cases} \tag{3.1}$$

with

$$L_\varepsilon u_\varepsilon = u_{\varepsilon t} - u_\varepsilon \cdot \mathrm{div}\big(a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon\big) - \gamma|\nabla u_\varepsilon|^{p(x,t)} - f(x,t), \tag{3.2}$$

$\beta_\varepsilon(\cdot)$ is the penalty function satisfying

$$\begin{aligned} &\varepsilon \in (0,1), \qquad \beta_\varepsilon(\cdot) \in C^2(\mathrm{R}), \qquad \beta_\varepsilon(x) \leq 0, \qquad \beta_\varepsilon(0) = -1, \\ &\beta'_\varepsilon(0) \geq 0, \qquad \beta''_\varepsilon(0) \geq 0, \qquad \lim_{x \to 0+} \beta(x) = \begin{cases} 0, & x > -0, \\ -1, & x = 0. \end{cases} \end{aligned} \tag{3.3}$$

With a similar method as in [8], we may prove that the regularized problem has a unique weak solution

$$u_\varepsilon(x,t) \in W(Q_T) \cap L^2(Q_T), \quad \partial_t u_\varepsilon(x,t) \in W\prime(Q_T),$$

satisfying the following integral identities

$$\begin{aligned} &\int_{t_1}^{t_2}\int_\Omega u_\varepsilon \cdot \phi_t - a(u_\varepsilon)u_\varepsilon|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\phi - \big(a(u_\varepsilon)-\gamma\big)|\nabla u_\varepsilon|^{p(x,t)}\phi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{t_1}^{t_2}\int_\Omega \big(\beta_\varepsilon(u_\varepsilon - u_0) - f(x,t)\big)\phi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_\varepsilon\phi \,\mathrm{d}x \int_{t_1}^{t_2} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\int_{t_1}^{t_2}\int_\Omega u_{\varepsilon t} \cdot \phi + a(u_\varepsilon)u_\varepsilon|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\phi + \big(a(u_\varepsilon)-\gamma\big)|\nabla u_\varepsilon|^{p(x,t)}\phi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{t_1}^{t_2}\int_\Omega \big(f(x,t) - \beta_\varepsilon(u_\varepsilon - u_0)\big)\phi \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.5}$$

We start with two preliminary results that will be used several times below.

**Lemma 3.1** *Let $M(s) = |s|^{p(x,t)-2}s$, then $\forall \xi, \eta \in \mathrm{R}^N$*

$$\begin{aligned} &\big(M(\xi) - M(\eta)\big)\cdot(\xi - \eta) \\ &\quad \ge \begin{cases} 2^{-p(x,t)}|\xi-\eta|^{p(x,t)}, & 2 \le p(x,t) < \infty, \\ (p(x,t)-1)|\xi-\eta|^2(|\xi|^{p(x,t)} + |\eta|^{p(x,t)})^{\frac{p(x,t)-2}{p(x,t)}}, & 1 \le p(x,t) < 2. \end{cases} \end{aligned}$$

*Proof* The proof can be found in [15]. □

**Lemma 3.2** (Comparison principle) *Assume $2 < \sigma < \frac{2p^+}{p^+-1}$, $p^+ \ge 2$, $u$ and $v$ are in $W(Q_T) \cap L^\infty(0,T;L^\infty(\Omega))$. If $L_\varepsilon u \ge L_\varepsilon v$ in $Q_T$ and $u(x,t) \le v(x,t)$ on $\partial Q_T$, then $u(x,t) \le v(x,t)$ in $Q_T$.*

*Proof* We argue by contradiction. Suppose $u(x,t)$ and $v(x,t)$ satisfy $L_\varepsilon u \ge L_\varepsilon v$ in $Q_T$ and there is a $\delta > 0$ such that for $0 < \tau \le T$, $w = u - v$ on the set

$$\Omega_\delta = \Omega \cap \big\{x : w(x,t) > \delta\big\}$$

and $\mu(\Omega_\delta) > 0$. Let

$$F_\varepsilon(\xi) = \begin{cases} \frac{1}{\alpha-1}\varepsilon^{1-\alpha} - \frac{1}{\alpha-1}\xi^{1-\alpha} & \text{if } \xi > \varepsilon, \\ 0 & \text{if } \xi \le \varepsilon, \end{cases} \tag{3.6}$$

where $\delta > 2\varepsilon > 0$ and $\alpha = \frac{\sigma}{2}$. Let a test function $\xi = F_\varepsilon(w) \in Z$ in (3.4),

$$\begin{aligned}
0 \geq & \int\int_{Q_T} w_t F_\varepsilon(w) + a(v)v\big(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\big)\nabla F_\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}t \\
& + \int\int_{Q_T} \big[a(u)u - a(v)v\big]|\nabla u|^{p(x,t)-2}\nabla u \nabla F_\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}t \\
& + \int\int_{Q_T} \big[a(u) - \gamma\big]|\nabla u|^{p(x,t)} - \big[a(v) - \gamma\big]|\nabla v|^{p(x,t)} \\
= & J_1 + J_2 + J_3 + J_4,
\end{aligned} \tag{3.7}$$

where $Q_{T,\varepsilon} = \{(x,t) \in Q_T | w > \varepsilon\}$,

$$\begin{aligned}
J_1 &= \int\int_{Q_T} w_t F_\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}t, \\
J_2 &= \int\int_{Q_T} a(v)v\big(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\big)\nabla w\,\mathrm{d}x\,\mathrm{d}t, \\
J_3 &= \int\int_{Q_T} \big[a(u)u - a(v)v\big]|\nabla u|^{p(x,t)-2}\nabla u \nabla F_\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}t, \\
J_4 &= \int\int_{Q_T} \big[a(u) - \gamma\big]|\nabla u|^{p(x,t)} - \big[a(v) - \gamma\big]|\nabla v|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Now, let $t_0 = \inf\{t \in (0,\tau] : w > \varepsilon\}$, then we estimate $J_1$ as follows

$$\begin{aligned}
J_1 &= \int\int_{Q_T} w_t F_\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \left(\int_0^{t_0} w_t F_\varepsilon(w)\,\mathrm{d}t + \int_0^{t_0} w_t F_\varepsilon(w)\,\mathrm{d}t\right)\mathrm{d}x \\
&\geq \int_\Omega \int_\varepsilon^w F_\varepsilon(\mathrm{s})\,\mathrm{d}s\,\mathrm{d}x \geq \int_{\Omega_\delta} \int_\varepsilon^w F_\varepsilon(\mathrm{s})\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned} \tag{3.8}$$

Let us first consider the case $p^- \geq 2$. By virtue of the first inequality of Lemma 3.1, we obtain

$$\begin{aligned}
J_2 &= \int\int_{Q_T} a(v)v\big(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\big)\nabla w\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int\int_{Q_T} a(v)v \cdot w^{-\alpha} 2^{-p(x,t)}|\nabla w|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \\
&= 2^{-p^+}\int\int_{Q_T} a(v)v \cdot w^{-\alpha}|\nabla w|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t > 0.
\end{aligned} \tag{3.9}$$

Noting that $\frac{p(x,t)}{p(x,t)-1} \geq \frac{p+}{p+-1} \geq \sigma^2 = \alpha > 1$ and applying Young's inequality, we may estimate the integrand of $J_3$ in the following way

$$\begin{aligned}
& \big|\big[a(u)u - a(v)v\big]w^{-\alpha}|\nabla w|^{p(x,t)-2}\nabla u \nabla w\big| \\
&= \left|\left[(\delta+1)w\int_0^1 \big(\theta u + (1-\theta)v\big)^\sigma\,\mathrm{d}\theta + d_0(u-v)\right]w^{-\alpha}|\nabla u|^{p(x,t)-2}\nabla u \nabla w\right| \\
&\leq \frac{C}{w^\alpha}\left[\frac{a(v)v}{C}|\nabla w|^{p(x,t)} + C_1\big(\delta, d_0, K, p^\pm\big)|w|^{p'(x,t)}|\nabla u|^{p(x,t)}\right]
\end{aligned} \tag{3.10}$$

$$\leq \frac{a(v)v}{2^{p^+-1}w^\alpha}|\nabla w|^{p(x,t)} + C_1\big(\delta, d_0, K, p^\pm\big)|u|^{p'(x,t)}.$$

Substituting (3.10) into $J_3$ and combining it with $J_2$, we obtain

$$J_3 \leq \frac{1}{2}J_2 + C\int\int_{Q_T} |\nabla u|^{p(x,t)-2}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.11}$$

Recall that $0 < \gamma \leq d_0$, $u \in W(Q_T) \cap L^\infty(0,T;L^\infty(\Omega))$. Then, we have

$$J_4 \leq \int\int_{Q_T} u^\sigma |\nabla u|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \leq C\int\int_{Q_T} |\nabla u|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t, \tag{3.12}$$

where $C$ is a positive constant. Thus, we insert the above estimates (3.8), (3.9), (3.11), and (3.12) into (3.7) and dropping the nonnegative terms, we arrive at

$$(\delta - 2\varepsilon)\big(1 - 2^{1-\alpha}\big)\varepsilon^{1-\alpha}\mu(\Omega) < C. \tag{3.13}$$

Secondly, we consider the case $1 < p^- \leq p(x,t) < 2, p^+ \geq 2$. According to the second inequality of Lemma 3.1, it is easily seen that the following inequalities hold

$$\begin{aligned}
&\big|\big[a(u)u - a(v)v\big]w^{-\alpha}|\nabla w|^{p(x,t)-2}\nabla u\nabla w\big| \\
&\quad= \left|\left[(\delta+1)w\int_0^1 \big(\theta u + (1-\theta)v\big)^\sigma\,\mathrm{d}\theta + d_0(u-v)\right]|w|^{2-\alpha}|\nabla u|^{p(x,t)-2}\nabla u\nabla w\right| \\
&\quad\leq \frac{a(v)v(p^{-1}-1)}{2w^\alpha}\big(|\nabla u| + |\nabla v|\big)^{p(x,t)}|\nabla \mathrm{w}|^2 + C_1\big(\delta, d_0, K, p^\pm\big)|w|^{2-\alpha}\big(|\nabla u| + |\nabla v|\big)^{p(x,t)} \\
&\quad\leq \frac{a(v)v(p^{-1}-1)}{2w^\alpha}\big(|\nabla u| + |\nabla v|\big)^{p(x,t)}|\nabla \mathrm{w}|^2 + C_1\big(\delta, d_0, K, p^\pm\big)\big(|\nabla u| + |\nabla v|\big)^{p(x,t)}.
\end{aligned}$$

Substituting the above inequality into $J_3$, we obtain

$$J_3 \leq \frac{1}{2}J_2 + C\int\int_{Q_T} \big(|\nabla u| + |\nabla v|\big)^{p(x,t)-2}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.14}$$

Similar to the case $p^- \geq 2$, estimate (3.13) still holds using (3.14) instead of (3.11). Note that $\lim_{\varepsilon\to 0}(\delta - 2\varepsilon)(1 - 2^{1-\alpha})\varepsilon^{1-\alpha}\mu(\Omega_\delta) = +\infty$, we obtain a contradiction. This means $\mu(\Omega_\delta) = 0$ and $w \leq 0$ a.e. in $\mathrm{Q}_\tau$. □

**Lemma 3.3** *Let $u_\varepsilon$ be weak solutions of* (3.1). *Then,*

$$u_{0\varepsilon} \leq u_\varepsilon \leq |u_0|_\infty + \varepsilon, \tag{3.15}$$

$$u_{\varepsilon_1} \leq u_{\varepsilon_2} \quad \textit{for } \varepsilon_1 \leq \varepsilon_2, \tag{3.16}$$

*where* $|u_0|_\infty = \sup_{x\in\Omega}|u_0(x)|$.

*Proof* First, we prove $u_\varepsilon \geq u_{0\varepsilon}$ by contradiction. Assume $u_\varepsilon \leq u_{0\varepsilon}$ in $Q_T^0$, $Q_T^0 \subset Q_T$. Noting $u_\varepsilon \geq u_{0\varepsilon}$ on $\partial Q_T$, we may assume that $u_\varepsilon = u_{0\varepsilon}$ on $\partial Q_T$. With (3.1) and letting $t = 0$, it is easy to see that

$$Lu_{0,\varepsilon} = -\beta_\varepsilon(u_{0,\varepsilon} - u_{0,\varepsilon}) = 1, \tag{3.17}$$

$$Lu_\varepsilon = -\beta_\varepsilon(u_\varepsilon - u_{0,\varepsilon}) \leq 1. \tag{3.18}$$

From Lemma 3.2, we arrive at

$$u_\varepsilon(x,t) \geq u_{0,\varepsilon}(x) \quad \text{for any } (x,t) \in Q_T. \tag{3.19}$$

Therefore, we obtain a contradiction.

Secondly, we pay attention to $u_\varepsilon(t,x) \leq |u_0|_\infty + \varepsilon$. Applying the definition of $\beta_\varepsilon(\cdot)$, we have that

$$L\big(|u_0|_\infty + \varepsilon\big) = 0, \qquad Lu_\varepsilon = -\beta_\varepsilon(u_\varepsilon - u_{0,\varepsilon}) \geq 0. \tag{3.20}$$

From (3.20), we obtain

$$u_\varepsilon(t,x) \leq |u_0|_\infty + \varepsilon \quad \text{on } \partial\Omega \times (0,\mathrm{T}) \tag{3.21}$$

and $u_\varepsilon(t,x) \leq |u_0|_\infty + \varepsilon$ in $\Omega$. Thus, combining (3.20) and (3.21) and repeating Lemma 3.2, we have

$$u_\varepsilon(t,x) \leq |u_0|_\infty + \varepsilon \quad \text{in } Q_T. \tag{3.22}$$

Thirdly, we aim to prove (3.16). From (3.1),

$$Lu_{\varepsilon_1} = \beta_{\varepsilon_1}(u_{\varepsilon_1} - u_{0,\varepsilon_1}), \tag{3.23}$$

$$Lu_{\varepsilon_2} = \beta_{\varepsilon_2}(u_{\varepsilon_2} - u_{0,\varepsilon_2}). \tag{3.24}$$

It follows by $\varepsilon_1 \leq \varepsilon_2$ and the definition of $\beta_\varepsilon(\cdot)$ that

$$\begin{aligned} &Lu_{0,\varepsilon_2} + \beta_{\varepsilon_1}(u_{\varepsilon_2} - u_{0,\varepsilon}) \\ &\quad = \beta_{\varepsilon_2}(u_{\varepsilon_2} - u_{0,\varepsilon}) - \beta_{\varepsilon_1}(u_{\varepsilon_1} - u_{0,\varepsilon}) \\ &\quad = \beta_{\varepsilon_2}(u_{\varepsilon_2} - u_{0,\varepsilon}) - \beta_{\varepsilon_1}(u_{\varepsilon_2} - u_{0,\varepsilon}) \geq 0. \end{aligned} \tag{3.25}$$

Thus, combining the initial and boundary conditions in (3.1) can be proved by Lemma 3.2. □

To prove this theorem, we need the following lemmas.

**Lemma 3.4** *The solution of problem* (3.1) *satisfies the estimate*

$$\|u_\varepsilon\|_{\infty,Q_T} \leq \|u_0\|_{\infty,\Omega} + \int_0^T \big\|f(x,t)\big\|_{\infty,\Omega}\,\mathrm{d}t + |\Omega| \cdot T = K(T) < \infty.$$

*Proof* Let us introduce the following function

$$u_{\varepsilon,M} = \begin{cases} M & \text{if } u_\varepsilon > M, \\ u_\varepsilon & \text{if } |u_\varepsilon| < M, \\ -M & \text{if } u_\varepsilon < -M. \end{cases} \tag{3.26}$$

The function $u_{\varepsilon,M}^{2k-1}$, with $k \in N$, can be chosen as a test function in (3.4). Let $t_2 = t+h, t_1 = t$ in (3.4), with $t, t+h \in (0,T)$. Then,

$$\begin{aligned} &\frac{1}{2k}\int_t^{t+h} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u_{\varepsilon,M}^{2k}\,\mathrm{d}x\right)\mathrm{d}t \\ &\quad + (2k-1)\int_t^{t+h}\int_\Omega a_{\varepsilon,M}(u_{\varepsilon,M})u_{\varepsilon,M}^{2(k-1)}|\nabla u_{\varepsilon,M}|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_t^{t+h}\int_\Omega \left[a_{\varepsilon,M}(u_{\varepsilon,M}) - \gamma\right]\cdot u_{\varepsilon,M}^{2k-1}|\nabla u_{\varepsilon,M}|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_t^{t+h}\int_\Omega \big(f(x,t) - \beta_\varepsilon(u_\varepsilon - u_0)\big)\cdot u_{\varepsilon,M}^{2k-1}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.27}$$

Dividing the last equality by $h$, letting $h \to 0$, and applying Lebesgue's dominated convergence theorem, we have that

$$\begin{aligned} &\frac{1}{2k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_{\varepsilon,M}^{2k}\,\mathrm{d}x + (2k-1)\int_\Omega a_{\varepsilon,M}(u_{\varepsilon,M})u_{\varepsilon,M}^{2(k-1)}|\nabla u_{\varepsilon,M}|^{p(x,t)}\,\mathrm{d}x \\ &\quad + \int_\Omega \left[a_{\varepsilon,M}(u_{\varepsilon,M}) - \gamma\right]\cdot u_{\varepsilon,M}^{2k-1}|\nabla u_{\varepsilon,M}|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_\Omega \left[a_{\varepsilon,M}(u_{\varepsilon,M}) - \gamma\right]u_{\varepsilon,M}^{2k}|\nabla u_{\varepsilon,M}|^{p(x,t)}\,\mathrm{d}x \\ &= \int_\Omega \big(f(x,t) - \beta_\varepsilon(u_\varepsilon - u_0)\big)\cdot u_{\varepsilon,M}^{2k-1}\,\mathrm{d}x. \end{aligned} \tag{3.28}$$

By Holder's inequality, we have

$$\left|\int_\Omega \big(f(x,t) - \beta_\varepsilon(u_\varepsilon - u_0)\big)\cdot u_{\varepsilon,M}^{2k-1}\,\mathrm{d}x\right| \le \|u_{\varepsilon,M}\|_{2k,\Omega}^{2k-1}\cdot\big\|f(\cdot,t) - \beta_\varepsilon(u_\varepsilon - u_0)\big\|_{2k,\Omega}. \tag{3.29}$$

Using Minkowski's inequality, we arrive at

$$\big\|f(\cdot,t) - \beta_\varepsilon(u_\varepsilon - u_0)\big\|_{2k,\Omega} \le \big\|f(\cdot,t)\big\|_{2k,\Omega} + \big\|\beta_\varepsilon(u_\varepsilon - u_0)\big\|_{2k,\Omega}.$$

From (3.15) and the definition of $\beta_\varepsilon(\cdot)$, we have that

$$\big\|f(\cdot,t) - \beta_\varepsilon(u_\varepsilon - u_0)\big\|_{2k,\Omega} \le \big\|f(\cdot,t)\big\|_{2k,\Omega} + |\Omega|. \tag{3.30}$$

Recall that $0 < \gamma < d_0$. Then, we use Lemma 3.1 to find

$$\int_\Omega \left[a_{\varepsilon,M}(u_{\varepsilon,M}) - \gamma\right]\cdot u_{\varepsilon,M}^{2k-1}|\nabla u_{\varepsilon,M}|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \ge 0. \tag{3.31}$$

Substituting (3.29) and (3.30) into (3.28), we arrive at the inequality

$$\begin{aligned} &\|u_{\varepsilon,M}\|_{2k,\Omega}^{2k-1}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{\varepsilon,M}\|_{2k,\Omega} + (2k-1)\int_\Omega a_{\varepsilon,M}(u_{\varepsilon,M})u_{\varepsilon,M}^{2(k-1)}|\nabla u_{\varepsilon,M}|^{p(x,t)}\,\mathrm{d}x \\ &\quad + \int_\Omega \left[a_{\varepsilon,M}(u_{\varepsilon,M}) - \gamma\right]\cdot u_{\varepsilon,M}^{2k-1}|\nabla u_{\varepsilon,M}|^{p(x,t)}\,\mathrm{d}x \\ &\le \|u_{\varepsilon,M}\|_{2k,\Omega}^{2k-1}\cdot\big\|f(\cdot,t)\big\|_{2k,\Omega} + \|u_{\varepsilon,M}\|_{2k,\Omega}^{2k-1}\cdot|\Omega|. \end{aligned} \tag{3.32}$$

Integrating over $(0,t)$ in (3.32) and dropping the nonnegative term (3.31), we arrive at

$$\|u_{\varepsilon,M}\|_{2k,\Omega} \le \|u_{\varepsilon,M}(\cdot,0)\|_{2k,\Omega} + \int_0^T \|f(\cdot,t)\|_{2k,\Omega}\,\mathrm{d}t + |\Omega|\cdot T, \quad \forall k \in N.$$

Then, as $k \to \infty$, we have that

$$\|u_{\varepsilon,M}\|_{\infty,\Omega} \le \|u_{\varepsilon,M}(\cdot,0)\|_{\infty,\Omega} + \int_0^T \|f(\cdot,t)\|_{\infty,\Omega}\,\mathrm{d}t + |\Omega|\cdot T = K(T). \tag{3.33}$$

If we choose $M > K(T)$, then

$$u_{\varepsilon,M}(\cdot,t) \le \sup\left|u_{\varepsilon,M}(\cdot,t)\right| \le K(T) < M$$

and therefore $u_{\varepsilon,M}(\cdot,t) = u_\varepsilon(\cdot,t)$. □

**Lemma 3.5** *The solution of problem* (3.1) *satisfies the estimates*

$$\int\int_{Q_T} a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \le K(T)|\Omega|^{\frac{1}{2}}, \tag{3.34}$$

$$d_0\int\int_{Q_T} |\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \le K(T)|\Omega|^{\frac{1}{2}}, \tag{3.35}$$

$$\int\int_{Q_T} u_\varepsilon^\sigma|\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \le K(T)|\Omega|^{\frac{1}{2}}. \tag{3.36}$$

*Proof* To prove Lemma 3.5, we proceed as in the proof of Lemma 3.4, and in (3.27) we take $k = 1$. We then obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_\varepsilon(\cdot,t)\|_{2,\Omega}\right) + \int_\Omega a_{\varepsilon,M}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x\\ &\quad + \int_\Omega \left[a_{\varepsilon,M}(u_{\varepsilon,M}) - \gamma\right]\cdot u_{\varepsilon,M}^{2k-1}|\nabla u_{\varepsilon,M}|^{p(x,t)}\,\mathrm{d}x\\ &\le \|f - \beta_\varepsilon(u_\varepsilon - u_0)\|_{2,\Omega}.\end{aligned}$$

Therefore, integrating in time over $(0,t)$, $\forall t \in (0,T)$,

$$\begin{aligned}&\|u_\varepsilon(\cdot,t)\|_{2,\Omega} + \int_0^t\int_\Omega a_{\varepsilon,M}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_\Omega \left[a_{\varepsilon,M}(u_{\varepsilon,M}) - \gamma\right]\cdot u_{\varepsilon,M}^{2k-1}|\nabla u_{\varepsilon,M}|^{p(x,t)}\,\mathrm{d}x\\ &\le \int_0^T \|f - \beta_\varepsilon(u_\varepsilon - u_0)\|_{2,\Omega}\,\mathrm{d}t\end{aligned}$$

and since the first and third terms on the left-hand side are nonnegative and recalling the L2-norm

$$\int\int_{Q_T} a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t \le K(T)|\Omega|^{\frac{1}{2}}. \tag{3.37}$$

From this we obtain (3.34). Since $a(u_\varepsilon) \ge d_0$, $a_{\varepsilon,M}(u_\varepsilon) \ge u_\varepsilon^\sigma$, (3.35), and (3.35) are immediate consequences of (3.34). □

**Lemma 3.6** *The solution of problem* (3.1) *satisfies the estimate*

$$\|u_{\varepsilon t}\|_{W'(Q_T)} \le C\bigl(\sigma, p^{\pm}, K(T), |\Omega|\bigr).$$

*Proof* From identity (3.5), we obtain

$$\begin{aligned}
&\int\int_{Q_T} u_{\varepsilon t}\xi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad= -\int\int_{Q_T} a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\xi \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int\int_{Q_T}\bigl[a(u_\varepsilon)-\gamma\bigr]|\nabla u_\varepsilon|^{p(x,t)}\xi \,\mathrm{d}x\,\mathrm{d}t + \int\int_{Q_T} f\cdot\xi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad= -A_1 - A_1 + A_1,
\end{aligned} \tag{3.38}$$

where

$$\begin{aligned}
&A_1 = \int\int_{Q_T} a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla\xi \,\mathrm{d}x\,\mathrm{d}t, \\
&A_2 = \int\int_{Q_T}\bigl[a(u_\varepsilon)-\gamma\bigr]|\nabla u_\varepsilon|^{p(x,t)}\xi \,\mathrm{d}x\,\mathrm{d}t, \qquad A_3 = \int\int_{Q_T} f\cdot\xi \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

First, we pay attention to $A_1$. Using Holder inequalities we obtain

$$\begin{aligned}
|A_1| &\le \int_0^t\int_\Omega a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-1}|\nabla\xi| \,\mathrm{d}x\,\mathrm{d}t \\
&\le 2\bigl\|a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-1}\bigr\|_{p'(x,t)}\|\nabla\xi\|_{p(x,t)}.
\end{aligned}$$

When $\int_0^t\int_\Omega (a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-1})^{\frac{p(x,t)}{p(x,t)-1}} \,\mathrm{d}x\,\mathrm{d}t \ge 1$, we arrive at

$$|A_1| \le 2\biggl(\int_0^t\int_\Omega \bigl(a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-1}\bigr)^{\frac{p(x,t)}{p(x,t)-1}} \,\mathrm{d}x\,\mathrm{d}t\biggr)^{\frac{1}{p'^{+}}} \cdot \|\nabla\xi\|_{p(x,t)}. \tag{3.39}$$

Moreover, when $\int_0^t\int_\Omega (a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-1})^{\frac{p(x,t)}{p(x,t)-1}} \,\mathrm{d}x\,\mathrm{d}t < 1$, we obtain

$$|A_1| \le 2\biggl(\int_0^t\int_\Omega \bigl(a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-1}\bigr)^{\frac{p(x,t)}{p(x,t)-1}} \,\mathrm{d}x\,\mathrm{d}t\biggr)^{\frac{1}{p'^{-}}} \cdot \|\nabla\xi\|_{p(x,t)}. \tag{3.40}$$

Combining (3.39) and (3.40), and using Lemma 3.5, we arrive at

$$|A_1| \le \bigl(2\bigl[\bigl(K^2(T)+1\bigr)^{\sigma/2} + d_0\bigr]\bigr)^{\frac{1}{p^{\pm}-1}} K^2(T)|\Omega| \cdot \|\xi\|_{W(Q_T)}. \tag{3.41}$$

Secondly, we calculate $A_2$ and $A_3$. Following a similar procedure as (3.41), we obtain

$$\begin{aligned} |A_2| &\le 2\Big[\big(K^2(T)+1\big)^{\sigma/2}+d_0\Big]^{\frac{1}{p^{\pm}-1}}K^2(T)|\Omega|\cdot\|\xi\|_{W(Q_T)} \\ &\quad + 2\gamma^{\frac{1}{p^{\pm}-1}}K^2(T)|\Omega|\cdot\|\xi\|_{W(Q_T)}, \end{aligned} \tag{3.42}$$

$$|A_3| \le 2|f|_\infty|T|\cdot\|\xi\|_{W(Q_T)}. \tag{3.43}$$

Substituting (3.41), (3.42), and (3.43) into (3.38), we conclude that

$$\begin{aligned} \int\int_{Q_T} u_{\varepsilon t}\xi\,\mathrm{d}x\,\mathrm{d}t &\le 4\Big[\big(K^2(T)+1\big)^{\sigma/2}+d_0\Big]^{\frac{1}{p^{\pm}-1}}K^2(T)|\Omega|\cdot\|\xi\|_{W(Q_T)} \\ &\quad + 2\gamma^{\frac{1}{p^{\pm}-1}}K^2(T)|\Omega|\cdot+2|f|_\infty|T|\cdot\|\xi\|_{W(Q_T)}. \end{aligned}$$

Then, we obtain Lemma 3.6. □

## 4 Proof of the main results

In this section, we are ready to prove Theorem 2.1 and Theorem 2.2. From (3.15), Lemma 3.5, and Lemma 3.6, we see that $u_\varepsilon$ is bounded and increasing in $\varepsilon$, which implies the existence of a function $u$, such that, as $\varepsilon \to 0$

$$u_\varepsilon \to u \quad \text{a.e. in } \Omega_T, \tag{4.1}$$

$$\nabla u_\varepsilon \to \nabla u \quad \text{weakly in } L^{p(x,t)}(Q_T), \tag{4.2}$$

$$\frac{\partial}{\partial t}u_\varepsilon \to \frac{\partial}{\partial t}u \quad \text{weakly in } W'(Q_T), \tag{4.3}$$

$$\mathrm{a}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}D_iu_\varepsilon \to A_i(x,t) \quad \text{weakly in } L^{p'(x,t)}(Q_T), \tag{4.4}$$

$$|\nabla u_\varepsilon|^{p(x,t)-2}D_iu_\varepsilon \to W_i(x,t) \quad \text{weakly in } L^{p'(x,t)}(Q_T), \tag{4.5}$$

for some functions $u \in W(Q_T)$, $A_i(x,t) \in L^{p'(x,t)}(Q_T)$, $W_i(x,t) \in L^{p'(x,t)}(Q_T)$.

**Lemma 4.1** *For almost all* $(x,t) \in Q_T$,

$$A_i(x,t) = a(u)W_i(x,t), \quad i = 1,2,\dots,N.$$

*Proof* In (4.4) and (4.5), letting $\varepsilon \to 0$, we have that

$$\int\int_{Q_T} a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla\xi\,\mathrm{d}x\,\mathrm{d}t = \sum_i\int\int_{Q_T} A_i(x,t)\cdot D_i\xi\,\mathrm{d}x\,\mathrm{d}t, \tag{4.6}$$

$$\int\int_{Q_T} |\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla\xi\,\mathrm{d}x\,\mathrm{d}t = \sum_i\int\int_{Q_T} W_i(x,t)\cdot D_i\xi\,\mathrm{d}x\,\mathrm{d}t. \tag{4.7}$$

By Lebesgue's dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0}\int\int_{Q_T}\big[a\big(u_\varepsilon\big)-a(u)\big]A_i(x,t)\cdot D_i\xi\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.8}$$

Hence, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int\int_{Q_T} a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}D_iu_\varepsilon D_i\xi - a(u)W_i(x,t)D_i\xi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \lim_{\varepsilon\to 0}\int\int_{Q_T}\bigl[a(u_\varepsilon)-a(u)\bigr]\cdot|\nabla u_\varepsilon|^{p(x,t)-2}D_iu_\varepsilon D_i\xi \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \lim_{\varepsilon\to 0}\int\int_{Q_T} a(u)\bigl(|\nabla u_\varepsilon|^{p(x,t)-2}D_iu_\varepsilon - W_i(x,t)\bigr)D_i\xi \,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned}$$

This completes the proof of Lemma 4.1. □

**Lemma 4.2** *For almost all* $(x,t)\in Q_T$,

$$W_i(x,t) = |\nabla u|^{p(x,t)-2}D_iu, \quad i=1,2,\dots,N.$$

*Proof* In (3.5), choosing $\xi = \Phi\cdot(u_\varepsilon - u)$ with $\Phi\in W(Q_T)$, $\Phi\geq 0$, we have

$$\begin{aligned}
&\int\int_{Q_T}\partial_t u_\varepsilon\cdot(u_\varepsilon-u)\cdot\Phi + \Phi\cdot a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla(u_\varepsilon-u)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int\int_{Q_T}(u_\varepsilon-u)\cdot a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla\Phi - f(x,t)(u_\varepsilon-u)\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned}$$

It follows that

$$\lim_{\varepsilon\to\infty}\int\int_{Q_T}\Phi\cdot a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla(u_\varepsilon-u)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.9}$$

On the other hand, from $u_\varepsilon, u\in L^\infty(Q_T)$, $|\nabla u|\in L^{p(x,t)}(Q_T)$, we obtain

$$\lim_{\varepsilon\to\infty}\int\int_{Q_T}\Phi\cdot a(u)|\nabla u|^{p(x,t)-2}\nabla u\nabla(u_\varepsilon-u)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{4.10}$$

$$\lim_{\varepsilon\to\infty}\int\int_{Q_T}\Phi\cdot\bigl[a(u_\varepsilon)-a(u)\bigr]|\nabla u|^{p(x,t)-2}\nabla u\nabla(u_\varepsilon-u)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.11}$$

Note that

$$\begin{aligned}
0 &\leq \bigl(|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon - |\nabla u|^{p(x,t)-2}\nabla u\bigr)\cdot(\nabla u_\varepsilon - \nabla u) \\
&\leq \frac{1}{d_0}\bigl[a(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon + \bigl[a(u_\varepsilon)-a(u)\bigr]\cdot|\nabla u|^{p(x,t)-2}\nabla u\bigr]\cdot(\nabla u_\varepsilon-\nabla u) \\
&\quad - \frac{1}{d_0}a(u)|\nabla u|^{p(x,t)-2}\nabla u\cdot(\nabla u_\varepsilon - \nabla u).
\end{aligned} \tag{4.12}$$

By (4.9)–(4.12), we obtain

$$\lim_{\varepsilon\to\infty}\int\int_{Q_T}\bigl(|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon - |\nabla u|^{p(x,t)-2}\nabla u\bigr)\cdot(\nabla u_\varepsilon-\nabla u)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.13}$$

Then, the proof of Lemma 4.2 is complete. □

**Lemma 4.3** *As* $\varepsilon \to 0$, *we have*

$$\beta_\varepsilon(u_\varepsilon - u_0) \to \xi \in G(u - u_0). \tag{4.14}$$

*Proof* Using (3.15) and the definition of $\beta_\varepsilon$, we have

$$\beta_\varepsilon(u_\varepsilon - u_0) \to \xi \quad \text{as } \varepsilon \to 0.$$

Now, we prove $\xi \in G(u - u_0)$. According to the definition of $G(\cdot)$, we only need to prove that if $u(x_0, t_0) > u_0(x_0)$,

$$\xi(x_0, t_0) = 0.$$

In fact, if $u(x_0, t_0) > u_0(x_0)$, there exist a constant $\lambda > 0$ and a $\delta$-neighborhood $B_\delta(x_0, t_0)$ such that if $\varepsilon$ is small enough, we have

$$u_\varepsilon(x, t) \geq u_0(x) + \lambda, \quad \forall (x, t) \in B_\delta(x_0, t_0).$$

Thus, if $\varepsilon$ is small enough, we have

$$0 \geq \beta_\varepsilon(u_\varepsilon - u_0) \geq \beta_\varepsilon(\lambda) = 0, \quad \forall (x, t) \in B_\delta(x_0, t_0).$$

Furthermore, it follows by $\varepsilon \to 0$ that

$$\xi(x, t) = 0, \quad \forall (x, t) \in B_\delta(x_0, t_0).$$

Hence, (4.13) holds, and the proof of Lemma 4.3 is complete. □

*The proof of Theorem* 2.1. Applying (3.15), (3.16), and Lemma 4.3, it is clear that

$$u(x, t) \leq u_0(x), \quad \text{in } \Omega_T, \qquad u(x, 0) = u_0(x), \quad \text{in } \Omega, \qquad \xi \in G(u - u_0),$$

thus (a), (b), and (c) hold. The remaining arguments of the existence part are the same as those of Theorem 2.1 in [8] by a standard limiting process. Thus, we omit the details. □

*The proof of Theorem* 2.2 We argue by contradiction. Suppose $(u, \xi_1)$ and $(v, \xi_2)$ are two nonnegative weak solutions of problem (1.1). Define $w = u - v$,

$$F(w) = \begin{cases} -\frac{1}{\alpha - 1} w^{1-\alpha} & \text{if } w > 0, \\ 0 & \text{if } w \leq 0, \end{cases} \tag{4.15}$$

and let $\xi = F_\varepsilon(w) \in Z$ be a test function in (3.4),

$$\begin{aligned} 0 \geq & \int \int_{Q_T} w_t F(w) + a(v)v\big(\big|\nabla u|^{p(x,t)-2}\nabla u - \big|\nabla v|^{p(x,t)-2}\nabla v\big)\nabla F_\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}t \\ & + \int \int_{Q_T} \big[a(u)u - a(v)v\big]|\nabla u|^{p(x,t)-2}\nabla u \nabla F_\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}t \\ & + \int \int_{Q_T} \big[a(u) - \gamma\big]\big(\big|\nabla u|^{p(x,t)} - \big|\nabla v|^{p(x,t)}\big)\,\mathrm{d}x\,\mathrm{d}t \\ & + \int \int_{Q_T} \big[a(u) - a(v)\big]|\nabla u|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t - \int \int_{Q_T} (\xi_1 - \xi_2)F(w)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.16}$$

Now, we prove

$$\int_\Omega (\xi_1 - \xi_2)F(w)\,\mathrm{d}x\,\mathrm{d}t \leq 0. \tag{4.17}$$

On the one hand, if $u_1(x,t) > u_2(x,t)$, then using (3.16) yields

$$u_1(x,t) > u_0(x). \tag{4.18}$$

From (2.1) and (4.18), it is easy to see that

$$\xi_1 = 0 < \xi_2. \tag{4.19}$$

Combining (4.18) and (4.19) and the fact that $\alpha = \frac{1}{2}\sigma > 1$, (4.16) is obtained.

On the other hand, if $u_1(x,t) < u_2(x,t)$, it is easy to see that $F(w) = 0$. Equation (4.16) still holds.

Using (4.16) in (4.15) and dropping the nonnegative term, (4.15) becomes

$$\begin{aligned} 0 \geq & \int \int_{Q_T} w_t F(w) + a(v)v\big(\big|\nabla u|^{p(x,t)-2}\nabla u - \big|\nabla v|^{p(x,t)-2}\nabla v\big)\nabla F_\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}t \\ & + \int \int_{Q_T} \big[a(u)u - a(v)v\big]|\nabla u|^{p(x,t)-2}\nabla u \nabla F_\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}t \\ & + \int \int_{Q_T} \big[a(u) - \gamma\big]\big(\big|\nabla u|^{p(x,t)} - \big|\nabla v|^{p(x,t)}\big)\,\mathrm{d}x\,\mathrm{d}t \\ & + \int \int_{Q_T} \big[a(u) - a(v)\big]|\nabla u|^{p(x,t)}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

By the above inequality and combining the initial and boundary conditions in (1.1), the uniqueness of the solution can be proved following the similar proof of (3.7)–(3.14). □

## 5 Conclusion

In this paper, an initial Dirichlet problem of degenerate parabolic variational inequalities in the following form

$$\begin{cases} \min\{Lu, u(x,0) - u_0\} = 0, & (x,t) \in Q_T, \\ u(x,t) = 0, & (x,t) \in \Gamma_T, \\ u(x,0) = u_0, & x \in \Omega, \end{cases}$$

is studied. The existence and uniqueness of the solutions in the weak sense are proved by the energy method and a limit process. The localization property of weak solutions is also discussed.

**Acknowledgements**
The authors are sincerely grateful to the reviewer and the Associate Editor who handled the paper for their valuable comments.

**Funding**
This work was supported by the Guizhou Provincial Education Foundation of Youth Science and Technology Talent Development (No. [2016]168), the Foundation of Guizhou Minzu University (NO.GZMUSK[2021]ZK02) and the Doctoral Project of Guizhou Education University (NO.2021BS037).

**Availability of data and materials**
Not applicable.

## Declarations

**Competing interests**
The authors declare that they have no competing interests.

**Author contribution**
YDS was a major contributor in writing the manuscript. TW performed the validation and formal analysis. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 8 February 2022 Accepted: 4 October 2022 Published online: 12 November 2022

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