Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Iterative methods for vector equilibrium and fixed point problems in Hilbert spaces

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Abstract

In this paper, algorithms for finding common solutions of strong vector equilibrium and fixed point problems of multivalued mappings are considered. First, a Minty vector equilibrium problem is introduced and the relationship between the Minty vector equilibrium problem and the strong equilibrium problem is discussed. Then, by applying the Minty vector equilibrium problem, projection iterative methods are proposed and some convergence results are established in Hilbert spaces. The main results obtained in this paper develop and improve some recent works in this field.

MSC: 65K15; 47J25; 90C33

Keywords: Vector equilibrium problem; Fixed point problem; Iterative algorithm; Convergence analysis

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space and X be a nonempty subset of $\mathcal{H}$. Let $f : X \times X \to \mathbb{R}$ be a bifunction satisfying $f(x,x) = 0$ for all $x \in X$. The scalar equilibrium problem consists in finding $x^* \in X$ such that

$$\text{(EP)} \quad f\left(x^*, y\right) \geq 0, \quad \forall y \in X.$$

As pointed out by Blum and Oettli [1], (EP) provides a unifying framework for several important problems, such as the optimization problem, saddle point problem, Nash equilibrium problem, fixed point problem, variational inequality and complementarity problem.

Let Z be a real Hausdorff topological vector space and C a convex cone of Z. Let $f : X \times X \to Z$ be a vector-valued bifunction satisfying $f(x,x) = 0$ for all $x \in X$. In 1997, Ansari et al. [2] introduced the following vector equilibrium problems: find $x^* \in X$ such that

$$\text{(SVEP)} \quad f\left(x^*, y\right) \in C, \quad \forall y \in X;$$

or find $x^* \in X$ such that

$$\text{(WVEP)} \quad f\left(x^*, y\right) \notin -\operatorname{int} C, \quad \forall y \in X,$$

where $\operatorname{int} C$ denotes the topological interior of C. Provided the cone C is proper (i.e., $C \neq Z$), any solution of (SVEP) must be also a solution of (WVEP).

Clearly, each of these problems constitutes a valid extension of (EP).

Existence of solutions is a fundamental question for equilibrium problems. In the past decades, it has been intensively studied by many authors, and a large number of existence results have been obtained in the literature. In most cases, a monotonicity or coerciveness condition on the equilibrium function f and/or a compactness condition on the feasible set X are imposed. For details, we refer the readers to the monographs [3–5] and the references therein.

Algorithm method is another fundamental but very important question for equilibrium problems. In the last years, lots of effective methods for solving scalar equilibrium problems have been proposed. For details, we refer the readers to [6–21] and the references therein. Recently, methods for finding a solution of vector equilibrium problems have also been explored. In 2009, by using a scalarization method, Cheng and Liu [22] suggested a projection iterative algorithm for finding solutions of a weak vector equilibrium problem by solving a corresponding convex feasibility problem in an n-dimensional Euclidean space. In 2012, by applying a regularization technique, Li and Wang [23] proposed a viscosity approximation method for finding a common element of the set of fixed points of a nonexpansive mapping and of the set of solutions of a strong vector equilibrium problem in a Hilbert sapce. Later, Shan and Huang [24] extended this viscosity approximation method for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings, of the set of solutions of a generalized mixed vector equilibrium problem, and of the set of solutions of a variational inequality problem. In 2015, by applying the Gerstewitz nonlinear scalarization function, Wang and Li [25] presented a projection iterative algorithm for finding solutions of a strong vector equilibrium problem by solving a corresponding scalar optimization problem. Afterwards, by extending and developing the iterative method used in [25], Huang, Wang, and Mao [26] introduced a general iterative algorithm for solving a strong vector equilibrium problem. In 2018, Wang, Huang, and Zhu [27] further suggested a projection iterative algorithm for solving a strong vector equilibrium problem with variable domination structure. Very recently, by using the auxiliary principle, Chadli, Ansari, and Al-Homidan [28] also proposed an algorithm for bilevel vector equilibrium problems.

Motivated by the works mentioned above, in this paper, we shall investigate iterative methods for finding common solutions of strong vector equilibrium problems and fixed point problems of multivalued mappings. The organization of this paper is as follows. In Sect. 2, a Minty vector equilibrium problem is introduced and the relationship between the Minty vector equilibrium problem and the strong vector equilibrium problem is discussed. Some definitions and known results are also recalled in this section. In Sect. 3, by employing the Minty vector equilibrium problem, projection iterative methods are suggested for finding common solutions of a strong vector equilibrium problem and a fixed point problem of a multivalued mapping. Moreover, some convergence results are established under suitable conditions of cone continuity and convexity. The main results obtained in this paper generalize and improve the corresponding ones of Van, Strodiot, Nguyen, and Vuong [29], Iusem and Sosa [8], Shan and Huang [24], and Huang, Wang, and Mao [26].

2 Preliminaries

In this paper, let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. When $\{x^k\}$ is a sequence of $\mathcal{H}$, we denote strong convergence of $\{x^k\}$ to $x \in \mathcal{H}$ as $k \to \infty$ by $x^k \to x$ and weak convergence by $x^k \rightharpoonup x$.

It is easy to see that

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,$$

for all $x, y \in \mathcal{H}$ and all $t \in \mathbb{R}$.

Let K be a nonempty closed and convex subset of $\mathcal{H}$. For every element $x \in \mathcal{H}$, there exists a unique nearest point in K, denoted by $P_K(x)$, such that

$$\|x - P_K(x)\| = \inf\{\|x-y\| : y \in K\}.$$

Then P_K is called the metric projection of $\mathcal{H}$ onto K.

Lemma 2.1 *The metric projection has the following basic properties*:

(i) $\langle x - P_K(x), y - P_K(x)\rangle \le 0$ *for all* $x \in \mathcal{H}$ *and* $y \in K$;
(ii) $\|P_K(x) - y\|^2 \le \|x-y\|^2 - \|x - P_K(x)\|^2$ *for all* $x \in \mathcal{H}$ *and* $y \in K$;
(iii) $\|P_K(x) - P_K(y)\|^2 \le \langle x - y, P_K(x) - P_K(y)\rangle$ *for all* $x, y \in \mathcal{H}$;
(iv) $\|P_K(x) - P_K(y)\| \le \|x-y\|$ *for all* $x, y \in \mathcal{H}$.

Let X be a nonempty subset of $\mathcal{H}$ and $S : X \to 2^X$ a multivalued mapping. The fixed point problem associated with S can be formulated as

$$\text{(FPP)} \quad \text{Find } x^* \in X \quad \text{such that } x^* \in S(x^*).$$

Denote by $\mathrm{Fix}(S)$ the set of all fixed points of S, i.e., $\mathrm{Fix}(S)$ denotes the solution set of (FPP).

Remark 2.1 When $S : X \to X$ is a vector-valued mapping, the (FPP) reduces to finding $x^* \in X$ such that $x^* = S(x^*)$.

For dealing with (SVEP), in this paper, we need the following Minty vector equilibrium problem: find $x^* \in X$ such that

$$\text{(MVEP)} \quad f(y, x^*) \in -C, \quad \forall y \in X.$$

Denote by $\mathcal{S}_0$, $\mathcal{S}_E$, and $\mathcal{S}_M$ the solution sets of (EP), (SVEP), and (MVEP), respectively.

Next, we shall give some definitions and known results needed in this paper.

Definition 2.1 ([30]) Let E and Z be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty subset, and $C \subseteq Z$ a closed convex cone. A mapping $g : X \to Z$ is said to be

(i) C-upper semicontinuous (for short, C-u.s.c.) (resp. C-lower semicontinuous (for short, C-l.s.c.)) at $x_0 \in X$ if, for any neighborhood V of 0 in Z, there exists a neighborhood U of x_0 in E such that

$$g(x) \in g(x_0) + V - C, \quad \forall x \in U \cap X$$

$$\big(\text{resp. } g(x) \in g(x_0) + V + C, \forall x \in U \cap X\big);$$

(ii) C-u.s.c. (resp. C-l.s.c.) on X if it is C-u.s.c. (resp. C-l.s.c.) at every point $x \in X$;
(iii) C-continuous on X if it is both C-u.s.c. and C-l.s.c. on X.

Remark 2.2 From the above definition, it is easy to see that a mapping $g : X \to Z$ is C-u.s.c. at $x_0 \in X$ if and only if $-g$ is C-l.s.c. at x_0.

Remark 2.3 If $g : X \to Z$ is continuous on X, then it is C-continuous on X. Conversely, if g is C-continuous on X, then it is continuous only when the cone C has a closed convex bounded base (see [30, Theorem 5.3, pp. 22–23]).

The following lemma plays an important role in our convergence analysis.

Lemma 2.2 *Let E and Z be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty closed subset, and $C \subseteq Z$ a closed convex cone. Let $g, h : X \to Z$ be two vector-valued mappings. Let $\{x_\alpha\}$ and $\{y_\alpha\}$ be two nets in X such that $x_\alpha \to x_0 \in X$ and $y_\alpha \to y_0 \in X$. Let $\{w_\alpha\}$ be a given net in Z such that $w_\alpha \to w_0 \in Z$.*
(i) *If g is C-l.s.c. at x_0 and h is C-u.s.c. at y_0 and*

$$g(x_\alpha) \in h(y_\alpha) + w_\alpha - C \quad \textit{for all } \alpha,$$

then $g(x_0) \in h(y_0) + w_0 - C$;
(ii) *If g is C-u.s.c. at x_0 and h is C-l.s.c. at y_0 and*

$$g(x_\alpha) \in h(y_\alpha) + w_\alpha + C \quad \textit{for all } \alpha,$$

then $g(x_0) \in h(y_0) + w_0 + C$.

Proof (i) Suppose to the contrary that $g(x_0) \notin h(y_0) + w_0 - C$, i.e., $g(x_0) - h(y_0) - w_0 \notin -C$. Then, by the closedness of C, there exists some neighborhood V of the origin in Z such that

$$\big[\big(g(x_0) - h(y_0) - w_0\big) + V\big] \cap (-C) = \emptyset.$$

Notice that C is a convex cone. We can further obtain

$$\big[\big(g(x_0) - h(y_0) - w_0\big) + V + C\big] \cap (-C) = \emptyset. \tag{2.1}$$

For the above neighborhood V of the origin in Z, it is known from the theory of topological vector spaces that there exists a balanced neighborhood V' of the origin in Z such that $V' + V' + V' \subseteq V$. For the neighborhood V', since g is C-l.s.c. at x_0, there exists a neighborhood $U(x_0)$ of x_0 such that

$$g(x) \in g(x_0) + V' + C, \quad \forall x \in U(x_0) \cap X.$$

As $\{x_\alpha\} \subseteq X$ and $x_\alpha \to x_0 \in X$, there must exist some α_1 such that, for every $\alpha \geq \alpha_1$,

$$g(x_\alpha) \in g(x_0) + V' + C. \tag{2.2}$$

On the other hand, since h is C-u.s.c. at y_0, there exists a neighborhood $U(y_0)$ of y_0 such that

$$h(y) \in h(y_0) + V' - C, \quad \forall y \in U(y_0) \cap X.$$

Since $\{y_\alpha\} \subseteq X$ and $y_\alpha \to y_0 \in X$, there must exist some α_2 such that, for every $\alpha \geq \alpha_2$,

$$h(y_\alpha) \in h(y_0) + V' - C. \tag{2.3}$$

In addition, since $w_\alpha \to w_0$, there must exist some α_3 such that, for every $\alpha \geq \alpha_3$,

$$w_\alpha \in w_0 + V'. \tag{2.4}$$

Take any α_0 such that $\alpha_0 \geq \alpha_1$, $\alpha_0 \geq \alpha_2$, and $\alpha_0 \geq \alpha_3$. Notice that V' is a balanced neighborhood and C is a convex cone. Then, by (2.2), (2.3), and (2.4), we have, for any $\alpha \geq \alpha_0$,

$$\begin{aligned} g(x_\alpha) - h(y_\alpha) - w_\alpha &\in g(x_0) + V' + C - h(y_0) - V' + C - w_0 - V' \\ &= g(x_0) - h(y_0) - w_0 + V' + V' + V' + C + C \\ &\subseteq g(x_0) - h(y_0) - w_0 + V + C. \end{aligned}$$

This, together with (2.1), implies that

$$g(x_\alpha) - h(y_\alpha) - w_\alpha \notin -C,$$

a contradiction. Thus $g(x_0) \in h(y_0) + w_0 - C$.

(ii) Since g is C-u.s.c. at x_0, we know that $-g$ is C-l.s.c. at x_0. Similarly, since h is C-l.s.c. at y_0, we know that $-h$ is C-u.s.c. at y_0. In addition, since $w_\alpha \to w_0$, we have $(-w_\alpha) \to (-w_0)$. By the assumption, we further have

$$-g(x_\alpha) \in -h(y_\alpha) - w_\alpha - C \quad \text{for all } \alpha.$$

Then, it follows from item (i) that

$$-g(x_0) \in -h(y_0) - w_0 - C.$$

That is, $g(x_0) \in h(y_0) + w_0 + C$. □

Definition 2.2 ([30]) Let E and Z be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty subset, and $C \subseteq Z$ a closed convex cone. A mapping $g : X \to Z$ is called lower semicontinuous (for short, l.s.c.) (resp. upper semicontinuous (for short, u.s.c.)) on X if, for any $z \in Z$, the set

$$L(z) = \{x \in X : g(x) \in z - C\}$$

$$\bigl(\text{resp. } L(z) = \bigl\{x \in X : g(x) \in z + C\bigr\}\bigr)$$

is closed in X.

Lemma 2.3 ([31]) *If g is C-u.s.c. (resp. C-l.s.c.) on X, then it is u.s.c. (resp. l.s.c.) on X.*

Definition 2.3 ([30, 32]) Let E and Z be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty convex subset, and $C \subseteq Z$ a convex cone. A mapping $h : X \to Z$ is said to be

(i) C-convex if, for any $u_1, u_2 \in X$ and any $t \in [0,1]$, one has

$$h\bigl(tu_1 + (1-t)u_2\bigr) \in th(u_1) + (1-t)h(u_2) - C;$$

(ii) C-quasiconvex if, for any $z \in Z$, the set $\{u \in X : h(u) \in z - C\}$ is convex;

(iii) properly C-quasiconvex if, for any $u_1, u_2 \in X$ and for any $t \in [0,1]$, one has

$$\text{either} \quad h\bigl(tu_1 + (1-t)u_2\bigr) \in h(u_1) - C,$$
$$\text{or} \quad h\bigl(tu_1 + (1-t)u_2\bigr) \in h(u_2) - C.$$

(iv) properly C-quasiconcave if $-h$ is properly C-quasiconvex.

Remark 2.4 Obviously, if h is C-convex or properly C-quasiconvex, then it is C-quasiconvex.

Definition 2.4 ([33]) Let Z be a real Hausdorff topological vector space and $C \subseteq Z$ a convex cone. A nonempty set $M \subseteq Z$ is called upward directed if, for every $u_1, u_2 \in M$, there exists $u \in M$ such that $u_1 \in u - C$ and $u_2 \in u - C$.

The following theorem is useful in the convergence analysis of our algorithm, which provides the existence of maximal elements.

Theorem 2.1 ([34]) *Let E and Z be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty compact subset, and $C \subseteq Z$ a closed convex cone. Assume that $f : X \to Z$ is u.s.c. and $f(X)$ is upward directed. Then, there exists $\bar{x} \in X$ such that*

$$f(x) \in f(\bar{x}) - C, \quad \forall x \in X.$$

Now, we present an important local property of (SVEP), which says that local solutions of (SVEP) are indeed global ones.

Theorem 2.2 ([25]) *Let E and Z be two real Hausdorff topological vector spaces, $X \subseteq E$ a nonempty convex subset, and $C \subseteq Z$ a closed convex cone. Let $f : X \times X \to Z$ be a vector-valued bifunction such that, for each $x \in X$, $f(x,x) = 0$ and $f(x,y)$ is C-convex in y. If there exist an open set $U \subseteq E$ and $\bar{x} \in X \cap U$ such that $f(\bar{x}, y) \in C$, $\forall y \in X \cap U$, then $\bar{x}$ solves (SVEP).*

The following lemma uncovers the relation between (SVEP) and (MVEP).

Lemma 2.4 ([26]) *Let $\mathcal{H}$ be a real Hilbert space and X a nonempty closed convex subset of $\mathcal{H}$. Let Z be a real Hausdorff topological vector space and C a closed convex cone of Z. Let $f : X \times X \to Z$ be a vector-valued bifunction such that, for any $x \in X$, $f(x,x) = 0$ and $f(x,y)$ is C-convex in y and, for each $y \in X$, $f(x,y)$ is u.s.c. in x. Then, the solution set of (MVEP) is contained in the solution set of (SVEP), i.e., $\mathcal{S}_M \subseteq \mathcal{S}_E$.*

Definition 2.5 ([35]) Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and the induced norm $\| \cdot \|$. Let X be a nonempty closed convex subset of $\mathcal{H}$. Denote by $\mathrm{CCB}(X)$ the family of nonempty convex closed bounded subsets of X. A multivalued mapping $T : X \to \mathrm{CCB}(X)$ is said to be $*$-nonexpansive if

$$\left\| P_{T(x)}(x) - P_{T(y)}(y) \right\| \le \|x - y\|, \quad \forall x, y \in X,$$

where $P_{T(x)}(x)$ denotes the metric projection of x onto the nonempty convex closed bounded subset $T(x)$.

Remark 2.5 If $T : X \to X$ is a vector-valued mapping and nonexpansive, i.e., $\|Tx - Ty\| \le \|x-y\|$, for all $x, y \in X$, then T is clearly $*$-nonexpansive. In particular, the identity mapping I on X is $*$-nonexpansive.

Theorem 2.3 ([36, Demiclosedness Principle]) *Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and the induced norm $\| \cdot \|$. Let X be a nonempty bounded closed convex subset of $\mathcal{H}$. Denote by $\mathcal{K}(X)$ the family of nonempty compact convex subsets of X. Let $T : X \to \mathcal{K}(X)$ be a $*$-nonexpansive mapping. Then $\mathrm{Fix}(T)$ is convex and closed and $I - T$ is demiclosed at 0, i.e., for every sequence $\{x^k\} \subseteq X$ such that $x^k \rightharpoonup x$ and $d(x^k, T(x^k)) \to 0$ as $k \to \infty$, one has $x \in T(x)$.*

3 Main results

In this section, we shall propose a projection iterative algorithm for finding common solutions of (SVEP) and (FPP) and further investigate its convergence.

From now on, unless otherwise specified, let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and the induced norm $\| \cdot \|$. Let X be a nonempty compact convex subset of $\mathcal{H}$. Let $\mathrm{CCB}(X)$ be the family of nonempty convex closed bounded subsets of X and $\mathcal{K}(X)$ be the family of nonempty compact convex subsets of X. It is clear that $\mathcal{K}(X)$ is included in $\mathrm{CCB}(X)$. Let Z be a real Hausdorff topological vector space and $C \subseteq Z$ a closed convex pointed cone. Let $S : X \to \mathcal{K}(X)$ be a multivalued mapping and $f : X \times X \to Z$ a vector-valued bifunction satisfying $f(x,x) = 0$ for all $x \in X$.

Also, the following assumptions are supposed to be satisfied:

(A1) S is $*$-nonexpansive on X;

(A2) f is C-continuous on $X \times X$;

(A3) For each $x \in X$, $f(x,y)$ is C-convex in y;

(A4) For any given $y \in X$, for every $x^1, x^2 \in X$, there exists $x \in X$, $\|x\| \le \max\{\|x^1\|, \|x^2\|\}$, such that

$$f\left(x^1, y\right) \in f(x,y) - C \quad \text{and} \quad f\left(x^2, y\right) \in f(x,y) - C.$$

Remark 3.1 Assumption (A4) holds, for example, when $f(\cdot,y)$ is C-u.s.c. and properly C-quasiconcave on X, see Huang, Wang, and Mao [26, Proposition 3.1].

Algorithm 3.1 **Step 0.** (Initial step) Take an arbitrary point $e \in C$. Choose two sequences $\{\lambda_k\}$ and $\{\mu_k\}$ satisfying $\lambda_k \geq 0$ and $\lambda_k \to 0$ $(k \to \infty)$ and $\{\mu_k\} \subseteq [a,b]$ for some $a, b \in (0,1)$. Select an initial $x^0 \in X$. Set $k = 0$ and $\rho_0 = \|x_0\|$.

Step 1. Define

$$X^k = \{x \in X : \|x\| \leq \rho_k + 1\}.$$

Step 2. Find $y^k \in X^k$ such that

$$f(y,x^k) \in f(y^k,x^k) + \lambda_k e - C, \quad \forall y \in X^k.$$

Step 3. Compute $u^k \in X$ as

$$u^k = P_{L_f(y^k)}(x^k),$$

where $P_{L_f(y^k)}(\cdot)$ denotes the metric projection onto $L_f(y^k) = \{x \in X : f(y^k,x) \in -C\}$.

Step 4. Calculate x^{k+1} as

$$x^{k+1} = \mu_k x^k + (1-\mu_k)v^k,$$

where $v^k = P_{S(u^k)}(u^k)$.

Step 5. Compute ρ_{k+1} as

$$\rho_{k+1} = \max\{\rho_k, \|x^{k+1}\|\}.$$

Step 6. Set $k = k+1$ and return to **Step 1**.

Now, we start the convergence analysis of Algorithm 3.1.

Lemma 3.1 *Algorithm 3.1 is well-defined.*

Proof Clearly, $\rho_k = \max\{\|x^0\|, \|x^1\|, \ldots, \|x^k\|\}$, so we see that the sequence $\{\rho_k\}$ is nondecreasing. This implies that $X^k \subseteq X^{k+1}$ for all k. As x^0 belongs to X^0, all the sets X^k are nonempty and trivially closed. Noting that $X^k \subseteq X$ and X is compact, we obtain that the sets X^k are compact. By assumption (A4), we know that, for any given $x^k \in X^k$, the set $f(X^k,x^k) = \{f(y,x^k) : y \in X^k\}$ is upward directed. Moreover, by assumption (A2), the vector-valued mapping $f(\cdot,x^k)$ is C-u.s.c. and so is u.s.c.. Then, it follows from Theorem 2.1 that there exists $y^k \in X^k$ such that

$$f(y,x^k) \in f(y^k,x^k) - C, \quad \forall y \in X^k.$$

This, together with $\lambda_k \geq 0$ and $e \in C$, implies that y^k can be selected as the point desired in **Step 2**.

Clearly, the set $L_f(y^k)$ is nonempty as $y^k \in L_f(y^k)$. Moreover, $L_f(y^k)$ is convex as $f(y^k, \cdot)$ is C-convex. By assumption (A2), $f(y^k, \cdot)$ is C-l.s.c. and so is l.s.c.. It follows that the set $L_f(y^k)$ is closed. Thus, the metric projection of x^k onto $L_f(y^k)$ is existing and unique. Similarly, as the set $S(u^k)$ is nonempty, closed and convex, there exists unique a metric projection of u^k onto $S(u^k)$. Therefore, x^{k+1} is uniquely calculated by **Step 4**. □

Let $\{x^k\}$, $\{y^k\}$, $\{u^k\}$, $\{v^k\}$, and $\{\rho_k\}$ be the sequences generated by Algorithm 3.1. Let $L_\infty = \bigcap_{k=1}^{\infty} L_f(y^k)$.

Proposition 3.1 *For each* $x^* \in L_\infty \cap \mathrm{Fix}(S)$, *one has*

(i) $\|v^k - x^*\| \leq \|u^k - x^*\|$;

(ii) $\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^k - u^k\|^2 \leq \|x^k - x^*\|^2$;

(iii) $\|x^{k+1} - x^*\| \leq \|x^k - x^*\|$;

(iv) *the sequence* $\{\|x^k - x^*\|\}$ *is convergent.*

Proof Take any $x^* \in L_\infty \cap \mathrm{Fix}(S)$ and let it be fixed.

(i) Since $x^* \in \mathrm{Fix}(S)$, we have $x^* \in S(x^*)$. Thus $x^* = P_{S(x^*)}(x^*)$. By assumption (A1), S is $*$-nonexpansive on X. It follows that

$$\|v^k - x^*\| = \|P_{S(u^k)}(u^k) - P_{S(x^*)}(x^*)\| \leq \|u^k - x^*\|. \tag{3.1}$$

(ii) Notice that $x^* \in L_\infty \subseteq L_f(y^k)$ and u^k is the metric projection of x^k onto $L_f(y^k)$. Then, by the property of metric projection, one has

$$\langle x^k - u^k, x^* - u^k \rangle \leq 0.$$

It follows that

$$\begin{aligned}
\|x^k - x^*\|^2 &= \|x^k - u^k + u^k - x^*\|^2 \\
&= \|x^k - u^k\|^2 + \|u^k - x^*\|^2 - 2\langle x^k - u^k, x^* - u^k \rangle \\
&\geq \|x^k - u^k\|^2 + \|u^k - x^*\|^2.
\end{aligned}$$

Hence, we obtain

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^k - u^k\|^2 \leq \|x^k - x^*\|^2. \tag{3.2}$$

(iii) Noting that $x^{k+1} = \mu_k x^k + (1 - \mu_k) v^k$, we have

$$x^{k+1} - x^* = \mu_k (x^k - x^*) + (1 - \mu_k)(v^k - x^*). \tag{3.3}$$

Then, by (3.3), (3.1), and (3.2), one has

$$\begin{aligned}
\|x^{k+1} - x^*\| &\leq \mu_k \|x^k - x^*\| + (1 - \mu_k)\|v^k - x^*\| \\
&\leq \mu_k \|x^k - x^*\| + (1 - \mu_k)\|u^k - x^*\| \\
&\leq \|x^k - x^*\|.
\end{aligned}$$

(iv) Clearly, the sequence $\{\|x^k - x^*\|\}$ is nonnegative. By item (iii), it is also nonincreasing and thus convergent. □

Proposition 3.2 *For each k, let $w^k = P_{S(x^k)}(x^k)$. If $L_\infty \cap \mathrm{Fix}(S) \neq \emptyset$, then the sequences $\{\|x^k - w^k\|\}$, $\{\|x^k - u^k\|\}$, and $\{\|x^k - v^k\|\}$ all converge to 0.*

Proof Take any $x^* \in L_\infty \cap \mathrm{Fix}(S)$ and let x^* be fixed. Then, by applying successively (3.1) and (3.2), we can obtain

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|\mu_k(x^k - x^*) + (1-\mu_k)(v^k - x^*)\|^2 \\
&= \mu_k\|x^k - x^*\|^2 + (1-\mu_k)\|v^k - x^*\|^2 - \mu_k(1-\mu_k)\|x^k - v^k\|^2 \\
&\le \mu_k\|x^k - x^*\|^2 + (1-\mu_k)\|u^k - x^*\|^2 - \mu_k(1-\mu_k)\|x^k - v^k\|^2 \qquad (3.4) \\
&\le \mu_k\|x^k - x^*\|^2 + (1-\mu_k)\|x^k - x^*\|^2 - \mu_k(1-\mu_k)\|x^k - v^k\|^2 \\
&= \|x^k - x^*\|^2 - \mu_k(1-\mu_k)\|x^k - v^k\|^2. \qquad (3.5)
\end{aligned}$$

Notice that $\mu_k(1-\mu_k) \ge a(1-b) > 0$. We can conclude from (3.5) that

$$0 \le a(1-b)\|x^k - v^k\|^2 \le \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2.$$

This, together with the fact that the sequence $\{\|x^k - x^*\|\}$ is convergent, yields $\|x^k - v^k\| \to 0$ as $k \to \infty$. Further, by applying (3.4) and (3.2), we can obtain

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &\le \mu_k\|x^k - x^*\|^2 + (1-\mu_k)\|u^k - x^*\|^2 \\
&\le \mu_k\|x^k - x^*\|^2 + (1-\mu_k)\big[\|x^k - x^*\|^2 - \|x^k - u^k\|^2\big] \\
&= \|x^k - x^*\|^2 - (1-\mu_k)\|x^k - u^k\|^2.
\end{aligned}$$

Since $0 < 1 - b \le 1 - \mu_k$, we get

$$0 \le (1-b)\|x^k - u^k\|^2 \le \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2.$$

Again, from the convergence of the sequence $\{\|x^k - x^*\|\}$, we obtain $\|x^k - u^k\| \to 0$ as $k \to \infty$.

Noting that the multivalued mapping $S(\cdot)$ is $*$-nonexpansive, we have

$$\|w^k - v^k\| = \|P_{S(x^k)}(x^k) - P_{S(u^k)}(u^k)\| \le \|x^k - u^k\|.$$

It follows that

$$\|x^k - w^k\| \le \|x^k - v^k\| + \|v^k - w^k\| \le \|x^k - v^k\| + \|x^k - u^k\|.$$

From this, we get $\|w^k - x^k\| \to 0$ as $k \to \infty$. □

Now, we are ready to prove the convergence of Algorithm 3.1.

Theorem 3.1 *Assume that $L_\infty \cap \mathrm{Fix}(S) \neq \emptyset$. Then, the sequence $\{x^k\}$ generated by Algorithm 3.1 converges to some $\bar{x}$ belonging to the set $\mathcal{S}_E \cap \mathrm{Fix}(S)$.*

Proof For each k, let $w^k = P_{S(x^k)}(x^k)$. As $\{x^k\}$ is a sequence of the compact set X, it has a convergent subsequence $\{x^{k_j}\}$ such that $x^{k_j} \to \bar{x} \in X$ as $j \to \infty$. Similarly, since $\{y^{k_j}\}$ is a sequence of the compact set X, it also has a convergent subsequence. Without loss of generality, we may assume that $y^{k_j} \to \bar{y} \in X$ as $j \to \infty$.

To show the conclusion, we divide the proof into three steps.

(I) $\bar{x} \in \mathrm{Fix}(S)$, i.e., $\bar{x}$ is a fixed point of S.

By Proposition 3.2, we know that the sequence $\{\|x^k - w^k\|\}$ converges to 0 as $k \to \infty$. As a consequence, its subsequence $\{\|x^{k_j} - w^{k_j}\|\}$ also converges to 0 as $j \to \infty$. This yields $d(x^{k_j}, S(x^{k_j})) \to 0$ as $j \to \infty$. As $\{x^{k_j}\}$ converges to $\bar{x} \in X$, it also converges weakly to $\bar{x} \in X$. Notice that S is $*$-nonexpansive on X. We can conclude from the Demiclosedness Principle that $\mathrm{Fix}(S)$ is convex and closed, and $I - S$ is demiclosed at 0. Thus $\bar{x} \in S(\bar{x})$, i.e., $\bar{x}$ is a fixed point of S.

(II) $\bar{x} \in \mathcal{S}_E$, i.e., $\bar{x}$ is a solution of (SVEP).

We first show that $f(\bar{y}, \bar{x}) = 0$.

In fact, by Proposition 3.2, we get $\|x^{k_j} - u^{k_j}\| \to 0$ as $j \to \infty$. This, together with $x^{k_j} \to \bar{x}$, implies $u^{k_j} \to \bar{x}$ as $j \to \infty$. In addition, by the definition of $L_f(y^{k_j})$ and the fact $u^{k_j} = P_{L_f(y^{k_j})}(x^{k_j})$, which is the metric projection of x^{k_j} onto $L_f(y^{k_j})$, belongs to $L_f(y^{k_j})$, we can get $f(y^{k_j}, u^{k_j}) \in -C$. As f is C-l.s.c., it is l.s.c.. Thus, $f(\bar{y}, \bar{x}) \in -C$.

On the other hand, observe that $\rho_k = \max\{\|x^0\|, \|x^1\|, \ldots, \|x^k\|\}$ and so $x^k \in X^k$. Then, by **Step 2**, we have

$$0 = f\left(x^{k_j}, x^{k_j}\right) \in f\left(y^{k_j}, x^{k_j}\right) + \lambda_{k_j} e - C.$$

It follows that

$$f\left(y^{k_j}, x^{k_j}\right) + \lambda_{k_j} e \in C.$$

Since f is C-u.s.c. and $\lambda_{k_j} \to 0$, we can conclude from Lemma 2.2(ii) that $f(\bar{y}, \bar{x}) \in C$. Notice that the cone C is pointed. We have $C \cap (-C) = \{0\}$. Therefore $f(\bar{y}, \bar{x}) = 0$.

Next, we show $\bar{x} \in \mathcal{S}_E$.

Indeed, it is known from functional analysis that the set X is bounded as it is compact. Then the sequence $\{x^k\}$ is also bounded as it is contained in X. Moreover, by noting the fact that $\rho_k = \max\{\|x^0\|, \|x^1\|, \ldots, \|x^k\|\}$, we know that the sequence $\{\rho_k\}$ is bounded. Let $\bar{\rho} = \sup\{\rho_k\}$. Take any $\delta \in (0,1)$ and let $B(\delta)$ be the open ball in $\mathcal{H}$ centered at 0 with radius $\bar{\rho} + 1 - \delta$. Then, we have

$$\left\|x^k\right\| \leq \rho_k \leq \bar{\rho} < \bar{\rho} + 1 - \delta.$$

This indicates that $\bar{x}$ belongs to the interior of $B(\delta)$. We claim

$$f(y, \bar{x}) \in -C, \quad \forall y \in X \cap B(\delta), \tag{3.6}$$

which means that $\bar{x}$ is a local solution of (MVEP) on $X \cap B(\delta)$. It follows from Lemma 2.4 that $\bar{x}$ is also a local solution of (SVEP) on $X \cap B(\delta)$. Then, by Theorem 2.2, we know that $\bar{x}$ is further a global solution of (SVEP), i.e., $\bar{x} \in \mathcal{S}_E$.

Hence, it remains to show that (3.6) holds. In fact, by the definition of supremum, we can choose k_0 satisfying $\rho_{k_0} \geq \bar{\rho} - \delta$. Observe that $\{\rho_k\}$ is nondecreasing. We have $\rho_k + 1 \geq \bar{\rho} + 1 - \delta$ for all $k \geq k_0$. It follows that $X \cap B(\delta) \subseteq X^k$ for all $k \geq k_0$. Thus, for each $y \in X \cap B(\delta)$, we have $y \in X^k$ for all $k \geq k_0$. Further, by applying **Step 2**, we can obtain

$$f(y, x^k) \in f(y^k, x^k) + \lambda_k e - C.$$

As f is C-continuous on X, it is both C-u.s.c. and C-l.s.c. on X. Notice that $\lambda_k \to 0$ as $k \to \infty$. Then, it follows from Lemma 2.2(i) that

$$f(y, \bar{x}) \in f(\bar{y}, \bar{x}) - C.$$

As $f(\bar{y}, \bar{x}) = 0$, we get $f(y, \bar{x}) \in -C$. Then, by the arbitrariness of y, we know that (3.6) holds.

(III) The whole sequence $\{x^k\}$ converges to $\bar{x}$ as $k \to \infty$.

Indeed, for each $\delta \in (0,1)$, we can obtain from (3.6) that

$$f(y, \bar{x}) \in -C, \quad \forall y \in X, \ \|y\| < \bar{\rho} + 1 - \delta.$$

Then, by the arbitrariness of δ, we have

$$f(y, \bar{x}) \in -C, \quad \forall y \in X, \|y\| < \bar{\rho} + 1. \tag{3.7}$$

Since $f(\cdot, \bar{x})$ is C-l.s.c., it is also l.s.c.. Then, by (3.7), we can further get

$$f(y, \bar{x}) \in -C, \quad \forall y \in X, \|y\| \leq \bar{\rho} + 1.$$

For each k, by **Step 2** of Algorithm 3.1, we know that $y^k \in X^k$. Hence $y^k \in X$ and $\|y^k\| \leq \rho_k + 1 \leq \bar{\rho} + 1$. It follows that $f(y^k, \bar{x}) \in -C$. This indicates that $\bar{x} \in L_\infty$ and so $\bar{x} \in L_\infty \cap \mathrm{Fix}(S)$. Hence, by Proposition 3.1(iv), the sequence $\{\|x^k - \bar{x}\|\}$ is convergent. This, together with the fact that the subsequence $\{\|x^{k_j} - \bar{x}\|\}$ converges to 0, implies that the whole sequence $\{\|x^k - \bar{x}\|\}$ also converges to 0. It means $\{x^k\}$ converges to $\bar{x}$ as $k \to \infty$. □

Remark 3.2 In [24], Shan and Huang studied the iterative method for finding common solutions of a generalized mixed vector equilibrium problem and fixed point problems of an infinite family of nonexpansive mappings and a variational inequality problem. They suggested a viscosity approximation method and established a convergence result Theorem 3.1 under suitable conditions of continuity and convexity. However, Theorem 3.1 of Shan and Huang [24] is very different from Theorem 3.1 of this paper. More precisely,

(i) in Theorem 3.1 of Shan and Huang [24], the mappings associated with the fixed point problems are all vector-valued, while in Theorem 3.1 of this paper, the mapping associated with the fixed point problem is set-valued;

(ii) the method to generate the approximating sequence $\{x^k\}$ is very different. In fact, the approximating sequence $\{x^k\}$ is produced in Theorem 3.1 of Shan and Huang [24] by using a viscosity approximation method, while it is generated in Theorem 3.1 of this paper by using a projection method;

(iii) the assumptions in Theorem 3.1 of this paper are weaker than those in Theorem 3.1 of Shan and Huang [24]. Indeed, the monotonicity condition imposed on the equilibrium mapping in Theorem 3.1 of Shan and Huang [24] is removed in Theorem 3.1 of this paper. Moreover, the continuity condition of the equilibrium mapping in Theorem 3.1 of Shan and Huang [24] is weakened to that of cone continuity in Theorem 3.1 of this paper.

From Theorem 3.1, we can obtain the following convergence result.

Corollary 3.1 *Assume that $\mathcal{S}_M \cap \mathrm{Fix}(S) \neq \emptyset$. Then, the sequence $\{x^k\}$ generated by Algorithm 3.1 converges to some point $\bar{x}$ belonging to the set $\mathcal{S}_E \cap \mathrm{Fix}(S)$.*

Proof It is clear that $\mathcal{S}_M \subseteq L_\infty$. Then, the nonemptiness of the intersection set $L_\infty \cap \mathrm{Fix}(S)$ can be derived immediately from the assumption. Hence, Theorems 3.1 yields the conclusion. □

If $S : X \to X$ is a vector-valued mapping and nonexpansive, then it is clearly $*$-nonexpansive. Moreover, for each $x \in X$, we have $P_{S(x)}(x) = S(x)$. And so **Step 4** in Algorithm 3.1 is reduced to

Step 4′. Calculate x^{k+1} as

$$x^{k+1} = \mu_k x^k + (1 - \mu_k) S(u^k).$$

Thus, we have the following iterative method for finding a common solution of (SVEP) and a fixed point problem with a nonexpansive vector-valued mapping.

Algorithm 3.2 The iterative steps are the same as those in Algorithm 3.1 except for **Step 4** which is replaced by

Step 4′. Calculate x^{k+1} as

$$x^{k+1} = \mu_k x^k + (1 - \mu_k) S(u^k).$$

Corollary 3.2 *Let $S : X \to X$ be a vector-valued nonexpansive mapping. Let $f : X \times X \to Z$ be a vector-valued bifunction satisfying $f(x,x) = 0$ for all $x \in X$. Assume that the assumptions (A2)–(A4) are satisfied. Let $\{x^k\}$ and $\{y^k\}$ be the sequences generated by Algorithm 3.2. If $L_\infty \cap \mathrm{Fix}(S) \neq \emptyset$, then $\{x^k\}$ converges to some point of $\mathcal{S}_E \cap \mathrm{Fix}(S)$;*

If $S = I$ (the identity mapping on X), then S is a vector-valued mapping, which is trivially nonexpansive and $\mathrm{Fix}(S) = X$. Moreover, for each $x \in X$, we have $P_{S(x)}(x) = x$ and $d(x, S(x)) = 0$. And so **Step 4** in Algorithm 3.1 is reduced to

Step 4″. Calculate x^{k+1} as

$$x^{k+1} = \mu_k x^k + (1 - \mu_k) u^k.$$

So, we have the following iterative method for solving (SVEP).

Algorithm 3.3 The iterative steps are the same as those in Algorithm 3.1 except for **Step 4** which is replaced by

Step 4″. Calculate x^{k+1} as

$$x^{k+1} = \mu_k x^k + (1 - \mu_k)u^k.$$

Corollary 3.3 ([26]) *Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let X be a nonempty compact convex subset of $\mathcal{H}$. Let Z be a real Hausdorff topological vector space and $C \subseteq Z$ a closed convex pointed cone. Let $f : X \times X \to Z$ be a vector-valued bifunction satisfying $f(x,x) = 0$ for all $x \in X$. Suppose that the assumptions* (A2)–(A4) *are satisfied. Let $\{x^k\}$ and $\{y^k\}$ be the sequences generated by Algorithm 3.3.*

(i) *If $L_\infty \neq \emptyset$, then $\{x^k\}$ converges to a solution of (SVEP);*

(ii) *If (SVEP) has no solution, then $\{x^k\}$ diverges.*

Proof (i) The conclusion follows immediately from Corollary 3.2 and the above explanation.

(ii) Assume that the sequence $\{x^k\}$ converges to some point $\bar{x} \in X$. Notice that the condition $L_\infty \neq \emptyset$ (missing in this item) is used in the proof of item (i) only to deduce that $\|x^k - u^k\| \to 0$ as $k \to \infty$. When $\{x^k\}$ converges, we can prove that this fact also occurs. And then, by following the arguments as in the proof of item (i), we can show that $\bar{x}$ is a solution of (SVEP), which contradicts the hypothesis of this item. And so $\{x^k\}$ does not converge.

Hence, there remains to show that, when $\{x^k\}$ converges, $\|x^k - u^k\| \to 0$ as $k \to \infty$. In fact, if $\{x^k\}$ converges, then, by **Step 4″** of Algorithm 3.3 and the assumptions on $\{\mu_k\}$, we have

$$\left\|x^{k+1} - x^k\right\| = \left\|\mu_k x^k + (1 - \mu_k)u^k - x^k\right\| = (1 - \mu_k)\left\|x^k - u^k\right\| \geq (1 - b)\left\|x^k - u^k\right\|.$$

As $(1 - b) > 0$ and $\{x^k\}$ converges, we conclude that $\|x^k - u^k\| \to 0$ as $k \to \infty$. □

In Theorems 3.1, if $\mathcal{H}$ is a finite-dimensional Euclidean space $\mathbb{R}^n$, then the compactness condition imposed on the feasible set X can be weakened by a closedness one.

For this, we need the following lemma.

Lemma 3.2 *Let $\{x^k\}$, $\{y^k\}$, $\{u^k\}$, and $\{v^k\}$ be the sequences generated by Algorithm 3.1. If $L_\infty \cap \operatorname{Fix}(S) \neq \emptyset$, then the four sequences are all bounded.*

Proof Take arbitrary $x^* \in L_\infty \cap \operatorname{Fix}(S)$ and let it be fixed. Then, by Proposition 3.1(iv), we know that the sequence $\{\|x^k - x^*\|\}$ is convergent, and so bounded. As a consequence, the sequence $\{x^k\}$ is bounded. Further, we can conclude the boundedness of the sequences $\{u^k\}$ and $\{v^k\}$ by Proposition 3.1, (ii) and (i), respectively.

On the other hand, since $\{x^k\}$ is bounded, there exists some $r > 0$ such that $\|x^k\| \leq r$ for all $k \in \mathbb{N}_+$. This yields

$$\rho_k = \max\left\{\left\|x^0\right\|, \left\|x^1\right\|, \ldots, \left\|x^k\right\|\right\} \leq r, \quad \forall k \in \mathbb{N}_+.$$

This indicates that all the sets X^k, $k \in \mathbb{N}_+$ are contained in the closed ball centered at 0 with radius $r + 1$. Hence, for each $k \in \mathbb{N}_+$, $\|y^k\| \leq r + 1$ as $y^k \in X^k$. It means that the sequence $\{y^k\}$ is bounded. □

Theorem 3.2 *Let $\mathcal{H}$ be the n-dimensional Euclidean space $\mathbb{R}^n$ and X a nonempty closed convex subset of $\mathcal{H}$. Let $\{x^k\}$ and $\{y^k\}$ be the sequences generated by Algorithm 3.1. If* $\mathrm{Fix}(S) \cap L_\infty \neq \emptyset$, *then $\{x^k\}$ converges to some point $\bar{x}$ belonging to $\mathcal{S}_E \cap \mathrm{Fix}(S)$.*

Proof Notice that the compactness condition of X is used essentially to obtain the compactness of the subset X^k ($k \in \mathbb{N}_+$) in the proof of Lemma 3.1. When $\mathcal{H}$ is the n-dimensional Euclidean space $\mathbb{R}^n$ and X is a nonempty closed convex subset of $\mathcal{H}$, the set X^k ($k \in \mathbb{N}_+$) is clearly compact as it is a nonempty bounded and closed subset of X. And then, by repeating the arguments as in the proof of Lemma 3.1, we can show that Algorithm 3.1 is well-defined.

Next, we shall prove that the conclusion is true.

In fact, since $\{x^k\}$ is a bounded sequence in the n-dimensional Euclidean space $\mathbb{R}^n$, it must have a convergent subsequence $\{x^{k_j}\}$ such that $x^{k_j} \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^n$. As $\{x^{k_j}\} \subseteq X$ and X is closed, we have $\bar{x} \in X$. Similarly, since $\{y^{k_j}\}$ is another bounded sequence of $\mathbb{R}^n$, it also has a convergent subsequence. Without loss of generality, we may assume that $y^{k_j} \to \bar{y} \in X$ as $j \to \infty$. Then, by following the rest of the arguments as in the proof of Theorem 3.1, we can show that $\bar{x} \in \mathcal{S}_E \cap \mathrm{Fix}(S)$ and $x^k \to \bar{x}$ as $k \to \infty$. □

Remark 3.3 In Theorem 3.2, the set X can be unbounded.

The following corollary indicates that, when $Z = \mathbb{R}$ and $C = \mathbb{R}_+ = [0, +\infty)$, the assumption (A4) in Theorem 3.2 can be omitted.

Corollary 3.4 *Let $\mathcal{H}$, X, and S be the same as in Theorem 3.2. Let Z be the real number space $\mathbb{R}$ and $C = [0, \infty)$ be the standard ordering cone of Z. Let $f : X \times X \to Z$ be a bifunction satisfying $f(x, x) = 0$ for all $x \in X$. Suppose that assumptions* (A1)–(A3) *are satisfied. Let $\{x^k\}$ and $\{y^k\}$ be the sequences generated by Algorithm 3.1. If $L_\infty \cap \mathrm{Fix}(S) \neq \emptyset$, then $\{x^k\}$ converges to some point $\bar{x}$ belonging to $\mathcal{S}_0 \cap \mathrm{Fix}(S)$.*

Proof Notice that assumption (A4), together with (A2), is used in Theorem 3.2 to guarantee the validity of **Step 2** in Algorithm 3.1. That is to guarantee the existence of y_k in **Step 2** of Algorithm 3.1. When $Z = \mathbb{R}$, $C = \mathbb{R}_+ = [0, +\infty)$, we can use only assumption (A2) to prove this fact occurs, but without assumption (A4). And then, the conclusion follows immediately from Theorem 3.2.

Indeed, when $Z = \mathbb{R}$, $C = \mathbb{R}_+ = [0, +\infty)$, the C-continuity of a vector-valued mapping reduces to the usual continuity of a real-valued function. Moreover, for each $k \in \mathbb{N}_+$, X^k is clearly compact. Thus, the real-valued function $f(\cdot, x^k)$ can attain its maximum value on X^k, i.e., there exists some point $y^k \in X^k$ such that

$$f(y, x^k) \le f(y^k, x^k) \quad \text{for all } y \in X^k.$$

That is,

$$f(y, x^k) \in f(y^k, x^k) - C \quad \text{for all } y \in X^k.$$

From this, it is easy to see that y^k satisfies **Step 2**. □

Remark 3.4 If $S = I$ (the identity mapping on X) and $e = 1$, then the above Corollary 3.2 reduces to the main result Theorem 3.3 of Iusem and Sosa [8].

Remark 3.5 For finding common solutions of (EP) and (FPP), Van et al. [29] suggested an extragradient-type method and provided an important convergence result Theorem 3.11. However, Theorem 3.11 of Van et al. [29] is different from Corollary 3.4 of this paper in the following aspects:

(i) the method to produce the approximating sequence $\{x^k\}$ is very different. In fact, the approximating sequence $\{x^k\}$ is generated in Theorem 3.11 of Van et al. [29] by using an extragradient-type method, while it is generated in Corollary 3.4 of this paper by using a projection method;

(ii) the assumptions in Corollary 3.4 of this paper are weaker than those in Theorem 3.11 of Van et al. [29]. More precisely, besides all the conditions included in Corollary 3.4 of this paper, the condition that S is lower semicontinuous on X is additionally needed in Theorem 3.11 of Van et al. [29]. Further, the condition $\mathrm{Fix}(S) \cap L_\infty \neq \emptyset$ in Corollary 3.4 of this paper is weaker than $\mathcal{S}_* \neq \emptyset$ in Theorem 3.11 of Van et al. [29], where $\mathcal{S}_* = \{x^* \in S(x^*) : f(y, x^*) \in -C, \forall y \in X\} = \mathcal{S}_M \cap \mathrm{Fix}(S)$.

4 Conclusions

The main purpose of this paper was to investigate iterative methods for finding common solutions of strong vector equilibrium problems and fixed point problems of multivalued mappings. With the help of a Minty vector equilibrium problem, projection iterative methods were suggested and some convergence results were established in Hilbert spaces under suitable conditions of cone continuity and convexity. The main results obtained in this paper generalize and improve the corresponding ones of Van, Strodiot, Nguyen and Vuong [29], Iusem and Sosa [8], Shan and Huang [24], and Huang, Wang, and Mao [26].

Acknowledgements

The authors thank the anonymous referees and editors for their valuable suggestions and comments to improve this work.

Funding

This work was supported by the National Natural Science Foundation of China (11661055, 71971102, 12061045), the Natural Science Foundation of Jiangxi Province (20212BAB201028), and the Science and Technology Research Project of Jiangxi Educational Committee (GJJ210866).

Availability of data and materials

There is no additional data required for the finding of results of this paper.

Declarations

Competing interests

The authors declare no competing interests.

Author contribution

All authors have equal contribution to this article. All authors read and approved the final manuscript.

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Publisher's Note

Received: 24 July 2020 Accepted: 2 September 2022 Published online: 18 October 2022

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