Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Equal-norm Parseval K-frames in Hilbert spaces with a new inequality

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Abstract

The focus of this paper is mainly on the frames of operators or K-frames on Hilbert spaces in Parseval cases. Since equal-norm tight frames play an important role for transmitting robust data, we aim to study this topic on Parseval K-frames. We find that each finite set of equal-norm of vectors can be extended to an equal-norm K-frame. We also find a correspondence between Parseval K-frames and the set of all closed subspaces of a finite Hilbert space. Furthermore, we provide a construction of dual equal-norm K-frames.

MSC: Primary 42C15; secondary 46C99; 41A58

Keywords: Parseval frames; K-frames; K-dual frames

1 Introduction

The concept of frames in Hilbert spaces was first introduced by Duffin and Schaeffer [11] in 1952 to study some profound issues in nonharmonic Fourier series. Frames play such a significant role in both pure and applied mathematics that are considered as a fundamental research area in mathematics, computer science and quantum information. In addition to their former application they are also applied in some other fields, such as signal processing, image processing, data compression and sampling theory.

Frames are redundant sets of vectors in a Hilbert space that yield one natural representation for each vector in the space, but they may also have infinite different representations for a given vector [2–5, 8, 9, 11, 18–20]. Recently, several new applications for (uniform tight) frames have been developed. The first one was developed by Goyal, Kovaĉević, and Vetterli [14–16] that uses the redundancy of a frame to mitigate the effect of losses in packet-based communication systems. Modern communication networks transport packets of data from a source to a recipient. These packets are sequences of information bits of a certain length surrounded by error-control, addressing, and timing information that assure that the packet is delivered without errors. This is accomplished by not delivering the packet if it contains errors. Failures here are due primarily to buffer overflows at intermediate nodes in the network. Hence, to most users, the behavior of a packet network is not characterized by random loss, but by unpredictable transport time This is due to a protocol, invisible to the user, that retransmits lost packets. Retransmission of packets takes much longer than the original transmission in many applications, retransmission of lost packets is not feasible and the potential for a large delay is unacceptable. Another recent important application of uniform normalized tight frames is in multiple-antenna code design [17]. Much theoretical work has been done to show that communication systems that employ multiple antennas can have very high channel capacities [12, 21].

Frames for operators or K-frames have been introduced by Găvruţa in [13] to study the nature of atomic systems for a separable Hilbert space with respect to a bounded linear operator K. In fact, K-frames are more general than the classical frames and due to the higher generality of K-frames, many properties of frames may not hold for K-frames, such as the corresponding synthesis operator for these frames is not surjective and hence, the frame operator is not invertible (see [22]).

In this paper, we will prove some fundamental results for the Parseval K-frames, which have already been proved for tight frames in [6] and we will show that each finite set of K-norm vectors can be extended to become a K-norm of a K-frame.

Throughout this paper, H, H_1, and H_2 are separable Hilbert spaces and $\mathcal{B}(H_1,H_2)$ is the collection of all the bounded linear operators of H_1 into H_2. If $H_1 = H_2 = H$, then $\mathcal{B}(H,H)$ will be denoted by $\mathcal{B}(H)$. The notation $W \prec H$ means that W is a closed subspace of H, also P_W is the orthogonal projection from H onto the closed subspace $W \prec H$ and $\{H_j\}_{j\in\mathbb{J}}$ is a sequence of Hilbert spaces, where $\mathbb{J}$ is a subset of $\mathbb{Z}$. When $\dim H < \infty$, we say that H is the finite Hilbert space.

2 Review of operator theory and *K*-frames

In this part, we aim to review some topics and notations about bounded operators and frames on Hilbert spaces.

Let U be an operator with closed range (or, $\mathcal{R}(U)$ be closed), then there exists a *right-inverse operator* $U^\dagger$ (pseudoinverse of U) in the following senses.

Lemma 2.1 ([7]) *Let $U \in \mathcal{B}(H_1,H_2)$ be a bounded operator with closed range $\mathcal{R}(U)$. Then, there exists a bounded operator $U^\dagger \in \mathcal{B}(H_2,H_1)$ for which*

$$UU^\dagger x = x, \quad x \in \mathcal{R}(U).$$

Lemma 2.2 ([7]) *Let $U \in \mathcal{B}(H_1,H_2)$. Then, the following assertions hold*:

(1) $(U^*)^\dagger = (U^\dagger)^*$.
(2) *The orthogonal projection of H_2 onto $\mathcal{R}(U)$ is given by $UU^\dagger$.*
(3) *The orthogonal projection of H_1 onto $\mathcal{R}(U^\dagger)$ is given by $U^\dagger U$.*
(4) $\ker U^\dagger = \mathcal{R}^\perp(U)$ *and* $\mathcal{R}(U^\dagger) = (\ker U)^\perp$.
(5) *On $\mathcal{R}(U)$ we have $U^\dagger = U^*(UU^*)^{-1}$.*

The operator $U : H \to H$ is called a *unitary operator* if $U^* = U^{-1}$. In this case, it is obvious that $\|U\| = 1$. The following lemma characterizes all orthonormal bases by unitary operators.

Lemma 2.3 ([7]) *Let $\{e_j\}_{j\in\mathbb{J}}$ be an orthonormal basis for H. Then, the orthonormal bases for H are precisely the sets $\{Ue_j\}_{j\in\mathbb{J}}$, where $U : H \to H$ is unitary.*

Suppose that W, V are closed subspaces of H with

$$\dim W = \dim V = m,$$

then, we can check that there is a unitary operator U on H so that $U|_W = P_V$. Indeed, if we choose $Ue_j = e'_j$, where $\{e_j\}_{j=1}^m$ and $\{e'_j\}_{j=1}^m$ are orthonormal bases for W and V, respectively, then U is unitary.

Assume that T and S are operators on H. The operator S is called *unitarily equivalent* to T if there is a unitary operator U on H such that $S = UTU^*$.

Lemma 2.4 ([10]) *Let $L_1 \in \mathcal{B}(H_1, H)$ and $L_2 \in \mathcal{B}(H_2, H)$ be on given Hilbert spaces. Then, the following assertions are equivalent*:

(1) $\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$;

(2) $L_1L_1^* \leq \lambda^2 L_2L_2^*$ *for some* $\lambda > 0$;

(3) *there exists a mapping* $U \in \mathcal{B}(H_1, H_2)$ *such that* $L_1 = L_2U$.

Moreover, if those conditions are valid, then there exists a unique operator U such that

(a) $\|U\|^2 = \inf\{\alpha > 0 \mid L_1L_1^* \leq \alpha L_2L_2^*\}$;

(b) $\ker(L_1) = \ker(U)$;

(c) $\mathcal{R}(U) \subseteq \overline{\mathcal{R}(L_2^*)}$.

Definition 2.1 (frame) Let $\{f_j\}_{j\in\mathbb{J}}$ be a sequence of members of H. We say that $\{f_j\}_{j\in\mathbb{J}}$ is a frame for H if there exist $0 < A \leq B < \infty$ such that for each $f \in H$,

$$A\|f\|^2 \leq \sum_{j\in\mathbb{J}} \left|\langle f, f_j\rangle\right|^2 \leq B\|f\|^2. \tag{1}$$

The constants A and B are called frame bounds. If the right-hand side of (1) holds, we say that $\{f_j\}_{j\in\mathbb{J}}$ is a Bessel sequence with bound B. We say that $\{f_j\}_{j\in\mathbb{J}}$ is an A-tight frame for H, if we have

$$\sum_{j\in\mathbb{J}} \left|\langle f, f_j\rangle\right|^2 = A\|f\|^2,$$

for each $f \in H$. If $A = 1$, the set $\{f_j\}_{j\in\mathbb{J}}$ is called a *Parseval frame*.

It is easy to check that if $\{f_j\}_{j\in\mathbb{J}}$ is a frame for H with bounds A and B, then by (1), the set $\{Pf_j\}_{j\in\mathbb{J}}$ is a frame for PH with the same bounds, where P is an orthonormal projection on H. We say that two frames $\{f_j\}_{j\in\mathbb{J}}$ and $\{g_j\}_{j\in\mathbb{J}}$ are *unitarily equivalent* if there is a unitary operator U on H such that $g_j = Uf_j$. In this case, we treat both the frames as the same.

Let $\{f_j\}_{j\in\mathbb{J}}$ be a Bessel sequence, then the *synthesis* and the *analysis* operators are defined by

$$T : \ell^2(\mathbb{N}) \to H, \qquad T\{c_j\}_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}} c_j f_j, \tag{2}$$

and

$$T^* : H \to \ell^2(\mathbb{N}), \qquad T^* f = \left\{\langle f, f_j\rangle\right\}_{j\in\mathbb{J}}. \tag{3}$$

Now, the frame operator is defined by $S = TT^*$ and this is an invertible and positive operator on H and also (see [7])

$$Sf = \sum_{j\in\mathbb{J}} \langle f, f_j\rangle f_j,$$

and

$$\langle Sf,f\rangle=\sum_{j\in\mathbb{J}}\bigl|\langle f,f_j\rangle\bigr|^2,$$

for each $f\in H$. The following result characterizes all Parseval frames as a projection of an orthonormal basis from a larger space.

Lemma 2.5 ([6]) *A set $\{f_j\}_{j\in\mathbb{J}}$ in H is a Parseval frame if and only if there are a larger Hilbert space $M\supseteq H$ and an orthonormal basis $\{e_j\}_{j\in\mathbb{J}}$ for M so that the orthonormal projection P of M onto H satisfies $f_j=Pe_j$ for each $j\in\mathbb{J}$.*

In the next part, we review notations of K-frames from [1, 13, 22].

Definition 2.2 (K-frame) [13] Let $\{f_j\}_{j\in\mathbb{J}}$ be a sequence of members of H and $K\in\mathcal{B}(H)$. We say that $\{f_j\}_{j\in\mathbb{J}}$ is a K-frame for H if there exist $0<A\leq B<\infty$ such that for each $f\in H$,

$$A\bigl\|K^*f\bigr\|^2\leq\sum_{j\in\mathbb{J}}\bigl|\langle f,f_j\rangle\bigr|^2\leq B\|f\|^2. \tag{4}$$

We say that $\{f_j\}_{j\in\mathbb{J}}$ is an *equal-norm-K-frame* for H, when $\{f_j\}_{j\in\mathbb{J}}$ is a K-frame and $\|f_j\|=c$ for every $j\in\mathbb{J}$. When $c=\|K\|$, we call $\{f_j\}_{j\in\mathbb{J}}$ a *K-norm-frame*. We say that $\{f_j\}_{j\in\mathbb{J}}$ is a tight K-frame for H, if we have

$$\sum_{j\in\mathbb{J}}\bigl|\langle f,f_j\rangle\bigr|^2=A\bigl\|K^*f\bigr\|^2,$$

for each $f\in H$. If $A=1$, the set $\{f_j\}_{j\in\mathbb{J}}$ is called a Parseval K-frame.

Since every K-frame is a Bessel sequence, then the synthesis and analysis operators are defined by (2) and (3). However, the frame operator $S=TT^*$ is not invertible on H. Also, $S:\mathcal{R}(K)\to S(\mathcal{R}(K))$ is invertible when the operator K has closed range ([22]). Hence, we can construct a Parseval frame for $\mathcal{R}(K)$ by the following result.

Proposition 2.1 *Let $\{f_j\}_{j\in\mathbb{J}}$ be a K-frame for H and K has closed range. If $S(\mathcal{R}(K))=\mathcal{R}(K)$, then*

(I) *$\{S^{\frac{1}{2}}P_{\mathcal{R}(K)}f_j\}_{j\in\mathbb{J}}$ is a Parseval frame for $\mathcal{R}(K)$.*
(II) *$\{S^{\frac{1}{2}}P_{\mathcal{R}(K)}f_j\}_{j\in\mathbb{J}}$ is a Parseval $P_{\mathcal{R}(K)}$-frame for H.*

Proof (I). Since the operator S is invertible and positive on $\mathcal{R}(K)$, $S^{-1}|_{S(\mathcal{R}(K))}$ has a unique positive root as $S^{-\frac{1}{2}}$ and this operator commutes with S^{-1} and so with S. Now, for each $f\in\mathcal{R}(K)$ we have

$$\begin{aligned}f&=S^{-1}S|_{\mathcal{R}(K)}f\\&=S^{-\frac{1}{2}}S|_{\mathcal{R}(K)}S^{-\frac{1}{2}}f\\&=\sum_{j\in\mathbb{J}}\bigl\langle S^{-\frac{1}{2}}f,P_{\mathcal{R}(K)}f_j\bigr\rangle S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_j\\&=\sum_{j\in\mathbb{J}}\bigl\langle f,S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_j\bigr\rangle S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_j.\end{aligned}$$

Therefore,

$$\|f\|^2 = \langle f,f\rangle = \sum_{j\in\mathbb{J}} \bigl|\bigl\langle f, S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_j\bigr\rangle\bigr|^2.$$

This completes the proof.

(II). Take $f \in H$, hence we can write $f = f_1 + f_2$, where $f_1 \in \mathcal{R}(K)$ and $f_2 \in \mathcal{R}(K)^{\perp}$. Then,

$$\bigl\langle S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_i, f_2\bigr\rangle = 0.$$

Thus, we compute that

$$\begin{aligned}\sum_{j\in\mathbb{J}} \bigl|\bigl\langle f, S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_j\bigr\rangle\bigr|^2 &= \sum_{j\in\mathbb{J}} \bigl|\bigl\langle f_1, S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_j\bigr\rangle + \bigl\langle f_2, S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_j\bigr\rangle\bigr|^2 \\ &= \sum_{j\in\mathbb{J}} \bigl|\bigl\langle f_1, S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_j\bigr\rangle\bigr|^2 + \sum_{j\in\mathbb{J}} \bigl|\bigl\langle f_2, S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_j\bigr\rangle\bigr|^2 \\ &\quad + 2\sum_{j\in\mathbb{J}} \operatorname{Re}\bigl\langle f_1, S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_j\bigr\rangle\bigl\langle S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_j, f_2\bigr\rangle \\ &= \sum_{j\in\mathbb{J}} \bigl|\bigl\langle f_1, S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_j\bigr\rangle\bigr|^2.\end{aligned}$$

Now, by item (I), we conclude that

$$\sum_{j\in\mathbb{J}} \bigl|\bigl\langle f, S^{-\frac{1}{2}}P_{\mathcal{R}(K)}f_j\bigr\rangle\bigr|^2 = \|P_{\mathcal{R}(K)}f\|^2.$$

□

Remark 2.1 Assume that $F = \{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame for H and K has closed range. It is obvious that $\ker KK^* = \ker K^*$, therefore we obtain $\mathcal{R}(KK^*) = \mathcal{R}(K)$. If S is the frame operator of F, then $\langle Sf,f\rangle = \|K^*f\|^2$. Hence, $S = KK^*$ and we conclude that $\mathcal{R}(S) = \mathcal{R}(K)$.

The following theorem gives a characterization of K-frames using linear bounded operators.

Lemma 2.6 ([13]) *Let $\{f_j\}_{j\in\mathbb{J}}$ be members of H. Then, $\{f_j\}_{j\in\mathbb{J}}$ is a K-frame if and only if there exists a linear bounded operator $T : \ell^2(\mathbb{J}) \to H$ such that $f_j = T\delta_j$ and $\mathcal{R}(K) \subseteq \mathcal{R}(T)$, where $\{\delta_j\}_{j\in\mathbb{J}}$ is an orthonormal basis for $\ell^2(\mathbb{J})$.*

In the proof of Lemma 2.6, we can check that the operator T is the synthesis operator (see Theorem 4 in [13]).

Proposition 2.2 *Let $\{f_j\}_{j\in\mathbb{J}}$ be a Parseval K-frame for H and $\{\lambda_k\}_{k=1}^n$ be the eigenvalues for the frame operator S, where $n = \dim H$. Then, $\lambda_k > 0$ for each $k = 1, 2, \dots, n$ and*

$$n\bigl\|K^*e_k\bigr\|^2 = \sum_{j\in\mathbb{J}} \|f_j\|^2,$$

where $\{e_k\}_{k=1}^n$ is an orthonormal basis for H.

Proof First, we note that the equality:

$$\sum_{j=1}^{n} \lambda_j = \sum_{i\in\mathbb{J}} \|f_i\|^2 \tag{5}$$

holds for K-frames similar to the proof of Theorem 1.1.12 in [7] for ordinary frames. Suppose that $\{\lambda_k\}_{k=1}^{n}$ are the eigenvalues of S and $Sf = \lambda_k f$ for some $0 \neq f \in H$. Since S is self-adjoint, then $\lambda_k \in \mathbb{R}$ for every $k = 1, 2, \ldots, n$ and also there is an orthonormal basis $\{e_k\}_{k=1}^{n}$ for H such that each e_k is an eigenvalue vector for S. Since $S = KK^*$, therefore $\|K^*f\|^2 = \lambda_k\|f\|^2$ and we conclude that $\lambda_k > 0$ for each $k = 1, 2, \ldots, n$ and also by (5), we can obtain

$$\frac{n\|K^*f\|^2}{\|f\|^2} = \sum_{j\in\mathbb{J}} \|f_j\|^2.$$

This completes the proof. □

Now, if $\{f_j\}_{j=1}^{m}$ is an equal-norm-Parseval K-frame for H, then by Proposition 2.2, for each $k = 1, \ldots, n$, we can obtain

$$\|f_j\|^2 = \frac{n}{m}\left\|K^*e_k\right\|^2, \quad (j = 1, \ldots, m).$$

Definition 2.3 ([1]) Let $F := \{f_j\}_{j\in\mathbb{J}}$ be a K-frame and $G := \{g_j\}_{j\in\mathbb{J}}$ be a Bessel sequence for H with synthesis operators T and Θ, respectively. We say that G is a K-dual for F if

$$T\Theta^* = K. \tag{6}$$

In this case, from (6) we can write for each $f \in H$,

$$Kf = \sum_{j\in\mathbb{J}} \langle f, g_j\rangle f_j, \tag{7}$$

and it is easy to see that G is a K^*-frame for H. In the following, we can begin to characterize all K-dual of a K-frame.

Lemma 2.7 ([1]) *Let $F := \{f_j\}_{j\in\mathbb{J}}$ be a K-frame. Then, $\{g_j\}_{j\in\mathbb{J}}$ is a K-dual of F if and only if $\{g_j\}_{j\in\mathbb{J}} = \{V\delta_j\}_{j\in\mathbb{J}}$, where $\{\delta_j\}_{j\in\mathbb{J}}$ is the standard orthonormal basis of ℓ^2 and $V : \ell^2 \to H$ is a bounded operator such that $TV^* = K$. In this case, $\{g_j\}_{j\in\mathbb{J}}$ is in fact a Parseval V-frame.*

3 Main results

The following result that is a general case of Theorem 3.2 in [6], shows that each finite set of K-norm vectors can be extended to become a K-norm of a K-frame.

Theorem 3.1 *If $F := \{f_j\}_{j=1}^{m}$ is a set of K-norm vectors in H, then there is a K-norm-frame for H that contains the set F.*

Proof Let $1 \leq j \leq m$ and $\{e_{ij}\}_{i\in\mathbb{I}}$ be an orthonormal basis for H that contains the vector u_j, where $u_j := \frac{f_j}{\|K\|}$. For any $f \in H$ we have

$$m\|K^*f\|^2 \leq m\|K\|^2\|f\|^2 = \sum_{j=1}^{m}\sum_{i\in\mathbb{I}}\left|\langle f, \|K\|e_{ij}\rangle\right|^2.$$

Therefore, $\{\|K\|e_{ij}\}_{j=1,i\in\mathbb{I}}^{m}$ is a K-frame for H with bounds m and $m\|K\|^2$ and is made up of K-norm vectors. □

Corollary 3.1 *If K has closed range, $F := \{f_j\}_{j=1}^m$ is a set of K-norm vectors in H and $\|K\|\|K^\dagger\| = 1$, then there is a K-norm-tight frame for $\mathcal{R}(K)$ that contains the set F.*

Proof Chose $f \in \mathcal{R}(K)$, such that

$$\|K\|\|f\| = \|K\|\left\|(K^\dagger)^*K^*f\right\| \leq \left\|K^*f\right\|,$$

hence we obtain $\|K^*f\| = \|K\|\|f\|$. Now, by Theorem 3.1, if $\{e_{ij}\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{R}(K)$, then $\{\|K\|e_{ij}\}_{j=1,i\in\mathbb{I}}^m$ is a K-norm-tight frame for $\mathcal{R}(K)$. □

The next result is the same as Lemma 2.5 for Parseval K-frames.

Theorem 3.2 *Let $\{f_j\}_{j\in\mathbb{J}}$ be a Parseval K-frame for H and K be closed range. Then, there is a larger Hilbert space $M \supseteq \mathcal{R}(K^\dagger)$ and an orthonormal basis $\{e_j\}_{j\in\mathbb{J}}$ for M so that the orthonormal projection P of M onto $\mathcal{R}(K^\dagger)$ satisfies $f_j = KPe_j$ for each $j \in \mathbb{J}$.*

Proof Assume that $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame for H. Via Lemma 2.6, there exists an orthonormal basis $\{e'_j\}_{j\in\mathbb{J}}$ in which $f_j = Te'_j$ for all $j \in \mathbb{J}$. Since $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame, we have $KK^* = TT^*$ and by Lemma 2.4 we can obtain $\mathcal{R}(T) = \mathcal{R}(K)$. Therefore, $f_j \in \mathcal{R}(K)$. Suppose that $f \in \mathcal{R}(K^\dagger)$, by Lemma 2.2 we have

$$\sum_{j\in\mathbb{J}}\left|\langle f, K^\dagger f_j\rangle\right|^2 = \sum_{j\in\mathbb{J}}\left|\langle (K^\dagger)^*f, f_j\rangle\right|^2 = \left\|K^*(K^\dagger)^*f\right\|^2 = \|f\|^2.$$

Hence, $\{K^\dagger f_j\}_{j\in\mathbb{J}}$ is a Parseval frame for $\mathcal{R}(K^\dagger)$. Hence, by Lemma 2.5, there exists a larger Hilbert space $M \supseteq \mathcal{R}(K^\dagger)$ and an orthonormal basis $\{e_j\}_{j\in\mathbb{J}}$ for M so that $K^\dagger f_j = Pe_j$ for each $j \in \mathbb{J}$ or $f_j = KPe_j$. □

The next result is a general case of Theorem 3.4 in [6] for K-frames. In the following, we let $\dim H = n$ with an orthonormal basis $\{e_j\}_{j=1}^n$.

Theorem 3.3 *Let K be closed range, P be an orthonormal projection on H onto $\mathcal{R}(K^\dagger)$ such that KP be a rank-m. Define*

$$\mathcal{K} = \{\textit{all Parseval K-frames for KPH}\}$$

$$\mathcal{M} = \{W \prec H, \dim W = m\}.$$

Then, there exists a natural one-to-one correspondence between $\mathcal{K}$ and $\mathcal{M}$.

Proof For each $F := \{f_j\}_{j=1}^n \in \mathcal{K}$, by Theorem 3.2, there is an orthonormal basis $\{e'_j\}_{j=1}^n$ such that $f_j = KPe'_j$ for any $1 \leq j \leq n$. Define a unitary operator U_F on H by $U_F e_j = e'_j$. Hence, we have $f_j = KPU_F e_j$, which is unitarily equivalent to $f'_j := U_F^* KPU e_j$. Define

$$\Phi : \mathcal{K} \longrightarrow \mathcal{M},$$

$$\Phi\{f_j\}_{j=1}^n = U_F^* KPU_F H.$$

It is clear that the operator Φ is well defined. Assume that $W \in \mathcal{M}$, thus there exists a unitary operator U on H such that $UW = KPH$. Hence, we obtain $U^*KPUH = W$, while $\{KPe_j\}_{j=1}^n \in \mathcal{K}$, which corresponds to W. This means that Φ is surjective. Finally, suppose that $G := \{g_j\}_{j=1}^n \in \mathcal{K}$ and V_G is a unitary operator on H such that $U_F^* KPU_F H = V_G^* KPV_G H$ which gives

$$V_G e_j = e''_j, \qquad g_j = KPe''_j = KPV_G e_j.$$

Since $U_F^* KPU_F e_j = V_G^* KPV_G e_j$ for each $1 \leq j \leq n$, we have $U_F^* f_j = V_G^* g_j$. Hence, f_j and g_j are unitarily equivalent for each j. Hence, F and G are two identical K-Parseval frames, then Φ is injective. □

Theorem 3.4 *Let $\{f_j\}_{j\in\mathbb{J}}$ be a Parseval K-frame for H with the synthesis operator T and $\{g_j\}_{j\in\mathbb{J}}$ be a Parseval K^*-frame for H with the synthesis operator Θ. If $\{g_j\}_{j\in\mathbb{J}}$ is a K-dual for $\{f_j\}_{j\in\mathbb{J}}$, then we have*

$$\|(T^* - \Theta^* K^*)f\|^2 = \|KK^* f\|^2 - \|K^* f\|^2, \quad (\forall f \in H).$$

Moreover, if K has closed range, then we can obtain the following inequality:

$$\|(KK^*)^{-1}\|^{-2}(1 - \|K^\dagger\|^2) \leq \|T - K\Theta\|^2 \leq \|K\|^2(\|K\|^2 - 1).$$

Proof Since $T\Theta^* = K$ and $TT^* = KK^*$, we obtain $T(T^* - \Theta^* K^*) = 0$. Therefore, for every $f, g \in H$ we have

$$\langle T^* g, (T^* - \Theta^* K^*)f\rangle = \langle g, T(T^* - \Theta^* K^*)f\rangle = 0.$$

Hence, we compute that

$$\begin{aligned} \|\Theta^* K^* f\|^2 &= \|T^* f - T^* f + \Theta^* K^* f\|^2 \\ &= \|T^* f - (T^* - \Theta^* K^*)f\|^2 \\ &= \|T^* f\|^2 + \|(T^* - \Theta^* K^*)f\|^2 \\ &\quad - \langle T^* f, (T^* - \Theta^* K^*)f\rangle - \overline{\langle T^* f, (T^* - \Theta^* K^*)f\rangle} \\ &= \|T^* f\|^2 + \|(T^* - \Theta^* K^*)f\|^2. \end{aligned}$$

However, via the hypothesis, we have $\|T^* f\|^2 = \|K^* f\|^2$ and $\|\Theta^* f\|^2 = \|Kf\|^2$. For the second part, the right inequality is evident. Since K has closed range, it is easy to check that the operator $KK^*:\mathcal{R}(K)\to\mathcal{R}(K)$ is invertible. Hence, for any $0\neq f\in\mathcal{R}(K)$ we have $\|f\|^2\leq\|(KK^*)^{-1}\|^2\|KK^*f\|^2$. On the other hand, via Lemma 2.2, we obtain

$$\|K^*f\|^2=\|K^\dagger KK^*f\|^2\leq\|K^\dagger\|^2\|KK^*f\|^2.$$

Hence, we conclude that

$$\|T-K\Theta\|^2=\|T^*-\Theta^*K^*\|^2\geq\frac{\|(T^*-\Theta^*K^*)f\|^2}{\|f\|^2}\geq\|(KK^*)^{-1}\|^{-2}(1-\|K^\dagger\|^2).$$

This completes the proof. □

In the next result, which is a general case of Proposition 3.3 in [6], we define $K^\natural$ as a left inverse of K and $\{\delta_j\}_{j=1}^m$ is the orthonormal basis for ℓ^2.

Theorem 3.5 *Let $F=\{f_j\}_{j=1}^m$ be an equal-norm-Parseval K-frame for the finite Hilbert space H with the synthesis operator T such that $T^*H\perp U^*H$ and $K^\natural f_j\perp U\delta_j$ for each $1\leq j\leq m$, where $U\in\mathcal{B}(\ell^2,H)$ and $K^\natural|_F$ are two isometries. Then, $\{f_j\}_{j=1}^m$ has infinitely many dual equal-norm K-frames.*

Proof Choose $V=aU^*$, where $a\neq 0$. Since $(K^\natural T+U)T^*=K^*$, via Lemma 2.7, if $g_j=(K^\natural T+V^*)\delta_j$ for $1\leq j\leq m$, then $\{g_j\}_{j=1}^m$ is a dual K-frame of $\{f_j\}_{j=1}^m$.

Assume that $P:\ell^2\to T^*H$ is an orthonormal projection so that $P\delta_j=T^*f_j$ for any $1\leq j\leq m$ and $\{e_k\}_{k=1}^n$ is an orthonormal basis for H where $n=\dim H$. Since $T\delta_j=f_j$, therefore for each $1\leq j\leq m$,

$$g_j=K^\natural f_j+V^*(I-P)\delta_j.$$

Fix $k=1,\dots,n$. It is easy to check that $\langle K^\natural f_j,V(I-P)\delta_j\rangle=0$ and also via Proposition 2.2, we can obtain

$$\|f_j\|^2=\frac{n}{m}\|K^*e_k\|^2,$$

and

$$\|P\delta_j\|^2=\|T^*f_j\|^2=\|K^*f_j\|^2=\frac{n}{m}\|K^*e_k\|^2\lambda_k,$$

where λ_k is the eigenvalue of the frame operator S_F. Since

$$\langle\delta_j,P\delta_j\rangle=\langle\delta_j,P^2\delta_j\rangle=\langle P\delta_j,P\delta_j\rangle=\|P\delta_j\|^2,$$

and also

$$\|(I-P)\delta_j\|^2=\langle\delta_j-P\delta_j,\delta_j-P\delta_j\rangle=1-\|P\delta_j\|^2,$$

thus, for each $j=1,\dots,m$, we conclude that

$$\|g_j\|^2=\langle K^\natural f_j+V(I-P)\delta_j,K^\natural f_j+V(I-P)\delta_j\rangle$$

$$
\begin{aligned}
&= \left\| K^{\natural} f_j \right\|^2 + \left\| V(I-P)\delta_j \right\|^2 \\
&= \|f_j\|^2 + a^2 \left\| (I-P)\delta_j \right\|^2 \\
&= \frac{n}{m} \left\| K^* e_k \right\|^2 + a^2 \left(1 - \frac{n}{m} \left\| K^* e_k \right\|^2 \lambda_k \right) \\
&= a^2 + \frac{n}{m} \left(1 - a^2 \lambda_k \right) \left\| K^* e_k \right\|^2. \qquad \square
\end{aligned}
$$

Acknowledgements
We gratefully thank the referees for carefully reading the paper and for the suggestions that greatly improved the presentation of the paper. The authors already appreciate the collaboration of the editorial board.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author contribution
The two authors contributed equally to the writing of this paper. They read and approved the final version of the paper.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 5 August 2021 Accepted: 2 September 2022 Published online: 24 September 2022

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